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thinking visually about algebra

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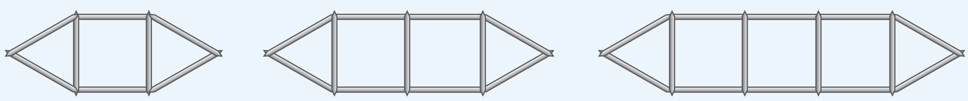
Many introductions to algebra in high school begin with teaching students to generalise linear numerical patterns. This article argues that this approach needs to be changed so that students encounter variables in the context of modelling visual patterns so that the variables have a meaning. The article presents sample classroom activities, together with sample work from students in the author's Year 7 classroom at a Catholic school in the South East of Melbourne.

My previous practice

In previous years, I used to teach algebra and patterns by teaching students to find the rule from tables of values and then, as an 'application', taught them to 'find the rule' from a visual pattern. Even then, generating a table of values from the visual pattern was used as an intermediate step. It is fair to say that too few students learnt how to go from diagram to algebraic rule directly. This is the approach taken in many textbooks. This approach is illustrated in Figure 1, reproduced from *Maths Xpress* (Pagon & Griffith, 2009, pp. 281-282).

Board Example

Write a formula to predict the total number of toothpicks when given the shape number in the pattern below.



Picture 1 Picture 2 Picture 3

1. Draw a table that shows each shape number and the number of toothpicks in each shape. Look for the common difference; the common difference is 3 since the number of toothpicks used goes up by 3 each.

Shape Number	No. of toothpicks
1	8
2	11
3	14

2. Define pronumerals to represent the variables. ▶ Let T = total number of toothpicks.
Let n = the shape number.

3. The formula needs to predict the number of toothpicks in a shape number. Look for a rule that uses the shape number to predict the number of toothpicks. ▶ The formula is $T = 3n + 5$

Shape 1: $8 = 3 \times 1 + 5$
Shape 2: $11 = 3 \times 2 + 5$
Shape 3: $14 = 3 \times 3 + 5$

Write the formula.

Figure 1: Numerical approach to finding a rule from a visual pattern, reproduced from *Maths Xpress* (Pagon & Griffith, 2009, pp. 281-282).

The Problem

I began suspecting that something was amiss with this approach when, at the end of the lesson sequence, I observed the students' responses to the 'garden beds' task (see Figure 2) from the Maths300 lesson library. Most students could find the formula: $n = 2s + 6$ (where n represents the number of tiles and s the size of the garden) but too few could answer the following question:

> Explain with the use of diagrams how someone else can understand your rule and how they can use your rule. Pretend that this person has not done the activity. (For a 'good' answer, see Figure 3).

I did want my students to be able to connect together multiple representations of the same problem (cf Swan, 2005, p. 16).

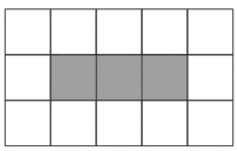


Figure 2: Tiling around 3 garden beds.

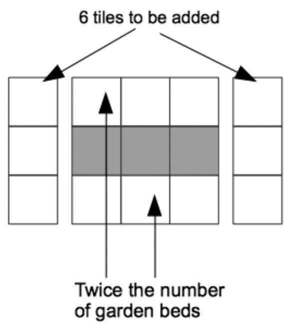


Figure 3: Visual explanation of the rule.

In a discussion of their longitudinal study with students starting in Grade 6, Becker and Rivera (2006) state they had found that students who derived their formulas from a table of numbers, whom they called predominantly numerical generalisers, had a tendency to base their generalisations on limited information. Furthermore, they had "no sense of what the parameters in particular formulas represent" (p. 466). To my way of thinking, this is a problem as I like to think of algebra as a language for modelling the real-world. I have also observed that many students in later years are able to find the equation of a straight line using formulas, but cannot relate the parts of the formula to the graphed variables. They simply do not expect a graph to tell them a story about the relationship between two quantities.

In contrast, those students who generalised from visual patterns, predominantly figural generalisers, were better able to justify their formulas and "to connect their symbols and variables to the patterns that generate the figures." (Becker & Rivera, p. 466)

What a visual justification looks like

Becker and Rivera (2009) give this example from a student who had studied the pattern in Figure 4 and derived the formula: $s = n \times 4 + 1$: There is one in the middle so that could be plus 1, and on the sides they all have the picture number [pointing to arms], so square tiles equal $n \times 4$ plus 1. Tiles are arranged to form pictures like the ones below.

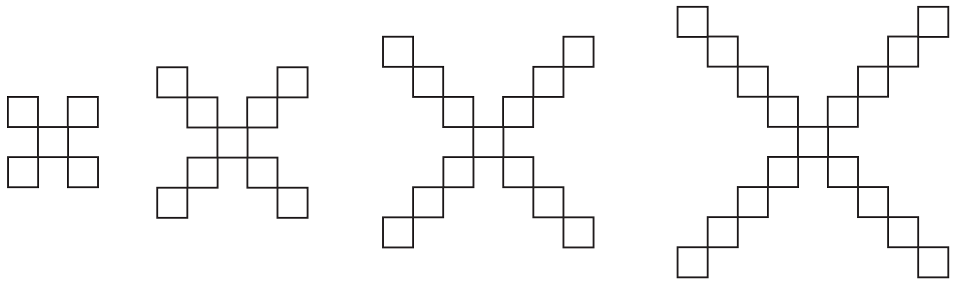


Figure 4: Square tiles pattern (reproduced with permission from Becker & Rivera, 2009).

Desirable as it may be, students tend to find it more difficult to reason visually than they do numerically. In a study of 54 Portugese students, Barbosa, Vale and Palhares (2012) made that same observation and speculated that it was "perhaps due to their greater experience working with numeric contexts".

The aim here is not to do away with the students' numerical reasoning but to add to it. We want the students to understand what the variables in their formulas stand for.

Discovering a better way

In 2009, I attended the *Psychology of Mathematics Education* conference in Thessaloniki, Greece. There, I had the fortune to meet Dr Ferdinand Rivera and Dr Joanne Rossi Becker who presented two sessions on their work with students on algebra.

The result of my conversations with Dr Rivera and Dr Becker was a task with which I now begin Patterns and Algebra. The task is made up of four pages like the one in Figure 6. It is sometimes hard to convince teachers that students can handle this task before any formal teaching of this topic. However, I have found attempts to introduce visual patterns after the students had learnt to work with tables of values to be ineffective. This seems to be caused by the fact that finding a linear equation from a table of values can be learnt procedurally. The procedure is easy enough to implement and the student loses the incentive to learn to think in a new way.

I copy the task on A3 paper so that groups of students can 'pour over' the problems. Each group sits around one table with each student in the group fulfilling a specific function, as shown in Table 1. Every few minutes, the presenter of one group updates the class on the group's progress and explains their strategy. This process is adapted from one described by Williams (2014).

Table 1: Roles of group members.

Role	Description
Organiser	Makes sure people speak in an orderly manner, not interrupting
Presenter	Presents to the class when asked by the teacher
Encourager	Makes sure everyone is contributing their ideas
Scribe	Writes down the result of the discussions

Conversing with students has shown me that their initial understanding of the pattern involves adding a number of circles at each stage to the ones 'already there' from the previous one. This is what Barbosa, Vale and Palhares call "the recursive strategy". This is the reason behind question 3: How many circles do you expect to see in the 10th and 20th stages? This question forces them to think of a stage that would require too much effort to 'work up to'. Instead, they feel the need to think of a formula, in informal terms at first, for the number of circles in the 20th stage. A colleague of mine has commented that this is an important step for moving students from counting, a lower-order thinking task, to algebraic generalisation, a high-order thinking task.

Students create the patterns

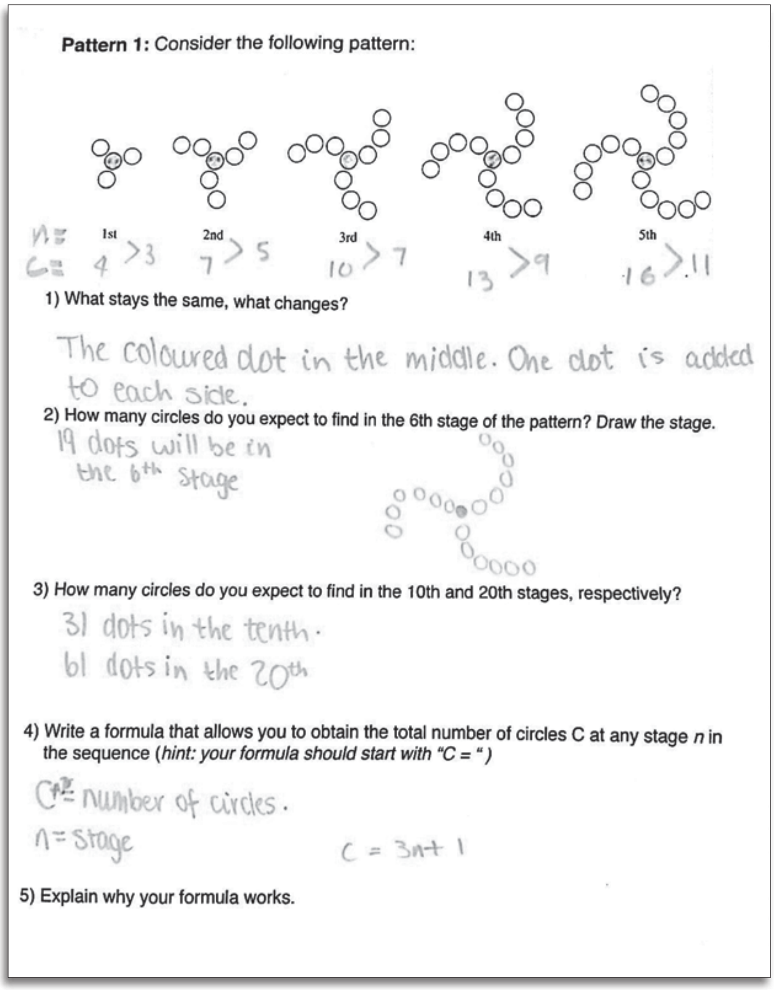
In the lesson following the task, I drew 4 different patterns on the board, all made with matches. I asked the students to find a formula relating the number of matches needed to the number of shapes to be drawn in each stage of the sequence. Figure 5 shows one such pattern from the Heinemann textbook (Coffey et al, 2011, p. 268). I found that I needed to demonstrate the visual (figural) justification as virtually all students were working numerically. Some students found it helpful to manipulate physical matches on their way to generalising the pattern.

Figure 5: Patterns made with matches.

Having managed to derive rules from patterns, students were ready to generate their own patterns from formulas. Figure 7 shows two snapshots from the whiteboard with alternative patterns corresponding to two different formulas.

In a subsequent lesson, I wrote the equation $y - 3x + 2$ and asked students to create a pattern with geometric shapes. Figure 8 shows samples of these patterns. As with the patterns made with matches, one can see their attempt at keeping part of the pattern constant: two shapes on top for '+ 2'.

Constructing patterns to fit a formula and demonstrating those patterns to one another seemed to bring about the 'aha' moment for many students.



Pattern 1: Consider the following pattern:

n = 1st 2nd 3rd 4th 5th

C = 4 >3 7 >5 10 >7 13 >9 16 >11

1) What stays the same, what changes?

The coloured dot in the middle. One dot is added to each side.

2) How many circles do you expect to find in the 6th stage of the pattern? Draw the stage.

19 dots will be in the 6th stage

3) How many circles do you expect to find in the 10th and 20th stages, respectively?

31 dots in the tenth.
61 dots in the 20th

4) Write a formula that allows you to obtain the total number of circles C at any stage *n* in the sequence (*hint: your formula should start with "C = "*)

C = number of circles.
n = stage

C = 3n + 1

5) Explain why your formula works.

Figure 6: Students' work with visual patterns.

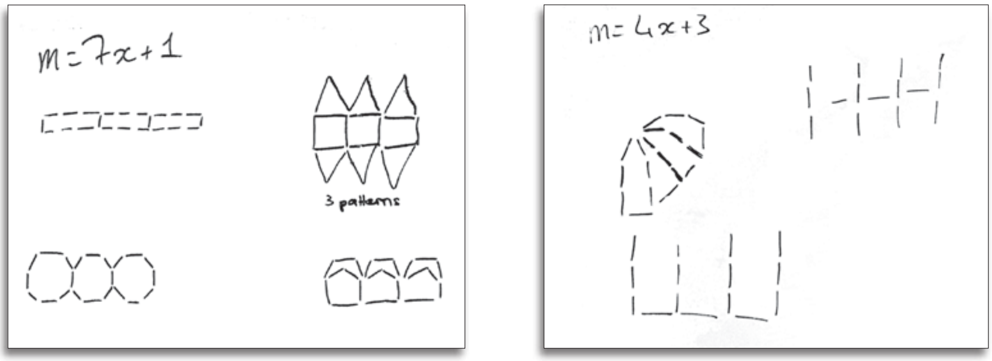


Figure 7: Patterns generated by students.

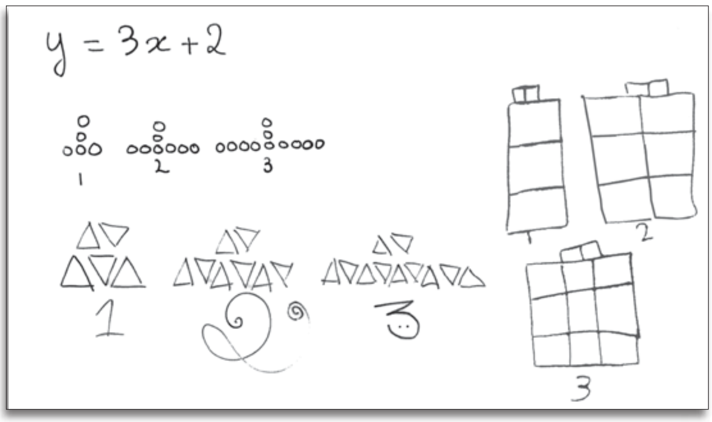


Figure 8: Student-generated patterns using geometric shapes.

Later, I presented the students with some patterns and asked them questions that would summarise all that we had learnt up to that time. Figure 9 shows the explanation given by Lucy who had previously been a purely numerical thinker.

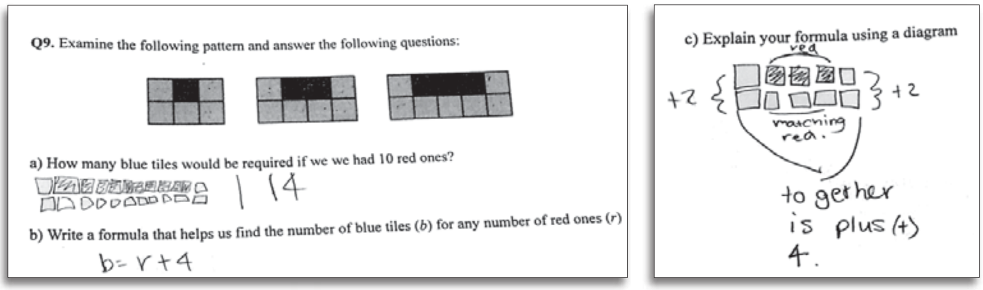


Figure 9: A student's explanation of a visual pattern.

Conclusion

In summary, there are many ways for teachers to help students make generalisations from visual patterns. Textbooks encourage students to generate tables of values and then to deduce rules from those tables. This approach is not hard for students to grasp and can be deferred. If students are to develop multiple strategies for thinking about such problems, we need to help them relate formulas to visual patterns. This, in turn, will give the variables some meaning as the algebra becomes an alternative representation of a tangible situation.

This article has, however, not addressed some important topics such as non-linear patterns and connections between diagrammatic, graphical and numerical representations of linear and other patterns.

References

Barbosa, A., Vale, I. & Palhares, P. (2012). Pattern Tasks: Thinking Processes Used by 6th Grade Students. *Relime vol.15* no.3 México nov. 2012. Retrieved August 20, 2014, from http://www.scielo.org.mx/scielo.php?pid=S1665-24362012000300002&script=sci_arttext&tlng=en

Becker, J. R. & Rivera, F. (2006), Establishing and Justifying Algebraic Generalization at the Sixth Grade Level. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, Vol. 4*, pp. 465-472. Prague: PME.

Becker, J. R. & Rivera, F. (2009), Justification Schemes of Middle School Students in the Context of Generalization of Linear Patterns. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the Joint Meeting of 33rd International Group for the Psychology of Mathematics Education, Vol. 5*, pp. 9-16. Thessaloniki, Greece: PME.

Coffey, D., Strasser, D., Phillipls, G., Nolan, J., Matheson, A., Koisch, P., Ardley, J., Boucher, K., Dimitriadis, G., Pan, A. (2011). *Heinemann Maths Zone 7 – VELS Enhanced.* Melbourne: Pearson.

Maths 300 (subscription required for access). *Garden Beds Lesson*, retrieved August 21, 2014, from: http://www.maths300.esa.edu.au/index.php?option=com_content&view=article&id=122

Pagon, T. & Griffith, S. (2009), *Maths Xpress 7 Teach Guide.* Milton:QLD: John Wiley & Sons Australia Ltd.

Swan, M. (2005). *Improving Learning in Mathematics: Challenges and Strategies.* Retrieved September 3, 2014, from: https://www.asms.sa.edu.au/wp-content/uploads/2013/04/Improving_learning_in_maths-UK.pdf

Williams, G. (2014). *Optimistic Problem-Solving Activity: Enacting Confidence, Persistence, and Perseverance. ZDM, 46* (3), 407 – 422