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AAMT—supporting and enhancing the work of teachers

Facilitating the Development of Proportional Reasoning through Teaching Ratio

Linda Parish
Australian Catholic University
<linda.parish@acu.edu.au>

If the ability to reason proportionally seems to be a good indication of likely success in further mathematical pursuits (Lamon, 1999), how do children develop this ability, and how can teachers facilitate this? In this present study, six ratio/rates task-based assessment questions were trialled on ten students from Grades 5 to 9 in an attempt to describe the developing understanding of students within this construct of rational number. Tentative points of growth (or stages of understanding) are suggested, with some implications for the classroom teacher.

This study was part of a Master of Education project to design an assessment interview to identify the points of growth, or stages of development, in children's thinking about ratio problems. Ratio is a sub-construct of rational number (Kieren, 1976) that requires proportional reasoning. The purpose of 'growth points' is to describe the developmental pathway children typically follow, and to inform teachers of where children have reached developmentally so that further learning can be targeted to their zone of proximal development. Two questions are explored in this paper: how do children typically develop ratio understandings, and can teachers use this knowledge in order to facilitate the development of proportional reasoning?

Theoretical Background

Proportional reasoning has been called the backbone, the cornerstone, the gateway to higher levels of mathematics success, and is considered as a "capstone" of primary school mathematics (Kilpatrick, Swafford & Findell, 2001; Lamon, 1999; Lesh, Post & Behr, 1988). Proportional reasoning involves "making multiplicative comparisons between quantities" (Wright, 2005, p. 363), together with "the ability to mentally store and process several pieces of information" (Lesh, et al, 1988, p. 93). An example of such a problem is, if three lollies cost ten cents, how much will twelve cost? According to Lamon (1999), "proportional reasoning is one of the best indicators that a student has attained understanding of rational numbers" (p. 3).

Kieran, 1976, cited in Clarke, Sukenik, Roche, and Mitchell, (2006), Lamon, (1999), and Wright, (2005) identified five sub-constructs of rational number – fractions as part-whole comparisons, fractions as measure, fractions as an operator, fractions as quotients, and fractions as ratio, or part-part comparisons. There are functional differences in each of these sub-constructs, but they are inter-related and it is believed that, if fractions are taught with a holistic approach, they can provide many contexts and representations that promote higher order thinking and develop proportional reasoning (Lesh et al, 1988).

This paper considers the sub-construct of ratio and rates. Ratio is a part-part comparison. For example, '3 lollies for 10 cents' describes a ratio between an amount of money and the amount of confectionery that can be bought with that amount of money. A ratio becomes a rate when it implies a constant, indicated by per. For example, 80km per hour describes a ratio between a measure of distance and a measure of time where for every one hour, 80 kilometres is travelled, implying that in three hours 240 kilometres will

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have been travelled. Rates can also be varying, such as monetary exchange rates, or the rate of acceleration, but these types of rates are not considered here.

The Assessment Interview

Within the sub-construct of ratio, Lamon (1993) identified four types of problems that are semantically distinct. The questions included in the interview were chosen, and/or designed, based on these four semantic types of ratio problems (see Figure 1). Ten students of mixed ability were interviewed, five from Grade 5, four from Grade 6, and one Year 9 student.

Lamon's Ratio Semantic Types	***Summary of Assessment Interview Questions***
1. Part-Part-Whole	**Questions 1 & 2:**
where the 'whole' is described in terms of two or more 'parts' of which the whole is composed.	Green Paint = 1 blue: 3 yellow. Christmas M & Ms – there are half as many greens as reds (adapted from Witherspoon, 2002)
2. Associated Sets	**Questions 3 & 4:**
where the relationship between two elements is only defined within the problem situation itself.	Ghost Drops – 3 for 10c – confectionary and money. Oranges and Lemons – combined to make a punch 2:3 vs. 3:5 – which is more orangey? (Lamon, 1999, p. 181)
3. Well-Chunked Measures	**Question 5:**
where two measures are compared to give a third, inclusive measure.	Adelaide Trip – kilometres per hour = speed. 160km in 2 hours; 360km in 4 hours – how long will it take to get to Adelaide (600km)?; at what speed am I travelling?
4. Stretchers and Shrinkers	**Question 6:**
problems where the ratio between two measures is preserved, or fixed, when a figure is enlarged (stretched) or scaled down (shrunk).	Two rectangles – 6 x 8 → ? x 12 – what is the height of the enlarged rectangle? (Lamon, 1993, p. 44)

Figure 1, Semantic Types and Related Problems

Observations from Question 1 – Part-Part-Whole

Green Paint = 1 blue: 3 yellow. "If I added two more blues, how much more yellow would I need?" established the students' concept of "homogeneity" (Lo & Watanabe, 1997, p. 219), a recognition that a relationship exists that needs to be preserved – for every one blue, three yellows are required – which is a necessary and important element of proportional thinking (Lo & Watanabe, 1997). Half the students interviewed showed this implicit understanding. The others, except for one, applied an additive strategy – if you add two to the blue, you will need to also add two to the yellow – and one appeared to simply guess, "you'd add two, maybe three" yellow.

The second part of this question, "How much yellow and blue paint do I need to make 28 litres of green paint?" measured students' ability to reverse the process of 'building-up', to a 'breaking-down', by recognising the ratio as a unit in itself. Of the five who correctly answered the first question, four recognised the ratio-unit of four litres, "there are 4 litres altogether and 4 x 7 = 28, so there'd be 7 blue and therefore 3 x 7 (21) yellow," with one describing it as "28 divided into quarters; ¼ is blue, ¾ is yellow." One employed what Lo and Watanabe (1997), in their research on developing ratio and proportion schemes, describe as a *ratio-unit/build-up strategy*: recognising 1:3 as a ratio-unit and 'building up' from that – 1:3 → 2:6 → 3:9 →… (see Figure 4). One girl was stumped with this question, not recognising the unit of four, "I counted the yellows by 3, and counted how many times I counted the 3, but then I got lost." Interestingly, one Grade 6 boy who had used an additive strategy in the first part of this question, recognised the ratio-unit and calculated correctly explaining, "I thought 4 x what = 28?, and then I did ... 7 x 3 = 21."

The final part of this question (how much yellow and blue paint would I need to make 10 litres?) enabled an assessment of a higher order thinking strategy, using a non-integer scalar proportion. Only Nathan (Year 9) got this question correct, although Michael (Grade 5), after much consideration, decided he could use the same process as before – divide 10 into quarters (which is 2½), but then incorrectly calculated 2½ x 3 as 6½. Two girls, who had been correct up to this stage, both said, "It can't be done, you'd have to have 8 litres or 12 litres." The boy who had been correct calculating the 1:3 = 7:21 ratio but had previously used an additive strategy, applied a mixture of both for this question – (starting with the initial 1:3) "I doubled it, which was 2:6, then I did 3 blues, which gave me 9 litres (3:6), then I added ½ to each (3½:6½) to make 10 litres"

Observations from Question 2 – Part-Part-Whole

Christmas M&Ms. "I have nine red and green M&Ms, there are half as many greens as reds, how many reds and how many greens do I have?" (adapted from Witherspoon, 2002) assessed students' ability to interpret and understand the common *language* of ratio, 'half as many', 'three times as many', 'half as many again'. Six students answered this question quickly and correctly, with one using an algebraic-type explanation, "If I can find the amount of red and then halve that ..., because red + ½ red = 9." One Grade 5 girl was correct with the 6 and 3, but was confused with the term 'half as many', deciding it was functionally the same as 'twice as many'. Three others simply halved the 9.

Observations from Question 3 – Associated Sets

Ghost Drops cost 3 for 10c: how much would 15 cost; how many could I buy for 80c? Associated Set problems were identified by Lamon (1993) as eliciting more relative thinking in more students than the other semantic types. She concluded that this was because of the highly pictorial and/or manipulative nature of these tasks. It was certainly true that students thought more relatively with the Ghost Drops problem, all students except for two solved it with relative thinking, although none of them used pictures or tally marks to solve it. They used either Lo and Watanabe's (1997) *ratio-unit/build-up strategy* (3:10 → 6:20 → 9:30 → etc.) (three students), or scalar reasoning (3:10 = 15: ?, and 3:10 = ? :80) (five students). One employed a functional reasoning strategy (i.e., finding a unit value for one Ghost Drop – if 3 = 10c, then 1 = 10 ÷ 3c), but then ignored the 'extra bit' (0.33 cents). Tanya (Grade 5), who for everything else either used visual judgement or 'just guessed', solved the Ghost Drop problem with a form of patterning – listing 3s, skip counting until she reached 15 then counted down her list in 10s to 50 for the first part of the question, then continued to count to 80 in 10s and wrote down corresponding 3s and added them up to solve the second part of the question (see Figure 2).

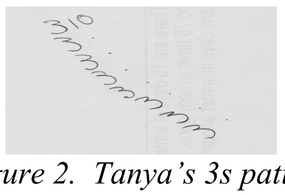


Figure 2. Tanya's 3s pattern.

One student did not approach the Ghost Drop problem 'logically', ignoring some of the given values (15 Ghost Drops = 15 x 10 = $1.50; and 80c would buy 80 ÷ 3 = 26 Ghost Drops with 2c left over).

Observations from Question 4 – Associated Sets

Oranges and Lemons – 2 orange: 3 lemon & 3 orange: 5 lemon, which is more 'orangey'? The Oranges and Lemons task (Lamon, 1999) elicited more additive thinking or visual judgement than any other task (seven students), even though some described it in terms of ratio (2 to 3, and 3 to 5). Nathan (Year 9) and Briony (Grade 6) compared each ratio correctly, but in different ways (see Figure 3).

Michael (Grade 5) solved this problem by forming two fractions that he could compare easily, "2/5 x 3 = 6/5 and 3/8 x 3 = 9/8, and 1/5 is more than 1/8 so 2/5 (A) would be more orangey."

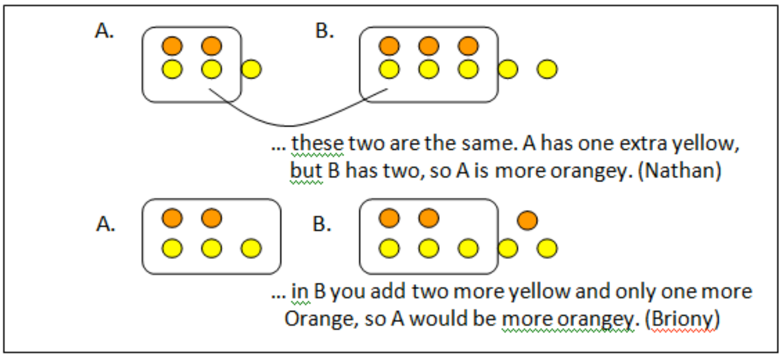


Figure 3. Ratio comparisons.

Observations from Question 5 – Well-Chunked Measures

The Adelaide Trip. This question was more difficult in that it used larger numbers (Lo & Watanabe, 1997), as well as a non-integer scalar relationship. The larger numbers elicited a different approach from students who had been comfortable with multiplicative reasoning in previous questions. To determine how long it would take to get to Adelaide, all students reverted to a ratio-unit/build-up strategy, with some being limited to a ratio-unit of 2hrs = 160km, while others recognised the ratio-unit of 1hr = 80km and ½ hr = 40km. No-one recognised the structure 1:80 = ?:600, but the reality is that a *ratio-unit/build-up* approach is probably the more efficient strategy to use in this instance anyway, suggesting that 'more sophisticated' does not automatically equate to 'most efficient'. However, it may also be worth considering supplying calculators at this stage of the interview to remove the constraint of manipulating larger numbers mentally. Would different strategies be employed if these constraints were removed?

The second part of the Adelaide Trip question asks, "What average speed am I travelling?" Only one Grade five student recognised the speed element of this question (Lamon, 1993), even Michael, who showed very strong reasoning skills in all other questions was totally stumped. The five students (from Grades 6 and 9) who did get this question correct, all knew the answer straight away, recognising that "if I did 160km in 2 hours, then I did 80km in one hour, which is 80km/hr." Two others just guessed "because the speed limit is 100km/hr".

Observations from Question 6 – Stretchers and Shrinkers

Enlarged Rectangle. Lamon, from her findings in her 1993 study, suggested that *Stretchers and Shrinkers* problems should only be introduced after students have developed multiplicative thinking. It was certainly very obvious in this interview that it was only those students who showed strong multiplicative reasoning in previous questions that saw Lamon's (1993) Enlarged Rectangle task as a relative problem, but even this was not predictable – Nathan (Year 9) had answered all previous questions correctly, but reverted to an additive strategy for this one.

Proposed Points of Growth (or Stages of Understanding)

The development of ratio and proportional reasoning is difficult to measure, as growth in understanding does not appear to be necessarily linear across all problem types, and, indeed, not all ratio problems are solved most efficiently by the more sophisticated strategy. However, the following are general observations.

Students with little or no understanding of ratio problems may attempt a visual judgement or just guess, but even at this level these guesses may be reasonable, or they may be completely off the mark in terms of the required goal of the task.

As students begin to recognise the significance of the numbers in ratio problems they will initially try to 'keep the balance' by adding equivalent amounts to each value. This is an additive strategy, and does not maintain 'relativeness'.

In the Ghost Drop problem, 'How much would 15 Ghost Drops cost?', Tanya wrote a list of threes – adding them together as she wrote them down – and stopped when she reached 15 (see Figure 1). Inhelder and Piaget (1958, cited in Lamon, 1993) called this preproportional reasoning because, "children achieved correct answers without recognising the structural similarities on both sides of the proportion equation" (p. 41).

Students who recognise the need to preserve an equivalent relationship between two values are starting to think relatively, and use a multiplicative strategy to 'build up' to a new value. Lo and Watanabe (1997) coined the phrase *'ratio-unit/build-up method'*, as these students were able to consider the ratio '3 for 10c', for example, as a composite unit (Kilpatrick et al, 2001), and then build this unit up (see Figure 4).

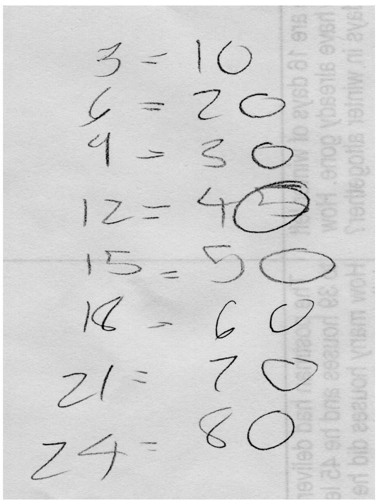


Figure 4. Ratio-unit/Build-up Strategy.

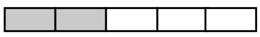
The next stage seems to be recognition of functional and scalar relationships. In functional reasoning a student is able to determine a unit value and multiply by this. For example, in the problem "'3 for 10c' and '10 for 35c', which is better value?", students can determine that if 3 cost 10c then one Ghost Drop will cost 3.33 cents, and 10 of these will be 33.3c, which is less than 35c. Scalar reasoning determines the multiplicative relationships between A and C in A/B = C/D. For example, 3 Ghost Drops (A) for 10c(B) = ? Ghost Drops (C) for 80c(D): D = B x 8, therefore C = A (i.e., 3) x 8.

The most sophisticated level of ratio reasoning observed was being able to manipulate large and/or non-integer scalars (Lo & Watanabe, 1997), for example, in the Adelaide Trip question, 160km(A) in 2 hours(B) = 80km(C) in 1 hour(D) = 600km(E) in ? hours (F). E = C x 600/80, therefore F = D x 600/80. Based on these findings points of growth (or stages of understanding) in the learning of fractions as ratio could be described as (see Figure 5):

	Growth Point	**Description**
0	Not Apparent	- Cannot comprehend the required 'goal' of the task.
1	Visual/Ignore	- Uses visual judgement or just guesses; ignores or does not consider individual values.
2	Additive	- Early attempts at quantifying, but using constant additive strategies rather than multiplicative relationships.
3	Pre-proportional Reasoning	- Pattern recognition and replication, but without recognising the multiplicative structure (non-reversible).
4	Ratio-Unit/Build-Up	- Recognises the ratio as a unit and can build up this unit maintaining the relative structure of the individual values (reversible).
5	Functional & Scalar Reasoning	- Can determine a unit value and multiply by this; can determine the multiplicative relationship between A & C in A/B = C/D.
6	Quantitative Proportional Reasoning (Lamon, 1993)	- Uses algebraic-type methods to represent and solve complex proportion problems.

Figure 5. Proposed Points of Growth (in understanding) for Fractions as Ratio

Some Implications for Teaching

Traditionally ratio as proportion is not taught before secondary school, and yet all the primary students interviewed were able to connect with at least some of the ratio interview tasks. Many of the tasks simulated familiar situations for the students, especially ratio as a rate (for example, cost per item, kilometres per hour), and all the primary school students appeared to have some intuitive understanding of how to solve at least some of the tasks in the assessment interview. Indeed, they tended to use more intuition than the secondary student who often tried to remember a rule or formula he knew he had been taught. Lamon (1999) argues that young children can see and understand part-part comparisons more naturally than part-whole comparisons. For example, they will describe  as 2/3 rather than 2/5. Vergnaud (1983) stated, "It is difficult and sometimes absurd to study separately the acquisition of interconnected concepts." (p. 127). So one question to consider is, is it necessary, or at all beneficial, to postpone ratio instruction until post-primary school? Do children need to be able to think multiplicatively before they can understand ratio, or does learning about ratio help them to think multiplicatively? For example, eight out of the ten students interviewed in this study understood the term 'twice as many' (Christmas M&Ms). Of these eight, only half recognised the multiplicative structure of the Green Paint question. If problems like the Green Paint question were used as an introduction to ratio instruction with a ratio of A:B = 1:2, *together with* the description ‘there are twice as many Bs as As’, children using an additive method for expanding the ratio (see Figure 5), would quickly recognise that adding the same to each value does not maintain the ratio of ‘twice as many Bs’, and would need to re-evaluate, and explore number patterns that do maintain this relationship.

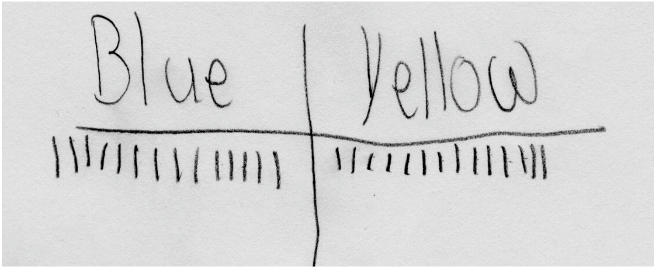


Figure 5. Additive method for calculating how many blues and yellows required to make 28L green paint.

The *ratio-unit/build-up* strategy, using Ratio Tables and/or Double Number Lines (see Figure 6) proved a useful tool in early work with proportional situations (Kilpatrick et al, 2001). The use of these could help children organise their thinking and help promote the move from additive to multiplicative reasoning.

Ghost Drops	Cost
3	10
6	20
9	30

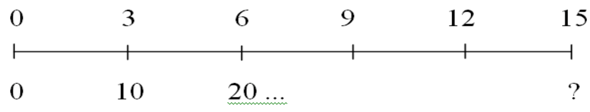


Figure 6. Ratio Table and Double Number Line.

Many children seem to intuitively organise their thinking this way. For those who do not, these tasks could be useful tools to introduce and discuss. Lamon (1993) also talked about using problem types that lend themselves to being re-presented with manipulatives or pictures.

One of the most interesting observations made during the study was that most of the students interviewed were not proficient with their multiplication facts, and they all became frustrated with this realisation when their thinking processes were interrupted by having to stop and work out, for example, 3 x 8. Ratio/proportion problems that require multiplicative thinking, give students a reason for knowing their multiplication facts, apart from ‘tables challenges’ and other rote exercises. Something that provides meaning and purpose to a students’ learning appears to be a great motivator.

There is much children’s literature available that could be used to initiate discussion about relative size. Stories such as Counting on Frank (Clements, 1991) and The Librarian Who Measured the Earth (Lasky, 1994), and books like Incredible Comparisons (Ash, 1996), all provide contexts for discussing ratio and/or rates, and build children’s concept of proportional understanding (Thompson, Austin & Beckman, 2002).

Conclusion

In terms of proposed points of growth for understanding fractions as ratio, this exploration of children's understanding of the sub-construct of ratio and rates under the enormous umbrella of 'proportional reasoning' is merely the tip of the iceberg. For this study only a small sample of students were interviewed; there may well be other types of strategies – correct and incorrect – that students typically employ, that would describe other stages of growth in children's thinking. The questions chosen for the assessment interview may also be limiting in assessing all of the possible stages of growth. However, the points of growth identified here may give teachers a starting point for identifying and understanding students' thinking in terms of proportional reasoning with ratio problems, and a greater understanding of where and how to move each student forward in their learning.

Due to the findings of primary school children's intuitive understandings of ratio, the consideration of beginning 'fractions as ratio' instruction earlier than secondary level of schooling may be something worth exploring – as long as this instruction allows for a development of the intuitive understandings, and not just an introduction of procedures and ratio formulae. Teacher understanding of the concept of ratio is another area not discussed that has a huge impact on children's learning. All are important areas for future research.

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