Document extract

Title of chapter/article	Inspiring Teachers; Challenging Children
Author(s)	Peter M. Neumann
Copyright owner	The Australian Association of Mathematics Teachers Inc.
Published in	Capital Maths: Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers
Year of publication	2017
Page range	25–32
ISBN/ISSN	978-1-875900-80-0

The Australian Association of Mathematics Teachers Inc.
ABN 76 515 756 909
POST GPO Box 1729, Adelaide SA 5001
PHONE 08 8363 0288
FAX 08 8362 9288
EMAIL office@aamt.edu.au
INTERNET www.aamt.edu.au

HANNA NEUMANN MEMORIAL LECTURE:

INSPIRING TEACHERS; CHALLENGING CHILDREN

PETER M. NEUMANN
The Queen's College, Oxford University

Introduction: Inspiring teachers

My presidential address to The Mathematical Association (UK) in April 2016 was entitled 'Inspiring Teachers'. The concept was a talk that might trace my mathematical experiences from grammar school sixth form, through my development in retirement as a contributor to masterclasses for the UK Mathematics Trust and the Royal Institution, learning from the students and from the inspiring teachers at whose masterclasses I assist, to the small understanding of Key Stage 2 Mathematics that I am gradually acquiring through an hour a week with some Year 6 students and their remarkable teachers in a local primary school.

Three months later I was greatly honoured to be invited to deliver the Hanna Neumann Memorial Lecture to the AAMT biennial conference 2017. Late in October 2016 a request came for a title. I was stumped. But just then there arrived out of the blue—by airmail—a delightful letter from Deanne Whittleston of Sydney. It was dated 2 November 2016 and I have permission to quote from it:

> She [Hanna Neumann] was remarkable for many reasons but to me she was remarkable as she managed to get me through Pure Mathematics I and in all my years of study (13 at school, 5 at ANU, 2 at Canberra University and 5 studying law in Sydney) she was the best teacher I ever had. How she did it I do not know.
> In my Pure Maths I class at ANU in 1967, there were about 200 students (only 4 of whom were women, of whom I was one). Hanna taught that class as though she knew us all intimately—she was incredible.

The lovely coincidence of the arrival of this letter just when I was at a loss for a lecture title tempted me to re-use that of my earlier address—though I was clear in my mind that the actual address would have to be very different. The title is deliberately ambiguous. So is the second half that I have added later.

Another point about the title is that no abstract is needed; yet another, that the words can be permuted in several ways, such as 'Challenging Children Inspiring Teachers' and 'Teachers Inspiring Challenging Children'. And of course, as so often, when a title is requested long in advance, what was needed was something that would restrict the content on the day as little as possible.

As I have written in *The Mathematical Gazette* (November 2016), my lectures are designed to be ephemeral oral presentations. They are not designed to be written down and published as articles. Please bear that in mind gentle reader, and judge accordingly. If you find something of value here I shall be delighted; if not, I shall not be surprised.

Challenging children 1: A Year 6 discovery

Eleven-year old Nathan on 9 June this year:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Choose four numbers, one from each row, one from each column. Whatever the choice you make, they add to 34.

I have no idea where this came from. Possibly from one of his family. Quite probably a discovery of his own. The context was that two weeks earlier, on the last Friday before the week-long half-term holiday, eleven high-ability Year 6 children, a teacher and I, had started from Durer's famous 4×4 magic square and had investigated other, mostly smaller, magic squares. These squares captured the children's imagination. And so it was that Nathan, refreshed by his holiday, came up with his discovery.

It was new to me. The response had to be "Lovely! Can you explain to me WHY it works? What about squares of other sizes?" By the end of the session (45 minutes) the children had investigated squares of sizes down to 1×1 (one of the girls did this) and up to 10×10, and three or four of them had understood why it works. Those were challenged to write their explanations down. But writing mathematical explanations is, in my somewhat limited experience, not something that eleven-year olds, even bright ones, put high on their agenda.

Challenging children 2: A Year 10 masterclass

Year 10 children are fourteen or fifteen years old. The masterclasses offered by the UK Mathematics Trust and by the Royal Institution of Great Britain are aimed at a group consisting of two of the ablest children in each of twenty to thirty schools. Since average cohorts will be between three and four classes per year (smaller in independent schools) the clientele of these masterclasses might be expected to come from the top 2% or 3% of the ability range. Of course, ability does not work quite that way; even so, the children will be among those who (in England and Wales, possibly also in Scotland and Northern Ireland) are deemed to be 'Gifted & Talented' (in a semi-formal political sense). These masterclasses are intended to show able children that there is mathematics outside of their syllabuses, that mathematics is not done and dusted, that there are areas where mathematicians in industry and in universities are still struggling to gain understanding. Here is an example of a Year 10 masterclass on combinatorics of words, that hides some deep group theory in which there are still many open research problems.

We start with an alphabet {a*, b, c…*}, as few or as many letters as we wish, and we focus on words. In this context words are any strings of letters such as;

a, aa, aaabaccba, bbababaaaab,

They are meaningless—all that is of interest is the combinatorics associated with linear arrangements of symbols. The *length* of a word *w* is defined simply to be the number of letters in *w*. The examples above have lengths 1, 2, 9 and 11 respectively. An important convention is that we allow length 0, no letters! But this word needs to be seen on the page, so we write 1 for the 'empty' word.

Before we move on to transformation rules, the main topic, let's digress briefly (but usefully).

Challenge

Think of a good way to list the words in the two-letter alphabet {*a, b*}.

It does not take children long to realise that a list starting

a, aa, aaa, aaaa, aaaaa,…

as a dictionary might, will never reach words that involve the letter *b*. And then some children have the idea of *length-lex* (also known as *shortlex*) listing:

1*, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb,*
aaaa, aaab, aaba, aabb, abaa, abab, abba, abbb, baaa, baab…

Exercises

1. What comes next after *abababb*?
2. What is the 64^{th} word? The 100^{th} word? The 2017^{th} word?
3. Can you find a general rule?

Once we are all happy that we know what is meant by 'words', that a word (in this area of mathematics) is just a meaningless string of letters, of any length, including length 0, then we move on.

In this branch of mathematics, we have *transformation rules* to change words into other words. We want to find out what effect these changes have.

First example

Focus on words *w* in the one-letter alphabet {*a*}. Choose a number—let's say 4. Then we have two rules:

- *Expansion rule:* Choose any word *u* and *expand w* by inserting *uuuu* between two adjacent letters of *w*, or at the front or at the back of *w*.
- *Contraction rule:* If we see *uuuu* somewhere in *w* (a part of *w* consisting of 4 consecutive non-overlapping instances of some word *u*) we may contract *w* by deleting it and closing up.

So for example, with $u = aa$ expansion may change aaa to $aa\underbrace{aaaaaaaa}a$, and with $u = aaa$ contraction can change $aa\underbrace{aaaaaaaaaaaa}aaa$ to *aa aaa*.

Equality of words

We say that words v and w have the same value, and we write $v = w$ if word v can be changed to word w by expansions and contractions any number of times (using the same or different words u), in any order. Since every expansion may be undone by a contraction, and vice-versa, if word v can be changed to word w by expansions and contractions any number of times then also w can be changed to v.

Exercise

Organise the following words into groups so that those in a group have the same value:

1, *a, aaa, aaaaa, aaaa, aaaaaaaaaaa* (11),
aaaaaaaaaaaaa (13), *aaaaaaaaaaaaaaaa* (16)

Problem

For this transformation rule, how many *different* values are there?

Answer

For this transformation rule there are 4 different values: any word has the same value as one of 1, *a, aa, aaa*, and these are different.

Similar examples of transformation rules

Stay with the one-letter alphabet {a}, but what if we replace 4 by 5? Now we have words in the one-letter alphabet {a} and our transformation rules are insertion or deletion of *uuuuu*. Equality of values is defined as before: we write $v = w$ if word v can be changed to word w by expansions and contractions of this kind.

Problem

How many different values are there now?

It is not hard to see that the answer is answer 5.

Problem

What is the answer if we replace 4 by 3 instead of by 5? Or by 12? Or by n?

Summary of first examples of transformation rules

Focus on words in the one-letter alphabet {a}; permitted transformations are insertion or deletion of u^n (that is, n consecutive non-overlapping instances of a word u); write $v = w$ if word v can be changed to word w by expansions and contractions; and the basic problem is, how many different values are there? The answer is that words v and w have the same value if their lengths leave the same remainder when divided by n. That is, any word has the same value as just one of these words:

$$1, a, aa, aaa \ldots a^{n-1}$$

So, there are *n* different values.

Let's move on to larger alphabets

Focus on words *w* in the two-letter alphabet {a, b}. For our next transformation rule we again choose a number—let's say 2 . Then our rules are:

Expansion rule

Choose any word *u* and expand *w* by inserting *uu* between two adjacent letters of *w*, or at the front or at the back of *w*.

Contraction rule

If we see *uu* somewhere in *w* (a part of *w* consisting of 4 consecutive non-overlapping instances of some word *u*) we may contract *w* by deleting it and closing up.

As before, we write $v = w$ if word v can be changed to word *w* (or, equivalently, *w* changed to *v*) by expansions and contractions (any number of times, in any order, using the same or different words *u*).

Problem

With this transformation rule for words in the two-letter alphabet {a, b}, how many different values are there?

Important note

When this topic is used for a Year 10 masterclass, it is usually spread over two sessions, each of 70 or 80 minutes, on successive days. The break is planned to come here. The three-part exercise about the length-lex listing of words in the two-letter alphabet, together with this problem, are offered as "homework", and are discussed at the start of the second day.

Have you found, given a little time, that any word *w* has the same value as one of these:

$$1, a, b, ab, ba, aba, bab?$$

But *are these all different*?

Perhaps about 400 or 500 children have been exposed to this question (a question posed in different language to first-year undergraduates at my university) and the number who have worked out that the answer is 'no' is positive but small (around 10):

$$ba = \underline{aa}ba = aaba\underline{bb} = a\underline{abab}b = \mathrm{ab}$$

Thus $ba = ab$, and it then follows quickly that $aba = b$ and $bab = a$. Therefore, for this example of a transformation rule for words in a two-letter alphabet there are values. Listed in length-lex order they are 1, *a*, *b*, *ab*.

If time permits we investigate what happens when we keep the same rule but work with words in larger alphabets. It does not take the majority of the children long to realise that they can use what they have just discovered (or been shown) to treat words in $\{a, b, c\}$. by focussing first on those that do not involve *c*, then those that do not involve *b*, then words that do not involve *a*, and the upshot is that, with the rule 'insert or delete *uu*', any word has the same value as one of:

$$1, a, b, c, ab\ ac, bc, abc$$

and these are all different, so there are 8 values. And if the alphabet has 26 letters then there will be 2^{26} different values; if the alphabet has m letters then there will be 2^m different values.

Next, we return to words *w* in the two-letter alphabet $\{a, b\}$ and we investigate the rule where 2 is replaced by 3. That is:

Expansion rule

Choose any word *u* and expand *w* by inserting *uuu* between two adjacent letters of *w*, or at the front or at the back of *w*.

Contraction rule

If we see *uuu* somewhere in *w* (a part of *w* consisting of 4 consecutive non-overlapping instances of some word *u*) we may contract *w* by deleting it and closing up.

As before, we write $v = w$ if word *v* can be changed to word *w* (or, equivalently, *w* changed to *v*) by expansions and contractions (any number of times, in any order, using the same or different words *u*). And as before, the fundamental problem is: how many different values are there now? The answer now (two-letter alphabet, insertion or deletion of three consecutive instances of words *u*) is that there are 27 different values of words. *But* this would be a hard problem for third-year or fourth-year university students!

The research context

Start with two positive whole numbers *m, n*. We work with words *w* in an alphabet that has *m* letters; *n* is known as the *exponent*.

Expansion rule

Choose any word *u* and expand *w* by inserting u^n between two adjacent letters of *w*, or at the front or at the back of *w*.

Contraction rule

If we see u^n somewhere in *w* (a part of *w* consisting of *n* consecutive non-overlapping instances of some word *u*) we may contract *w* by deleting it and closing up.

As before, we write $v = w$ if word *v* can be changed to word *w* (or, equivalently, *w* changed to *v*) by expansions and contractions (any number of times, in any order, using the same or different words *u*). Call the resulting list (organised in length-lex order) of different values $B(m, n)$.

The Burnside Problem

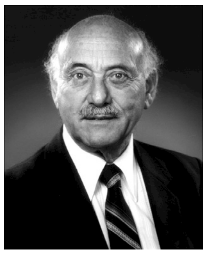
Is the list $B(m, n)$ finite? If so, how long is it?

William Burnside and his problem.
'On an unsettled question in the theory of discontinuous groups'
Quarterly Journal of Mathematics, 1902:
is $B(m, n)$ finite? Now, in 2017, much is known, much remains unknown.

At this point in a Year 10 masterclass I would spend fifteen or twenty minutes showing the children something of what we know, something of what is still unknown despite great efforts by many mathematicians. The fact that (in spite of the clue in the title of Burnside's paper) this is all a part of modern group theory remains hidden.

But what is the hidden group theory? In fact, $B(m, n)$ forms a group. It is the "freest" group generated by the m letters subject only to the condition that $u^n = 1$ for every word u in the m generators (that is, every element of the group). Technically, $u^n = 1$ is an identical relation in $B(m, n)$ in the sense of B. H. Neumann, *Identical relations in groups*, PhD thesis, Cambridge (1935), and article published in *Mathematische Annalen* (1937).

BHN in Cardiff, circa 1937.



BHN in Canberra, circa 1970.

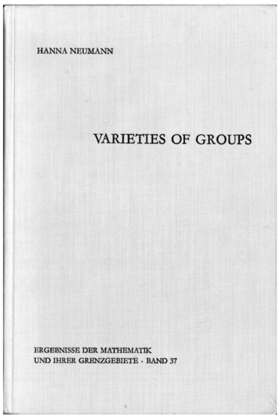
There is more hidden group theory, though. The groups that satisfy a given set of identical relations form what is called a variety of groups.

Example

The abelian variety consisting of the groups in which $ab = ba$ (or equivalently $a^{-1}b^{-1}ab = 1$) for all elements a, b.

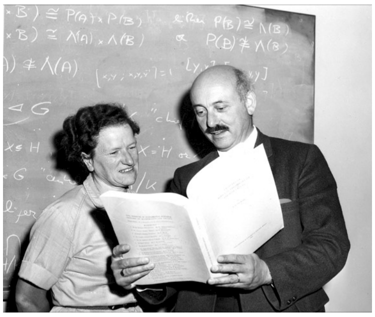
Example

The Burnside variety consisting of all groups in which $u^n = 1$ for all elements u.



The 1967 monograph.

Hanna in Canberra, 1959.



The inaugural Vice-President and President of AAMT.

Conclusion: The Burnside Problem for Exponent 5

Take words in a two-letter alphabet $\{a, b\}$. Words v, w are deemed to have equal value if one can be obtained from the other by insertion or deletion of words of the form *uuuuu* (written u^5). The resulting list of different words is known as $B(2, 5)$:

Is $B(2, 5)$ finite?

It is unknown. That way fame and fortune lie!