mathematics:
learn, lead, link

Proceedings of the 25th Biennial Conference of the
Australian Association of Mathematics Teachers Inc.

Edited by N. Davis, K. Manuel & T. Spencer

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PREFACE

“Mathematics: Learn, Lead, Link” is an appropriate theme for the 25th Biennial Conference of the Australian Association of Mathematics Teachers. In a time of upheaval and change at every social scale, all representative bodies and their individual members must learn how to manage new and diverse challenges. That learning must be translated into transformational leadership that inspires and supports practitioners to make effective changes in their local spheres. I believe that this conference is a very important link in the network of change that many hope will lead to a future general populace that is more mathematically aware and competent.

Among the major themes that are very publicly connected to education are national security, financial literacy, environmental change, STEM, the apparent failure of our students in international rankings, national standards and the professionalism and quality of our teaching force, and the continuing marginalisation of Aboriginal Australians. It is true that schools have a fundamental role to play in dealing with social problems. However, calls from the public to solve this or that ill by delivery of additional curriculum are multitudinous and often seem simplistic, without genuine understanding of the complexities of education.

Fortunately, our conference speakers, as a community, with diverse interests, affiliations and experience, have the new ideas and the vital information that can support transformational learning and change. Coming together as a large and diverse group enables us to make links to fellow educators and to share knowledge and innovation. It is undoubtedly true that educators today must continue to learn and act upon their learning, for public good, as well as personal benefit. Every individual needs to understand more deeply the ways in which we can act to lead others and be an effective part of the network of change.

That which encourages me greatly in this endeavour is the tremendous support that educators give to each other, which I hope you experience in your workplace. In the context of this conference, I am grateful for the willing and generous help from many colleagues, most of whom I have never met, in carrying out my small role in the proceedings.

That which challenges me, is how we can best extend important ideas beyond our community of educators, to those who influence the thinking of other groups in society. I am conscious then, of a responsibility to make the most of the opportunity provided by the conference. I must think deeply about what influence I might have as an educator in my particular station; what learning is most essential to be an effective participant, and leader, in my community. It is obvious to me that I can make many links to colleagues at the conference. I believe that from these links I could develop new understandings that may enable me to create meaningful connections with people, and exert a positive influence, however small, on society at large.

Neil Davis
Proceedings Lead Editor

Review process

Presentations at AAMT 2015 were selected in a variety of ways. Keynote and major presenters were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or a workshop. Seminars and workshops were selected as suitable for the conference based on the presenters' submission of a formal abstract and further explanation of the proposed presentation.

Authors of seminar and workshop proposals approved for presentation at the conference were also invited to submit written papers to be included in these proceedings. These written papers were reviewed without any author identification (blind) by at least two reviewers. Reviewers were chosen by the editors to reflect a range of professional settings. Papers that passed the review process have been collected in the 'Papers' section of these proceedings.

The panel of people to whom papers were sent for review was extensive and the editors wish to thank them all:

Judy Anderson
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KEYNOTES

MORE THAN MATHEMATICS: DEVELOPING EFFECTIVE PROBLEM SOLVERS

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Complex, loosely-defined problems encountered in both the workplace and everyday life demand more than technical proficiency in mathematics. They also require broader capabilities including formulating problems, devising and implementing solution approaches, creativity, teamwork, project management, and communication skills. Significantly, these skills are often needed for any challenging mathematical problem — independent of whether it originates in the 'real world' or not.

This paper explores two questions. How do we meaningfully prepare our students with these skills in a mathematical setting? How can we develop and broaden their abilities and confidence in posing and solving mathematical problems? In this discussion I will draw on my experiences in working as an industrial mathematician, training workplace-ready students, and teaching a new course specifically designed to build mathematical thinking and problem-solving skills in pre-service teachers through games and puzzles.

Introduction

It is a privilege to give this lecture, which celebrates and honours Hanna Neumann. Biographical accounts show her to be a remarkable and inspirational person, who made significant contributions to both mathematics and mathematics education throughout her distinguished career, first in Britain (1938–63) and then Australia (1963–71). (I wholeheartedly recommend the biography by Newman and Wall (1974) which can be read online.) Among her many accomplishments, Hanna was the first woman appointed as a professor of mathematics in an Australian university (at the Australian National University in 1964). Hanna was also instrumental in the formation of the Australian Association of Mathematics Teachers (AAMT). The great esteem in which both communities still hold Hanna is evident, with the AAMT and the Australian Mathematical Society including named lectures at their society meetings in her honour.

Hanna strove to show that "doing and thinking about mathematics can be joyous human activities" and to make mathematics accessible and valued—for both its intrinsic beauty and its practical application—by all (Newman and Wall, 1974). She believed that "the community had to be educated to create a more favourable climate (one in which mathematics is not feared) for the learning of mathematics—especially among girls."

As Hanna herself said to a joint meeting of the Mathematical Association of South Australia and the Australian Mathematical Society in 1971 (Neumann, 1973):

> Speaking about teaching to trained teachers makes me very conscious of our, that is the university teachers', lack of training — I wish I had not got to say this! But this is by-the-way; perhaps I have been sufficiently long occupied, indeed at times pre-occupied, with the teaching side of my job that this undertaking is not entirely ludicrous.

With this firmly in mind I will discuss how we can develop, strengthen and broaden our students' mathematical skills and deepen their appreciation of the power of mathematics by linking to both engaging real-world problems and recreational puzzles. I will start with inspiration from my mathematical area of Operations Research.

Operations research: The science of making smarter decisions

Operations research (OR) is the scientific approach to decision making. In OR, there is not a single technique that can be used to solve all mathematical problems that arise. Rather, an OR practitioner like myself will select the most appropriate technique from across the mathematical sciences, including mathematical modelling, probability and statistics, optimisation, stochastic processes, simulation and game theory.

Operations research is a very practical discipline. From its origins in British military applications during World War I, OR has expanded to find significant implementation in many other areas. For example, some of the special issues of the journal *Interfaces*, which emphasises the impact of OR in organisations, have focused on:

- energy industry
- freight transportation and logistics
- mining
- humanitarian applications
- sports analytics and scheduling
- health care
- military applications
- finance
- marketing
- forest products industry.

This short list illustrates the breadth of OR; it would be easy to fill the page with more examples. (Try www.LearnAboutOR.co.uk for teacher resources of OR in everyday life.)

While mathematical analysis is at the heart of solving OR problems, Operations Research is more than mathematics. Taha (2011) comments that:

> OR is both a science and an art. It is a science by virtue of the mathematical techniques it embodies, and it is an art because the success of the phases leading to the solution of the mathematical model depends largely on the creativity and experience of the OR team.

Broadly, the five phases of implementing Operations Research in practice are:

1. *Defining the problem* and the objective(s). This is especially challenging for complex or ill-defined real-world problems. It usually entails observing the current system and comprehending a lot of domain-specific knowledge.
2. *Constructing a mathematical model*, including identifying any assumptions and simplifications.

3. *Solving the mathematical model*, which might require mathematical techniques ranging from elementary to sophisticated, or computer-based approaches for mathematically intractable problems.
4. *Validating the model*, typically by using historical data, covering a variety of situations, to determine if the model is an accurate representation of reality. A ‘common sense’ check is also recommended as is careful investigation of any ‘surprising’ solutions. Also ‘does the model solve the original problem?’.
5. *Implementing the solution or recommendations*, which includes translating and summarising techniques, analysis, outcomes and findings into appropriate and usually non-technical language for the ‘customer’.

This methodology is rarely linear and it is likely that phases will be revisited as the process of finding a solution evolves.

Industry problems[1] are nearly always too complex for one person to work on. In addition, the multi-disciplinary nature almost always requires teamwork with professionals in other fields.

Case study: Keeping trains on track and on time

My earliest experience in tackling industrial OR problems was though my PhD research (2002–09), which investigated operationally feasible methods to produce integrated train timetables and track maintenance schedules so that, when evaluated according to key performance criteria, the overall schedule is the best possible.

A train timetable specifies a path, with timing information, for each train through the network. A train timetable is represented graphically with a ‘string line diagram’. Figure 1 is one of the earliest train timetables (1885), from Paris to Lyon, and shows time on the horizontal axis and distance on the vertical axis. Existing methods schedule track maintenance once the train timetable has been determined, which almost certainly produces sub-optimal solutions.

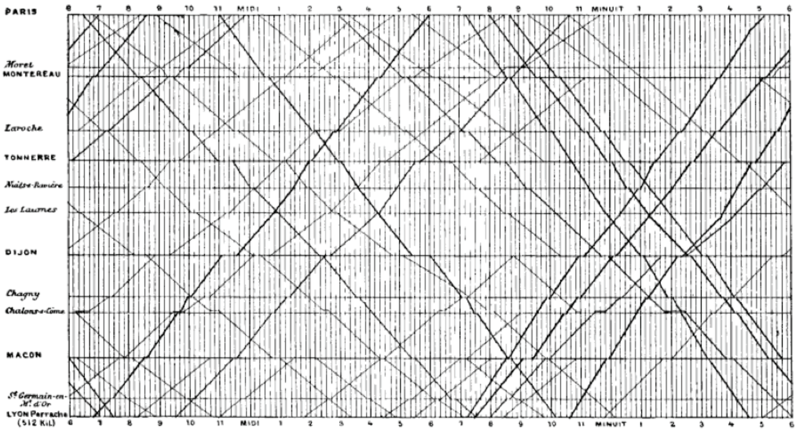

Figure 1. String line diagram (train timetable) from Paris to Lyon (1885).

After carefully observing the manual creation of timetables in train control centres, talking with train controllers and analysing historical data, I was able to formulate an integer linear programming model to mathematically describe the scheduling process. I

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[1] The term ‘industry problems’ is used to describe problems arising from outside of academia, including from business, industry, government and non-corporate entities.

painstakingly applied a sophisticated mathematical technique to find provably optimal solutions but found that it was not operationally feasible for realistic-sized problems. I then devised a fast probabilistic computer algorithm to efficiently generate thousands of alternative schedules in minutes from which good solutions could be selected. I benchmarked by simulating and comparing with the existing manual process. I wrote thousands of lines of computer code. I gave presentations to both industry and academia, and interviews to media. I wrote reports targeted at industry, mathematical papers for academia, and ultimately a PhD thesis. My work contributed to a larger project with an expected value of $7.6 million (2006 report) to the Australian rail industry and which was commercialised into industry-ready software. However, the research had only isolated uptake in industry, partly because we neglected to fully appreciate both human behaviour and the organisational shift required to implement our work. As Taha (2011) notes, "… while mathematical modelling is a cornerstone of OR, unquantifiable factors (such as human behaviour) must also be accounted for."

Despite the somewhat unsatisfying conclusion, this case study demonstrates the breadth of skills needed to meaningfully tackle challenging mathematical problems.

Maths in careers: Enhancing job prospects

A recent report commissioned by the Office of the Chief Scientist, *Australia's STEM workforce: a survey of employers,* and released on 29 April 2015, found that the top five skills of STEM[2] employees that are of highest importance to employers are: active learning (ability to learn on the job), critical thinking, complex problem-solving, creative problem-solving, and interpersonal skills. Respondents also listed skills important in their workplace in addition to those in the survey. The report found that "overwhelmingly, communication skills were most predominantly mentioned, followed by writing, project management, marketing, financial and leadership skills." When recruiting workers, academic results were considered of far less importance than interpersonal skills (with nearly half of respondents rating this as very important).

While the importance of technical skills (which was not specifically included in the 'skills and attributes' survey) was considered fundamental, more than 80% of employers agreed that people with STEM qualifications are valuable to the workplace even when their major field of study is not a prerequisite for their role. STEM employees were considered among the most innovative and adaptable in workplaces.

The study of mathematics—to whatever level—builds skills transferable to other disciplines and occupations. For example, studying mathematics requires the ability to think clearly, pay attention to detail, follow complex reasoning and construct logical arguments—valuable skills to have in many contexts! (See the extensive list from the University of Warwick (2011) of transferable skills obtained through study of mathematics.)

Careers in maths: The hidden mathematician

Job advertisements for those with mathematics and statistics qualifications rarely list it in the position titles. Mathematically-trained professionals work across a diverse range of industry, business and government sectors. The yearly 'Maths Ad(d)s' publication[3] by

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[2] Science, Technology, Engineering, and Mathematics.

[3] Maths Ad(d)s booklets can be ordered from AMSI (careers.amsi.org.au/mathsadds) or downloaded as a free PDF.

the Australian Mathematical Sciences Institute (AMSI) gathers together a sample of job advertisements for careers requiring training in mathematics and statistics. Recent Maths Ad(d)s include: bioinformatician, remuneration analyst, software programmer, data scientist, epidemiologist, energy forecaster, credit risk modeller, market researcher, meteorologist, behavioural scientist, operations manager, quantitative analyst, teacher, and groundwater modeller. For more than 101 careers in mathematics, see the book with the same name published by the Mathematical Association of America (Sterrett, 2014).

Training our future mathematically-capable professionals

So, how do we meaningfully prepare our students to be mathematically capable and workplace ready? I offer two models with which I have had experience: one at tertiary level and one at secondary level.

The UniSA Mathematics Clinic

The Mathematics Clinic was started in 1973 by the Mathematics Department at Harvey Mudd College (HMC) in Claremont, California, to train industry-bound mathematics graduates. It is a sister program to the Engineering Clinic established in the early 1960s by the Engineering Department at HMC and was considered to be like medical clinics which provide interns with hands-on real-world experience in a supervised environment (Borrelli, 2010). The Clinic solicits real-world open-ended problems (called projects) from companies and government agencies. Over 150 Mathematics Clinic projects have been conducted at HMC[4].

The Mathematics Clinic[5] program at the University of South Australia (UniSA) is based on the HMC model and is the only one of its type in Australia. It is a year-long team-based sponsored project undertaken by final-year mathematics undergraduates. It provides a rigorous research experience in tackling mathematics problems sourced from industry. It is primarily an educational program, although it also delivers products (e.g. mathematical algorithms, reports, presentations, analyses, software).

In 2014 the UniSA Mathematics Clinic team developed mathematical techniques and prototype software to aid the Australian Army in determining both the number and type of new land combat vehicles required to replace the existing fleet. The Clinic project contributed to Land 400—the largest project the Army has ever undertaken. The Defence Science and Technology Organisation (DSTO) was the project sponsor.

A UniSA Clinic team typically consists of four students, an Academic Supervisor, an Academic Consultant who provides specialised mathematical expertise (which in 2014 was me), and a sponsor Liaison to monitor progress and provide domain-specific expertise. A student is appointed as the Project Manager; others take on leadership roles throughout the year, typically to lead the completion of a project deliverable.

The first deliverable is the Work Statement—the formal agreement between the team and the sponsor to accomplish certain tasks and produce certain products. It drives the process of defining the problem, as well as planning the management of and schedule

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[4] www.math.hmc.edu/clinic/

[5] The current UniSA Mathematics Clinic Director is Associate Professor Lesley Ward. More details at www.unisa.edu.au/IT-Engineering-and-the-Environment/Information-Technology-and-Mathematical-Sciences/Maths-Clinic/

for the entire project. Writing the Work Statement is often the first meaningful experience that students have in translating a complex real-world problem into a form that can be tackled mathematically. One particular challenge for the 2014 Clinic team was to identify how they could add value to Land 400 within the constraints of a year-long student project. The team decided to analyse the number of vehicles required for three specific types of missions (screening an object from the enemy, searching for a target within an area, and clearing an urban area of enemies). They also reviewed and compared several multi-criteria decision-making methods that could be used to select which vehicles to purchase, as different types of vehicles are deemed more (or less) suitable depending on which characteristic they are judged by.

Preparing the Work Statement requires intensive discussions between the liaison and the team, and usually a site visit. Our students discovered the acronym-rich world of Defence and by the end of the project became conversant in the jargon—a large shift from the start of the year!

The mathematical effort required to tackle the project is initiated and undertaken by the students with guidance from the academic advisors. The students meet regularly for work sessions and to divide up the tasks. The 2014 Clinic team needed to juggle four sub-projects and formed smaller work-teams for some tasks. Regular meetings are also held with the sponsor liaison and academic staff to provide updates and to seek clarification or guidance. Students are coached in chairing meetings, preparing agendas, managing projects, interacting with the sponsor, building and maintaining effective teams, dealing with inevitable problems and issues, and re-negotiating the scope of the project if needed.

Clinic teams give a formal mid-project written report and presentation. The project culminates with a professional-quality final report, presentation to the sponsor in their workplace, and delivery of project outputs. Clinic teams often write academic papers and give conference presentations. Students are trained in developing these skills, and in delivering and targeting a technical message to be accessible by different audiences.

The Mathematics Clinic is an enriching experience that builds not only professional skills, but student self-confidence. It demonstrates the link between mathematical knowledge and application, and provides exposure to a potential employer. It was a real joy to watch the 2014 Clinic students blossom into self-directed, capable, industry-ready mathematics graduates.

School to work

The 2010 SACE[6] 'School to Work' Mathematics and Science programs were designed to partner schools with industry or universities to devise innovative ways to make maths and science interesting and relevant, ensure students understand the value of maths and science, and help to connect students with maths and science career paths.

I was involved in the 'Developing Mathematicians' project[7] at St Michael's College in Adelaide which provided a structured year-long program to engage Year 10 students in genuine, relevant and challenging mathematical research experiences and build transferable skills in preparation for both the SACE Research Project and their future

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[6] South Australian Certificate of Education

[7] The project was led at St Michael's College by Dr Pauline Carter, Mark Winston and Carmen Swan. Dr Amie Albrecht was the supporting mathematician.

careers. The project deliberately took on a gendered focus by bringing together two single-gendered classes (25 boys and 24 girls) and including a female mathematician.

In the first two terms, students were guided through the train planning process, beginning with demographic analysis of the travel demand, designing train services to meet demand, and constructing detailed operational train timetables. Sessions were loosely directed with a workbook and centred around group discussion and a hands-on approach. Teachers, a university mathematician and old scholars assisted as ‘guides on the side’. Students experienced the complexities of authentic mathematical work.

Having developed skills and confidence, in the second half of the year students worked on problems from a local steel industry in a model that parallels the Mathematics Clinic. The teachers and mathematician did some pre-work to develop simplified realistic scenarios. Students designed mathematical algorithms and spreadsheets to determine material requirements for orders in fencing and roofing. They visited the site and eventually made final presentations to the industry liaison.

A large part of the success of the project was the authenticity brought by engaging students and teachers with mathematicians and industry clients. Several of the students continued working on the problems beyond the end of the project. We observed that female students developed a more confident and resilient approach to learning mathematics, were more receptive to tackling a challenge, and less reliant on teacher direction. Students’ understanding of mathematical careers were broadened.

The two models I have described are significant undertakings, but hopefully they provide inspiration for bringing elements of the experience into the classroom. Authentic problems can be obtained from many places; a teacher at St Michael’s College had a connection with the steel company. Enlist old scholars or perhaps link with the Mathematicians in Schools[8] initiative. Peruse the archive of stimulating projects from the Moody’s Mega Math Challenge[9] in the US, which is accompanied by an excellent handbook describing the modelling process (there are other similar competitions, both locally and globally). Draw on problems that students encounter in their own lives. I also encourage you to look for engaging questions in your own surroundings; Dan Meyer’s 101questions site[10] has an ever-growing collection of contributed photos and videos that provoke questions and are suitable for projects.

Puzzles and other pastimes

Engaging problems are not confined to the ‘real world’. The ‘Cheryl’s birthday’ puzzle recently captivated the attention of many, particularly via social media. It is as follows[11]:

> Albert and Bernard just became friends with Cheryl and they want to know when her birthday is. She gives them a list of 10 possible dates:
>
> | | | |
> |---|---|---|
> | 15 May | 16 May | 19 May |
> | 17 June | 18 June | |
> | 14 July | 16 July | |
> | 14 August | 15 August | 17 August |
>
> Cheryl then whispers to Albert which month her birthday is in, and whispers to Bernard which day her birthday is on.
>
> Albert: I don't know when Cheryl's birthday is, but I can be certain that Bernard does not know either.

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[8] www.mathematiciansinschools.edu.au

[9] m3challenge.siam.org

[10] www.101qs.com

[11] This is Rob Eastaway’s version, reworded for clarity. Source: www.robeastaway.com/blog.

> Bernard: At first I didn't know when Cheryl's birthday is, but now I do know.
> Albert: Then I also know when Cheryl's birthday is.
> So when is Cheryl's birthday?

This scenario is highly unlikely to occur in real life—your students will probably tell you just to ‘friend’ Cheryl on Facebook and then you’ll be notified when her birthday[12] comes along!—but puzzles such as these can be as stimulating and challenging as those that arise from real-world contexts. Like many people, I can be engrossed solving Sudokus, struggling with a Rubik’s Cube or a logic puzzle like ‘Cheryl’s birthday’, sliding tiles in Tetris, or suspecting that the three utilities problem is impossible. Many board and electronic games require logical thinking or strategising—and are also fun!

Companies like Google and Microsoft famously ask potential employees to solve brainteasers and maths riddles for a good reason; these puzzles require the types of ‘thinking skills’ that employers are looking for. So, can we intentionally develop these mathematical thinking skills in our students by using puzzles?

Puzzle-based learning

A few years ago, I was inspired by the puzzle-based learning approach of Michalewicz and Michalewicz (2008) which “focuses on getting students to think about framing and solving unstructured problems (those that are not encountered at the end of some textbook chapter)” with the purpose of increasing “the student’s mathematical awareness and problem-solving skills by solving a variety of puzzles and reflecting on their solution processes” (Meyer et al., 2014, p. ix). As the authors point out, the educational use of puzzles is not new, with a long history preceding that of their work and 20th century champions such as Gyorgy Polya and Martin Gardner. However, the puzzle-based learning course (currently taught at the University of Adelaide, where Z. Michalewicz is Emeritus Professor in Computer Science) was the first structured course using puzzles that I had seen, with a syllabus organised around problem-solving techniques and various mathematical topics. Students in the course at the University of Adelaide are assumed to have knowledge of SACE Stage 2 Mathematical Studies but it is not a prerequisite. Details of how the course runs are in Falkner et al. (2010).

Training our future teachers

At about the same time that I discovered puzzle-based learning, I was reflecting on the way in which we at UniSA teach mathematical content (as distinct from pedagogy) to our pre-service primary and middle teachers who choose mathematics as a specialisation, and whether it exemplifies the types of experiences I want them to have.

The mathematical prerequisite for entry into our primary and middle teaching degree is one semester of Year 11 mathematics (as a result of completing the SACE, although some students have done more mathematics). Despite having chosen it as an area of specialisation, many students describe themselves as lacking confidence or experiencing anxiety when doing mathematics—especially when tackling problems that are unfamiliar or challenging.

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[12] Cheryl’s birthday is July 16.

The importance of being stuck

In my experience, most students associate making mistakes and being stuck with an inability to do mathematics. In contrast, professional mathematicians spend much of their time quite comfortably feeling unsure of the next step. This is often hidden from students as Mason et al. (2010, p. ix) describe:

> Elegant solutions such as are found in most mathematics texts rarely spring fully formed from someone's brain. They are more often arrived at after a long and tortuous period of thinking and not thinking, with much modification and changing of understanding along the way, but most beginners do not realise this. ... Elegance can come later.

I want my students to feel stuck. I am glad when they are. Research about the brain shows that "when students make a mistake in maths, their brains grow, synapses fire, and connections are made. This finding tells us that we want students to make mistakes in maths class and that students should not view mistakes as learning failures but as learning achievements" (Boaler, 2015, p. xix). Encouraging our students to develop a growth mindset ('the more they work the smarter they will get') rather than a fixed mindset ('some people are naturally good at maths and some are not') is key in their mathematical development. It is important though, that this struggle is productive. We can assist by equipping our students with an awareness of the different phases and processes that take part in mathematical thinking, as well as acknowledging and discussing their emotional responses—both negative and positive.

A course in 'Developing Mathematical Thinking'

In 2014 at UniSA we offered a new elective course to pre-service primary and middle teachers called 'Developing Mathematical Thinking'. In its design, I drew together many of the elements I have already elaborated on—stimulating curiosity, promoting mathematical communication, developing confidence, increasing problem-solving skills—and an intention to use mainly puzzles, some of which are 'hands on'. Of particular interest were 'low-threshold, high-ceiling' activities "which pretty well everyone in the group can begin, and then work on at their own level of engagement, but which [have] lots of possibilities for the participants to do much more challenging mathematics" (McClure, 2011). (The NRICH Project[13] is a good source of activities.)

To articulate mathematical processes and phases of work, the course rests heavily on the framework provided in the book *Thinking mathematically* (Mason et al., 2010). Each week is dedicated to one or more themes. Students work, mostly in groups and at their own pace, on problems that require a range of mathematical techniques but that are purposely selected by me to reinforce the week's themes. I will occasionally teach a specific mathematical concept to the whole class, but it is more likely that students will teach each other as needed. The weekly themes include:

- *Specialising* (trying specific examples to get to grips with a problem) and *generalising* (detecting a pattern that holds for a wide class of specific cases).
- *The Entry phase* in which we work to understand the problem.
  - *What do I know* (from the question or from experience)?
  - *What do I want?* To find an answer? To prove something?

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[13] nrich.maths.org

  - *What can I introduce?* Definitions, notation, diagrams, tables, physical models, other ways to systematically record work.
  - Framing our own questions (for example, using 101qs.com as starters).
- The role of intuition.
- Strategies for ‘getting unstuck’.
- Working *systematically* to detect patterns or expose cases for which a theory might not hold.
- *The Attack phase* in which we try to solve the problem.
  - Making conjectures.
  - Justifying and convincing—yourself, a friend, an enemy.
  - What it means to prove something and why we need proof. (Detecting a pattern that holds for a few cases is not enough!)
- *The Review phase* which includes checking and extending work.
- Writing to record your thinking. Writing to explain your work to someone else.

We discuss these themes in class, and students reflect on them in their formative and summative assessment work. Although I could elaborate on many elements of this course, I will conclude by describing two—a small weekly ‘starter’ and the student-selected major projects.

Weekly starter: Exposing multiple problem-solving approaches

To encourage exploration of multiple valid ways to solve a problem and to promote mathematical discussion, each week starts with a ‘visual pattern’, usually drawn from Fawn Nguyen’s site www.visualpatterns.org. An example is shown in Figure 2.

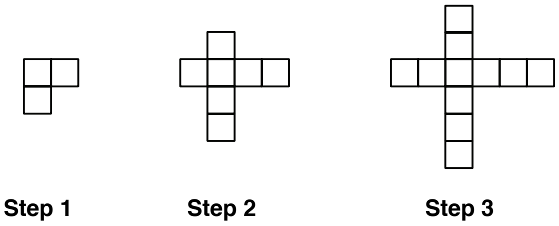
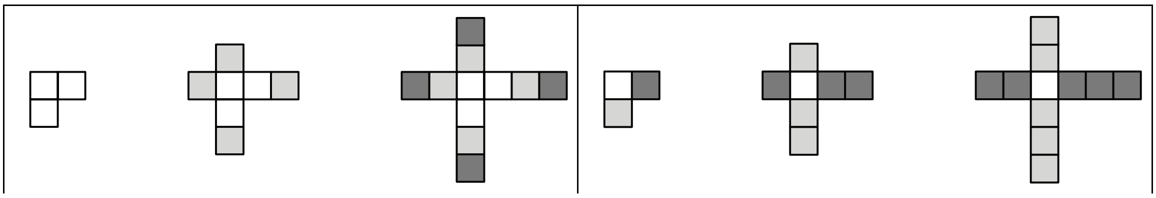


Figure 2. Visual pattern.

Students work individually on the following questions for about five minutes.

- Draw the figure in the next step.
- How many squares are in the figure you just drew?
- How many squares are in Step 43?
- What is the equation for this pattern? (How many squares in Step *n*?)

Students then discuss with their neighbours to help improve their approach. We then gather together many different approaches as a class, with students sharing their reasoning. Examples are shown in Figure 3. Each equation can be algebraically shown to be equivalent to $4n - 1$. We equally celebrate incomplete and complete approaches.



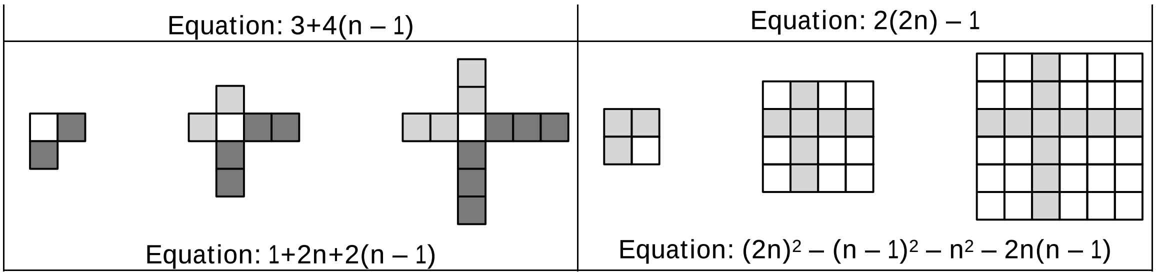


Figure 3. Four possible approaches to the equation for the pattern in Figure 1. The shading is intended to convey the way in which the pattern grows or the figure can be decomposed.

Student projects

The project is a major body of work, drawing together concepts from the entire course. It is an in-depth investigation by an individual student or pair of students on a topic of their own choosing. The project comprises several structured activities over the semester which together are worth more than half of a student's final grade.

Students discuss their choice of topic with me in Week 5. I provide a range of possible ideas and students can also propose their own. Here is a selection from 2014:

- Grid-based logic puzzles like Sudoku. How many different puzzles are there for a given grid size? How are they created to ensure only one solution? What are the best solving strategies?
- Board and card games like the Game of SET in which the goal is to form SETs where each attribute is 'all-same' or 'all-different' across three cards. One student investigated the maximum number of cards that do not contain a SET.
- Mathematical art like fractals, and algorithmic drawing in which artwork is created by following a sequence of rules (e.g. turn 90 degrees clockwise and draw a line with length from a pre-specified list). What are the rules that determine if the drawing is a closed loop or has other interesting properties?
- Physical puzzles like the Tower of Hanoi. What is the minimum number of moves required for *n* disks? How does this change if the number of towers changes? What if the rules for placing disks on towers are altered?
- Pattern and algebra-based questions like the Leap Frog Puzzle or extending the question 'How many squares on a chessboard?' to triangular grids.

In Week 8 students obtain feedback on a draft report from me and via peer review (which is supported in various ways). The purpose of the draft is to help develop and distil ideas; uncover any flaws in the problem-solving work; practice mathematical writing; and enable mid-course corrections, if needed. It encourages the process of revising existing work and incorporating new work. Students meet again with me in Week 10 to discuss their draft. The polished final report is submitted in Week 14 and accompanied by a final 10-minute oral presentation to the class.

Students are trained in giving mathematical talks through weekly smaller talks designed to progressively build skills and confidence. Students start by 'presenting' at their desks to a friend and with the support of their notes. They gradually advance to presenting at the whiteboard, to classmates they might not know, with props and carefully chosen examples. The length of the talks increases incrementally. By the time students give their final presentations, they are experienced at delivering quality talks.

While it is too early to measure the impact that this course has on their further mathematics study, students' reflections show the effects they most valued: "I was encouraged to use a variety of strategies to complete tasks and extend my thinking to a higher level", "I was constantly challenged but never felt I was out of my depth", "The course has developed my math knowledge and expanded it to approach problems differently" and "Working in groups not only broadened my knowledge but it also made me understand some other ways in which different people think".

For me, helping our prospective mathematics teachers develop into enthusiastic, confident and capable problem solvers gives me both great satisfaction and optimism for the impact they will have on their own students in the future.

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LINKING COMMUNITIES OF RESEARCH AND PRACTICE IN MATHEMATICS EDUCATION

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This paper explores different ways in which researchers and teachers can work together to develop new knowledge in mathematics education. A framework for analysing researcher–teacher relationships is presented and then used to compare ways in which I have worked with teachers in three research projects that differed in terms of (1) how the relationship was initiated, (2) how roles and relationships were negotiated, and (3) who chose the research topic and research questions. Through this comparison I hope to stimulate professional conversations about the possibilities and challenges of researcher–teacher collaboration.

Introduction

Most university-based mathematics educators would claim that their research is motivated by a desire to enhance the quality of mathematics teaching and learning in schools—yet education research is often criticised for its lack of impact on, and relevance to, classroom practice. A second criticism of research is that it often excludes teachers from decision-making about the research project, reflecting the unequal relationship between researchers and teachers who participate together in classroom based studies. This so-called “research–practice gap” has sometimes been explained by reference to the different practices that researchers and teachers use to understand and improve teaching and learning, and the different forms of knowledge that result from these practices. Formal research seeks to develop generalisations about educational phenomena, and these theoretical generalisations can seem far removed from the immediate needs of teachers who seek to address practical day-to-day problems (Wiliam, 2003). This tension between the different aims of formal research and teachers’ practical inquiry is also evident in the asymmetries of power between researchers and teachers, since it is most often teachers who are co-opted into the research agenda of university academics (Heid et al., 2006).

Instead of dwelling on the “research–practice gap”, I want to encourage participants in this conference to think about *how researchers and teachers can work together to develop both theoretical and practical knowledge in mathematics education*. To do this, I will outline a framework for describing different types of researcher–teacher relationships that asks (1) how and why a research partnership is initiated, (2) how roles and relationships are negotiated, and (3) who chooses the research topic and who ultimately benefits from the research. I will use this framework to compare and critique ways in which I have worked with teachers in three different types of research projects, illustrating three different ways of linking communities of research and practice.

Learning as social practice

The idea of *communities of practice*, introduced by Wenger (1998), can help build understanding of the professional work of researchers and teachers. A community of practice is characterised by mutual engagement of participants, negotiation of a joint enterprise (such as teaching mathematics in a primary or secondary school), and development of a shared repertoire of resources (language, tools, theories, rituals) for making meaning of practice. Mathematics teachers in schools and mathematics education researchers based in universities would claim membership of related, but distinct, communities of practice. Any community has "insiders" and "outsiders", but Wenger writes of the various ways in which communities can be connected across the boundaries that define them.

One form of connection is described as a *boundary encounter*. The briefest is a *one-on-one* conversation between two individuals from different communities to help advance the boundary relationship. For example, a teacher participating in a professional development workshop presented by a university researcher might approach the presenter afterwards to ask questions or discuss ideas for practice. A more sustained encounter involves *immersion* in the practice of another community via a site visit, such as when a researcher visits a school over a period of weeks or months to collect classroom data. Both of these cases involve only one-way connections between communities. A two-way connection can be established when *delegations* comprising participants from each community are involved in a boundary encounter. If such an encounter provides a forum for mutual engagement, then a new *boundary practice* starts to emerge and becomes a longer-term way of connecting communities in order to coordinate perspectives or to resolve common problems.

Another form of connection involves creation of a *periphery* that makes the boundaries between communities more permeable. People who do not intend to become full members of a community can nevertheless engage peripherally in some of the community practices.

I will use the notions of *boundary encounters* and *peripheries* to identify characteristics of productive collaborations between researchers and teachers when working on mathematics education projects.

A framework for analysing teacher–researcher relationships

Table 1 shows the framework I developed with colleagues to analyse teacher–researcher relationships in conducting joint research projects (Novotná & Goos, 2007). This framework emerged from discussions between researchers and teachers over several conferences of the International Group for the Psychology of Mathematics Education.

An important question to consider in terms of *beginning the researcher–teacher partnership* is how teachers enter into this process and who initiates the research. At times, a university-based researcher seeks out teachers to participate in a project that has already been planned. Occasionally a partnership might be initiated by a teacher who seeks out a university-based researcher. Alternatively, teachers could be encouraged or required by a school administration or government education department to enter into a university-based research project. In all these instances it is worth considering “why” as well as “how” such research partnerships are initiated.

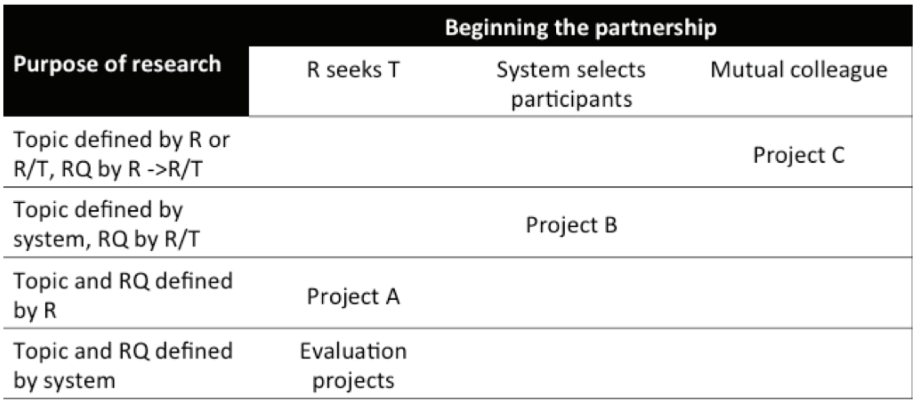
Table 1. Framework for analysing researcher–teacher relationships.

Beginning the partnership	Participants	Purposes of the research
Why? Researcher motivation Teacher motivation How? Researcher seeks teacher Teacher seeks researcher Education system selects participants	Roles & expectations Language & communication of findings Trust/relationships Communities & asymmetric needs	Topic (who chooses?) Research questions (whose?) Benefits (for whom?)

Other questions relate to how the *participants* will interact and the *purposes* of the research. The extent to which roles are shared between teacher and researcher is an issue: there may be benefits and disadvantages in either maintaining strong role separation or sharing/swapping roles (teacher-as-researcher or researcher-as-teacher). However roles are determined, expectations need to be made clear from the start as a foundation for building trust and mutual respect. Because these two communities use and value different forms of language, some thought needs to be given to how to communicate the findings of research to non-researcher audiences. The *purposes of the research* may depend upon how the partnership is initiated, since this often influences the choice of topic, negotiation of research questions, and realisation of any benefits for theory, practice, or policy development.

How have I worked with teachers in my own research?

To investigate the question driving this paper—*How can researchers and teachers work together to develop both theoretical and practical knowledge in mathematics education?*—I used the framework shown in Table 1 to analyse the funded research projects I have conducted since beginning my PhD research study. I eliminated projects that did not involve interaction with school teachers, for example, projects investigating learning and teaching in higher education where the teachers were academic colleagues. This left a list of 16 projects from 1994–2015, involving teachers in more than 150 schools. I developed an analytical matrix that partially categorised these projects along two dimensions of the framework presented in Table 1, “Beginning the partnership” and “Purposes of the research”; that is, I assigned projects to the matrix cells based on *how* the partnership was initiated and who chose the *topic* and *research questions*. The next step was to “fill in” the cells with more detail about the third dimension of the framework, “Participants”. To do this, I described participant *roles* as either separate, shared, or dual (if the researcher also had a role as a teacher educator or professional developer); developed a proxy indicator of *relationships* in terms of the frequency and duration of researcher–teacher interactions; and indicated who *communicated the findings* to research and professional *communities* in the form of publications and conference presentations.



Purpose of research	Beginning the partnership		
	R seeks T	System selects participants	Mutual colleague
Topic defined by R or R/T, RQ by R ->R/T			Project C
Topic defined by system, RQ by R/T		Project B	
Topic and RQ defined by R	Project A		
Topic and RQ defined by system	Evaluation projects		

Figure 1. Analytical matrix.
(R=researcher, T=teacher, RQ=research question)

A simplified version of the resulting matrix is presented in Figure 1. It identifies four clusters of projects. For the purpose of this paper I chose not to analyse the cluster comprising evaluation projects commissioned by government agencies for which I had no part in defining the topic or research questions. I then selected one project from each of the remaining clusters, and labelled them A, B and C. These projects illustrate the three approaches I have used to work with teachers on classroom-based research.

Project A: Research with pre-service and beginning teachers

The project

The first project is typical of research conducted by mathematics teacher educators with their pre-service students. This was a longitudinal study conducted in two waves, from 2000–2004 with successive cohorts of my own teacher education students (Goos & Bennison, 2008), with a follow up study from 2006–2008 in which some graduates from the first study participated as beginning teachers (Goos, 2014). The aims were to analyse processes through which a technology-enriched community of practice was established and maintained in the pre-service teacher education program, and to identify and analyse factors influencing the use of technology in mathematics classrooms. To address the first aim, I analysed bulletin board discussions between the pre-service teachers during and after the teacher education program. To address the second aim, I carried out longitudinal case studies of selected participants during their practicum sessions and in the early years of their professional experience after graduation.

The researcher–teacher relationship

This researcher–teacher *partnership was initiated* as soon as the pre-service students enrolled in my course and therefore became prospective research participants. I invited all my students to participate and explained that the case study component of the project would continue after they graduated.

There was no sharing of teacher and researcher roles amongst *participants*, but role boundaries became blurred in another way in that I had the dual roles of teacher educator and researcher. While participation was voluntary (in keeping with the University's ethical guidelines for research), it is possible that students' participation was motivated by their relationship with me as the teacher-educator. Despite the protection provided by ethics approval processes it can be difficult in these circumstances to negotiate the power relationship that exists between the researcher and the researched. Thus, expectations regarding participant roles—beyond the broadly defined activities outlined in the research information sheet and consent form—were never explicitly discussed. With regard to communicating the findings from the research, although I often helped the pre-service teachers to publish their technology integration work in professional journals (e.g., Quinn & Berry, 2006), I did not take advantage of opportunities for them to share their research experiences with a wider audience. Instead, I filtered their experiences through my own research perspective when I communicated findings from this project to the research and professional communities.

The *purposes of the research* arose from my experiences of teaching previous cohorts of students and my observations of the potential for technologically knowledgeable pre-service and beginning teachers to act as change agents in schools. This was my own motivation for the project. Thus the teacher-participants unknowingly influenced the topic and research questions without having any direct input into their formation. Several of these beginning teachers later approached me to volunteer for other research projects, which may indicate they gained some benefit from participation.

I would describe this researcher–teacher relationship as one where the researcher-as-teacher-educator is a more experienced member of the community of professional practice, while the pre-service and beginning teachers are newcomers (Goos & Bennison, 2008). Thus the relationship is concerned with helping pre-service teachers to gain entry to a professional community of mathematics teaching.

Project B: Research-based professional development

The project

This one-year professional development project was commissioned by a government department of education in 2009 (Goos, Geiger & Dole, 2014). Its purpose was to support teachers to embed numeracy across the middle years curriculum. The research aims of the project were to investigate changes in teachers' instructional practices, personal conceptions of numeracy, and confidence in numeracy teaching.

Twenty teachers in ten primary and secondary schools participated in the study, working through two action research cycles of numeracy curriculum implementation. The research team provided three whole-day professional development workshops throughout the year to support teachers' planning and evaluation, and visited teachers in their schools on two occasions between the workshops to offer further advice and feedback. The research data comprised surveys of numeracy teaching confidence, workshop tasks that elicited teachers' developing understanding of the numeracy model, lesson observations, interviews with teachers and their students, curriculum planning documents, and student work samples.

The researcher–teacher relationship

The researcher–teacher *partnership was initiated* by the government education department that recruited schools to participate. As a result, the research team did not meet the teachers until the first professional development workshop. It appeared that most of the teachers had volunteered to participate, and most seemed interested and engaged throughout the year of the project.

Clear role distinctions were maintained by *participants* who were either teachers or researchers. However, the researchers filled dual roles as professional developers who were expected to bring about change in teaching practice. Findings from this project were communicated to the research community via conference presentations and publications (e.g., Goos, Geiger & Dole 2011, 2014). The research team also invited four teachers each to serve as lead author of a set of four articles that were published together in *The Australian Mathematics Teacher*, one of AAMT's professional journals (e.g., Willis et al., 2012). The project thus gave these teachers a voice and provided them with opportunities to share their practical inquiry with colleagues in other schools.

The broad *purposes of the research* were determined by the education department and the deliverables were made explicit through a contractual agreement between the department and the university. However, the pedagogical focus of the project was jointly determined by the researchers and teachers. Thus, although the projects' broad research questions were defined by the researchers, the teachers identified their own personal goals in exploring different aspects of numeracy. Our interview data show that more than half of the teachers claimed they now had greater understanding of the cross curricular nature of numeracy.

In this second project, the researcher–teacher relationship could be described as a series of one-way *immersive boundary encounters* between two communities, because the researchers immersed themselves in the practices of the professional teaching community via site visits to schools. In addition, some teachers also engaged *peripherally* in publishing practices more common to research communities when they co-authored articles with the researchers in mathematics teaching journals. However, although there was some permeability in the boundaries around the two communities, no boundary practices emerged that could connect the researcher and teacher communities beyond the life of the project.

Project C: A collaborative research relationship

The project

The third project is a long-term partnership between myself and a secondary school mathematics teacher. The partnership began in 1994 with my PhD research, most of which was conducted in this teacher's classroom over a two year period. One of the aims of my doctoral study was to investigate the teacher's role in creating a classroom culture that supported students' mathematical thinking and sense-making (see Goos, 2004). I used the concept of a classroom community of inquiry to help me understand how this teacher structured learning activities and social interactions to develop his students' mathematical thinking.

The researcher–teacher relationship

After I completed my doctoral research, the teacher enrolled in his own PhD under my supervision. During his candidature he also took an active and high profile role in professional associations, culminating with his election as President of the Australian Association of Mathematics Teachers. In 2005, he embarked on a career change, leaving his job as a school teacher to take up a tenurable position as a university academic. We continue to collaborate on other research projects (e.g., Project B in this paper). The analysis that follows draws on an extended conversation that we recorded in preparation for writing a journal article about this researcher–teacher relationship (Goos & Geiger, 2006).

Initiation of the partnership came about when the teacher and I were introduced to each other by our former pre-service teacher education lecturer, who had become my PhD supervisor. At the time, the teacher had recently completed a Masters degree and was motivated to participate in my research by his desire to resume regular professional conversations with a university researcher. Thus there was some equity in the partnership from the start in terms of its initiation and our underlying motivations.

As *participants*, although we agreed to keep our roles separate, the nature and distinctiveness of these roles changed over time as we developed mutual trust. At the time I was a novice researcher as well as a novice teacher, and thus I was conscious of the kind of respectful relationship that needed to be established with this experienced teacher if the research was to be productive. The teacher explained how he valued my presence as someone who "only ever asked me *why* I was doing things, you never made any judgmental comments" (Goos & Geiger, 2006, p. 37). However, my efforts to understand did result in post-lesson discussions that led him to modify his teaching plans for the next lesson—thus making me more of a participant than a passive observer.

We explicitly negotiated issues related to power and what each of us wanted to achieve out of the collaboration as we began to write and present papers together at research conferences. He believed that "teachers' voices ... have to be heard if research is going to make a difference to teaching and learning in schools" (Goos & Geiger, 2006, p. 38), and he saw jointly authored publications as acknowledging his equal contribution to creation of the new knowledge reported therein. Likewise, I gained credibility with practising teachers through joint presentations at professional development conferences where he was well known because of his leadership and advocacy roles in teacher professional associations. This was how we introduced each other into the distinct sub-cultures of mathematics education to which we separately belonged—the community of educational researchers and the community of teachers—and how we learned to communicate with different audiences using the language of research and the language of practice. Thus our goals and needs, although different, were mutually recognised and valued.

Although the *purposes of the research* were determined initially by my interests in that I proposed the initial topics and research questions, this situation evolved into a more equal arrangement when the teacher enrolled in his own PhD under my supervision. In later projects we have jointly determined the research focus and questions, with mutually beneficial outcomes.

This third example shows how a two-way connection developed between researcher and teacher. As well as immersing ourselves in the practices of each other's communities, for example, by giving joint presentations at both research and professional conferences, we also created *boundary practices* that explored the different perspectives of each community. Our conversation about what we learned through working at the boundary is one example of such a practice. We also took advantage of our leadership roles in national mathematics teacher and mathematics education research organisations to share and discuss our boundary practices with members of our respective communities.

Conclusions and implications

This paper has considered how researchers and teachers could work together to develop the kinds of knowledge most valued by their respective communities of practice. Different kinds of connections between communities were discussed, including boundary encounters that might lead to longer lasting, two-way boundary practices, and peripheral participation by members of one community in some of the practices of another community. I used three of my own research projects as examples of these links. Project A, with pre-service and beginning teachers, involved a one-way connection where I made site visits to schools. My overall aim was to help the participants gain entry to the community of professional teaching practice. Project B, a professional development study with practising teachers to embed numeracy across the curriculum, also involved one-way connections and school site visits. Another form of connection in this project was the peripheral participation of teachers in some of the publishing practices that are common in the research community. Only in Project C was a two-way connection established between the teacher and researcher communities through the development of long-term boundary practices. This project was characterised by *mutuality* of the researcher's and teacher's motivations, roles, and purposes, and *complementarity* of their expertise and knowledge.

It is interesting to think about how Projects A and B, or other projects like them, could grow into the more equitable type of researcher–teacher partnership evident in Project C. For example, it is possible that research partnerships with my pre-service students and graduates will continue to evolve over time towards mutually agreed goals, or at least goals that are mutually beneficial. It is more difficult to work towards shared goals and decision-making in projects initiated and funded by education systems that define the research or professional development focus and select the schools that will participate. Instead, it may be possible for the researcher to develop longer lasting relationships with funding bodies that lead to follow up projects with broader scale and focus. This has been the case with Project B, where in subsequent projects I have worked with school principals, curriculum leaders, education systems and teacher registration authorities to develop numeracy curriculum frameworks and teaching resources that can be shared with larger numbers of teachers throughout an entire educational jurisdiction.

The framework shown in Table 1 could be used as a template for designing research projects that are beneficial for the communities of mathematics teachers and mathematics education researchers, and for negotiating researcher–teacher relationships and the expectations that these entail. The comparison of the three types of research project I presented above could also stimulate professional conversations within and between these two communities about the possibilities and challenges of researcher–teacher collaboration. To get the conversation started, I offer the following two questions:

1. In communicating the findings from research with teachers, who should speak for whom and to whom?

2. What conditions are needed for researchers and teachers to explore each other's roles and understand how their respective communities develop theoretical and practical knowledge?

These questions invite readers to consider how researchers and teachers might both develop professional knowledge and "principled practice" (Heid et al., 2006, p. 78).

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LINKING IT ALL TOGETHER

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This paper is a summary of the opening keynote at the AAMT conference 2015. It brings together various connections within mathematics through considering the use of rich tasks.

Introduction

Thank you so much for inviting me. I am delighted to be in Australia again, and in Adelaide, and I am greatly looking forward to conversations over the next few days about the work that you are doing in mathematics education.

It was over a year ago, at the BCME conference in Nottingham, when Will invited me. At that time I was the Director of NRICH[14]—a project which I know many of you already use. Since then I have changed job so the focus of my talk today will be slightly different from the title and abstract that I originally sent. In this plenary then I would like to combine some of what I had intended to talk about, with some new thoughts that have come to me as a result of my changed role.

Learners' views

Before I do that I am going to show you some short excerpts of English and Scottish children, mostly primary learners but not exclusively, talking about their experiences of mathematics. These film clips originated in a research project I did where I was interested in finding out what came to mind when learners thought about their mathematics lessons. I asked them several other questions too, which I will share with you later, but for now here are three or four short pieces for you to watch.

Whilst you are watching them, I would like you to consider two questions: firstly what are these children's views of mathematics, and secondly, if you asked your own students the same sort of question, would you get the same sort of responses or something different?

Mathematics to them is a set of mostly unconnected content. They talk about having 'done' fractions or decimals, for example. I put this somewhat down to the Numeracy Project in the UK, in which the year's plans were set out in unrelated chunks so learners

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[14] NRICH: nrich.maths.org

might do some work on multiplication on Mondays and Tuesday and then swap to, say, identifying polygons for the rest of the week. Little attempt was made to tell any stories that connected the mathematics. Not only does this lead to a rather random view of the subject, we also know from the work of Brown and Askew (2001) that the most successful teachers look for and make connections in their teaching. So it is not surprising that the children in my study thought of mathematics as an unconnected set of topics. And mostly arithmetic, at that.

There were a handful who talked about using mathematics to solve problems, but again this was quite rare. Mostly mathematics is seen to be about learning to replicate or identify or calculate—although later when I did ask them if they knew any grownups who used mathematics, many said yes; teachers and people who go shopping! In the UK our assessment criteria use the words fluency and reasoning and these are often interpreted very narrowly by teachers.

The learners have a clear idea of how ‘good’ they are at the subject and know which ability group they fit. No Dweck growth mindset there! (Dweck 2006). But you will notice that they are not unhappy—they are quite satisfied with their lessons although in hundreds of hours of film, no-one ever got really excited when talking about their mathematics experiences! There is something here about expectations of just getting through what they have to do—if they are in year 4 there is a year 4 program to be got through and in a way that is their job as a student. I think, of everything, this is the bit that saddens me most. I suspect if I had asked them to tell me about almost any other subject—art, music, science, history—they would have been able to recall something creative that they did, something that was personal to them, perhaps creating a picture, playing a piece of music with others or dramatising a historical event. But hardly anyone talked about any personal mathematics they had done, discovered or been proud of. No-one thought of mathematics as a creative subject, or one in which they could have some personal input, whereas we know that the creativity of mathematicians has led to solutions of some of the world’s problems.

Comparing curricula

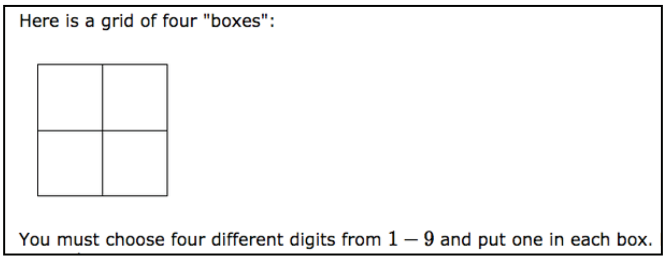
If we look at the Australian curriculum, we see that these ideas—creativity, enjoyment, confidence, connectedness, fluency, reasoning, problem posing and solving—are all embedded. Likewise in the UK, although we need two paragraphs to differentiate between aims and purpose. Nevertheless these ideas are thought important, and indeed when you read curriculum documents from across the world they all say much the same.

For me ‘connectedness’ is what mathematics is all about, though I would suggest that you are really lucky as a learner if you are let into that secret early on in your education. I am going to look at four ways in which mathematics can be thought of being connected, and these relate both to the curriculum aims and to our reactions to those film clips. And I hope you will be happy to do some mathematics along the way.

Making connections

Task 1: Reach 100

http://nrich.maths.org/1130



Here is a grid of four "boxes":

You must choose four different digits from 1 — 9 and put one in each box.

Make four two-digit numbers by reading top to bottom and left to right, then add them all up. Let us see who has made the largest sum? And the smallest?

I am hoping that this has provoked you into thinking about some questions you would like to ask. Here are some possibilities:

- What is the highest possible sum (and how do you know)?
- What is the lowest possible sum (and how do you know)?
- Can you make all the ones in between (what are your strategies and if they are not all possible why not?)
- Can you find a strategy for finding any sum? (e.g., work backwards?)
- What would happen of you changed the size of the grid?

Firstly the connection between fluency and reasoning. I am not sure how this is interpreted here, but in the UK it is very often taken to distinguish between algorithmic and factual recall speed on the one hand, and developing or following arguments on the other. My take on this is that you can and should promote them simultaneously. If we consider what knowledge and processes were need to get going on this task, we can see that here is an activity that supports fluency in the most restricted sense of the word, in that there is a lot of addition of two digit numbers involved. However to answer any of the subsequent questions there will, in addition to some trial and error, be some reasoning about place value and the relative importance of the position of the digits.

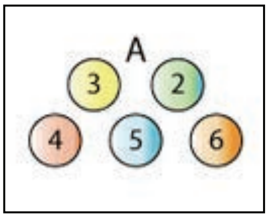
The phrase ‘low threshold high ceiling’ originated with Seymour Papert (1980). It means something that is accessible to all but with inbuilt challenge, and he used it to describe Logo. The phrase has been adopted by NRICH because it nicely describes these tasks which are suitable for a whole range of confidence and competence. Such rich tasks promote both fluency and reasoning, and make connections across domains, in this case number and algebra (though some would say that they are the same domain and that algebra is the structure of arithmetic). More than that (and to go back to the film clips), activities such as these, where learners are encouraged to pose their own problems, support confidence and creativity. They really are problems where you do not know what to do and there is no predetermined route to follow.

One way of using rich tasks is to let the learner just play around for a while in the environment, to get the feel of it, then offer a closed task which forces him/her to look at the structure of the environment—in this case the place value element. Then follow up with some multistep opportunities in which the learner has choices to make. Ruthven (1989) suggests that this exploration, codification and consolidation route is more profitable and fits better into a constructivist framework that the usual show, tell and practice routine that happens so often in many of our classrooms. Of course from a teacher's point of view the codification part, where the teacher makes the learning explicit and formalises it, makes great demands on the teacher who has to react to what is happening in the classroom rather than deliver a preplanned lesson.

Let us have a look at another rich task, one that is aimed at an older audience but which I hope you will see has a wide range of appeal.

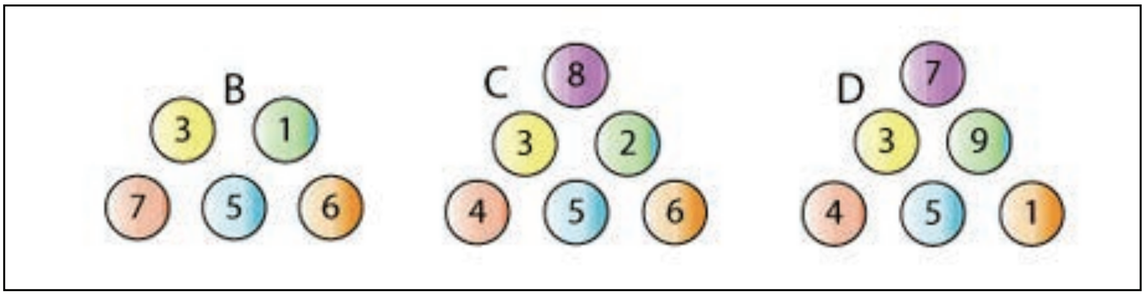
Task 2: Odds and evens

http://nrich.maths.org/4308
Here is a set of numbered balls. Assume I am playing against you.



I am trying to get an even sum and you are trying to get an odd sum. I go first and I pick two at random out of a bag—in this case 3 and 5. The sum is 8, this is even so I win a point. I replace the balls in the bag and you have a go. We keep going until we have each had, say, six goes. So my question is, is this a fair game and how do you know whether it is or it is not?

What if we chose a different selection of balls? Assuming we have to play using one of these sets, which is the fairest and how do you know?



I wonder how you worked this out. Perhaps you used a sample space, perhaps a tree diagram? Or perhaps you tried it out a few times and recorded your result? One of these is the fairest—but it is still not actually a fair game. Is it possible to choose a set of balls that would be totally fair? How would you go about it?

So here again we see the idea of playing around, being asked to do something specific which forces us to look at the structure of the given information, working up to a multistep problem. In each case the teacher's role is to help to formalise the learning and discoveries that have been made.

Rich tasks

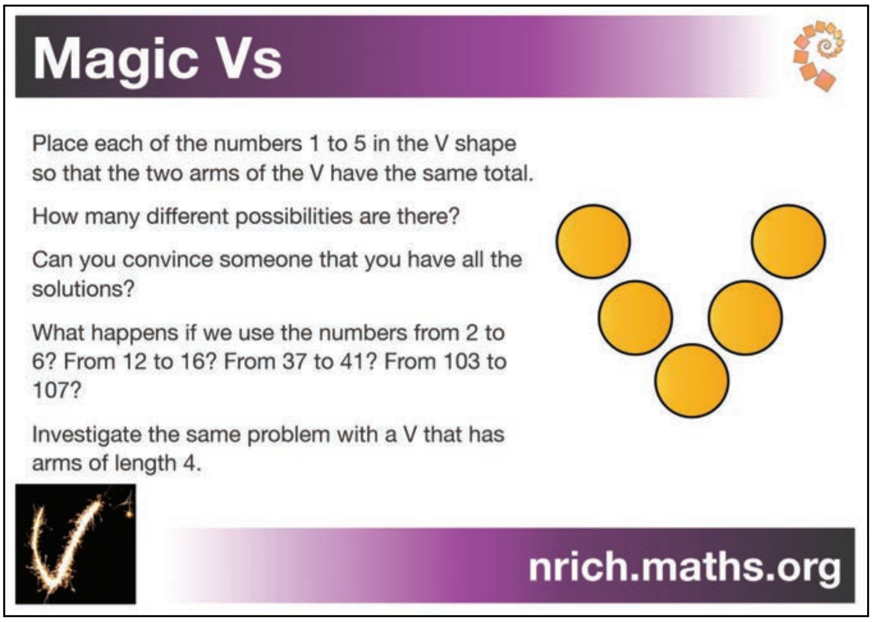
NRICH uses the term 'rich tasks' to describe activities such as these. Here are the characteristics of rich tasks, although not all tasks will exhibit all characteristics:

- combine fluency, problem solving and mathematical reasoning;
- are accessible, e.g., most students would be able to start (low threshold);
- promote success through supporting thinking at different levels of challenge (high ceiling);
- encourage collaboration and discussion (because talking often clarifies thoughts, and explaining and justifying are important parts of doing mathematics);
- use intriguing contexts or intriguing mathematics;
- allow for:
  - learners to pose their own problems,
  - different methods and different responses,
  - identification of elegant or efficient solutions,
  - creativity and imaginative application of knowledge;
- have the potential for revealing patterns or lead to generalisations or unexpected results;
- have the potential to reveal underlying principles or make connections between areas of mathematics.

Here are two other tasks which you may wish to try.

Task 3: Magic Vs

http://nrich.maths.org/6274



Magic Vs

Place each of the numbers 1 to 5 in the V shape so that the two arms of the V have the same total.

How many different possibilities are there?

Can you convince someone that you have all the solutions?

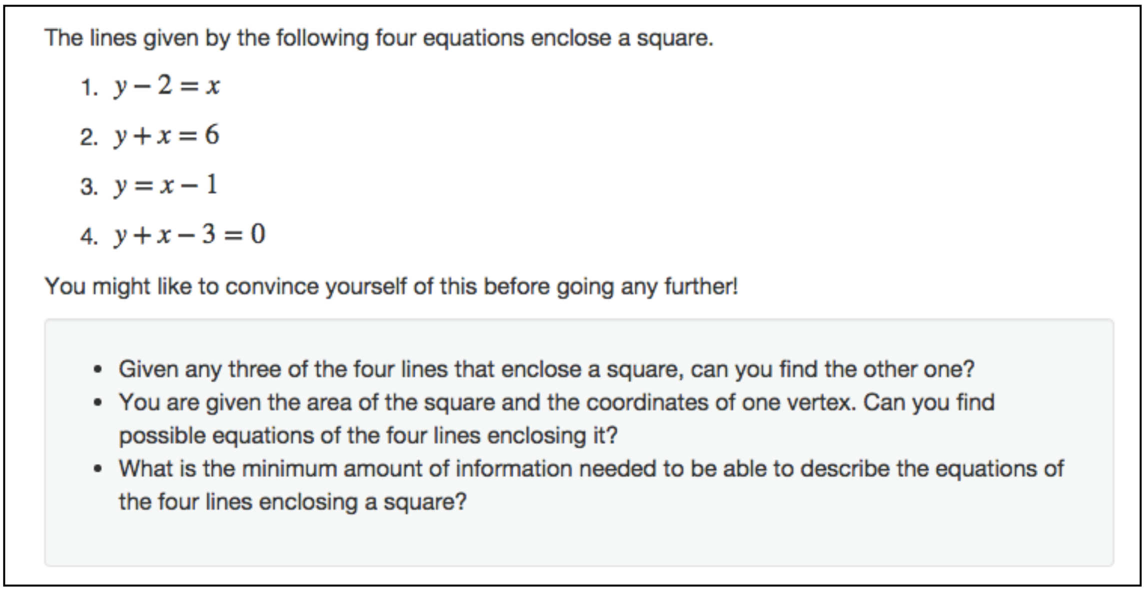
What happens if we use the numbers from 2 to 6? From 12 to 16? From 37 to 41? From 103 to 107?

Investigate the same problem with a V that has arms of length 4.

nrich.maths.org

This problem gives opportunities for children to make conjectures, prove these conjectures and make generalisations. They will be practising addition and subtraction, and applying their knowledge of odd and even numbers.

Task 4: Simultaneous squares



The lines given by the following four equations enclose a square.

1. $y-2=x$
2. $y+x=6$
3. $y=x-1$
4. $y+x-3=0$

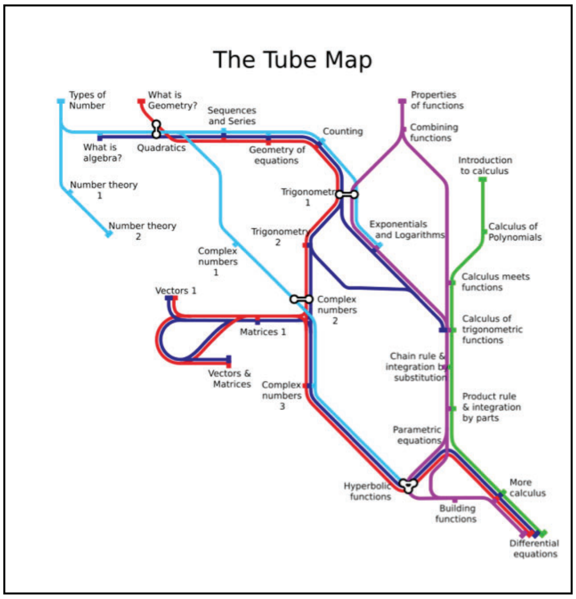
You might like to convince yourself of this before going any further!

- Given any three of the four lines that enclose a square, can you find the other one?
- You are given the area of the square and the coordinates of one vertex. Can you find possible equations of the four lines enclosing it?
- What is the minimum amount of information needed to be able to describe the equations of the four lines enclosing a square?

This problem provides an opportunity for students to draw together several different mathematical concepts that they have met previously, including the manipulation of linear equations, simultaneous equations, properties of straight-line graphs (including parallel and perpendicular lines), Pythagoras' Theorem, and the use of quadratic equations to solve geometrical problems.

Cambridge Maths

Both NRICH and my new project, Cambridge Maths, take all these connections very seriously. The Simultaneous Squares activity is taken from a new post-16 project called CMEP. The intention is to redesign the post-16 mathematics syllabus through making explicit the connections across and within mathematics. The big picture is based on the London Underground map and there are several possible routes through.



Cambridge Maths, of which CMEP will ultimately be a part, is a very ambitious project, funded by the University of Cambridge. The intention is to devise a mapping, or framework, of all the mathematics you might reasonably meet between the ages of 5 and 19. The framework will, as you might expect, make the most of the connections between the elements of mathematics. As well as the mapping there will be summative assessments at the end of various routes through the framework, a full set of resources to support those routes, and a comprehensive professional development offer to support teachers from novice to expert. The intention is that it would have international acceptance and in the long term would influence policy making, and especially in the UK! It is an amazing project to be part of and in the first instance I am working on the framework. I'm doing this through a web consultation and conversation and it would be great to have some input from Australian colleagues. You can follow the progress at www.cambridgemaths.org.

Task 5: Strike it Out

A final game. This is my all-time favourite. You need a 0–20 number line and someone to play with. Take turns to write an addition or subtraction calculation using three numbers from the line. Once you have written the calculation, cross the first two numbers off and circle the answer. The other player has to start their calculation with the circled number. So a set of responses could look like this:

3 + 4 = 7
7 – 5 = 2
2 + 15 = 17
17...

The winner is the player who goes last.

It is an engaging context for practising addition and subtraction, but it also requires some strategic thinking. It is easily adaptable, and can be used co-operatively or competitively.

Some questions to ask yourself:

- Is it possible to cross all the numbers off? Could you prove it?
- Is there a strategy for winning?
- What is the mathematical knowledge that is needed to play?
- Who would this game be for?
- What is the value added of playing the game?
- Could you adapt it to use it in your classroom?

In conclusion

Connectedness, creativity, confidence, fluency, problem solving—all these are possible through judicious choice of tasks. John Dewey summed it up beautifully:

"Give the pupils something to do, not something to learn; and if the doing is of such a nature as to demand thinking; learning naturally results."

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For more information

Email lynne.mcclure@cambridgemaths.org or join the consultation at www.cambridgemaths.org.

LEADING LEARNING: GOING BEYOND THE CONTENT

VAL WESTWELL
Department for Education and Child Development, SA

When any new curriculum is introduced it is almost inevitable that initially it will be used in a naïve way, with a focus on the content. In South Australia we wanted the Australian Curriculum to live up to its potential, as promised by the evidence base used to inform its development. We wanted to enable leaders and teachers to think about the curriculum in a way that supports students to become successful learners as described in the Melbourne Declaration.

This paper will describe a snapshot of some key elements of the SA Department of Education and Child Development (DECD) approach for leading achievable changes in teaching and learning. Central to the strategy was the creation of a suite of resources, the Leading Learning website. This website supports and challenges school leaders and teachers to think about the *Australian Curriculum* in ways that engage all learners to think creatively and critically.

The tools described in this paper can be found, along with many other resources, at http://www.acleadersresource.sa.edu.au.

Background

Almost 20 years ago, Seymour Papert (a renowned Mathematics educator) said that, ‘We need to produce people who know how to act when they’re faced with situations for which they were not specifically prepared’. Acknowledgement of this need and the implications for education has gathered momentum in systems across the world in recent decades. The 2013 OECD skills report cited *problem solving* and the *ability to learn* as skills needed by workers in the 21st century, ‘to help them weather the uncertainties of a rapidly changing labour market’.

In South Australia, focus on the need to develop young people who are powerful learners led to the establishment of the ‘Learning to Learn’ project. This project culminated in the publication of the South Australian Teaching for Effective Learning framework (TfEL, 2010). The mandated framework draws together the research evidence, the voices of educational experts from around the world, and South Australian expertise.

The TfEL framework is based on high-challenge, social constructivist pedagogy. It recognises that we are all learners and that the conditions needed for students’ learning are just as important to adult learners, including the teachers and leaders in our schools. The framework reflects two key concepts:

- *Learning for effective teaching* supports leaders and teachers to see themselves as learners, reflect on their practice and create opportunities to develop their professional expertise.
- *Teaching for effective learning* supports teachers to develop their practice in three domains:
  - Create safe conditions for rigorous learning
  - Develop expert learners
  - Personalise and connect learning.

In South Australia, we believe it is essential to bring together the Australian Curriculum (*what* we teach) with the pedagogy articulated in the TfEL Framework (*how* we teach) in order to develop successful, powerful, lifelong learners.

In December 2014, AAMT mirrored this position about the need to bring together the what and the how, when they stated; "It is more important than ever before for teachers to consider *how* they teach as well as what they teach—*what and how* cannot be separated when developing skills in key areas such as critical thinking, communication, and mathematical modelling" (AAMT, 2014).

Implementing the Australian Curriculum: Opportunity and challenge

With the introduction of the Australian Curriculum, many school leaders in South Australia reported their teachers felt overwhelmed by the apparent content requirements. Leaders and teachers saw this manifesting itself in an unintentional regression to a didactic coverage of curriculum content, rather than the responsive, high-challenge pedagogical approach that had begun to develop through their implementation of the principles of the TfEL framework.

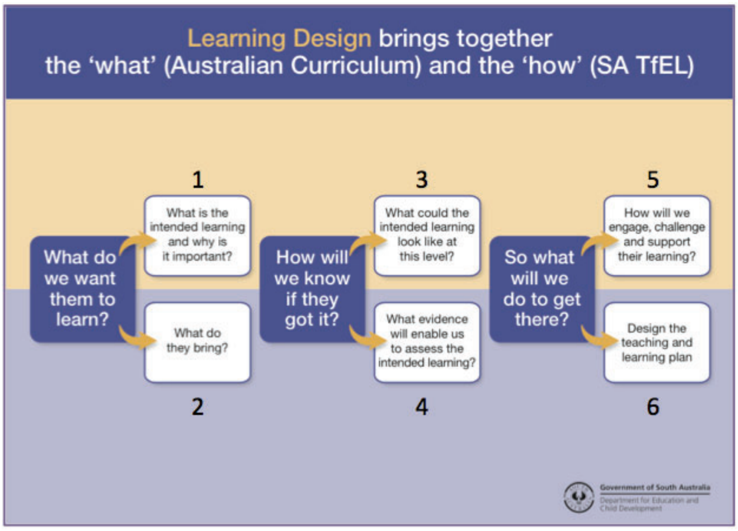
Leaders' and teachers' concerns presented us with an opportunity to engage them in exploration of how we could work with the *Australian Curriculum* and the TfEL framework and in doing so develop powerful, expert learners.

The Australian Curriculum was shaped to reflect 21st century global imperatives and Teaching and Learning Services (DECD) responded to the implementation of the curriculum to make the strategic intent visible and doable. In the case of mathematics, the strategic intent is captured in the *Australian Curriculum: Mathematics* Proficiencies (AC: Proficiencies) and we sought to raise their profile through modelling how they can be woven together with the content.

The Leading Learning website (DECD, 2013) and associated professional development opportunities have been created and continue to evolve to support our teachers, leaders and students to become designers of learning, through which the strategic intent of the *Australian Curriculum* and the principles of the Teaching for Effective Learning framework are brought together.

Transforming the way we think about the intended learning

The Teaching for Effective Learning framework is complemented by a number of resources, one of which is a Learning Design framework (Figure 1). This framework has been developed to support teachers to be designers of learning in which they bring together the 'what' (Australian Curriculum) and the 'how' (SA TfEL). There are considerable similarities between this framework and the 'Backwards Design' framework (McTighe & Wiggins, 2006). The notable difference is the way in which Learning Design considers what the student brings to the learning. Considering students strengths, challenges, dispositions, culture and aspirations are all essential aspects of SA Learning Design.



Learning Design brings together
the 'what' (Australian Curriculum) and the 'how' (SA TfEL)

What do we want them to learn?

1 What is the intended learning and why is it important?

2 What do they bring?

How will we know if they got it?

3 What could the intended learning look like at this level?

4 What evidence will enable us to assess the intended learning?

So what will we do to get there?

5 How will we engage, challenge and support their learning?

6 Design the teaching and learning plan

Government of South Australia
Department for Education and Child Development

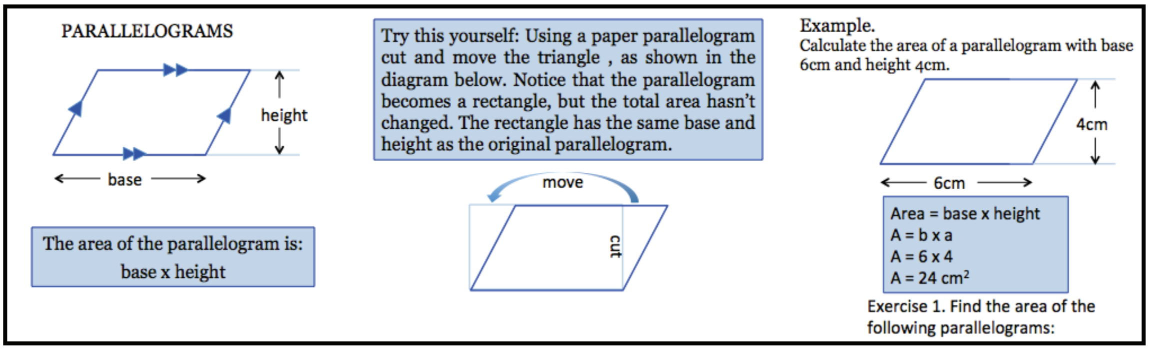
Figure 1. Learning design framework.

The first box of Learning Design asks teachers to consider, '*What* is the intended learning and *why* is this important?' Recent South Australian research suggests that this question is answered in two fundamentally different ways. One response reflects linear, transactional pedagogy that is primarily content driven. The other is a responsive, design-oriented pedagogy, which includes a broader range of learning intentions. The Leading Learning resource is designed to support a fundamental shift from transactional pedagogy to design-oriented, responsive pedagogy.

In a transactional model of mathematics pedagogy, the intended learning would identify only the content to be addressed with students. For example, the teacher could describe the intended learning as:

What?
Students learn to: Calculate the area of a parallelogram.
Why?
This learning is important because it is in the Year 7 curriculum.

With this learning intention, it would perhaps be appropriate to present students with information such as this:



PARALLELOGRAMS

The area of the parallelogram is: base x height

Try this yourself: Using a paper parallelogram cut and move the triangle , as shown in the diagram below. Notice that the parallelogram becomes a rectangle, but the total area hasn't changed. The rectangle has the same base and height as the original parallelogram.

Example.
Calculate the area of a parallelogram with base 6cm and height 4cm.

Area = base x height
A = b x a
A = 6 x 4
A = 24 cm^2

Exercise 1. Find the area of the following parallelograms:

Figure 2. Traditional textbook introduction

Figure 2 is representative of many textbook and worksheet introductions to parallelogram area. Notice that the learner is:

- *told* which measurements they will need to make
- *told* to use the terms base and height
- *told* the area formula in words, then in symbols
- *shown* one explanation of why the formula works
- *shown* how to apply the formula
- *asked to apply the given formula* to questions similar to the example shown.

For many students, this transactional pedagogy, which provides a 'learning diet' of *told, shown and asked to apply given procedures,* has many unintended and quite damaging learning outcomes. Inevitably, many students will begin to believe that to learn maths they must first be shown what to do. They learn that there is only one way to respond to each problem and that successful mathematics learning is about remembering and applying processes.

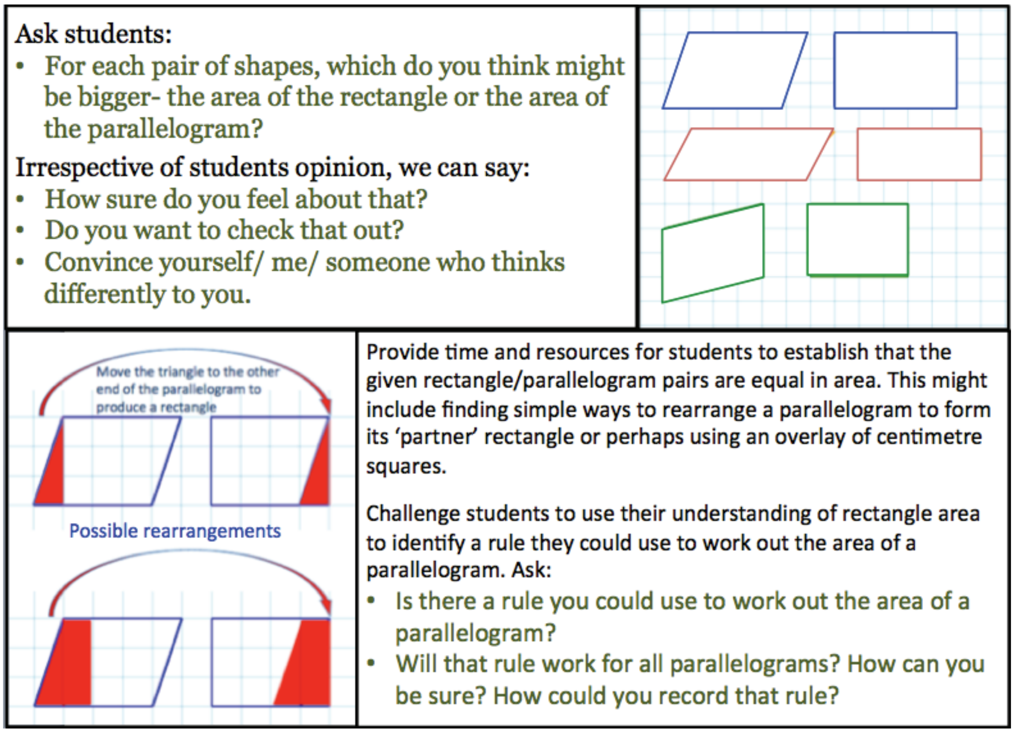
In contrast to this transactional pedagogy, in responsive, design-oriented maths pedagogy the intended learning would identify a broad range of skills and capabilities. For example, the teacher could describe the intended learning as:

What?
Students learn to:

- transfer and build on existing understanding
- look for, establish and express general rules
- develop skills in working collaboratively
- calculate the area of a parallelogram.

Why?
This learning is important because:
it uses the content of the Year 7 curriculum to develop skills in thinking and communicating mathematically and working collaboratively.

The pedagogy shift that is expressed here centres on using content as a vehicle for intentionally developing a broader base of skills and dispositions in students. There is no doubt that we want the students to learn the content, but it is insufficient on its own. To achieve this learning intention, perhaps we could begin by asking students to make a judgment; "For each pair of shapes, which do you think might be bigger: the area of the parallelogram or the area of the rectangle?" (Figure 3)



Ask students:

- For each pair of shapes, which do you think might be bigger- the area of the rectangle or the area of the parallelogram?

Irrespective of students opinion, we can say:

- How sure do you feel about that?
- Do you want to check that out?
- Convince yourself/ me/ someone who thinks differently to you.

Provide time and resources for students to establish that the given rectangle/parallelogram pairs are equal in area. This might include finding simple ways to rearrange a parallelogram to form its 'partner' rectangle or perhaps using an overlay of centimetre squares.

Challenge students to use their understanding of rectangle area to identify a rule they could use to work out the area of a parallelogram. Ask:

- Is there a rule you could use to work out the area of a parallelogram?
- Will that rule work for all parallelograms? How can you be sure? How could you record that rule?

Figure 3. Transformed 'Parallelogram task'.

This question requires students to activate their understanding of area, prior to any information being provided by the teacher or text. When using this question, the teacher does not ask students directly what they understand by the term area, but they do gain insight into the students understanding nonetheless. Even if the student has only a sense of the term area, rather than an ability to articulate the definition, the teacher can observe this 'sense' of area. Equally, it does not preclude students from articulating the definition.

In this pedagogical paradigm, the teacher is responsive to the insight that her students bring, therefore it is difficult to identify a precise pattern that her questioning would follow. However, it may be broadly similar to that in Figure 3. In this example, the learner is challenged to:

- *notice* a relationship;
- *convince* someone of their current opinion about the relationship;
- *transfer* what they already know about area calculation in rectangles to a new context;
- *establish* a rule for area calculation in parallelograms (perhaps verbally, then written words then using symbolic representation).

This learning experience consisting of *notice, convince, transfer* and *establish a rule* has the same outcome as the text book learning experience in relation to 'content learned'. However the additional learning outcomes in this model often include students' learning that:

- *they* can create new understanding *without* it being modelled to them;
- purposeful dialogue with peers can be mutually beneficial for learning.

When we provide students with the opportunity to make connections and construct understanding, they are more likely to develop conceptual understanding. Whilst, initially, this may appear to be a more time consuming learning process, we should consider that; “Conceptual understanding also supports retention. Because facts and methods learned with understanding are connected, they are easier to remember and use, and they can be reconstructed when forgotten” (National Science Research Council, 2001).

Hence developing students’ conceptual understanding through applying pedagogy such as this, could result in breaking the cycle of ‘re-teaching’ content year after year. We would still need to ensure that students have formal mathematical language and conventions, but when we provide this information after they have established conceptual understanding, we are adding useful information at the point of need. In this model of pedagogy we have instructed only that which we cannot support students to reason for themselves.

In the latter example, the development of the proficiencies is achieved through day-to-day learning of content. In the former, there is almost no alternative other than to teach the content and somehow bolt on the proficiencies after the fact.

Supporting South Australian teachers and leaders to transform tasks

Examples such as the parallelogram question, can be useful in communicating the nature of the shift in pedagogy that we intend for teachers to enact. However, in the same way as we want our students to view mathematics as a coherent whole, in which they identify principles behind the specific examples, we also want our teachers and leaders to be clear about the pedagogical principles that we are asking them to enact.

Providing teachers with individual examples, such as the parallelogram question, without challenging and supporting them to identify the principles applied to the design of the task, does no more than give them one classroom activity.

Many resources and much professional development does exactly this. Even ‘expert’ teachers who share their practice can fall in to the trap of sharing their activities and telling teachers what they do, without developing an understanding of the deeper learning intentions that are implicit for the expert. Whilst ‘smart borrowing’ *with* understanding can be useful in developing teacher practice, the use of activities, without understanding the deeper learning intention and the pedagogy required to achieve it, can be disempowering and de-professionalising. Leaders empower teachers when they support them to build their own bridges from their current to their next practice.

The ‘Transforming Tasks’ professional learning module in the Leading Learning resource (Figure 4) does exactly this. Teachers are asked to *suggest* possible transformations for a task, *notice the relationship* between the task ‘before’ and ‘after’ the transformation, and work with colleagues to *establish general principles* for transforming tasks for greater engagement with intellectual challenge.

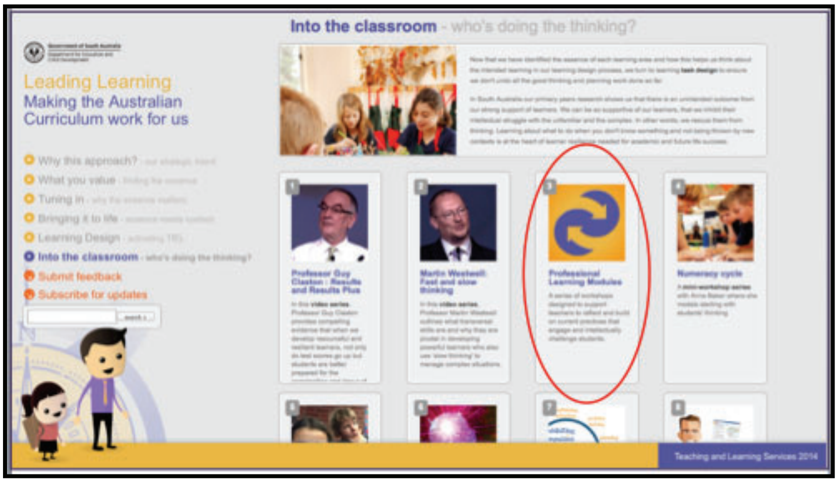


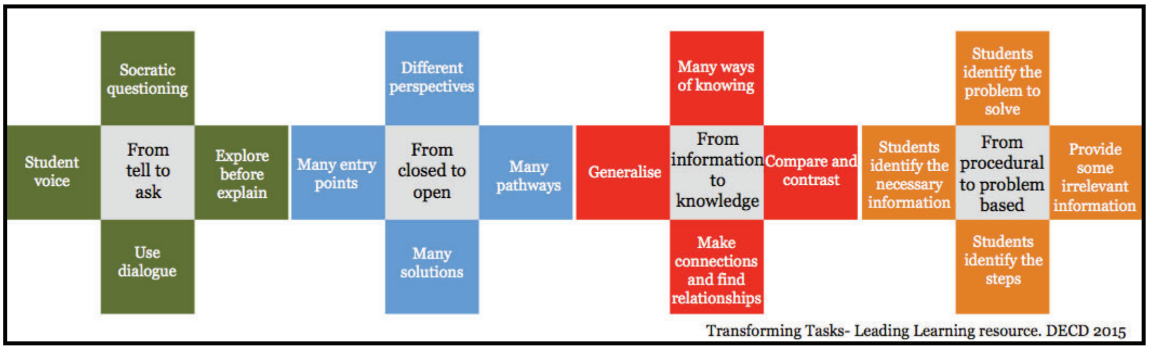
Figure 4. Leading Learning Resource. Into the classroom: Professional-learning module 'Transforming Tasks.

'Transforming Tasks': Strategies and techniques

The 'Transforming Tasks' module identifies four strategies that, together, articulate the nature of the pedagogical shift we aim to achieve.

These four key *strategies* are:

- From closed to open
- From information to knowledge
- From tell to ask
- From procedural to problem based



	Socratic questioning			Different perspectives			Many ways of knowing			Students identify the problem to solve	
Student voice	From tell to ask	Explore before explain	Many entry points	From closed to open	Many pathways	Generalise	From information to knowledge	Compare and contrast	Students identify the necessary information	From procedural to problem based	Provide some irrelevant information
	Use dialogue			Many solutions			Make connections and find relationships			Students identify the steps	

Transforming Tasks- Leading Learning resource. DECD 2015

Figure 5. Overview of strategies and techniques from "Transforming Tasks' http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom.

Each of the four transformation strategies could be achieved through applying many different techniques. Just four techniques were chosen to exemplify each strategy (Figure 5).

For each of the 16 techniques, sets of examples demonstrate how they each can be used to transform a traditional textbook or worksheet style task into a task that develops a broader skill set and provides greater engagement with intellectual challenge. These examples show each technique being used in isolation, so they allow practice to be changed in manageable steps. Of course, techniques can also be combined. In the parallelograms example (Figure 3) the teacher combined three techniques:

- Compare and contrast – Supporting the shift 'From information to knowledge'
- Explore before explain – Supporting the shift 'From tell to ask'
- Generalise – Supporting the shift 'From information to knowledge'

It is intended that these strategies and techniques be used alongside the TfEL framework to establish a common language for articulating intentional learning design.

During the creation of this resource, subject specialists in Mathematics, Science, English, History, Geography, the Arts and Technology worked together to establish techniques that were deemed appropriate for use in all learning areas. Having one set of strategies that can be applied across all learning areas is valuable when considering the demands on primary school teachers. Observation of the use of this resource suggests that having common strategies could be useful in supporting interdisciplinary collaboration in the high school setting. It also allows non-specialist leaders to share a common language with teachers.

Raising the profile of the Australian Curriculum: Mathematics Proficiencies: Making them visible and doable

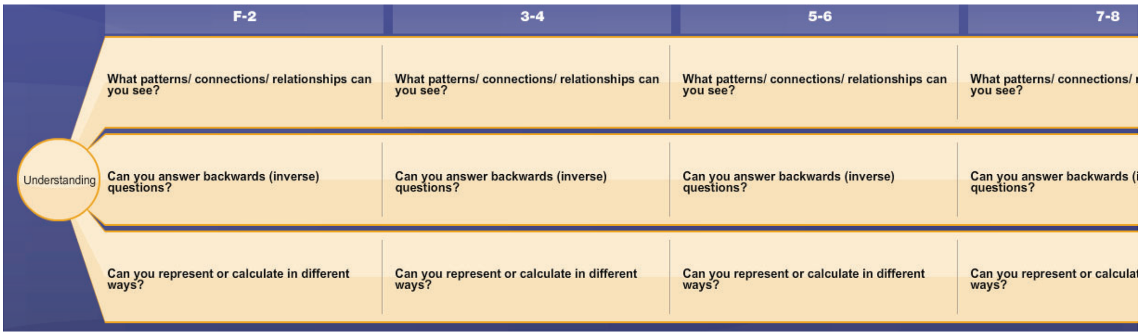
The verbs used in the Proficiencies describe the actions in which students can engage when learning and using mathematics content. To embed the Proficiencies in students learning experiences, teachers need to ask questions that activate those actions in their students. But what questions would achieve this? The Proficiencies describe the actions, but not the questions that can drive those actions.

The Bringing it to Life (BitL) tool was developed to bring the proficiencies to life in the classroom. The tool models questions that can be used to drive the actions described in the Proficiencies. It has three layers, which increase in the level of detail provided.

Questions have been develop for all four of the AC: Proficiencies, but for the purpose of this paper we will look at the AC Proficiency: Understanding. The first layer of the tool poses questions that help teachers to enact the emphasis of each Proficiency.

The questions, in the first layer of the BitL tool (Figure 7), identified to reflect the verbs in the 'Understanding' proficiency are:

- What patterns, connections, and relationships can you see?
- Can you answer backwards (inverse) questions?
- Can you represent, calculate or create in different way?



	F-2	3-4	5-6	7-8
Understanding	What patterns/ connections/ relationships can you see?	What patterns/ connections/ relationships can you see?	What patterns/ connections/ relationships can you see?	What patterns/ connections/ you see?
	Can you answer backwards (inverse) questions?	Can you answer backwards (inverse) questions?	Can you answer backwards (inverse) questions?	Can you answer backwards (questions?
	Can you represent or calculate in different ways?	Can you represent or calculate in different ways?	Can you represent or calculate in different ways?	Can you represent or calcula ways?

Figure 7. First layer of the mathematics BitL tool – Understanding.
http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

The following description of Conceptual Understanding, from the 'Adding It Up' report, reflects why, for this proficiency, we identified questions that highlight connectivity and flexibility in the use of mathematics:

> Students with conceptual understanding know more than isolated facts and methods… They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know.
> A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways… The degree of students' conceptual understanding is related to the richness and extent of the connections they have made.
> (Kilpatrick, Swafford & Findell, 2001)

Layer two of the BitL tool contains suggestions about the type of questions that teachers can use with students in order to activate each element of the proficiency. For example, specific questions under the heading, 'What patterns, connections and relationships can you see?' include:

- Which could be the odd one out?
- What is the connection between…?
- How are these (values/shapes/angles/questions/graphs, etc.) the same as each other?
- How are these (values/shapes/angles/questions/graphs, etc.) different to each other?

The questions can be used across all content areas, and in layer three, specific examples are provided of how each question could be brought together with a particular piece of content.

Teachers and leaders have used the BitL tool, both to design learning for their students and also to reflect on the nature of the questions in their existing tasks. Often teachers have commented how the tool has supported them to realise the extent to which they focus on questions reflecting the Fluency proficiency—the 'recall and remember' element of the mathematics curriculum.

Supporting teachers to embed the proficiencies in their learning design is vital. Without support from leaders of mathematics curriculum, many teachers can get locked into delivering mathematics content and in turn students get locked into being passive receivers of information. The BitL tool provides leaders and teachers with a resource to make the proficiencies visible and doable in the classroom.

Supporting effective practice

The 'Transforming Tasks' resources and the BitL tool are just two examples of discrete assets within the Leading Learning resource. While the resource allows leaders and teachers to dip in and out, an overall developmental narrative exists. This allows leaders to determine where their school sits in the developmental story and target tools that might be most effective for them. The developmental narrative, from rationale to practice, is framed in the navigation menu:

- Why this approach?
- What you value
- Tuning in
- Bringing it to life
- Learning Design
- Into the classroom

Assets within this resource include: videos of national and international educational leaders; custom designed animations to assist in describing and communicating the shift in pedagogy to stakeholders, including parents and students; tools to assist with enacting the pedagogy described in the TfEL framework; and the voices of South Australian teachers and leaders, sharing their practice.

The Leading Learning resource continues to be developed in response to research and feedback from teachers and leaders. Exciting new modules are currently planned for ‘Engagement’, ‘Developing executive function through curriculum’ and ‘SOLO’.

Conclusion

We know that transactional delivery of mathematical facts has limited sustained impact for many students. When we instruct appropriate mathematical processes, rather than supporting students to develop conceptual understanding we can leave a trail of misconceptions active in our students’ thinking. The instructed information can sit in disequilibrium with their intuitive understanding and when they have forgotten the instructed information they will return to their intuitive understanding. When students learn, with appropriate support, to construct understanding to establish relationships for themselves, they are better placed to reconstruct that understanding when the rules have been forgotten. When teachers design for learning to occur in this way, they are going beyond the delivery of content. They contribute to the development of successful lifelong learners.

The conditions needed for students’ learning are as important for adult learning. Just as with our students, transactional delivery of learning for teachers can have limited effect. The Leading Learning resource supports leaders and teachers to work together to construct their conceptual understanding, from which their next pedagogy can be established.

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PAPERS

FOCUS ON FORMULATION:
HOW THIS HAPPENS IN A CHINESE CLASSROOM

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Mathematical formulation has been given prominence in the 2012 PISA Mathematics Framework and in the draft 2015 Framework which have identified three key mathematical processes: formulating, employing and interpreting mathematics. This presentation will look at how a typical Chinese middle school teacher structures a lesson on simultaneous equations around the key idea of formulation. This emphasis placed by Chinese teachers in choosing problem contexts which foster mathematical formulation has lessons for Australian teachers. It may also help to explain why Chinese students perform so highly in PISA on this measure compared to students in other countries.

PISA mathematical processes

The PISA definition of mathematical literacy (Organisation for Economic Cooperation and Development (OECD), 2015) refers to an individual's capacity to formulate, employ, and interpret mathematics. These three words, formulate, employ and interpret, provide a useful and meaningful structure for organising the mathematical processes that describe what individuals do to connect the context of a problem with the mathematics and thus solve the problem. Items in the 2012 PISA mathematics survey were assigned to one of three mathematical processes, as they will be in 2015:

- formulating situations mathematically;
- employing mathematical concepts, facts, procedures, and reasoning; and
- interpreting, applying and evaluating mathematical outcomes.

Formulating situations mathematically

The term *formulate* in the PISA definition of mathematical literacy refers to:

> individuals being able to recognise and identify opportunities to use mathematics and then provide mathematical structure to a problem presented in some contextualised form. In the process of formulating situations mathematically, individuals determine where they can extract the essential mathematics to analyse, set up, and solve the problem. They translate from a real-world setting to the domain of mathematics and provide the real-world problem with mathematical structure, representations, and specificity. They reason about and make sense of constraints and assumptions in the problem. Specifically, this process of formulating situations mathematically includes activities such as the following:
>
> - identifying the mathematical aspects of a problem situated in a real-world context and identifying the significant variables;

- recognising mathematical structure (including regularities, relationships, and patterns) in problems or situations;
- simplifying a situation or problem in order to make it amenable to mathematical analysis;
- identifying constraints and assumptions behind any mathematical modelling and simplifications gleaned from the context;
- representing a situation mathematically, using appropriate variables, symbols, diagrams, and standard models;
- representing a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions;
- understanding and explaining the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically;
- translating a problem into mathematical language or a representation;
- recognising aspects of a problem that correspond with known problems or mathematical concepts, facts, or procedures;
- using technology (such as a spreadsheet or the list facility on a graphing calculator) to portray a mathematical relationship inherent in a contextualised problem (OECD, 2015, p. 10).

Australia's performance in PISA

In the report *PISA 2012: How Australia measures up* (Thomson, De Bortoli & Buckley, 2013), mean scores were presented in students' performances on the three process subscales, by country. In the Formulating process subscale, Australian students' mean score was 498, being 6 points above the OECD average of 492. Among countries whose mean scores were significantly higher than Australia were Shanghai-China (624, the top performing group for PISA 2012), Singapore (582), Chinese-Taipei (578), Hong Kong-China (568), Korea (562) and Japan (554). It is clear that on average Australian students do not perform as highly on the mathematical formulation component of PISA as we would hope. A similar profile of Australian students' performance is shown on the Employing process subscale. However, it should be noted that on the Interpreting process subscale, the mean score (514) of Australian students is significantly above the OECD average (497), although still significantly lower than Shanghai-China (579), Singapore (555), Hong Kong-China (551), Chinese Taipei (549), Korea (540) and Japan (531).

Is there something Australian teachers and schools can learn from these high performing countries? In the following section we offer a vignette of a "typical" mathematics classroom in Shanghai. What is evident from this one classroom is its sustained focus on mathematical formulation. While this Chinese classroom may not be typical of schools outside Shanghai, we can see how careful choice of mathematical problems and sustained attention to developing students' thinking pays off. What is quite clear is the mathematical formulation is demanding and far from mechanical.

A teaching episode from a Year 6 class in Shanghai

This lesson takes place in a Shanghai Year 6 class. The school is a local area school. However, the mathematical problem that the class is invited to work on may be more typical of Year 7 in many other Chinese schools (Ministry of Education, 2011). In this lesson, the topic is applications of simultaneous equations to real-world contexts, a topic that is typically treated later in Year 9 or 10 in Australia (Australian Curriculum, Assessment and Reporting Authority, 2014). The class has already discussed an earlier and simpler application in this lesson. In previous lessons, students have already been taught how to simplify and solve algebraic expressions. The focus for this lesson is on how to formulate a problem in one or more ways that are amenable to a solution. Understanding how to get a particular formulation is critically important, as is the capacity to explain what a particular formulation means. This segment of the lesson opens with the teacher presenting the main problem for the lesson which is scheduled to take 45 minutes.

> T (12:15–12.29): Here we come to the second problem (PPT shows the text): There is a boarding school in our city. At the beginning of the new semester, students will be arranged into dormitory. If we put 4 students into each room, there would be 5 empty rooms. If we put 3 students into a room, 100 students would have no room to live in. The question is: How many students live in this school's dormitory and how many rooms does the dormitory have? Please discuss in small groups for 5 mins. This one is a little difficult, please consult and discuss, with four people at each table. Let's see which group will get the most solutions. Please look at the problem first.

Method 1

After 5 minutes discussion in small groups, the teacher calls on the whole class to listen to what has been found. Student S1 is called on first.

> T (17:44–18:03): (The teacher wrote down "Method 1" on the blackboard.) OK, stop, now. Please tell the whole class your own ideas. (Many students raised their hands). First, S1, please show us your thoughts.

Student S1 offers a formulation of two simultaneous equations:

> S1 (18:04–19:02): First, assume that the School has a total of x students, and then be able to get: $4x - 20 = 3x + 100$. (The teacher wrote down $4x - 20 = 3x + 100$ on the blackboard and some students began to whisper and query this linear equation)

We note that this formulation presents two formulations for x, the total number of students. In a typical Australian lesson, it might be more usual to define, at the outset, y as the total number of rooms in the dormitory. And so producing two simultaneous equations involving x and y; namely $y = 4x - 20$ and $y = 3x + 100$. In any case, the teacher notes that some students are sceptical about S1's formulation and asks for silence.

> T (19:03–19:16): Please don't discuss, let's listen to her interpretation first, and we could help her find out if there something is wrong, what is the $4x$?

Student S1 ponders and does not respond

> S1 (19:16 – 19:24): (Silent, no answer)

This is now a critical point for the lesson. The formulation presented by S1 is obviously wrong. The teacher could have dismissed it as incorrect and moved to ask another student, but chooses to use this response to go over the key relationships.

> T (19:24–19:47): What is the 4's unit? The 4's unit is persons per room, rather than persons, right? (Teacher wrote down "4 persons/room" on the blackboard) What is the x unit? The x unit is the total number of persons, right? Because this is what she assumed. (Teacher wrote down x persons on the blackboard) So, what I want to ask is, does it make sense to multiply these two (the 4 and the x)?

Students generally respond agreeing that the original formulation does not make sense.

> S (19:47–19:50): No, it doesn't make sense.

What is notable about this segment is that the teacher tries to ensure that students think about whether the particular terms of S1's formulation are logical. All students are required to make their own evaluation before the class moves on.

> T (19:50–20:19): It doesn't make sense, right? So I think this place (4*x*) is wrong. (Teacher draws a circle round the 4*x* with chalk on the blackboard.) In the same way, this place (3*x*) is also wrong. (Teacher draws a circle round the 3*x* with chalk on the blackboard.) So this formula is not correct, right? OK? S1, what was your original answer? This formulation is wrong, but it doesn't matter (The teacher pointed to $4x - 20 = 3x + 100$ on the blackboard). We asked her tell us her original answer, try again.

It appears that the teacher knew that student S1 had another formulation, which the student gives as follows:

> S1 (20:19–20:41): (Teacher writes the following formulation on the blackboard)

$$\frac{x-4\times 5}{4}=\frac{x+100}{3}$$

The teacher's response to this second formulation by student S1 is quite different to the first. While this second formulation is not entirely correct, the teacher wants the whole class need to think carefully about how it can be made right.

> T (20:41–22:34): Have your seat, let's have a check together. This formula looks a little like correct, right? What is this (*x*)? It's the total number of persons. 4 is the number of persons per room. What is the 5? 5 is the number of empty rooms. So $x - 4 \times 5$ is persons living in the dormitory, right? (Some students indicate agreement briefly.) And then divided by 4. What does 4 mean? Persons/room. What could we get after dividing by 4? It is the total number of the rooms, right? (Some students indicate agreement briefly.) What about the other side? This side also means the number of the rooms, so it is the same, right? Let's check this formula and find out if there something is wrong? Look: the left side. What is the meaning of $x - 20$? What is the meaning of the 20? (Teacher pointed at 4×5 on the blackboard.) ... How about these 20 persons? There are 5 rooms nobody lives in, right? (Some students indicate agreement briefly. Teacher writes down 5 rooms nobody lives on the blackboard.) So the 20 is the number of persons who could live in the 5 rooms, right? (Some students indicate agreement briefly.) Then what is the meaning of −20? The total numbers of students minus the number of students who do not live in the dormitory. Do you think that makes any sense? Do you think there is something wrong? There is this number of occupied dormitory rooms which we don't know, plus 5 unoccupied rooms. Right? (The teacher draws a diagram with chalk on the blackboard, like this: □□□□□□□...××××) Originally, all the students could live in these rooms, (teacher points at the □s). The rooms on this side (teacher points at the ×s) are where nobody lives. If you subtract the number of students who don't live in the 5 rooms on this side (teacher points at these×s), and then divide by 4, does the result make sense? It is not correct, is it? Then you tell us how to do it? How could you do some modification based on her work? S2? (Student S2 is invited to state her formulation and gives it.)

In this extended exposition, the teacher goes over the problem herself, pausing only to ask if the students are following her line of thinking. She pauses several times to ask "Right?", almost rhetorically. Unlike the way in which she responded to an obviously incorrect first formulation, her treatment here is much more careful. "Let's have a check together" is used as a cue for the teacher to model for the students the kind of mathematical thinking that they need to analyse the nearly correct response given by S1. The teacher treats this second formulation as a serious attempt and possibly one that is representative of other students in the class. A key idea is to remind students that after dividing by 4, they can obtain an expression for the total number of the rooms. Again, students are invited to think about the meaning of each term. The teacher also introduces a simple diagrammatic representation of the room in the dormitory showing those rooms which are occupied and those which are empty. This diagram will be featured in subsequent stages of the lesson.

S2 (22:34–22:50):

$$\frac{x}{4}+5=\frac{x-100}{3}$$

Student S2 presents a correct formulation for the number of rooms, using the two conditions. But the teacher wishes now to make sure that S2 and the class are able to explain these relationships in context-specific language.

T (22:50–22:58): Please give us an interpretation about your linear equation. She (S2) has done some adjustment on it, right? Give an interpretation. (S2 refers to the first term.)

$$\frac{x}{4}$$

S2 (22:58–23:05) This is on behalf of the total number of people that are (accommodated) in rooms (a little slow and vague).

Student S2 needs some prompting:

T (23:05–23:32): What can we get from it? (The teacher points to $x/4$) This is the number of rooms, isn't it, that these people live in. (The teacher points at the □s), then divided by 4, then it is the number of rooms in these (□s), OK. Why do you add 5?

Student S2 responds correctly:

S2 (23:32–23:35): Because there are five empty rooms.

The teacher now asks S2 to explain the right side in context-specific language.

T (23:42–23:49 referring to the right side of the equation): What about the right side?

Student S2 correctly uses context-specific language to explain the right hand side.

S2 (23:49–23:55): $x - 100$ is all the people who live in the dormitory, then divided by 3 is the number of rooms.

Some teachers might simply commend the student before moving on to the next task. But mathematical formulation is more than being able to give the correct mathematical terms. All students need to be aware that any formulation needs to make sense in terms of the specific features of the context being considered. This teacher is clear about this.

Method 2

T (25:30–25:53 (The teacher writes "Method 2. Let the dormitory have y rooms" on the blackboard) Let me ask, Can anyone make a formulation different from the first formulation?

$$\frac{x}{4}+5=\frac{x-100}{3}$$

Student S3 gives a second formulation using y as the total number of rooms to obtain an expression for the total number of students. Again two simultaneous equations are presented involving y only, leaving implicit that $x = 4(y – 5)$, $x = 3y + 100$.

S3 (25:53–26:10): $4(y – 5) = 3y + 100$ (Teacher writes down $4(y – 5) = 3y + 100$ on the blackboard simultaneously)

Once again, the teacher asks this student to give a context-specific explanation:

> T (26:10–26:12 to S3): Would you please explain it?

Student S3 explains the meaning of y – 5:

> S3 (26:12–26:26): $y - 5$ is the number of rooms which have students living in them.

Before the teacher returns the class of the diagrammatic representation, it is important to make sure that students explain the meaning of y – 5 in terms of the context and are able to relate this term to the diagram that has been presented.

> T (26:26–26:39): That is to say, looking at the total number of occupied dormitory rooms, (Teacher points at the □□□□□□···×××××) right? Get rid of the 5 rooms…….And there are 4 people in each room, right? And then?

Student S3 then explains why it is necessary to multiply $(y - 5)$ by 4.

> S3 (26:43–26:48): This gives the total number of students who live in the dormitory.

The teacher sees no need to comment on this formulation and so asks S3 to comment on the right hand side.

> T (26:43–26:48): Oh, then multiply by 4, which is the total number of students who live in the dormitory, isn't it? OK, (Teacher draws a tick under the $4(y - 5)$, and what about the other side?

Student S3 now explains the meaning of 3y:

> S3 (26:48–26:53): 3y is the total number of students who could live in the dormitory.

Student S3 does need to be reminded about the need to add 100, but the teacher is concerned to have the class attend to this feature of the formulation.

> T (26:53–27:01): Oh, does each room have a person living in it? Is every room full of students? How many people live in each room?

This question is addressed to the whole class. Students respond together:

> Ss (27:01:27:02) 3 persons. (Teacher says 3 persons simultaneously with students).

The teacher then reiterates:

> T (27:02:27:11): But, this (pointing to 3y) is not the total number of students, is it? What do we need to add? The number of students who could not live in the dormitory, right? OK. (Teacher draws a line under 3y+100)

At this point, the teacher is concerned to have students understand clearly that the term 3y+100 expresses the number of students that are accommodated using a three-person per room rule and the 100 students who are left over. Students also need to understand that there are now two different formulations of the total number of students depending on whether each room accommodates 4 people or 3 people.

> T (27:11:27:45): What's this? (the teacher points to $4(y - 5)$) This is the total number of students who live in the dormitory of the school. This (the teacher points to $3y + 100$) is also the total number of students who lived in the dormitory of the school. We now get a formula according to this proper relationship, right? OK, please have your seat.

At the same time, the teacher returns to the equation resulting from Method 1:

$$\frac{x}{4}+5=\frac{x-100}{3}$$

The teacher points to the formulation on the left side and says: This is the total number of the dormitory rooms of school, right? The other (the right) side is also the total number of the dormitory rooms of school, isn't it? Does any student have an idea that is different? (27:45–28:22). The teacher asks various students if they are happy with the formulation. No disagreement is indicated. However, the teacher recognised that some students may not have written the left-hand side as 4(*y* – 5). So to clarify:

> T (28:22–29:06): Some of your classmates may have said: "This place can be changed." (Teacher points at the 4(*y* – 5), and asks "How could it be changed? There is a bracket here, right? Based on this, I could change it into 4*y* – 4×5 = 3*y*+100, is this one right?

Students agree with the possibility of expanding the left-side bracket:

> Ss (29:06–29:07): Students in unison say: Right.

The teacher then wishes to check that students understand what 4y – 4×5 means.

> T (29:07–29:19): We can see they are the same from the perspective of the formulation, right? How could we understand it from the perspective of the context? (Students hands go up) S5?.

Student S5 offers an explanation:

> S5 (29:19–29:30): 4*y* is the number of students there would be if every room contained 4 people. Then, 4×5 is the number of students that the vacant dormitory rooms would accommodate.

The teacher returns to the diagram showing the total number of rooms (□□□□□□...×××××). The teacher makes a special point of reminding students that 4*y* cannot logically be the total number of students. If every room had 4 students living in it, there would be too many students. The actual number of students is less than 4*y.* How many less is what the teacher wants students to think about.

> T (29:30–29:59): 4y is the number of students there would be if every room had 4 people living in it, right? And then subtract here (the teacher points to the empty rooms indicated by ×××××) the number of students that 5 empty rooms can accommodate, that is the actual number of students, isn't it? (Students are silent.)

The teacher then refers to the diagram showing all the rooms, □□□□□□⋯×××××and revisits the reasoning behind the formulation based on having 4 students in each room. Then the teacher considers the situation where 3 students live in each room.

> T (29:59–32:01): How many people live in each room? 3 persons (Teacher draws 3 crosses into every □ and into every×of the second one). What's the result? How many people are there outside? 100 persons. What do you think about it? Are there any new ideas? The teacher then asks students to consider the difference between the two representations.

Now that the teacher is confident that the students understand that there are two expressions for the total number of students, she wants them to look carefully at the two expressions. For this purpose, having the left side in expanded form is likely to be easier for students to think about than having 4(*y* – 5). Students need to develop a capacity to evaluate which form of a given formulation is more suitable for finding a value for one of the unknowns. Student S6 offers an explanation:

> S6 (32:01–32:10): The difference of them two should be *y* – 5. Oh, no, *y* + 5, *y* + 5 persons.

Some confusion between the number of rooms and number of persons is evident.

> T (32:10–32:11): What is *y*? (The teacher asks Student S6.)
>
> S6 (32:11:32–16): This side has *y* rooms, and every room is full.

The teacher now wants students to consider the difference between the two sides of the simultaneous equation in its expanded form: $4y - 20 = 3y + 100$. This is an interesting move that could enable some students to solve the equation visually.

> T (32:16:32:42): (The teacher pointed at the ×××××in the upper) If these are all full of students, right? How many people if every room is full? (Teacher wrote $4y$ on the blackboard.) That's 4y persons, (Teacher pointed at the ×××××in the down) and this is so such numbers rooms, right? Every room has 3 persons and all are full, right? That is 3y. (The teacher wrote $3y$ on the blackboard). What is the difference of the two numbers? How much? y persons? Can you work out y persons? Take a seat and think about it.

The teacher has referred to "y persons" based on the expressions involving y which are the formulations for the total number of students. And some students after looking at the equations above conclude that they are finding the number of people:

> Ss (32:42–32:46): There is a total difference of 120 people.

The teacher senses that students are locked into thinking about the number of people which is the topic of this formulation. But are they thinking of what y represents?

> T (32:46–32:51): Oh, there is a difference of 120 people, why? (Teacher wrote $4y - 3y = 120$ on the blackboard) what is the 120?

> Ss (32:51–33:01): Ur, Ur, 5 rooms.

This seems to be a point of difficulty for the class. Some students appeared to think that since the two expressions, namely $4y - 20$ and $3y + 100$ involving y concern the total number of students, the value for y obtained by solving these two simultaneous equations, $4y - 3y = 120$, represents the number of students. This confusion has lost sight of the original formulation where y designates the number of rooms. There was no evaluation of this result. If this had been done, then students could have seen that with 120 rooms, the total number of students would be 4 × 120 less 20 (i.e. 460 students); and using the other arrangement of having 3 students per room, the total number of students would be 3 × 120 plus 100, also giving 460 students.

In the next few minutes, the teacher asked students to consider which of the two formulations was easier to solve. Teacher and students agreed that the simultaneous equations involving y are easier to solve than the fractional formulations involving x. Students then looked at another problem for the remainder of the lesson.

What is clear and important about this segment of the lesson is how at various stages the teacher and students are able to focus in a prolonged way on alternative mathematical formulations of the problem. The actual employment of mathematics to solve the problem is assumed. Students are invited to consider whether Method 1 or Method 2 offers an easier path to the solution. It would have been interesting to see how well students were able to solve for x using the two formulations derived in Method 1. There was a need to involve students more in interpreting the result obtained in Method 2, and to evaluate this result. That appears to be a weak point. However, in terms of its consistent focus on mathematical formulation this lesson shows clearly the importance placed by this Chinese teacher on elements of mathematical formulation that are strongly emphasised by the PISA Framework (OECD, 2015).

Lessons for Australian teachers

From an Australian perspective, this lesson, while not without some features that we may question, provides a different perspective to mathematical formulation that may be presented in a typical multi-step approach to solving simultaneous equations. A multi-step approach to solving simultaneous equations from worded problems is succinctly presented by Vincent, Price, Caruso, Romeril and Tynan (2012) in their MathsWorld10 (Australian Curriculum edition) textbook as: Step 1: define the two variables; Step 2: Construct two different linear equations using the two variables; Step 3: Solve the equations simultaneously; and Step 4: Interpret the solution within the context of the problem (p. 215). This Chinese lesson is almost entirely focussed on the two opening steps, with Step 2 having the greatest potential to help students to think mathematically. For that to be achieved, one needs a carefully chosen and well-tried problem. The process of mathematical formulation, whether at Year 10 in this Australian textbook or earlier in the Chinese example, is not mechanical. It requires an approach to teaching which recognises the importance of:

- assisting students to think about the mathematical aspects of a problem situated in a real-world context and to identify variables for the total number of rooms and the total number of students;
- having students understand and explain the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically;
- representing a situation mathematically, using appropriate variables, symbols, and diagrams, and connecting the different symbolic expressions to a well-chosen diagram;
- ensuring that students understand and can explain the relationships between the context-specific language of a problem and the symbolic language needed to represent it mathematically; and
- modelling through demonstration and questioning how to translate a problem into mathematical language.

These features, clearly present in the Chinese lesson discussed here, need to inform any multi-step approach to ensure that mathematical formulation is treated seriously.

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FROM ARRAYS TO ALGEBRA

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By identifying the learning experiences that need to be developed in the primary school, and how secondary teachers can build on these experiences, ensures solid algebraic understanding and a smooth transition between primary and secondary mathematics. Using concrete materials, within a concrete–representational–abstract pedagogical approach, may be seen as one among many methods that contribute to the overall process of developing algebraic skills. Linking an area-based model to previous understandings involving numbers should assist the conceptual understanding of algebraic expansion and factorisation to ensure that students are not just fluent in algebraic manipulation but also understand how the processes work.

Introduction

As a secondary trained teacher who now finds herself predominantly teaching primary pre-service teachers, an interest in the important transition between primary and secondary mathematics teaching has been developed. By looking at the big picture, it is possible to identify the learning experiences that need to be developed in the primary years and how secondary teachers can build on these experiences to ensure a smooth transition and solid mathematical outcomes.

One of the big ideas to set children up for algebra in secondary schools is developing multiplicative thinking, and an important part of developing multiplicative thinking is developing an understanding of the importance of arrays as multiplicative models. As early as Year 2 the *Australian Curriculum: Mathematics* states that students should "recognise and represent multiplication as repeated addition, groups and arrays" (ACARA, 2014). In particular, arrays and regions assist in supporting the shift from additive thinking ('groups of' model) to multiplicative thinking ('factor-factor-product' model; Siemon, 2013), and eventually to proportional and algebraic thinking.

By laying appropriate groundwork, primary teachers can set children up for not only a deeper and more robust understanding of multiplication and division, but also mathematical success in the secondary years.

A developmental approach

When students are asked to use blocks, tiles or counters to make four groups of three they will often make a model that resembles Figure 1. This representation is correct, but if the aim is to move students from additive to multiplicative thinking, may not be the most efficacious. The model used in Figure 1 can encourage students to think about multiplication only as repeated addition, whereas an array or region model (Figure 2) not only demonstrates the strategy of repeated addition it also encourages other understandings such as the relationship between factors and multiples and the link of multiplication with area.

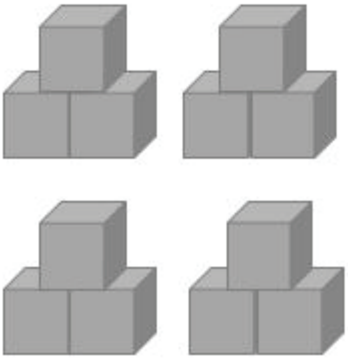


Figure 1. Four groups of three.

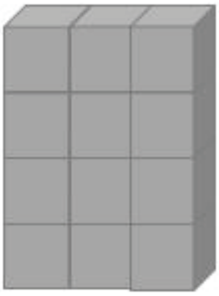
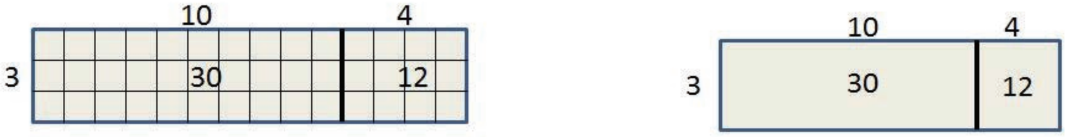


Figure 2. Array: four groups of three.

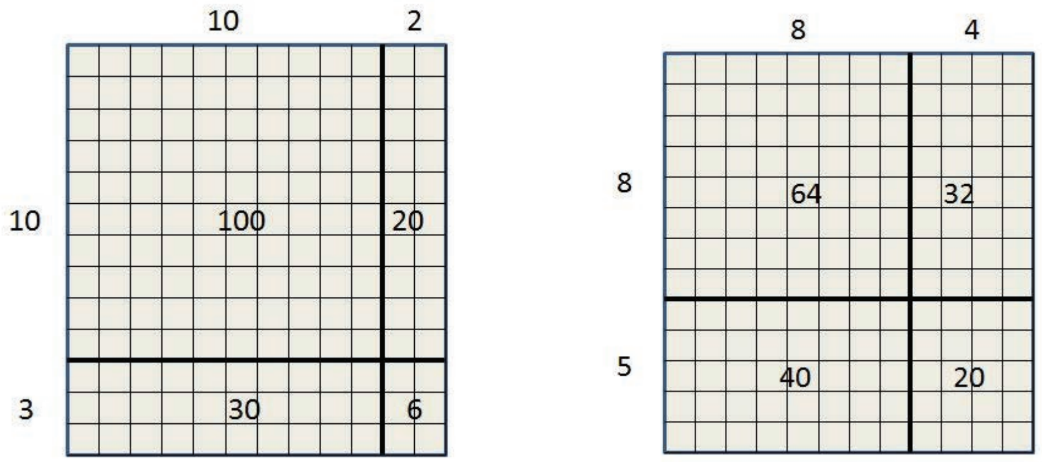
Array models can be extended into other multiplication situations. Representing a one-digit by two-digit multiplication as an array provides a visualisation of both the magnitude of a number and how it can be partitioned to promote understanding. It also demonstrates the link between multiplication, the distributive property and area (see Figure 3).



$$3 \times 14 = 3 \times 10 + 3 \times 4 = 30 + 12 = 42$$

Figure 3. Array representations of 3 × 14.

This idea can then be extended to two-digit by two-digit multiplication, or any-digit-by-any-digit multiplication. Students who do not use an array or region method to visualise multiplication and who have a limited understanding of the distributive property often think that 13×12 can be calculated by $10 \times 10 + 3 \times 2$, which is incorrect. Using the array model and identifying the area of the associated regions allows students to identify why this is not the case (see Figure 4).



$$\begin{aligned} 13 \times 12 &= 10 \times 10 + 3 \times 10 + 10 \times 2 + 3 \times 2 \\ &= 100 + 30 + 20 + 6 \\ &= 156 \end{aligned}$$

$$\begin{aligned} 13 \times 12 \quad &= 8 \times 8 + 5 \times 8 + 8 \times 4 + 5 \times 4 \\ &= 64 + 40 + 32 + 20 \\ &= 156 \end{aligned}$$

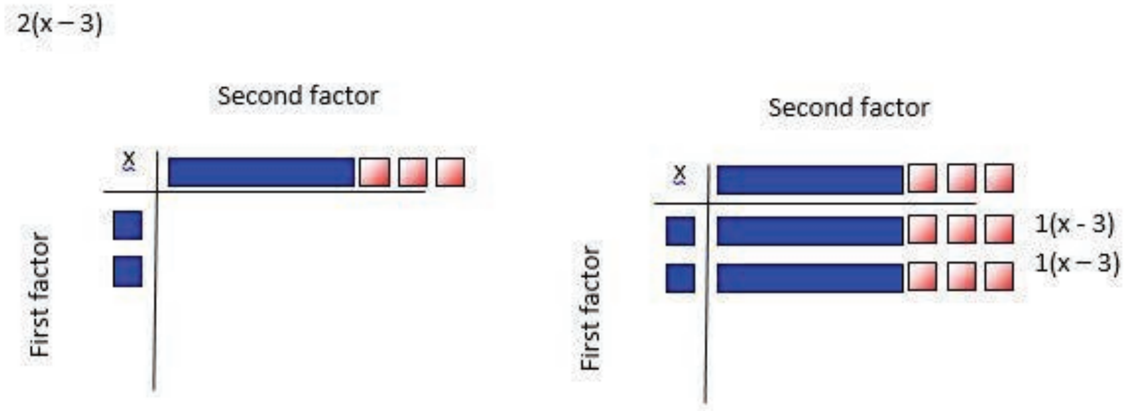
Figure 4. Array representations of 13 × 12.

Building on previous understandings is important. When students are able to build on prior knowledge by making connections to previous understandings, their learning will be more meaningful. For example, just as students should see that the process of multiplying by an any-digit number is just an extension of the process for multiplying by a one-digit number, they should understand that the process is the same no matter what the multiplier (Reys et al., 2012). This, in turn, leads to the process of generalising and allows students to apply this prior knowledge to algebraic representations, forging the links of factors and the distributive property from number and applying them to algebra.

This development of the understanding then continues into the secondary classroom. The use of a concrete, visual, area-based model, such as algebra tiles allows this seamless transition. Algebra tiles are a model that can facilitate this. Although algebra tiles are versatile, they are not intended alone to constitute an entire course in algebraic manipulation but instead are anticipated to be one of several ways in which conceptual understanding can be developed. To use algebra tiles, students need only to understand the properties of the additive identity ($a + 0 = a$) and the additive inverse ($a + (–a) = 0$).

Building on the array model for multiplication of any-digit numbers, linear and quadratic expansions and factorisation may be modelled with algebra tiles. They are also useful for investigating integer arithmetic, but that is beyond the scope of this paper.

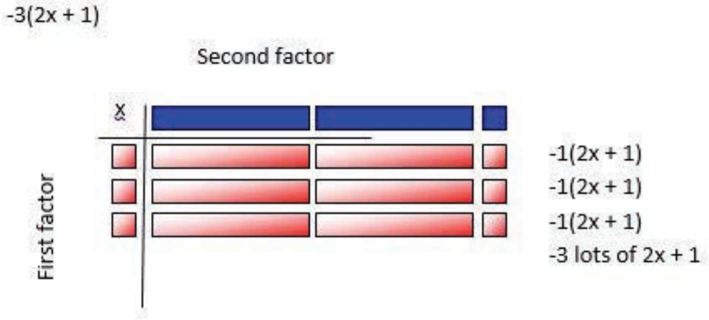
By looking at $2(x – 3)$ as two lots of $(x – 3)$ a model can be constructed using algebra tiles (see Figure 5). The use of the terms ‘factors’ and ‘multiple’ which were introduced when students were looking at arrays in the Number strand in the primary years are illustrated in a meaningful way.



The answer is 2x – 6, which is a rectangle with length x – 3 and width 2.

Figure 5. Expanding 2(*x* – 3).

In the same way, multiplying by a negative number may also be modelled (see Figure 6).



The answer is – 6x – 3 which is a rectangle of length 2x + 1 and width – 3.

Figure 6. Expanding –3(2*x* + 1).

Linear factorisation builds on previous understandings of the array models of prime and composite numbers that have been developed in the primary years. Students can use algebra tiles to consolidate that prime numbers (and the number 1) can only be represented by one array, whereas composite numbers may be represented by at least two different arrays (See Figures 7 and 8).

Figure 7. Seven is prime.

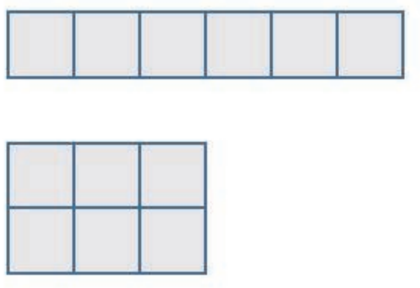
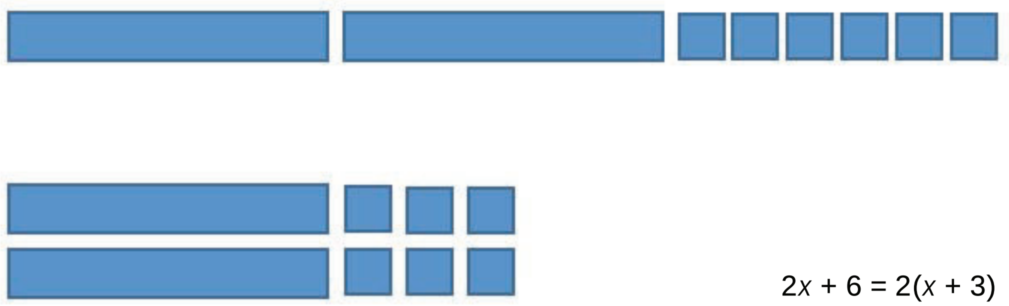


Figure 8. Six is composite.

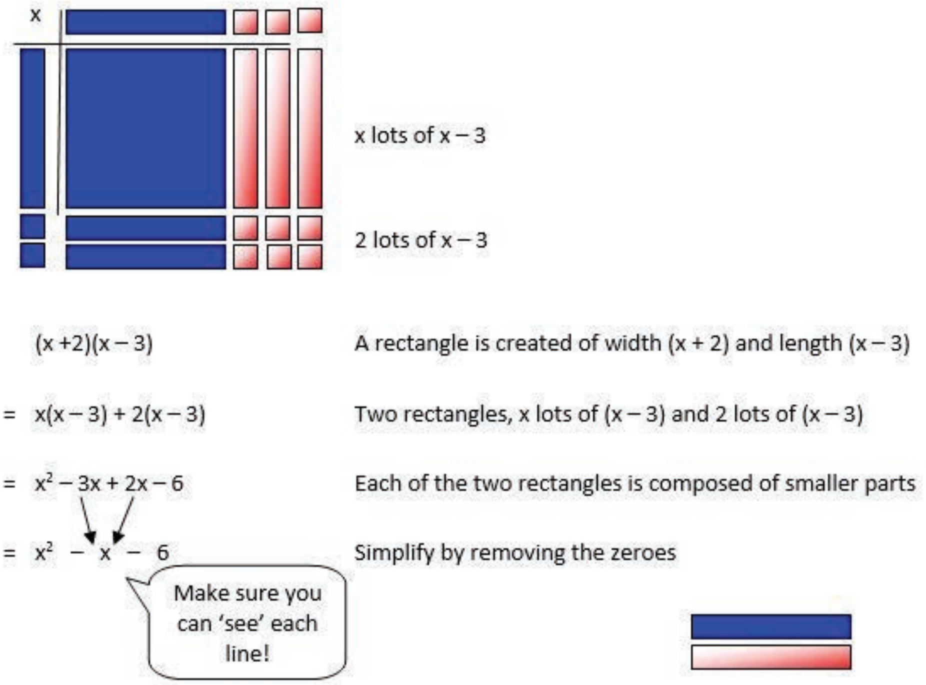
In the same manner, 2x + 5 can only be represented by one array, so it has no integral factors whereas 2x + 6 may be represented by more than one array which means it has integral factors (see Figure 9). Naturally, if there is a positive integral factor, then there is also a negative integral factor.



$2x + 6 = 2(x + 3)$

Figure 9. 2*x* + 6 may be represented by two arrays, so has an integral factor.

Quadratic expansions may also be easily represented using algebra tiles. The area representation parallels the formal expansion method for number using the distributive property. The multipliers (factors) may be set up on the frame (see Figure 10); $(x + 2)(x – 3)$ means all of $(x + 2)$ is to be multiplied by all of $(x – 3)$.



$(x+2)(x-3)$	A rectangle is created of width (x + 2) and length (x – 3)
$= x(x-3) + 2(x-3)$	Two rectangles, x lots of (x – 3) and 2 lots of (x – 3)
$= x^2 - 3x + 2x - 6$	Each of the two rectangles is composed of smaller parts
$= x^2 - x - 6$	Simplify by removing the zeroes

Figure 10. Expanding (*x* + 2)(*x* – 3).

Each line of formal mathematical notation is accompanied by a concrete representation, which allows students to 'see' the meaning of each line. Other investigations of perfect squares, difference of perfect squares and examples of other coefficients of x^2 can also be explored. For a detailed description of how to factorise quadratics using algebra tiles see Day (2014).

The solution of linear equations may also be modelled using algebra tiles (See Figure 11), especially if students in their primary school years have had a good grounding in the equals sign representing balance. Once again, formal notation can be developed alongside the concrete model.

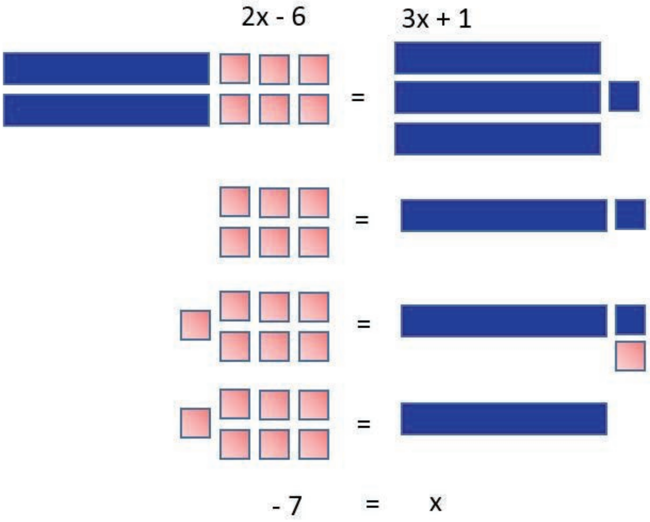


Figure 11. Solution of linear equation 2*x* – 6 = 3*x* + 1.

There are some limitations of the algebra tiles model. To represent a variable with concrete materials is one such obvious limitation. To endeavour to overcome this, it is important that the length attributed to the variable is not a multiple of the length attributed to a unit piece. In this way students are less likely to form a visual impression of a numerical value for the variable. Although there are some limitations of the model, the advantages of having a concrete, visual, area-based model far outweigh the limitations (Lovitt, personal communication, June 26, 2013).

The Australian Curriculum: Mathematics

The following are content descriptors from the *Australian Curriculum: Mathematics* (ACARA, 2013) that can be directly addressed by the use of algebra tiles.

- Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)
- Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)
- Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)
- Solve simple linear equations (ACMNA179)
- Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190)
- Factorise algebraic expressions by identifying numerical factors (ACMNA191)
- Solve linear equations using algebraic and graphical techniques. Verify solutions by substitution (ACMNA194)
- Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate (ACMNA213)
- Factorise algebraic expressions by taking out a common algebraic factor (ACMNA230)
- Expand binomial products and factorise monic quadratic expressions using a variety of strategies (ACMNA233)
- Solve problems involving linear equations, including those derived from formulas (ACMNA235)

With the Proficiency Strands of Understanding, Fluency, Reasoning and Problem Solving as the power driving the content of the *Australian Curriculum: Mathematics*, there is increased motivation for teachers to ensure that students are not just procedurally fluent in algebraic manipulation but also understand how the processes work and how to reason mathematically. By linking the area-based model to previous understandings involving number, secondary teachers can build on the work done in primary schools and assist students to gain an understanding of the concepts underpinning algebraic manipulation. Having students work with concrete materials provides an ideal forum for mathematical conversations which assist students to clarify their own ideas and share ideas with others. This leads to explanation, justification and communication, all of which contribute to mathematical reasoning.

Acknowledgment

Much of the work on algebra tiles is adapted with permission from Lovitt, Marriott and Swan (1979). In the words of Charles Lovitt: “The tiles are a model, a concrete, visual, area-based model, and all models have limitations ... but it is still one of the most powerful models I have found” (Lovitt, personal communication, June 26, 2013).

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LEARNING ENVIRONMENTS THAT SUPPORT THE DEVELOPMENT OF MULTIPLICATIVE THINKING

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Given the right learning environments primary aged children can and do develop the capacity to think multiplicatively. Through vignettes taken from interviews with a Year 5 class during a research project, the optimal conditions for the conceptual development underpinning multiplicative thinking is examined.

There is a nation-wide concern that we are not producing enough students with a mathematics background. Over the last few years there has been a continuing trend of students electing not to choose higher or even intermediate mathematics as a subject of study (Kennedy, Lyons & Quinn, 2014). There are many possible reasons for this. One of them is that many students choose not to be involved in higher mathematics due to a lack of success during primary and lower secondary years. The reasons behind this lack of success are many and varied, but certainly a major contributor would be an inability to cope with the content of the mathematical curriculum. One area of the mathematical curriculum that causes concern and is really a 'gate keeper' to higher mathematics is the capacity to think multiplicatively.

Multiplicative thinking is not just about multiplication facts

Most teachers would agree that basic multiplication facts are essential for further success in mathematics. They sit at the root of learning multi-digit multiplication, fractions, ratios, division, and decimals (Elkins, 2002; Kilpatrick, Swafford & Findell, 2001). Fluency with multiplication facts allows students to focus on more sophisticated tasks such as problem solving (Westwood, 2003). The authors of this paper whole-heartedly endorse the learning of multiplication facts, but students can be fluent with multiplication facts yet still be additive, rather than multiplicative thinkers. What is required is the development of the ability to apply these facts to the variety of situations that are founded on multiplication.

What is multiplicative thinking?

According to Siemon et al. (2006a, p. 28), multiplicative thinking is:

- a capacity to work flexibly and efficiently with an extended range of numbers (and the relationships between them);

- an ability to recognise and solve a range of problems involving multiplication and/or division including direct and indirect proportion; and
- the means to communicate this effectively in a variety of ways (e.g., words, diagrams, symbolic expressions, and written algorithms).

Whilst most teachers would recognise and acknowledge the first two dot points they may be a little more unfamiliar with the notion of communicating these ideas. However if one takes into consideration the Proficiency Strand requirements from the *Australian Curriculum: Mathematics* (ACARA, 2015) it should be noted that communication is an integral element of the Reasoning Proficiency and therefore fundamental to mathematics learning. It is these same communication skills that allow students to explain their understandings so that teachers can make judgments about their students' depth of knowledge for the purposes of assessment.

Why is multiplicative thinking important?

Multiplicative thinking is critical to mathematical success as it underpins the majority of work pursued in the area of Number and Algebra through the upper primary and lower secondary years of schooling. It is fundamental to the development of concepts and understandings such as algebraic reasoning (Brown & Quinn, 2006), place value, proportional reasoning, rates and ratios, measurement, and statistical sampling (Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006b).

Multiplicative thinking and primary school students

The development of multiplicative thinking is obviously important and this raises the question of whether it is being appropriately developed in primary schools. The research of Clark and Kamii (1996) revealed that 52% of Year 5 students were not sound multiplicative thinkers. This was supported through research conducted by Siemon, et al. (2006a) who found that up to 40% of Years 7 and 8 students performed below curriculum expectations in multiplicative thinking and at least 25% were well below expected level.

It is important to note the areas where students might find difficulties in developing multiplicative thinking. The following list is quite long, but what should be highlighted is that there are ways and means to remediate these difficulties that are developmentally suitable for primary school students and develop a conceptual rather than procedural understanding.

Some of the difficulties that students encounter are:

- seeing the relevance of the many-to-one count (Jacob & Willis, 2003);
- employing a row by column structure (an array) to work out a number of squares, and resorting to additive strategies (Batista, 1999);
- recognising the multiplicative situation to employ multiplicative thinking (Van Dooren, De Bock & Verschaffel, 2010);
- reconceptualising their early counting and additive understanding about number to understand multiplicative relationships (Sophian & Madrid, 2003; Wright, 2011);
- recognising that multiplicative thinking is distinctly different from additive thinking even though it is constructed by children from their additive thinking processes (Clark & Kamii, 1996);

- understanding the relationship between multiplication and division and being able to consistently employ the inverse relationship between the two operations (Jacob & Willis, 2003);
- understanding and employing the commutative property of multiplication;
- transitioning from using manipulative materials when the need to describe when the operations of multiplication and division became objects of thought rather than actions (Sophian & Madrid, 2003; Wright, 2011).

The remainder of this article will focus on students from a Year Five classroom and the conceptual understandings that were displayed when students responded to a written quiz and a one-to-one interview regarding multiplicative thinking.

One-to-one interviews

A series of one-on-one interviews were undertaken across several schools. Although acknowledging that one-to-one interviews are time consuming, this form of data collection was chosen as a way in which to:

- uncover children's thinking and help to understand why some students learn and others fail to learn (Heng & Sudarshan, 2013);
- enhance the knowledge and skills of teachers by developing a deeper understanding and awareness of the way that children construct their mathematical understandings (Heng & Sudarshan, 2013);
- illuminate student misconceptions (Heng & Sudarshan, 2013) and reveal conceptual understanding of a topic which may not necessarily be obvious through a written assessment task (Clements & Ellerton, 1995); and
- allow teachers to develop more realistic expectations of what the student can and cannot achieve (Clarke, Roche & Mitchell, 2011)

We would like to explore a selection of the comments made by some of the students during the interview, discuss how these comments have the potential to give an insight into the capacity to think multiplicatively, and highlight some of the teaching practices that may be helpful in assisting the students to develop this thinking. For the purposes of this article we will look in some depth at one of the interviews, with supplementary comments taken from observations from other interviews.

James (a pseudonym) approximately 11 years of age

When James answered the questions posed to him in the interview he was not the student who was the quickest to respond. In answering the question of 7 × 6, he took a considered moment and then answered the question correctly. Later on, when explaining a mental computation strategy, he made a multiplication fact error of $3 \times 8 = 16$, so his recall was not infallible. However, what he was able to do was to analyse his solution and determine that 16 could not have possibly been a correct answer. James may have made a multiplication recall error but his understanding of numbers allowed him to appreciate there was an error, locate that error and then fix it.

James was asked to use mental computation to work out the answer to 6 × 17 and to articulate how he would solve the problem. He reasoned as follows: "Six times seven is 42, so, then I think what's six times one? And I think six, and add it to 60, so 42 plus 60 is 102. The 60 came from six times one but it's actually a ten." James's explanation allowed us to see a number of important mathematical ideas and understandings in action. In order to achieve this computation mentally, he applied some basic facts (6 × 1 and 6 × 7), partitioning (seeing seventeen along place value partitioning lines as 10 and 7) and place value (the one is seen as a ten). When James was asked to illustrate his thinking using manipulative materials he was able to do so. Selecting from a range of materials James chose the Multibase Attribute Blocks (MAB) and modelled his understanding by creating six lots of seventeen, each made up of a long (ten) and seven ones. It should be noted that he actually talked about needing six tens and then about the required ones, not one ten and seven ones and another one ten and seven ones, and another...etc. His thinking suggested he was seeing the objects multiplicatively rather than additively.

James was then asked if, by knowing 6 × 17, did he know what 17 × 6 was? His answer to this was a very firm "Yes". When asked to explain how he knew, James expressed his understanding by saying it was "the same thing backwards", and then proceeded to explain how he was still multiplying 6 by 10 and 6 by 7, hinting at an understanding of the commutative property of multiplication. He then went on to use smaller numbers to illustrate his understanding. He selected counters to illustrate five multiplied by three and constructed a multiplicative array. That is he arranged the counters as three rows of five counters in a rectangle (Figure 1).

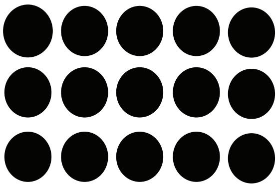

Figure 1. A three by five array.

He then said "You can also look at it as three columns and five rows" and turned the rectangle to illustrate a five by three array (Figure 2).

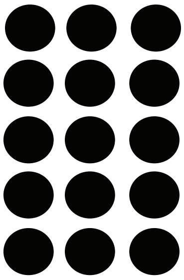

Figure 2. A five by three array.

Again, this level of understanding strongly indicated multiplicative thinking. Other students when faced with constructing a model to illustrate three by five created three sets of five (Figure 3) and when asked if there was another way of showing this did not employ the array, perhaps indicating that additive thinking was dominant.

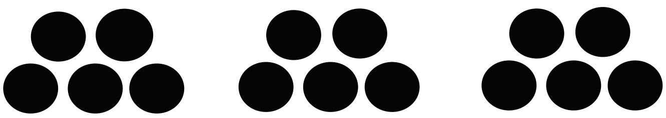

Figure 3. Three sets of five.

When asked if multiplying 89 by three gave the same answer as 80 by three and nine by three James was unequivocal in answering in the affirmative. He stated that splitting it up to "three times 80 and three times nine" was easier than trying to work out 89 times three or 89 and 89 and 89 (James indicated a developing understanding of the limitations of repeated addition/additive thinking). Although he did not name it, nor was he required to, James showed himself to be completely comfortable in partitioning a number and employing the distributive property of multiplication over addition. Likewise when James was asked to explain the inverse relationship between multiplication and division he did so without hesitation.

James was asked the question: "My friend Paul says that if you know that 6 × 17 is 102, you can work out the answer to 102 ÷ 6. Is he correct?" His reply was "that (pointing to 102) is a multiple of six and 17… I know a multiple divided by a factor is another factor." Not only had James displayed an application of the inverse property of multiplication and division, his articulation of it was impressive. It should be worth noting that several students from the same classroom as James produced well-informed and articulate answers and were obviously well used to verbalising their reasoning and understanding.

At a different point of the interview James used the term "add a zero" when he was multiplying by ten. When probed about the notion of "adding a zero" he was able articulate that by adding a zero you were actually "timesing by 10". Initially using the phrase "add a zero" may suggest a naïve understanding of what happens to a number when multiplied by ten. However, when asked what the answer was to 3.6 multiplied by 10, James did not over-generalise this rule, he stated: "I can't add a zero onto the end of this 'cause that just makes the fraction longer." He was then asked if that made the number any different, to which he replied, "No it doesn't make it any different, so I just move the six one space up, past the… um… decimal point… to make it 36." James showed a sound understanding in that he talked about moving the digit one place rather than moving the decimal point, and he was very aware that adding a zero to 3.6 did not alter the magnitude of the number, that is, 3.6 is equal to 3.60. Another observation which can be made was that James related the numbers to the right of the decimal point as being fractions, something which not all students did. Later he expressed that two point six meant two and six-tenths. He further explained that "timesing by ten is basically having a number line, like you'd have the millions, one hundred thousands, ten thousands, down to one and then into the fractions". Although he may have used the term number line a bit loosely, James strongly indicated a knowledge of the place value system.

When the conversation was steered towards division as a multiplicative situation James seemed equally relaxed. He referred to the inverse relationship between multiplication and division and how he employed multiplication when presented with division problems. He also explained that division does not always provide a smaller answer and used the examples of dividing by one and dividing by fractions. Again he talked about moving the number (digit) when dividing by ten and correctly articulated that the digit moves to the left and also mentioned that the resulting number was ten times smaller. When shown the card 8 ÷ 0.1 and asked to predict the answer James immediately answered with zero point eight (0.8). When asked to check this answer using a calculator and seeing that the answer was 80, James responded by talking about being exposed recently in the classroom to some reasoning. He said that "the 80 is a bigger digit but it's actually smaller pieces." He was asked to clarify and after a little hesitation he said, "You'd have how many ones go into eight, so you'd have eight ones. So you'd have eight ones and the ones are the parts. (A prompt: "How many parts?"). There's 10... there would be ten." James has reasoned that 10 'pieces' per whole multiplied by eight gives a total of 80 'pieces'. As he had a conceptual understanding regarding fractions, decimals, place value and multiplication he was able to reason a construct by making connections through and with these understandings.

Teaching experiences

After the student interviews were carried out and analysed, it was decided to spend some time discussing the teaching experiences that James' teacher could identify as contributing to the development of multiplicative thinking in this particular Year Five classroom. A taped semi-structured interview was undertaken and the following points emerged from this interview:

- much time was spent on developing place value concepts and the role zero plays in place value;
- there was a focus on discussions in the mathematics classroom;
- the development of mathematical vocabulary was seen as important;
- development of multiplication facts was seen as a reasoning activity as well as a fluency activity;
- parents were involved in assisting students with their multiplication facts (and were shown how to do this);
- division was introduced before multiplication and making connections between the two was seen as a priority;
- students were given permission to learn at their own developmental stage;
- fractions and decimal fractions were investigated together;
- a lot of time was spent on building mathematical connections; and
- a collaborative culture existed between the teachers in the school and there was a school vision of how to develop mathematical concepts.

As evidenced by the student interviews it was clear that this teacher was successful in developing multiplicative thinking in this Year Five class. One interesting point that arose from the teacher interview was the notion that the use of arrays was highlighted in the Year Four classes in the school and that the Year Five teacher was able to build on the work done in the previous year. This whole school approach was mentioned on several occasions during the teacher interview and was obvious during the student interviews, as the few students who took an entirely algorithmic approach to multiplication were all new to the school.

Conclusion

We are not suggesting that James is necessarily a *typical* Year Five student, but he *is* a Year Five student. It should be noted that to a lesser or greater extent, all of the understandings he articulated were also articulated by his peers. What James and his peers showed the authors was that multiplicative thinking is achievable in the primary school setting. The many mathematical ideas which underpin multiplicative thinking are part of the primary curriculum and sound pedagogical practices can help the students towards developing a conceptual understanding of this vital piece of the mathematics jigsaw.

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ADDRESSING ALTERNATIVE CONCEPTIONS IN MATHEMATICS USING DISCREPANT EVENTS

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The following report presents an analysis of eight mathematics lessons in which students displaying alternative conceptions were lead to reconceptualise their own understanding. Similarities were noted between the approach used by the teacher for juxtaposing discrepant events with questioning in each of these incidents and Cognitive Change Theory, an approach used by science educators to address scientific misconceptions by creating cognitive conflict. Initial data indicates that this approach may be applicable during the explore phase of challenging lessons, encouraging student to think through their own ideas and accommodate new information.

Introduction

Educators often consider why students answer questions in mathematics incorrectly, particularly when those answers are commonly given by students in a wide variety of circumstances (Swan, 2001). While some incorrect answers are simply errors or miscalculations, others are thought to be set within deeper levels of knowledge and more problematic for learners to overcome (Ryan & Williams, 2007). The term *misconception* is used by some researchers to describe situations in which a learner's understanding is considered to be in conflict with accepted meanings and understandings of mathematics (Barmby, Bilesborough, Harries, and Higgins, 2009). Other researchers object to this term because, while these ideas are technically incorrect, from the student's viewpoint, the ideas expressed are logical (Sneider and Ohadi, 1998). In this paper the term *alternative conceptions* is used, expressing the viewpoint that these ideas form a natural part of the development of mathematical understanding (Swan, 2001). This paper reflects the viewpoint held by Hansen (2014), that rather than trying to avoid the development of these ideas by students in the first place, effective teaching identifies, exposes and confronts these ideas, enabling students to restructure their own thinking.

One important assumption underpinning this paper is that students actively construct their own understanding of mathematics and that, to be effective, students need to think deeply about mathematics, connect ideas and be challenged. This paper espouses a connectionist (Askew, Brown, Rhodes, Johnson & William, 1997) or parapositional (Adam & Chigeza, 2014) disposition, that effective teachers focus heavily on the connections between concepts and use multiple teaching approaches as appropriate to the context.

A second important assumption is that when new information fits with what we already know it is assimilated within our existing conceptual understanding, but when it does not, we either reject the new information or accommodate it by changing our cognitive structures (Piaget & Inhelder, 1969). Posner, Strike, Hewson & Gertzog (1982) posited a theory for conceptual change in which an alternative conception which conflicts with new information, is modified or is no longer considered useful and is rejected as untenable.

This paper seeks to apply the lens of conceptual change theory to several mathematics lessons in which students displaying alternative conceptions were led to reconceptualise their ideas.

Background

Hattie (2009) expresses the belief that a learner's construction of knowledge and ideas is more important than the knowledge or ideas themselves. This construction, termed conceptual understanding by Bereiter (2002), connects both surface and deep knowledge to create a schema by which a learner interprets new ideas. Wenning (2008) explains that learners interpret new experiences and information in the light of these existing schema, grafting new understandings onto prior conceptions. He theorises that when new concepts do not fit within a learner's schema these are likely to be forgotten or even rejected, leading to his conclusion that addressing a learner's alternative conceptions in science is critical for the development of understanding.

Conceptual change theory (Posner et al., 1982) proposes a process by which a learner's existing conceptions may be replaced by more robust ideas. According to this theory, new information which conflicts with a learner's pre-existing schema is introduced to create cognitive conflict or disequilibrium (Piaget & Inhelder, 1969; Resnick, 1983), so that "the learner recognises inconsistencies between existing beliefs and observed events" (Swan, 2005, p. 7). As these inconsistencies are recognised, learners choose which of their ideas make the most sense and accommodate those that conflict by constructing new connections and changing their conceptual understanding (Piaget & Inhelder, 1969).

One approach that shows promise for applying conceptual change theory to the field of science education is Erilymaz's (2002) protocol for conceptual change discussion. In this model, teachers use challenging problems to identify students' alternative conceptions. This identification, termed 'exposure' by Erilymaz, involves predicting the results of scientific experiments as well as attempts at solving problems. Exposure is followed by conducting the scientific experiments, or 'discrepant events', during which students observe outcomes that conflict with their predictions. Teachers use questioning to help students focus on this discrepancy, increasing the level of cognitive conflict and encouraging students to grapple with the inconsistencies observed. At a definable point during this discussion cognitive conflict reaches disequilibrium, whereby students reject their initial predictions and accommodate the new information, addressing their own alternative conceptions in the process.

Erilymaz's approach, while originally designed for science educators, shares some common features with the Launch–Explore–Summarise (LES) structure designed by

Lappan, Frey, Fitzgerald, Friel and Phillips (2006) for using challenging tasks within mathematics lessons. Both approaches begin with the posing of a challenging problem. The explore phase from the LES structure, in which students experiment with and discuss ways to solve the mathematical problem posed is also mirrored in Erilymaz's structure, with scientific experimentation used as discrepant events to provoke student exploration and cognitive conflict. Both approaches also draw on student discussion, with a plenary summarising phase led by the teacher that draws together different student ideas and formalises knowledge. Just as challenging tasks following the LES structure have been found to encourage students to think deeply about mathematics and "connect different aspects of mathematics together, to devise solution strategies for themselves and to explore more than one pathway to solutions" (Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche & Walker, in press, p. 6), so Erilymaz argues that posing problems combined with discrepant events in science lessons can encourage students to connect different concepts, explore ideas and change their own conceptions.

Longfield (2009), extends Erilymaz's work on discrepant events to mathematics and history lessons, concluding that these were useful for motivating students to re-examine their thinking, becoming active participants in their own learning and creating new knowledge for themselves. Longfield also suggests that this approach to conceptual change theory may be useful for addressing alternative conceptions in mathematics.

Theoretical approach

Within conceptual change theory three steps for learning are considered essential (e.g., Mayer, 2008; Luciarello, 2014). The following three steps for learning form the basis for the lesson analysis in this report:

1. The learner recognises an anomaly in their thinking, with a rising awareness that his/her current conception is inadequate to explain observable facts. This step involves using discrepant events to create cognitive conflict, thereby "motivating students to re-examine their thinking about previously held ideas and beliefs" (Longfield, 2009, p.266).
2. The learner actively constructs a new model that is able to explain the observable facts.
3. The learner uses the new model to find a solution to a problem.

Methodology

Eight mathematics lessons in which the researcher used the LES structure to pose and explore challenging problems with students in each of the grades from prep to grade seven were recorded using three video cameras and four microphones. As literature on how to assess conceptual change is limited (Jonassen, 2006), 20 incidents were selected from these lessons in which all of the following steps took place: an alternative conception was identified, discrepant events were juxtaposed with questioning to prompt accommodation, the student actively created a new model of understanding and the student generalised this model to solve both the initial challenging problem and a new, more difficult problem.

The selected incidents included:

Table 1. Incidents selected for examination using cognitive change theory.

Prep:	Three incidents: Two involving number conservation (to five) and one involving partitioning of single digit numbers.
Grade one:	One incident: Relative size of numbers to 10.
Grade two:	Three incidents: One involving number conservation and two involving partitioning of two-digit numbers into tens and ones.
Grade three:	Three incidents: One involving relative size to 10, one with relative size to 100 and one with relative size to 1000.
Grade 4:	Three incidents: One involving the equivalence of halves, one involving the comparative size of triangles and rectangles and one involving the use of the term “quarters” to mean any fraction that was not halves.
Grade 5:	Three incidents all involving decimal numbers: One in which students thought tenths were the same size as ones, one in which students thought tenths were the same size as halves, and one in which students thought that 0.7 was the same as 1/7.
Grade 6:	One incident: Involving the commutativity of multiplication.
Grade 7:	Three incidents all involving Proportional Reasoning: One involving the equivalence of halves and two involving the equivalence of thirds.

For the purpose of this paper, discussion is limited to three incidents: one from each of Prep, Grade 3 and Grade 4. For each of the three incidents selected, the video and audio recordings were synced to create an audio-visual record that captured multiple viewpoints. Each incident was then transcribed, with additional annotations included for student actions as well as still images captured from the video feed to illustrate expressions and movements. Similarities between the incidents were noted and examined, to identify actions, questions, statements and expressions that met any of the criteria described in the previous paragraph. An analysis of the juxtaposition of discrepant events with questioning is included in the results section.

Results and discussion

Data from the audio-visual record and transcripts indicates that the following five phases were present in each of the incidents analysed:

Phase 1: An alternative conception held by one or more students was identified by the researcher and confirmed at least twice using a challenging problem.

Phase 2: Students predicted the results of a discrepant event before it was carried out and then observed a different outcome to that predicted. The researcher juxtaposed multiple discrepant events (minimum 13 events) with questioning to build cognitive conflict.

Phase 3: The student discarded his or her alternative conception and accommodated the new information to create a new conceptual model. The researcher prompted students to explain this change in ideas.

Phase 4: The student used his or her new model to successfully solve the initial challenging problem.

Phase 5: The student generalised this new model to answer at least one more difficult question that required the new understanding.

Further analysis focused on the interplay between discrepant events and questioning and the creation of cognitive conflict during phases two and three of the selected incidents. Findings are summarised in Table Two below.

Table 2. Discrepant events, questioning, cognitive conflict and accommodation in three incidents during which students discarded their alternative conceptions.

Discrepant events observed:	**Most commonly occurring questions:**
Incident 1: Prep students thought that five counters would no longer be five when they were moved. One student predicted four different amounts when the same five counters were shaken in a cup (initially six, then three, four and finally two). Following the incident one student stated, "It will still be the same amount—none fell out." A second student added further explanation stating, "You're not magic!"	
13 discrepant events: • Shaking the counters in a cup, predicting how many there would be, tipping these out and counting them (5 times). • Placing five blocks on a desk and moving these around into different spatial arrangements. Predicting how many there would be, counting these and discussing why there were still five (4 times involving different arrangements, with multiple times counting each, 8 events altogether).	• How many will there be now? • So you think the number of counters will change each time? • How many are there really? You count them for me. • Did it change? • It's still five? But I thought it was going to change? • Did I shake it wrong? Is there something else I could try? • How come it didn't change? • Is there a way that you could move the blocks so that there wouldn't be five?
Incident 2: Grade three students thought that 100 would be half way on a number line between one and 1000. The line was constructed from masking tape on the floor (5m long). Following the incident, the students successfully self-corrected their initial prediction and successfully placed several three-digit numbers on the line in their correct positions.	
39 discrepant events: • Students were given 200, 300 and 400 to place on their line, then 900, 800, 700, 600 and 500. They observed that there was not enough space and moved the 100 to the ¼ position between one and 1000 (8 events). • Students and teacher stepped out the line, counting in hundreds to observe that there was a large space between the 1 and the 100. Students moved the 100 a little closer to the one but maintained a relatively larger space between the 1 and 100 compared to the other hundreds (4 events). • Students were asked to explain why they left a larger space between one and 100. They stated that they were leaving room for the tens. Students were given 10, 20... 90 to place on their line. They moved the 100 back to the middle of the line (9 events). • Students were given 110, 120, 130 and 140 to place on their line. They expressed confusion and tried to move the 200 closer to the 1000. Students were given 210 and 310 to place on their line. Students stated that the line was too short (6 events). • The students counted in tens between each hundred, identifying how many tens were in each and considered if there was enough space for all of the tens (9 events). • Students removed the tens from the line and stepped it out again (3 events) before finally moving the 100 to the correct position.	• Where do you think (this number) goes? Note: This question was asked more than 30 times with different numbers and in slightly different ways throughout the incident. • How about 200, 300, 400... 900? • Does that look right to you? (Asked when students started to look quizzically at the line) • Which bit looks funny? How come it looks funny? • Can you make it look right please? You move the blocks until it looks right to you. • Tell me about this space? • How many tens are there in here? How many in here? • How about this space? Aren't there any tens in here? • What do you notice about all the spaces? • How big is 100 compared to 1000? Is it a really big number? Is it about half way? Well where do you think it goes then?

	• That looks pretty different to where you originally had the 100. Tell me about why you changed your mind.
Incident 3: One grade four student thought that when two congruent, right-angled triangles were joined to make an isosceles triangle, this was larger than when the same two triangles were joined to form a rectangle. This focused on informal representations of area rather than calculating area of triangles. Following the incident the student solved the initial problem and also applied this solution to area problems for other shapes.	
32 discrepant events • Both the isosceles triangle and rectangle were formed on the desk using four congruent, right-angled triangles (two for each shape). The boy thought the triangle was bigger. The pieces of the rectangle were shifted to form a second isosceles triangle identical to the first. The boy stated, “Now they’re the same”. This position was not maintained when the pieces were shifted back to form the two different shapes. The child explained this by stating, “The triangle is always bigger than the rectangle.” (2 events) • Swapping one of the right-angled triangular pieces from the rectangle with an identical piece from the triangle. The boy thought that the triangle would still be bigger (1 event). • The pieces of the rectangle were rearranged to form a parallelogram. The boy decided that this was confusing and picked up the triangle to cover the parallelogram so that he could compare them. He stated with some surprise, “They’re kind of the same size.” The teacher then rearranged the pieces in the triangle to exactly overlay the parallelogram, at which point the boy said, “Now they’re the same” (2 events). • The pieces were moved apart by 2 cm. The boy decided that now they were smaller. The teacher drew his attention to look at the whole area by reminding him that he got to “Eat both bits of cake” in each situation. He maintained his position, saying, “They’d still be the same size if they’re together, but if they’re separated that means this one (touches the joined shape), that means this one’s bigger.” (2 events) • The teacher moved the separated pieces by tiny amounts, asking the boy to choose which was bigger each time until the pieces touched, forming the original triangle instead of a parallelogram. He maintained that the pieces were smaller until they touched, at which point this changed to “bigger”. The pieces were separated (“smaller”), then joined (“bigger”) multiple times (14 events). • Each piece was picked up and rotated rapidly in the air. The boy stated, “You’re just shifting it”. Both pieces were picked up and rotated simultaneously. He maintained that they were just being shifted. Then the pieces were joined together rapidly to form each of the different shapes examined so far. The boy stated, “You’re making it bigger”. The teacher drew his attention to this discrepancy by asking, “Bigger than the two pieces on their own?” While he answered, “Yes”, the pitch of his voice rose indicating that he was questioning his idea (6 events). • The paper was returned to the desk and shifted to form a variety of shapes. The teacher asked, “Am I changing the amount of paper?” The boy decided that the amount of paper wasn’t changing. Following some more pointed questions, the boy stated, “They’re the same size.” When the situation was changed he maintained, “This one looks bigger, but they’re the same size” (5 events).	• Which one’s the biggest now? • How about if I shift them like this, which one’s the biggest now? • Now they’re the same? Ok, how about if I shift this one? Which is the biggest now? • This one’s bigger now? How about if I swap these bits? • The triangle’s bigger than this one? • How about if I move them like this? • So now they’re kind of the same? • What if I move the pieces apart? • So if the pieces are touching it’s the same size, but if they’re separated you think it’s smaller? • Now it’s bigger? • Now it’s smaller? • (picks up a piece and rotates it in the air) Am I changing the size of it? • How about this one? Am I changing the size of it? • How about if I put the pieces together, but you get to have both pieces either way? • So now it’s bigger than the two pieces on their own? • Am I changing the amount of paper? • So I have more paper if I move them like this? • How much paper is there? • Does it matter how I arrange the pieces? Does that change the amount of paper? • So you think they look different, but actually they’re the same size?

Of particular note to the researcher was the nature of the questioning within phases two and three. As a student's alternative conception was identified, the questions became more pointed, exposing the disparity between prediction and observation. During a 30-second discussion in the grade four incident, the teacher asked, "Now they're the same? Ok, how about if I shift this one? Which is the biggest now? This one's bigger now? How about if I swap these bits? The triangle's bigger than this one?" These questions required the student to make a choice and then observe the outcome of that choice. Successive questions formed sequences which appeared to narrow the available options and produce a logical process by which a student's idea could be evaluated. The questions in these phases both created and increased the cognitive conflict as a student confronted his or her own conceptions. At an identifiable moment during each incident this conflict appeared to peak, reaching a tipping-point of disequilibrium whereby the student acknowledged the disparity between his or her observations and preconceptions and then resolved this disparity by reconceptualising his or her own ideas.

Initially this use of narrow, pointed questions within an otherwise challenging lesson structure seemed incongruous. However, these questions appeared to provide students with a logical process for confronting and changing their own conceptions. If conceptual understanding links both surface and deep learning as learners construct their own understandings (Bereiter, 2002), perhaps approaching alternative conceptions from a connectionist perspective (Askew et. al, 1997) and integrating pointed questions and disparate events into a challenging problem provides a way forward.

Conclusion

This paper presents data that indicates that conceptual change theory can be applied to challenging mathematics lessons, enabling students to change their own minds and alter their own mathematical conceptions. Discrepant events and questioning can be juxtaposed to create cognitive conflict, leading students to discard their alternative conceptions and accommodate new information. The five phases identified in this report were found to be common in twenty discussions across eight recorded lessons in which students who displayed alternative conceptions demonstrated cognitive change.

Some cautions are wise at this point, including considering who is doing the thinking—the teacher or the student. It is important within this process to ensure that the student is genuinely changing his or her own thinking, rather than leading student to the "right" answer. Balance and sensitivity is needed on the part of the teacher to ensure that questions scaffold student thinking only as much as is necessary to build cognitive conflict to the point of disequilibrium and achieve cognitive change. For alternative conceptions to be genuinely addressed, students need to change their own minds.

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LEARNING WITH CALCULATORS: DOING MORE WITH LESS

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It seems that calculators continue to be misunderstood as devices solely for calculation, although the likely contributions to learning mathematics with modern calculators arise from other characteristics. A four-part model to understand the educational significance of calculators underpins this paper. Each of the four components (representation, calculation, exploration and affirmation) is highlighted and illustrated, mostly with relatively unsophisticated modern calculators such as those widely accessible to students in Years 6–10, but also recognising some calculator features not available to younger Australian students. Intelligent use of calculators at these levels of schooling offers many opportunities for students to develop a solid understanding of key aspects of mathematics through their own actions, provided our apparent obsession with calculators as merely ‘answering devices’ is overcome.

Introduction

Although the hand-held calculator is now almost fifty years old, it was not until around forty years ago that the first versions became available in schools in Australia. Unsurprisingly, in view of the name ‘calculator’, these were used mostly to undertake awkward calculations, including those of interest to scientists and engineers. Indeed, the first calculators to be used routinely in schools from the late 1970’s were called ‘scientific’ calculators, apparently for that reason. At that stage, a calculator was described as ‘scientific’ when it included functions that had previously required the use of printed tables, such as those for logarithms, exponentials and trigonometric functions. Calculators offered a way for students to undertake numerical calculations efficiently and accurately. The efficiency arose from the speed of computation, compared with alternative methods, while the accuracy arose from superiorities of calculators over four- or five-figure tables in printed books.

A key purpose of this paper is to highlight the limitation of continuing to regard calculators solely as devices for handling (or even for avoiding) arithmetic. Much has changed, both in education and in technology, since the introduction of calculators, but it seems that much educational thinking about calculators continues to be stuck in the 1970s. Summarising the extensive and overwhelmingly positive research on calculators, Ronau et al (2011) noted that efforts are now needed to focus attention on effective uses of calculators in classrooms, towards which this paper is offered.

Developments in technology, education and assessment

There have been many developments in technology over the past forty years. The use of sophisticated technology has become commonplace for many Australians. This technology includes: many kinds of Internet resources; computer software, including apps, some of which are specifically for mathematics; devices such as laptops, tablets and smartphones; classroom technologies such as digital projectors, interactive whiteboards and wireless Internet. All of these have come at a cost, sometimes with equity implications, so that some schools and their students are equipped with essentially all of these resources, while others have access to very few of them.

Assessment has moved more slowly, partly for equity reasons, but mostly for security reasons, especially in high-stakes assessment, so that very few of these technological developments have been entrenched into curricula and most are prohibited from use in examinations. The exception is the hand-held calculator, versions of which are acceptable for external examination purposes throughout Australia and hence are likely to be available to essentially all students for almost all of the time. This paper explores the potential of calculators for supporting the learning of mathematics, especially in the primary, middle and early secondary years.

Developments in calculators

Calculators have also developed considerably over the past forty years. At a macro level, two major changes have included the development of graphics calculators (through the addition of a graphics screen and many mathematical capabilities to scientific calculators) and CAS calculators (through the further addition of computer algebra systems to graphics calculators). In addition, and importantly, calculators for younger students have been developed. Developments of these kinds reflect an important, but often unrecognised, change. Rather than being tools for scientists and engineers, calculators have been developed almost entirely as tools for mathematics education since they were first invented.

Changes to calculators have had educational purposes. This is especially evident in the development of more sophisticated calculators, such as graphics calculators and CAS calculators, both of which are best regarded as custom-designed technologies for mathematics education, not for professional scientists and engineers. But it is no less evident in the development of less sophisticated calculators, including those for students in the middle years and early secondary years of schooling. Amongst the developments that have clearly been designed by manufacturers to meet the needs of learners are the following:

- Consistent use of the conventional rule of order so that entering $2 + 3 \times 4$ into a calculator and generating the result will give 14 (instead of 20, which is still the case for many calculators and phones not designed for education).
- Improvements in user interfaces, such as multi-line displays allowing users to see what has been input into the calculator, as well as the result. Similarly, typical calculators can show earlier calculations and calculator inputs efficiently.
- The use of conventional mathematical symbolism. Perhaps the best examples of this are the representation of fractions with a horizontal vinculum and of powers in the form of superscripts.

- Mathematical functionality that matches typical school mathematics curricula. The development of surd and fraction capabilities and the development and improvement of capabilities for statistics are the most obvious examples for scientific calculators. The development of graphing and geometric capabilities are perhaps the most obvious examples for graphics calculators.

These and other similar changes were not developed with the needs of professional scientists and engineers in mind; rather they were consciously developed to suit students learning mathematics.

In addition to functionality refinements and improvements, calculators have become significantly less expensive over recent decades. Roughly speaking, a scientific calculator for school in the mid 1970s cost around one day of average weekly male earnings; these days, a considerably improved scientific calculator for schools will cost around an hour of average weekly male earnings, a remarkable drop in forty years.

A model for calculator use in education

A key purpose of this paper is to describe and defend a model for calculator use in mathematics education. This is essentially an analytical matter, rather than an empirical one, and requires some consideration of how students learn, as well as how they are taught, as well as a study of calculator capabilities.

It also requires a consideration of what mathematics is—what it comprises. Indeed, the development of calculators has raised this question to some prominence, as there seems to be a widespread view (at least in the wider community, if not in the education community) that numerical calculation is the essential characteristic of mathematics. Many people seem to make the association of mathematics with arithmetic, in both directions: seeing lots of numbers is interpreted by many people as mathematics (even when that is clearly not the case, as in accounting). Conversely, mathematical activity is regularly (mis)understood as producing answers in the form of numbers.

The model described here is based on extensive work completed to support the educational use of scientific calculators by Kissane & Kemp (2013). In the following four sections, different ways in which calculators might be used are described. It is not suggested that students are restricted to using their calculator in only one of these ways at any time; in fact, frequently more than one use will be involved in a single activity.

Representation

Calculators represent mathematical objects and concepts increasingly well, at least in the sense that the representation on a screen is increasingly faithful to the representation on other media. Figure 1 shows some examples of this claim.

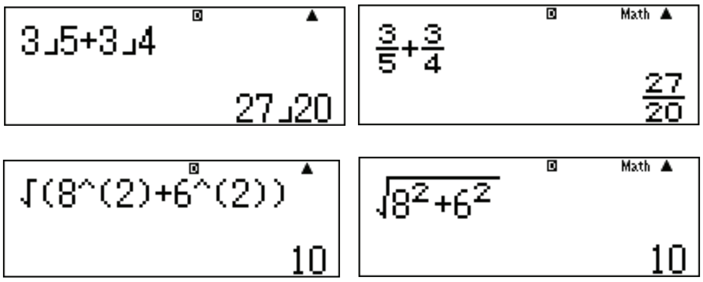
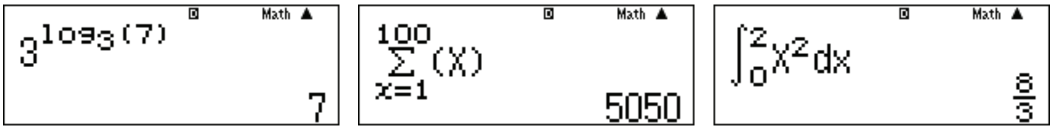


Figure 1. Developments in representing fractions, powers and radicals on scientific calculators.

This is clearly beneficial to younger learners, removing previous disconnections of calculator representation and representation in textbooks or whiteboards, and eliminating the task of needing to interpret what is on the screen or juggle with syntax problems. For older learners, more sophisticated calculators similarly are increasingly likely to use conventional representations, rather than more complicated expressions, as illustrated in the calculator screens in Figure 2.



$3^{\log_3(7)}$

7

$\sum_{x=1}^{100}(X)$

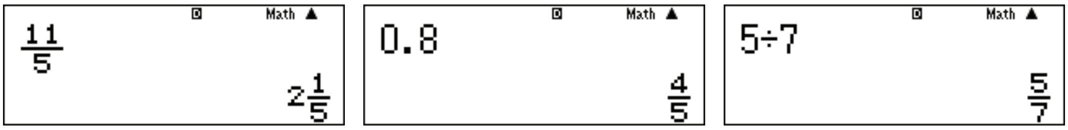
5050

$\int_0^2 X^2 dx$

$\frac{8}{3}$

Figure 2. Calculator representation of more sophisticated mathematical expressions.

A calculator does more than merely represent mathematical objects, however. By the nature of its operation, a calculator allows for objects to be 're-presented', or presented again. As shown in Figures 1 and 2, calculators dynamically respond to inputs (shown on the first line of the display) with outputs (shown on the second line of the display). While this has routinely been dismissed as 'calculation' in the past, it is important to recognise that it is a much more important activity than that, and a source of stimulation and thus learning for thoughtful or curious users. Figure 3 shows some examples for fractions and decimals, which are key ideas in the middle years of schooling.



$\frac{11}{5}$

$2\frac{1}{5}$

0.8

$\frac{4}{5}$

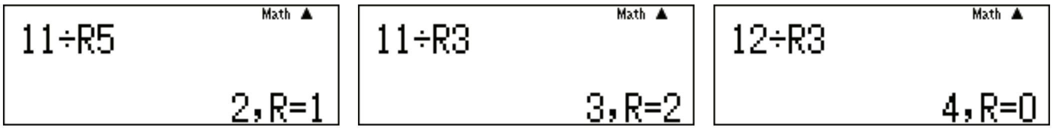
5÷7

$\frac{5}{7}$

Figure 3. Re-presenting fractions, decimals and division.

With help from teachers, students can be supported to become both more thoughtful and more curious and use such representations offered by calculators to increase their understanding about Number: fractions and decimals are merely different ways of representing the same number; the same number can be represented with fractions in many different ways; divisions can be represented as fractions; and so on.

For younger students, for whom decimals and fractions are not yet comfortable ideas, integer division operations on calculators provide a different representation, as shown in Figure 4. In this case, the symbolism using R for remainder is not universal in mathematics, although the concept of division with remainder is important, and a necessary precursor to the later idea of a division producing a fraction.



11÷R5

2,R=1

11÷R3

3,R=2

12÷R3

4,R=0

Figure 4. Representing division of whole numbers with remainders.

In some cases, calculators routinely transform representations from one form to another, while in other cases, users need to assist this process (such as through the use of a fractions to decimals key, for example). More sophisticated transformations give rise to opportunities to learn other aspects of mathematics, as illustrated in Figure 5.

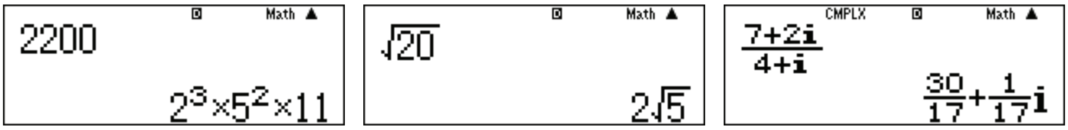


Figure 5. Further examples of re-presentation on calculators.

It is of central importance that when integers are represented as a product of their prime factors, only one representation is possible (which is quite different from representing a decimal as a fraction). The representation of √20 as 2√5 is sometimes referred to as ‘simplifying’ (although in one sense, it might not be unreasonable thinking of it as ‘complicating’!); such exact transformations on calculators are regrettably concealed from many younger learners by Australian examination rules, although they are routinely available to students with graphics calculators and CAS calculators. While the third screen in Figure 5 shows an example of relevance only to more sophisticated students studying complex numbers, it illustrates the same point that calculators routinely transform inputs to outputs that can be used as good starting points for students to learn about the mathematics involved.

Space precludes a fuller treatment of this matter. Users of graphics calculators will be well aware, however, of the immense power of representing functions in multiple ways, in the form of symbols, tables and graphs—the so-called ‘rule of three’—and the many learning opportunities these provide for students. While scientific calculator screens are too small to represent graphical objects, many can represent functions in two of these ways, using symbols and tables, as shown in Figure 6.

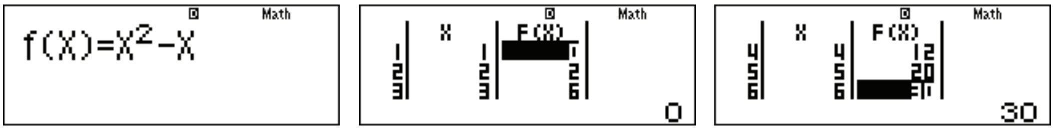


Figure 6. Representing a function symbolically and numerically.

Since the 'new maths' changes in the 1960s, the idea of a function has been regarded as of central importance in school mathematics, so it is not surprising that a device to support learners might be designed to represent functions. In Figure 6, a function has been defined and tabulated and the table can be scrolled easily to overcome the limitations of screen size, allowing students to engage with the representation of a function as a set of ordered pairs. (Scientific calculators with such capabilities are not generally approved for high stakes examination purposes in Australia, however.)

Computation

While it is unhelpful to restrict thinking about calculators to their capacity to undertake numerical computation, it is important to recognise nonetheless that such a capacity is helpful for students; indeed, this was the initial attraction of calculators to mathematics education nearly forty years ago. One possible reason for the widespread confusion of computation with mathematics is that it is of practical importance, and almost everyone recognises that mathematics is 'useful' for everyday purposes. A calculator allows students to obtain numerical answers to understood problems, and is the sensible means of doing so when it is too difficult, tedious or inappropriate to use alternative methods, such as mental arithmetic, approximation or by-hand computational methods. All modern curricula, including the Australian Curriculum (ACARA, 2015), expect students to develop expertise with a range of computational methods, including use of a calculator.

Arithmetical computation is known well enough to not require illustration here, beyond noting that students with a calculator have the means to undertake accurately and efficiently any practical computation that they understand. Hence practical mathematics with real measurements and real data can be undertaken, rather than be restricted to a pretend world where measurements are always in whole numbers and angles restricted to the (remarkably) few for which exact trigonometric ratios are known. Indeed, without such a computational capability, students are unlikely to have access to mathematical modelling of contexts of interest to them, or statistical analysis of realistic data from everyday sources.

It is also worth noting that computation on calculators may offer more than arithmetical convenience, however. Calculators may render some previous computational methods obsolete, or at least redefine them to be of interest mostly as historical curiosities, rather than essential components of learning. Obvious examples of this are long division and the extraction of square roots by hand. Less obvious, so contestable, examples might include Gaussian elimination and numerical integration.

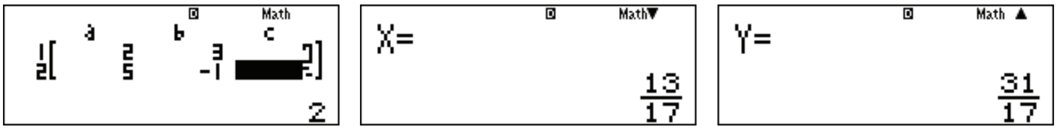


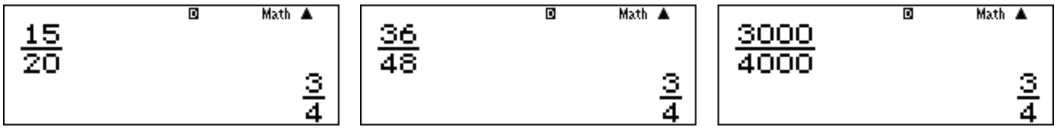
Figure 7. Solving simultaneously $2x + 3y = 7$ and $5x - y = 2$ on a calculator.

Few would argue that it is not important for students to understand the logic of Gaussian elimination to solve systems of equations such as that shown in Figure 7. Yet the extent to which students should be restricted to undertaking it by hand, and need to spend the considerable time to develop fluency in what is essentially a tedious and error-prone algorithm (instead of spending time on other aspects of mathematical thinking) is much more contestable.

Exploration

While faithful representation of mathematical ideas is helpful—even provocative—and a capacity for numerical calculation is essential, it is the role of the calculator as an exploratory device that offers the most promise for educational gain. As calculators are small, personal and responsive, students can use them to explore mathematical ideas and relationships for themselves in a low-risk and efficient environment. Good teachers have long known that purposeful engagement with mathematical ideas is a key element of learning. A calculator provides opportunities of that kind.

To illustrate, consider the concept of equivalent fractions, of central importance to young learners. With suitable encouragement, students can see that there are many fractions that have the same value, as shown in Figure 8. A productive task here is to challenge students to find still other fractions that are equivalent to three quarters.



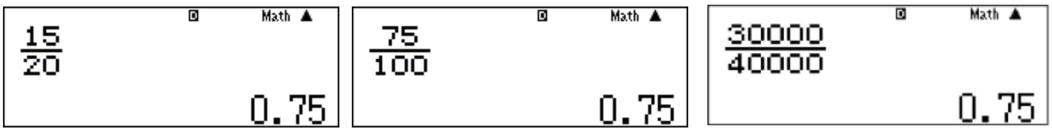
$\frac{15}{20}$ $\frac{3}{4}$

$\frac{36}{48}$ $\frac{3}{4}$

$\frac{3000}{4000}$ $\frac{3}{4}$

Figure 8. Some fractions that are equivalent to three quarters.

In addition, students can easily check for themselves (using a fractions to decimals key) that each of these has the same decimal representation, as do other equivalent fractions, as illustrated in Figure 9. The calculator provides a fertile and responsive environment for students to explore equivalent fractions for themselves.



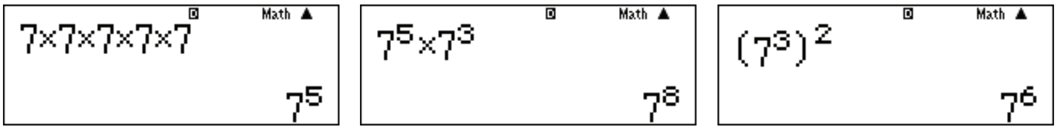
$\frac{15}{20}$ 0.75

$\frac{75}{100}$ 0.75

$\frac{30000}{40000}$ 0.75

Figure 9. Equivalent fractions have the same decimal representation.

As another example, consider students learning about indices. A calculator allows students to explore the meaning of powers, and its consequences, especially when factors of numbers can be efficiently revealed. Figure 10 shows some examples in which a factor command has been used to decompose a result in ways that seem likely to help students make sense of what is going on (as distinct from merely calculating answers).



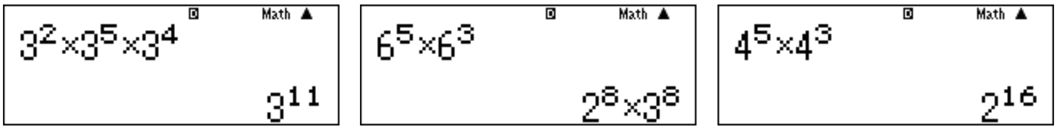
$7\times7\times7\times7\times7$ 7^5

$7^5\times7^3$ 7^8

$(7^3)^2$ 7^6

Figure 10. Exploring and understanding powers.

Exploring powers on the calculator will provide students with an opportunity to see some of the important relationships for themselves. This can be left to chance with undirected exploration, or may be provoked by suitable tasks. For example, the calculator screens shown in Figure 11 all offer opportunities for productive reflection or discussion by students, with a focus on understanding what is happening. Notice in the second and third screens that the results obtained are not what many students might expect, but potentially lead to a deeper understanding of rules for indices.



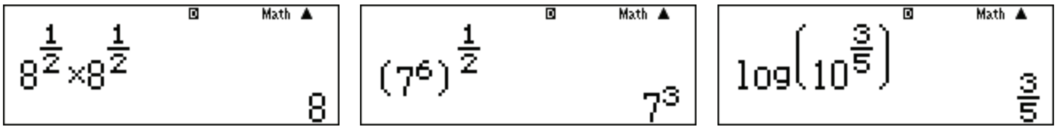
$3^2\times3^5\times3^4$ 3^{11}

$6^5\times6^3$ $2^8\times3^8$

$4^5\times4^3$ 2^{16}

Figure 11. Further explorations of integral powers.

At a still later stage, students can explore powers further and in more sophisticated ways to see for themselves the generality of the relationships involved when fractional powers (much less intuitively meaningful than integral powers) are used.



$8^{\frac{1}{2}}\times 8^{\frac{1}{2}}$

8

$(7^6)^{\frac{1}{2}}$

7^3

$\log\left(10^{\frac{3}{5}}\right)$

$\frac{3}{5}$

Figure 12. Exploring fractional indices.

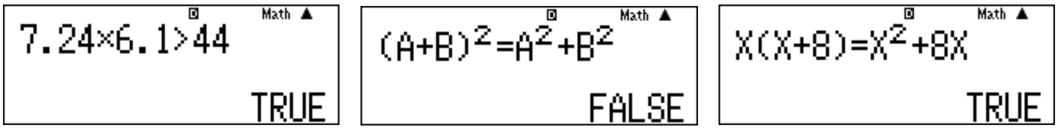
In Figure 12, the calculator is being used to explore the concepts and the relationships involved. There is no suggestion here that the calculator is the ultimate preferred tool for undertaking most calculations with indices, but rather it is an intermediate tool for helping to make sense of them. At some point, students will also encounter more formal treatments of the mathematics involved (such as formal arguments for rules). In addition, they should later recognise that $7^5 \times 7^3 = 7^8$, rather than use their calculator to determine such a product. The calculator is a means to an end, not the end itself.

Affirmation

The fourth aspect of the model for calculator use involves affirmation, concerned with users reassuring themselves in some sense about the quality of their mathematical thinking. Sound use of a calculator should always involve the user having some sense of what to expect or some intuitions about a likely result. Good teachers have routinely encouraged students to think about results before entering them in their calculators, as a form of checking, but it is possible to be more systematic on this matter.

An early form of affirmation, still sometimes seen in printed material, is for students to complete calculations mentally or by hand and to then check their efforts with a calculator. When the focus is on developing manual expertise with computation, this might be a defensible, although low-level, activity. Context is important here: while it would not be appropriate to use a calculator as an alternative to learning or checking recall of tables (such as 7 × 8), it may be a sensible use in developing and practicing mental or approximate methods of computation (to handle tasks like 7.24 × 6.1).

There are more sophisticated uses of verification, however. Some calculators have a Verify mode, explicitly designed for students to check their thinking in some sense. Figure 13 shows some examples of this, in which a response of 'True' or 'False' is shown for the given inputs. In these examples, the user is checking their own thinking, not relying on the calculator to generate the original responses.



$7.24\times 6.1>44$

TRUE

$(A+B)^2=A^2+B^2$

FALSE

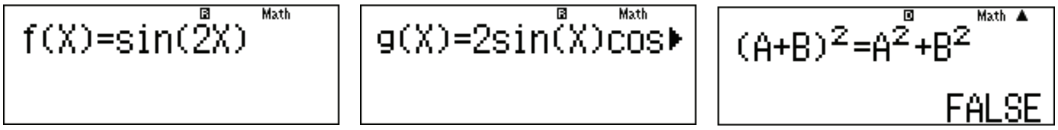
$X(X+8)=X^2+8X$

TRUE

Figure 13. Using Verify mode to check thinking.

In the first example, the calculator is being used to refine approximate mental computation skills. In the other two examples, the key idea is that of an identity, one meaning of an equals sign. In a class of students, instructed to give different values to the variables involved, the critical idea of a variable is involved. (Indeed, if some students assign a value of zero to either A or B, the exceptional case of equality will be produced, and the expression declared true, for an interesting discussion point).

This idea of affirmation might also be invoked more generally, even when a Verify mode is not available. On a graphics calculator, identities can be powerfully explored by comparing tables and graphs for functions thought to be the same. This is more difficult on a scientific calculator, but the example in Figure 14 offers a possible mechanism (using a calculator with a capacity to tabulate a pair of functions simultaneously, a nice feature available on all graphics calculators, but generally not available on calculators for Australian schools).



f(X)=sin(2X)

g(X)=2sin(X)cos▸

$(A+B)^2=A^2+B^2$

FALSE

Figure 14. Exploring the identity: sin 2*x* = 2 sin *x* cos *x*.

These examples are not meant to suggest that the calculator's purpose is to undertake such computations, and nor is the calculator sufficient to establish the mathematical results involved. Rather, the calculator is being used to affirm students' thinking—or to disconfirm it, of course—and thus help to provide a means of making sense of the mathematics involved. Ultimately, a formal argument is needed to establish an identity; the calculator is used to motivate the need for such an argument.

Calculators and curriculum in Years 6 to 10

The few references to technology made by the *Australian Curriculum: Mathematics* (ACARA, 2015) seem to suggest that digital technologies are regarded as computational devices, through the repetition of the phrase, ‘with and without digital technologies’ in Content Descriptions. Yet a careful reading (by the author) of the Number substrands for Years 6 to 10A, suggests that learning related to 37 of the 38 Content Descriptions can be supported and enhanced through the careful use of a modern scientific calculator, many more than are flagged in the official documentation. It seems opportune for teachers and students to take advantage of these potentials, using tools that are widely available to students.

Conclusion

Significant changes to calculators since the 1970s have rendered them as purpose-built devices to support the learning of mathematics, in addition to undertaking computations. The model described in this paper is offered to clarify the precise ways in which that educational promise might be fulfilled, through exploiting the calculator’s capabilities for representation, computation, exploration and affirmation.

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THE ‘BIGGEST LOSER’ GAMBLING ISSUES PROJECT

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The Mathematical Association of Victoria has just finished trialling its cross–curricular gambling issues project. The four curriculum areas covered in this two- to three-week project are English, Social Education (Civics and Citizenship), Health Education and Mathematics.

This paper outlines the mathematics unit, twelve lessons which cover all the probability and statistics content of Year 9 and 10 of the Australian Curriculum. Three key messages are independent events (‘chance has no memory’, misconceptions), expectation equals probability times payout, and variability (how the spread of a punter’s return on the ‘pokies’ drifts away from the break-even point as the number of trials increases).

Why get involved?

The Mathematical Association of Victoria took on this research project for the following reasons:

- Gambling is an issue—and mathematics education is important to understanding and attitudes.
- There was some dissatisfaction with the mathematics of previous attempts, where ‘odds’, ‘bookies odds’ and ‘payouts’ were confused and where important mathematical content was overlooked.
- The cross-curricular cooperation required for such a project had not previously been sufficiently supported by parallel units of work embedded in the required content of each curriculum area. Few teachers felt confident to take on the whole task themselves.
- Assessment is important. Assessment of the effect of the project on student attitudes to gambling has been lacking in previous projects. The trial used pre- and post-activity questionnaires for this purpose. (The statistical analysis of results is at present incomplete.) ‘Biggest Loser’ also incorporates assessment against the requirements of the Australian Curriculum in all four curriculum areas.
- Involvement has been encouraged and supported by the Victorian Responsible Gambling Foundation.

The mathematics unit Biggest Loser

The mathematics unit covers all the probability and statistics content of Year 9 and 10 of the Australian Curriculum. For Year 10A standard deviation could easily replace the inter-quartile range of box plots. Interpretation of bi-variate data and correlation is included in the Social Education unit. In the 2014 pilot program one school used the mathematics unit within its program for the Victorian Certificate of Applied Learning (VCAL)

A typical lesson in the mathematics unit involves simulated gambling, firstly with dice or cards and later with random numbers and a spreadsheet. The lesson culminates in an important discussion of results and, where appropriate, the relevant theoretical probability. Teachers conduct their lessons in ways that suit the availability of technology to their classrooms and the extent to which they might encourage open student investigation. Worksheets are provided for teachers who would prefer more focussed student work.

The unit's title, 'Biggest Loser', makes it clear that, in gambling situations, the punters lose and the 'house' or 'bookie' wins. Key contexts are poker machines and sports gambling (football, women's sports and racing). Real payout data is provided for sports gambling lessons, but the teacher can easily access data that is more recent or of more interest to the students.

Three recurring themes in the mathematics unit are independent events, long-term expectation and understanding of variability.

Independent events

Gambling simulations, firstly with dice or cards and later with random numbers in spreadsheets, can reveal common misconceptions and contradictions that many students (and adults) can reveal in their understanding of independent events. The message we want to get across is 'Chance has no memory', with students eventually releasing themselves from ideas such as:

- A run of losses won't continue for long. It will even out soon.
- Better to go for a large prize than a smaller one, since the gain is greater if you win.
- There is a pattern in the results, if only I can find it.
- Many people are winning: it's my turn soon.
- I am a lucky person but haven't had a lucky day for a while. I feel lucky now.
- I have the power to 'will' my numbers to come up.
- When I lose it is just bad luck—or the machines are rigged.
- I deserve to win. Maybe I will today.
- Using skill in gambling I can get good results.

The last lesson in the mathematics unit offers students initial results from six poker machines and asks them to write about which of the machines they would choose for the next set of results. Their responses reveal how well they now understand the concept of independence.

Long-term expectation

The gambling context is ideal for introducing the important concept of expectation. The key equation is:

Expected long term return to the punter = payout × probability

where 'payout' is the amount returned to the punter for a winning bet of $1. (The author finds it disappointing that this equation is not mentioned in the *Australian Curriculum: Mathematics*; discussion of 'risks worth taking' cannot proceed without it.)

In the 'Lucky Colours' simulation punters bet $1 on their choice of four possible colours.

- With a payout of $4 the game is agreed to be 'fair', with

  Expected return = $\$4 \times \frac{1}{4}$ = 100%
- With a payout of $3, the game is agreed to be 'unfair', with

  Expected return =$\$3 \times \frac{1}{4}$ = 75%

  and thus an anticipated overall 25% loss on bets placed.

Extended simulation allows students to see how close this theory is matched by the outcomes of simulation. The teacher provides real data from various gambling sites and students calculate to find interesting patterns and variations in the long term expected losses. An example is the following data taken from weekly betting on AFL football.

Fremantle $1.16 vs Carlton $5.25

Given reasonable assumptions, the probability of Fremantle winning is thus

$$\frac{5.25}{1.16+5.25}=0.819$$

Hence the expected return on a $1 bet is $1.16 × 0.819 = 95 cents. Over a large number of bets the betting agency thus plans to gain 5% of all bets placed.

A more complex example is early betting on the 2015 Australian Open Tennis Women's Tournament. A well-known betting agency offered a $3 payout for the favourite, Serena Williams. The bet to payout ratios for all players summed to 1.275702, rather than to 1, so using assumptions based on this information the probability of Serena winning can be adjusted to 1/3 ÷ 1.275702 = 0.26129

Thus a 'fair' payout would have been 1/0.26129 =$3.83,

and the expected return to punters on the Australian Women's Open is

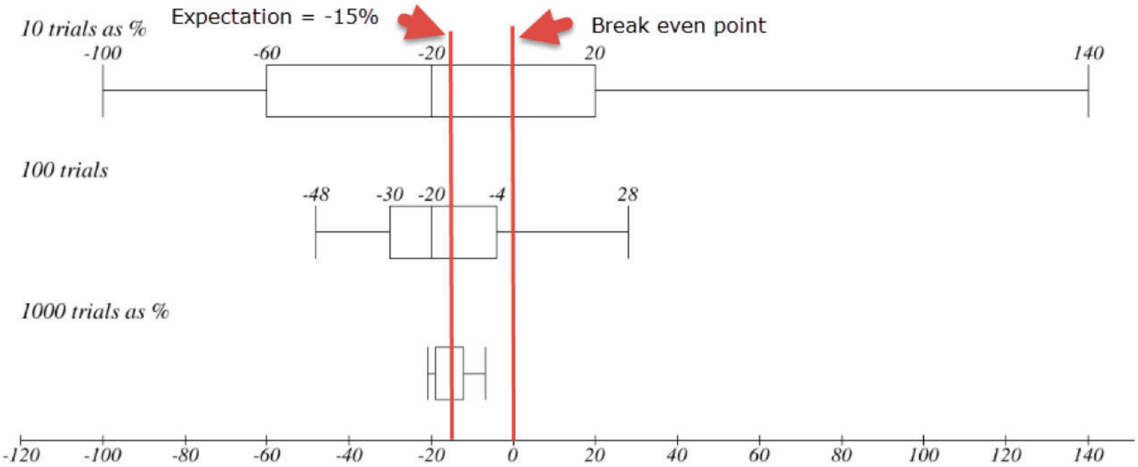
1/1.275702 ≈78%, a 22% expected loss on the total bets placed.

Understanding of variability

Understanding of variability is a concept vital to informed citizenship yet ignored in all but the most senior secondary courses that cover binomial probability distributions. Two things an informed citizen should know are:

- a random sample of no more than 2000 is sufficient to obtain, with about 1% accuracy, the results for the whole population—no matter how much larger it is than the sample;
- the longer a punter plays an unfair gambling game the less chance they have of breaking even.

In one of the key mathematical activities a simulated poker machine is set at 85% expectation and, with the press of the spreadsheet's recalculate button, a new set of results for 1000 $1 bets is obtained. Students repeat this simulation enough times for reliable box plots, such as the following, to be developed.



In this simulation the payout was set at $4, and 23 data points were collected. For the first box plot:

No wins out of 10 gave a loss of $10, or –100% of $10. This was the minimum value.

At the other extreme, 6 wins out of 10 gave a profit of $24 – $10 = $14 or 140% of $10.

With the probability of a win set at 85% of 1/4 = 0.2125 it is not surprising that, in the 23 trials, there was never a case of more than 6 wins out of 10.

The key conclusion for the students is that as the number of games increases the variability tightens so much about the expected value that the possibility of 'breaking even' becomes more and more remote. The teacher can note that the spread of the distribution is inversely proportional to the sample size, so the 1000 games box is approximately one tenth the width of the 10 games box. The key message is:

Short term gain? (a possibility) Long term pain! (a virtual certainty).

The Australian Curriculum: Mathematics

The gambling issues unit has been designed as a vehicle for covering all the key concepts of Years 9 and 10 of the Australian Curriculum. (For Year 10A students standard deviation can be employed instead of box plots to interpret variability.) Two additional simulations—followed by theoretical analysis—are also required:

Australian Curriculum	The Gambling Issues context
2 step chance experiments – with replacement	Two games of Lucky Colours, with various payouts – simulated results confirmed by theoretical analysis through tree diagrams
2 step chance experiments – without replacement	A quinella bet in a well handicapped race (Stawell Gift?) With 8 runners, the probability of picking the first two (in any order) is 1/28

Evaluating the trials

Qualitative feedback from participating teachers has indicated some success in dealing with misconceptions that students have had in their attitudes to gambling. Quantitative analysis involves data from pre- and post-unit questionnaires and questionnaires from control groups of Year 10 students in the same school. At the time of writing the results of this analysis are not yet available.

The product

The units have been edited in response to feedback from trial schools and in anticipation that the materials may be used more widely—at other year levels and in other states of Australia. The materials comprise approximately 35 lessons, with worksheets and spreadsheets as appropriate, together with copies of relevant files of key reports and data. For information on availability of these materials contact the Mathematical Association of Victoria. For discussion, talk to Dr Ian Lowe, MAV Professional Officer.

Further reading

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Useful websites

http://www.qgso.qld.gov.au/products/reports/aus-gambling-stats

The Queensland Government Statistician's Office is the source of Australian Gambling Statistics (AGS). This is a comprehensive set of statistics related to gambling in Australia, covering the entire range of legalised Australian gambling products. The publication has been produced since 1984, and is compiled annually by the Queensland Government Statistician's Office in co-operation with all Australian State and Territory governments. This 30th edition incorporates information on all legalised gambling in Australia up to and including 2012–13.

http://www.responsiblegambling.vic.gov.au

The Victorian Responsible Gambling Foundation is an independent Victorian statutory authority whose objectives, functions and powers are detailed in the Victorian Responsible Gambling Foundation Act 2011. It is a first contact point for data and research on gambling in Victoria. Similar authorities exist in some other States and Territories.

http://www.gamblingandracing.act.gov.au

The ACT Gambling and Racing Commission.

http://www.iga.sa.gov.au/aboutiga.aspx

South Australia's Independent Gambling Authority.

http://www.ilga.nsw.gov.au

Independent Liquor and Gaming Authority, NSW.

http://www.justice.qld.gov.au/corporate/about-us/liquor-gaming/gaming

About gaming in Queensland, Queensland Government Department of Justice and Attorney General.

http://www.treasury.tas.gov.au/domino/dtf/dtf.nsf/v-liq-and-gaming

The Tasmanian Government Department of Treasury and Finance Liquor and Gaming Office.

EXPLORING IDEAS CONTAINED IN THE MATHEMATICAL CONCEPT OF A LIMIT AND UTILISING ITS MEANING IN TEACHING

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The concept of limit is an example of a concept created from mathematicians' thinking, but it also has reasons for its appearance, and contains interesting ideas behind it. It has the potential to develop into a knowledge system and relate to real life. The problem in teaching concepts like this is how to approach them and what aspects need to be considered by students to understand the inner ideas of a concept and how they can apply them. Using a constructivist approach and the support of several dynamic models, this study developed tasks that support students not only in constructing the concept of limit but also in understanding reasons for its existence, the ideas contained in it, and in applying those ideas in solving problems creatively.

Introduction

The concept of limit is one of the most basic concepts of calculus. Difficulties in solving physical problems and mathematical paradoxes provided the impulse for the development of this concept. However, it took mathematicians a long time to have a complete and exact definition. Elements of this idea appear early in the mathematics curriculum in schools, such as concepts of 'transition through limit' and thinking of 'infinite and continuous' which are implicit in the calculation of the circumference of a circle as the limit of the perimeter of the inscribed polygons. Thus, some elements of calculus are introduced implicitly through students' learning of algebra and arithmetic. These factors allow us to clarify the fundamental difference between algebra and calculus by looking at the kind of thinking, methods and technical characteristics involved. The emergence of the concept of limit has formed the foundations for the development of calculus. All the major concepts of calculus such as derivative, continuity, integration, convergence and divergence are defined by the language of limit. Limit also serves as the fifth operation which signals the 'transition through limit'. These show the importance of the concept of limit and its complexity. Instead of taking a textbook approach to these topics, this paper endeavours to change the presentation of the concept of limit to explain the connections between the concept of limit in mathematics and its contact with real life. Ignoring these issues can cause difficulties for students who may often wonder: What is the interpretation of the concept of limit in reality? Does it have practical applications or not? These concerns, if not answered, hinder the implementation of students' learning tasks. In this paper we focus on the following research questions:

Question 1: Which examples can help illustrate the concept of limit?
Question 2: How do students construct knowledge of the limit of sequence by experimenting on dynamic mathematical models?
Question 3: How do students discover the ideas contained in the concept of limit and connect them to real life?

Research framework

The idea of approaching the mathematical concept

According to Thai (2015), how students receive and explore mathematical content is not confined by the framework of a school program, but also through different types of activities in various ways, including learning outside the school. Therefore, in the teaching of mathematics, besides a traditional academic approach which values the logic of mathematics as a deductive science, there needs to be a complementary approach which values and draws on the experiences of students. Students' experiences can be used as a 'bridge' to mathematical language and can provide anchor points for developing and appreciating mathematical ideas.

How can we express the logic of mathematical sciences and use students' experience in learning? Each concept has a scientific basis forming it, so showing its existence and clarifying its role will help us solve the above-mentioned problems. However, to help students learn actively, activities assigned by teachers should be constructed on the experience of students. This is necessary primarily to help students see the concept in more familiar terms, thus reducing abstraction. Also, students usually perform better if they understand the task and if they are assisted to discover knowledge by themselves.

Aspects need to be considered when teaching concepts

According to Thai (2015), it is not enough to teach students a concept in isolation. They need to know its location and its role in the system. This helps students understand the importance of the concept in the system and to see its relationship with other concepts. This is also a way to create a sustainable knowledge. Therefore, when considering each element of knowledge, Thai (2015) argues that each element of knowledge should be referenced by three dimensions:

- Where is it in any position in the overall picture of mathematical science, and is it consistent with the trend of the development of mathematical science?
- To meet the goal of teaching mathematics in schools, what is its role in developing students' capacity to learn and do mathematics?
- How are its relationships integrated with other disciplines?

We believe that teaching the concept of limit needs to reflect on the above aspects. First, we must see that the concept of limit has importance in mathematics. The concept of limit is used to construct other concepts of calculus such as continuous functions, derivatives, integrals and so on. The concept and its applications have provided us with many mathematical tools. Second, learning the concept of a limit in calculus helps students see things from the perspective of a state of change. This contributes to developing their thinking capacity. Third, the concept of limit and the tools of calculus create opportunities for students to see relationships as more integrated and interdisciplinary, especially through concepts and problems associated with Physics and to solve those problems in practice.

The development of mathematical concepts in the minds of students

The problem about whether the ideas 'really are' mathematics is addressed by Nunes, Schliemann and Carraher (1993) who drew on Vergnaud's (1990) theory of concepts to argue that concepts developing in a learner's mind need always to be seen to have three aspects: invariants, representations and situations. 'Situations' make the concept meaningful. This aspect appears to be broad enough to include social situations (such as selling, sporting events, maths classes) and real or imagined problem-situations (Carraher, 1991, p. 178). 'Invariants' refer to the properties or relations associated with the concept, such as symmetry, commutativity, conservation of equality. 'Representations' are based on the set of symbols (linguistic or non-linguistic) used to communicate or discuss invariants: "representations always involve keeping some features of the concept in focus, while losing sight of others" (Nunes et al., 1993, p. 145). This triple aspect of mathematical concepts allows Nunes et al. (1993) to argue that the invariants must be constant across thinking in different contexts, whereas its representations and especially related situations depend on particular contexts.

In creating learning environments, we use representations, including special uses of computer software to create dynamic models in teaching mathematics. Mathematical software as *Geometer's Sketchpad* (GSP) can create dynamic images displaying the results of each change of variable. Well-planned use of such software is intended to provide opportunities for students to explore, to predict general properties, and to propose hypotheses that facilitate the creation of new ideas and help to solve problems.

Research design

In this research design, the following factors are key: the teacher design schedule, the teaching design idea, the design of mathematical tasks, data collection and analysis.

Teacher design schedule

In designing our teaching experiment and preparing teaching and learning activities, we adopted a constructivist learning approach following Confrey (1991) and Nam and Stephens (2014). The following four points indicate how we have combined the key ideas of a teaching experiment within a constructivist-learning framework. The teacher:

- needs to focus on the important key conceptions of the limit of a function, not focusing too much on general teaching strategies or overall descriptions of the limit. For example, in introducing the limit of a function, we focus on such activities which make clear the process of confirming a given limit;
- needs to have a clear plan on how to respond to students' incorrect answers;
- should have a long-term plan to consistently develop students' deep understanding of the limit of a function; and
- should utilise concrete examples that are familiar and easy for students to understand to help them understand the limit of a function and its relationships.

Teaching design

The concept of limit is a difficult concept to teach, and there is no simple way to help students understand the definition all at once. Our teaching concept design is to start by creating activities to attract students' attention, to motivate their learning, and to help them see the possibility of new knowledge. Next, create activities for students to understand concepts intuitively, and then to conduct activities to help students gradually understand a more precise definition. In the implementation of these activities, students obtain certain ideas related to the concept. Therefore, we designed some open-ended questions to provide students with an opportunity to propose innovative ideas. The plan is to create an 'open space' for students to see some hidden ideas behind the concept. Once students have developed the concept of a sequence having a limit of zero, we ask them to carry out a further task which is aimed to assist students in the process of forming the concept. The remaining activities ask students to use their knowledge of limit to construct new knowledge in the form of a formula for the lateral surface area of cone, and its volume; and to provide opportunities for students to apply the knowledge learned, while confirming the role of limit in practice.

Design of mathematical tasks

It is important to choose the tasks that are suitable for senior high school mathematics students. Tasks should be designed to encourage students' positive thinking and they need to connect the knowledge and experience of students. To help students understand the concept of existence of a limit, we designed the following tasks.

Task 1

A given model of a regular polygon *n* ($n \geq 3$) has its edges inscribed in a circle. The value of *n* can be changed by dragging the tip of parametric slider *n*. By changing the value of the parameter *n,* it is possible to predict the result of area of polygon when *n* tends to infinity.

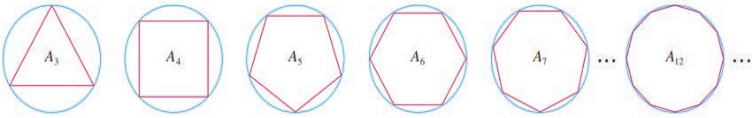

Figure 1. Model of a regular polygon inscribed in a circle.

By manipulating models in this way, students realise that when *n* tends to infinity then the shape of a polygon nearly coincides with that of a circle. Therefore, the polygonal area is approximately the area of the circle, but the difference between them is always a very small value. Figure 1 shows the special relationship between the area of a circle and the area of the n^{th} polygon. In order to involve students in the lesson, the teacher can ask, "What relationship do these concepts in mathematics express?"

Task 1 is designed to help students appreciate the existence of a concept in mathematics from a 'dynamic' point of view, which is that a circle can be considered as the limiting shape of regular polygons that are inscribed in it when the number of edges tends to infinity. This view is the basis for recognising that the circumference of a circle can be considered as a limit of the perimeter of a regular polygon. These images will serve as a stepping stone to help students construct the concept of the volume of a cone from the volume of a pyramid.

Task 2

Introduction

Given a sequence of numbers $u_n = \frac{(-1)^n}{n}$ displayed on a GSP screen. Each red dot in the graph describes two values, *n* and u_n. Red dots can be erased by pressing the Esc key on computer keyboard one or several times. The red dot (at *E*) is the current value of *n*. This value can be changed by dragging the tip of the parametric slider *n*

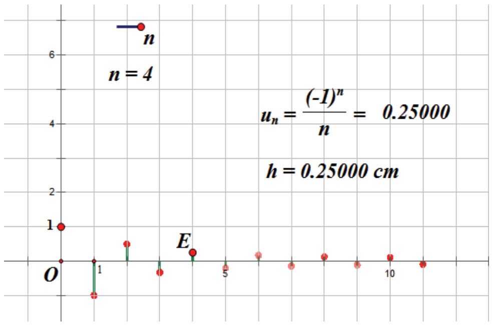

Figure 2. The limit of sequence of numbers

Four investigative questions

Question 1. By changing the value of parameter *n,* comment on the change of u_n when *n* becomes larger.

This question is to help students understand the concept intuitively that when *n* tends to positive infinity, then the sequence tends to zero. We let students conduct the following activities step by step to describe it correctly.

Question 2. Give five values of *n* so that the distance between u_n and 0 is less than $\frac{1}{100}$. Then complete Table 1 showing correlative values between *n* and $|u_n - 0|$.

n					
$\lvert u_n - 0 \rvert$					

From what term onwards is the distance between u_n and 0 less than $\frac{1}{100}$?

This question is to help students define specific values of *n* such that the distance between u_n and 0 becomes smaller than $\frac{1}{100}$ This is a buffer step in implementing and understanding subsequent activities and also aims to help students get number intuition about the concept.

Question 3. Find two natural numbers *M* which are larger than 100, and use them to complete Table 2 showing correlative values between *n* and *M.*

M = __	After which term u_n, is $\lvert u_n - 0 \rvert < \frac{1}{M}$ = __?
M = __	After which term u_n, is $\lvert u_n - 0 \rvert < \frac{1}{M}$ = __?

Question 3 is to help students build confidence that there always exists some N so that the distance between u_n and 0 is smaller. This activity is a prerequisite for the next question.

Question 4. If $\frac{1}{M}$ is changed to an arbitrary small positive real number ε, can we always find a natural number N so that for all $n > N$, the distance between u_n and 0 is less than ε? Please explain your thinking.

Question 4 is to help students understand the nature of a formal definition of the concept of a limit.

Task 3

Try to explain the meaning of equality in the statement, $\lim u_n = a$. What does equality $\lim u_n = a$ remind you of from Task 2?

Task 3 aims to create new ideas during the process of formulation of the concept. Task 3 is to help students realise that the equation should be understood differently to a normal equation that students have learned in Algebra. It is not an ‘invariant’ relation of the equality of two numerical expression, but requires a new and flexible way of looking at what equality means in this context: the quantity changes around a and gets closer to a but it may never actually reach the value a. The second part of question provides opportunities for students to realise that hiding behind the concept of limit is the key idea that a number a can be seen as the result of a limit of a sequence. To help students to think about the applications of mathematics in real life, we pose the following situations where they have to use their knowledge to explain and solve; and from there recognise the relationship of a limit with real life.

Task 4

In real life, we often say “the probability that a son is born is $\frac{1}{2}$.” Does this mean that for every 10 couples who each have a child, five of the children will be boys? Please explain.

Task 5

Given a model of a regular pyramid with its base inscribed in a circle of radius R, the number of edges can be changed by dragging the tip of parametric slider n. By changing the value of parameter n,

a. Please comment on changes to the pyramid when the number of edges of polygon tends to infinity.
b. Calculate the volume of the resultant cone if the height of the cone is h.

Question (a) is intended to help students realise that when n tends to infinity, the regular pyramids tends to resemble a cone. Adopting a dynamic point of view enables students to see a cone gradually emerging from a pyramid when the number of edges tends to infinity. This prompts students to see that the volume of the cone can be seen as a limit of the volume of a pyramid when the number of edges tends to infinity. This leads intuitively to building a formula for the volume of a cone without having to make calculations based on the dynamic model.

Task 6

Given a cone with a circle as its base with centre *I* and radius *R*, the height of cone can be changed by moving the point *S* on a line perpendicular to the plane containing the circle.

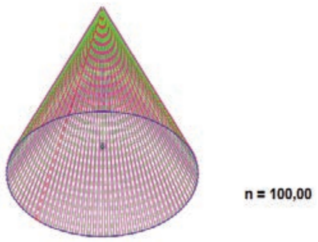

Figure 4. Model of a cone.

a. Change the position point *S* to reduce the height of the cone and comment about what happens to the cone if point *S* gradually merges to point *I*. State the relationship between the cone and the bottom circle.
b. Can you then use this result predict a formula for lateral surface area of cone?

Question (a) is intended to help students realise that the bottom circle is limit of the cone when *S* moves closer to zero. Question (b) requires students to use this dynamic idea to build up a formula for the lateral surface area of a cone.

Research results

In this section, we focus on the four results: interactions among students, teacher support to students facing difficulty, teacher handling of students' correct results, and teacher responses to students' incorrect results.

Interaction among students

Students belonging to the same study group presented their findings in order to contribute to a whole group conclusion. Each group evaluated other groups' results. Access to a dynamic model provided an opportunity for students to build knowledge and generate new ideas

Teacher support for students facing difficulty

In monitoring the implementation of group activities, if students had difficulties answering these questions, it was necessary for the teacher to pose additional questions or suggestions to support students' thinking (Nam & Stephens, 2014))

When performing Task 1, for example, when some groups did not mention results, we asked them to operate on the dynamic model by increasing the value of *n*, the number of edges of a polygon, and then to compare the area of the resulting polygon with the area of a circle. When performing Task 2 with a dynamic model some student teams only considered relatively small values of *n*. In such cases, we posed the question "If we make *n* larger, what will change?" But, by taking the slider off the screen, a difficulty occurred for some students. It could be resolved by asking students to consider unit reduction on the horizontal axis by moving the red dot to the left.

When performing Task 3, to support students discovering the key idea, we posed questions like: “What is the left hand side? What is the right hand side?” The purpose of these questions is to help students see that “the left-hand side is the limit of a sequence, and the right hand side is a number”. From here, students could realise that they can see a number as a result of a limit of sequence. From this perspective students could appreciate a key mathematical idea which allows them to construct a new number, e, such that $e = \lim\left(1+\frac{1}{n}\right)^n$. These results can give students access to an important operation in calculus, that is, ‘transition through the limit’. With this idea, when learning about the limit of a function, which is a continuous function, students can get the idea that discontinuous functions are considered as the limit of continuous functions. These results help the students to see the relationship between objects that may not seem to be relevant, such as a specific number and a sequence, or apparent opposites such as continuous and discontinuous functions.

When performing Task 4, which suggested that if ten children are born, five will be boys, many groups said that it is wrong. Of course, students realised that this is one possible outcome, but they face difficulty in explain how the probability would result to $\frac{1}{2}$. To support students, the teacher posed questions like, “What kind of outcome is referred to when we speak of a probability of $\frac{1}{2}$?” With this question, two groups identified this as a ‘statistical probability’. From here, students realised that $\frac{1}{2}$ is the limit $\lim\frac{n_s}{n}$, where n_s is the number of couples who give birth to a son, n is the number of couples have children, and n is very large number.

When performing Task 5, there were some difficulties in leading groups to identify the volume of pyramid. To assist students to calculate the volume of cone, we asked, “When the number of bottom edges tends to infinity then what becomes of the pyramid?” This suggestion helps students to recognise that the volume of cone is limit of the volume of pyramid when the number of bottom edges gradually tends to infinity”. Moreover, students already know the formula of the volume of pyramid is $V = \frac{1}{3}Bh$, where B is the bottom area, h is the height of the pyramid. However, the base of the cone is a circle of radius R, so the volume of the cone is $V = \frac{1}{3}\pi R^2 h$.

When performing Task 6, to help students have different points of view, we added the question: “Can we have different views on the relationship between the circle and the cone?” This question asks students to give responses like, “The bottom circle is the limit of the cone when its height h tends to 0 or the slant height l gradually to R.” The last perspective is the basis for the student's idea that “The bottom area of the circle is limit of the lateral surface area of cone when slant height tends to radius R”.

Teacher processing of correct results

When performing Task 1, the majority of groups correctly predicted the outcome, in which case, we asked them to explain their answer. Typical answers were: “When the number of edges tends to infinity then the polygon seems to coincide with the circle”.

When performing the Task 6, when students were asked to explain the formula of the lateral surface area of the cone, one group explained, “When increasing the height of the cone from 0 to h, we can see a bottom radius (slant height) turn back into the slant height, while others remain the same radius. In the case $l = R$, $S_{xq} = \pi R^2 = \pi RR$. We have "the lateral surface area of the cone is πRl." The perception, again an intuitive result derived from the dynamic model, is not a guarantee of being correct, but its value gave students a way to build their knowledge. Such awareness helps students see the dialectical relationship between mathematical knowledge and what students can reconstruct by themselves, even when they have forgotten the formula.

Responding to students' incorrect results

In conducting Task 2, three groups said: "There will be a time when the points $(n{:}u_n)$ are located on the horizontal axis", and so concluded that "the distance between u_n and 0 becomes smaller and smaller and finally the distance is equal to 0". To help students recognise this kind of mistake, we asked students to enlarge the units on the vertical axis and to see how their image changes. Increasing the units on the vertical axis helped students recognise that when we magnify the vertical unit, points $(n{:}u_n)$ are actually not on the horizontal axis. When performing Task 4, one group, when asked to explain its conclusion, answered, "Because the formula for calculating the probability that $n(B)$ is the number of pairs of boys, n is the total number of pairs."

Conclusion

The concept of limit is an abstract and sometimes difficult concept. But understanding this concept is essential because it not only plays an important role in mathematics, it is the basic concept to construct the system of calculus, as well as a tool to help us understand and explain things in real life. By teaching using the above tasks, we argue that a dynamic model provided an opportunity for students to build intuitive knowledge and to connect ideas from different areas such as geometry and probability. The dynamic model also helped students to see the variety of applications of the concept of limit such as: the base circle is the limiting shape of a cone when its height tends to zero; or the base circle is the limiting shape of a cone as its top approaches the centre of the circle. Aside from the correct answers, wrong or incomplete answers were also anticipated in our teaching design. These wrong or incomplete answers provided important opportunities for teachers to direct appropriate questions for students to move them towards a more correct understanding.

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THE IMPORTANCE OF FRACTIONAL THINKING AS A BRIDGE TO ALGEBRAIC REASONING

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This presentation will discuss how Year 6 primary school students create algebraic meaning and syntax through their solutions of standard fraction problems. Sample solutions will show how students use 'best available' symbols to move beyond arithmetic calculation and to create innovative chains of algebraic reasoning. Several efficient and successful multiplicative methods are used to achieve this goal, in contrast to less efficient methods, usually additive, which may work only with simple fractions. Teachers need to recognise the underlying algebraic meaning emerging from students' solutions and help all students use more efficient strategies and build their own bridges to algebra.

Introduction

Many researchers argue that a deep understanding of fractions is important for a successful transition to algebra. The National Mathematics Advisory Panel (NMAP, 2008) stated that the conceptual understanding of fractions and fluency in using procedures to solve fractions problems are central goals of students' mathematical development and are the critical foundations for algebra learning. Teaching, especially in the primary and middle years, needs to be informed by a clear awareness of what these links are before introducing students to formal algebraic notation.

Eighteen Year 6 students from an eastern suburban metropolitan school in Melbourne were assessed using a paper and pencil test. These students were chosen as they were deemed by their teachers to be highly successful in mathematics. The paper and pencil assessment included three fraction tasks (Pearn & Stephens, 2014) and two relational thinking tasks (Mason, Stephens & Watson, 2009; Stephens & Ribeiro, 2012). This paper aims to identify and examine students' responses to the three fraction tasks that demonstrate clear links between algebraic thinking and students' solutions to fractional tasks involving reverse processes.

Previous research

According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). Many researchers believe that much of the basis for algebraic thought rests on a clear understanding of rational number concepts (Kieren, 1980; Lamon, 1999; Wu, 2001) and the ability to manipulate common fractions. There is also research documenting the link between multiplicative thinking and rational number ideas (Harel & Confrey, 1994).

Siegler and colleagues (2012) used longitudinal data from both the United States and United Kingdom, to show that, when other factors were controlled, competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of students' attainment in algebra and overall mathematics performance five or six years later. They controlled for factors such as whole number arithmetic, intelligence, working memory, and family background. We need to extend these important findings to highlight for teachers those specific areas of fractional knowledge that impact directly on algebraic thinking.

Research conducted by Lee (2012) and Lee and Hackenburg (2013) with 18 middle school and high school students showed that fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations. They concluded that "Teaching fraction and equation writing together can create synergy in developing students' fractional knowledge and algebra ideas" (p. 9). Their research used both a Fraction based interview and an Algebra based interview. The two interview protocols were designed so that the reasoning involved in the Fraction based interview provided a foundation for solving problems in the Algebra interview. In both interviews students were asked to draw a picture as part of the solution. For the Fraction tasks they were also asked to find the answer whereas in the Algebra tasks they were asked to write an appropriate equation but not solve it. Examples of one of each of the Fraction and Algebra Tasks are shown in Table 1 below.

Table 1. Examples of tasks used by Lee and Hackenburg.

Fraction Task	Algebra Task
Tanya has \$84, which is $\frac{4}{7}$ of David's money. Could you draw a picture of this situation? How much does David have?	Theo has a stack of CDs some number of cm tall. Sam's stack is $\frac{2}{5}$ of that height. Can you draw a picture of this situation? Can you write an equation?

It is important point to note that the thinking required to solve these types of fractional tasks is similar to the kind of thinking required to "solve for *x*" in a corresponding algebra equation. This will be discussed later. Both the Fraction Task and the Algebra Task from the Lee and Hackenburg study (2013) shown in Table 1 cannot be solved additively, for example, by saying "I have to add another three-sevenths". Lee and Hackenburg do not discuss the range of possible methods that students might use to solve the fraction task, presenting instead an example of a picture and associated comments by one student. Students are not required to solve the algebra equation ($S = \frac{2}{5}T$ where S and T represent the number of CDs that Sam and Theo have).

Stephens and Pearn (2003) identified Year 8 proficient fractional thinkers as students who demonstrated a capacity to represent fractions in various ways, and to use reverse thinking with fractions to solve problems. This research also showed that effective reverse thinking depends on a capacity to apply multiplicative operations to transform a known fraction to the whole. This capacity will later be fundamental to the solution of algebraic equations.

In this study we identify algebraic thinking in terms of students' capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then to operate multiplicatively on both in order to find the whole. Jacobs, Franke, Carpenter, Levi and Battey (2007) also emphasise the need to "facilitate students' transition to the formal study of algebra in the later grades (of the elementary school) so that no distinct boundary exists between arithmetic and algebra" (p. 261). Three distinct aspects of algebraic thinking identified by Jacobs et al. (2007) and by Stephens and Ribeiro (2012) are important for this study. They are students' understanding of equivalence, transformation using equivalence, and the use of generalisable methods.

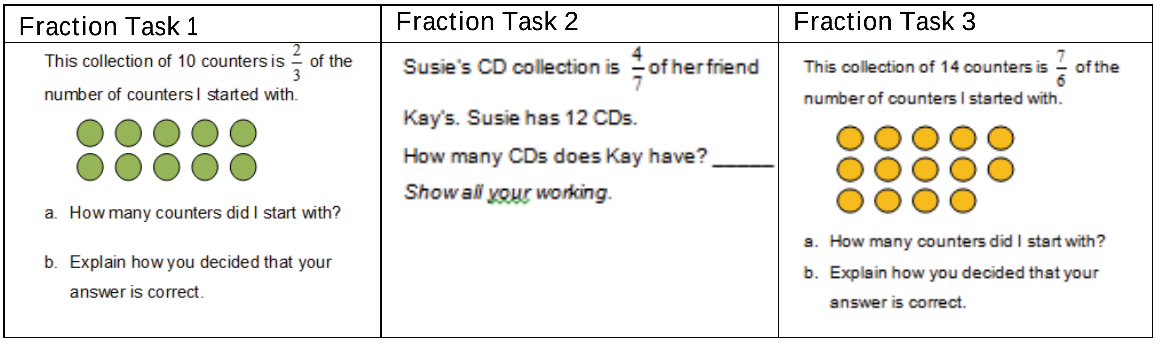
This study

Unlike the Lee and Hackenburg study (2013) which used both a fraction interview and a separate algebra interview, this study is based on analyses of students' performances in a single paper and pencil test. This test included relational thinking tasks using numbers and fractions, some corresponding algebraic (symbolic) tasks, and three fraction questions (Figure 1). The fraction questions required students to scale-up a given fraction to make a whole accompanied by equivalent changes in the quantities represented by the particular fraction. The relational thinking and symbolic tasks are shown in Tables 2 and 3.

Eighteen Year 6 students from an eastern suburban metropolitan school in Melbourne were assessed at the start of Year 6 using this paper and pencil test. These students were chosen by their teachers as deemed to be highly successful in mathematics. The paper and pencil assessment included three fraction tasks (Pearn & Stephens, 2014) and two relational thinking tasks based on Stephens (2010). All 18 students were given the same three fraction tasks while, due to time limitations available for the written questionnaire, ten students were given the multiplication component and eight were given the division component tasks.

The tasks

The three fraction items specifically required students to use reverse or reciprocal thinking in which their task is to find a whole collection when given a part of a collection and its fractional relationship to the whole. We devised these three items to offer students opportunities to use more explicit algebraic thinking which was not needed in the earlier task relating to one-half. Specifically these three tasks required students to relate a given fraction to an equivalent number of objects, and when transforming the fraction to make a whole to carry out corresponding operations on the number of objects. These elements of algebraic thinking are explicitly embedded in the relational thinking tasks shown below in Tables 2 and 3. There, students need to use equivalence, compensation and/or transformation, and generalisation.



Fraction Task 1	Fraction Task 2	Fraction Task 3
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.	Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? ______ *Show all your working.*	This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.

Figure 1. The three fraction questions.

Each of the three fraction questions (Figure 1) was marked out of three. One mark was given for a correct response with no explanation, or if there was some evidence of correct diagram with an initial representation which the student did not take further (starting point). Two marks were given for a correct answer with limited explanation and three marks were given for a correct answer with adequate explanation. Zero was given when the question was not attempted or an incorrect response was given.

Table 2. Sample items from the relational thinking tasks.

	Question 1 Task 1	Question 1 Task 4
Multiplication	36 × 25 = 9 × □	3 ÷ 4 = 15 ÷ □
Division	$\frac{2}{3} \times \_ = 1$	$\frac{7}{6} \div \_ = 1$

The four items in Question 1 of both the multiplication and division relational thinking test (Table 2) students were scored out of three marks. Students were asked to “write a number in the box to make a true statement and to explain their working briefly”.

Table 3. Sample items from the multiplication and division component tasks.

	Question 2 Item 2	Question 2 Item 4	Question 2 Item 5
	When you make a correct sentence what is the relationship between the numbers in Box A and Box B	What can you say about *c* and *d* in this mathematical sentence?	What can you say about *a* and *b* in this mathematical sentence?
Multiplication	5 × □ = 10 × □ Box A Box B	$c \times 2 = d \times 14$	$a \times \frac{3}{4} = b \times 1\frac{1}{2}$
Division	3 ÷ □ = 15 ÷ □ Box A Box B	$c \div 8 = d \div 24$	$a \div \frac{2}{7} = b \div \frac{4}{7}$

Question 2 of both the multiplication and division tests focused on students understanding of equivalence relationships with two unknown numbers represented by Box A and Box B, and by symbolic representations of two unknowns (see Table 3). Question 2 consisted of five items, four items worth three marks with one item worth two marks. The total possible score for the multiplication or division component was 26 marks. In Question 2 for both multiplication and division questionnaires the tasks included both whole numbers and fractions (Table 3). In Question 2, students were required to explain the relationships between the respective unknown numbers or the given symbolic representations (*c* and *d*; or *a* and *b*) .

Results

Table 4 shows the range of scores for the three fraction tasks. Fraction Tasks 2 and 3 were found to be more difficult than Fraction Task 1 as shown in the mean scores.

Table 4. Scores for the three fraction tasks ($n = 18$).

	Score 0	Score 1	Score 2	Score 3	Mean
Fraction Task 1	0	0	4	14	2.78
Fraction Task 2	4	0	4	10	2.11
Fraction Task 3	4	0	5	9	2.06

Seven students scored nine out of a possible nine for the three fraction tasks. Six of these seven students scored between 23 and 26 on either the multiplication or the division items. The student who scored nine on the three fraction tasks only scored 17 out of a possible 26 marks as her explanations for the division items were not adequate.

Table 5. Scores for the multiplication/division algebraic thinking items ($n = 18$).

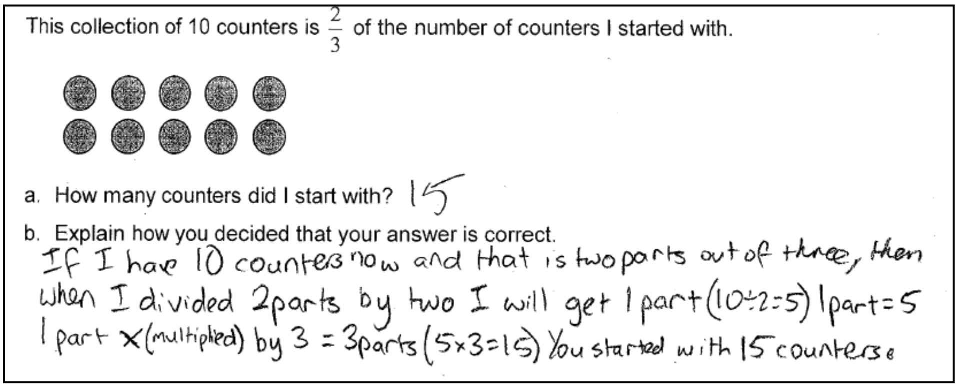
Questions 1 & 2	Score < 10	11≤ score ≤ 17	18≤ score ≤ 22	23 ≤ score ≤ 26
Multiplication	2	1	2	5
Division	1	5	0	2

Analysis and discussion

In this section of this paper we will examine the strategies used in the three fraction tasks by the seven students who had a perfect score on the fraction tasks. All but one of these students scored between 23 and 26 on the multiplication /division algebraic thinking items. Since our focus in this paper is to identify instances of fractional thinking that could be classified as algebraic; namely, understanding of equivalence, transformation and/or compensation using equivalence, and use of generalisable methods, these seven students offered the best chance to show this.

The seven students identified above showed subtle differences in the chains of reasoning they employed and the symbols they used. All seven clearly recognised the connection between the number of objects and the fractional parts given in the problem. Responding to Fraction Task 1 (Figure 1) Student 5 wrote: “If I have 10 counters now and that is two parts out of three ...” In some cases, the connection between the number of objects and the fractional parts of the problem was implied. For example, when solving Fraction Task 1 (Figure 1) Student 3 wrote $10 = \frac{2}{3}$ meaning that the ten dots represented two-thirds of the whole group. Similarly when solving Fraction Task 2 (Figure 1) Student 1 wrote idiosyncratically that “12 = 4” implying that 12 CDs was the same as four-sevenths of the whole group. Student 4 gave a similar response for Fraction Task 3 when she wrote $14 = \frac{7}{6}$ in this case meaning that the 14 dots represented seven-sixths of the whole group.

After making the connection between the number of objects in the group and the fractional part, students determined the number of objects in the unit fraction by dividing by the numerator of the given fraction. Student 5 (Figure 2) gives an elaborated verbal response, using equivalence between the 10 counters and “two parts out of three”. After that the student argues confidently from “two parts” to “one part” to “three parts” without needing to state the fractional value represented by each “part”. As each “part” is transformed so is the number of counters represented by each “part”, until the student correctly concludes: “You started with 15 counters”.



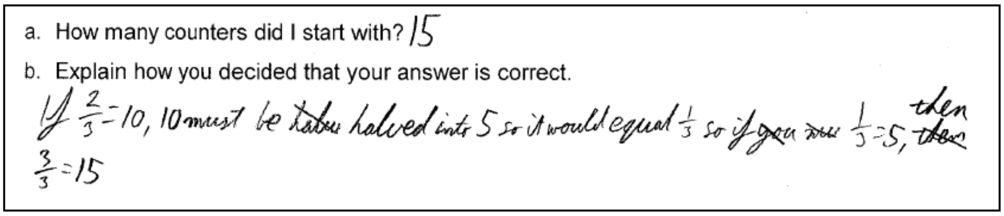
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.

a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct.
If I have 10 counters now and that is two parts out of three, then when I divided 2 parts by two I will get 1 part (10÷2=5) 1 part=5
1 part × (multiplied) by 3 = 3 parts (5×3=15) You started with 15 counters.

Figure 2. Student 5 response to Fraction Task 1.

Student 6 (Figure 3) starts with a symbolic statement equating two-thirds as ten. The symbolism maybe called idiosyncratic but the mathematical meaning is perfectly clear. Like the previous student this student moves symbolically from two-thirds to one-third to three-thirds, at each step referring to the equivalent number of counters.



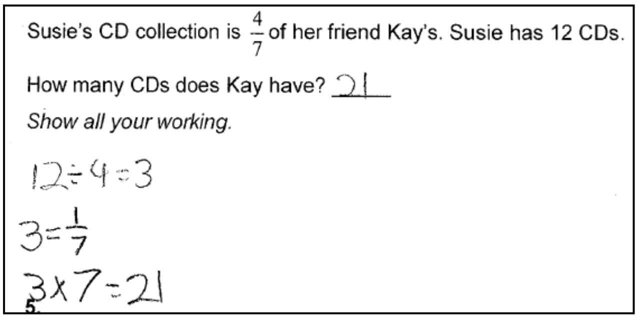
a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct.
If $\frac{2}{3}$=10, 10 must be ~~taken~~ halved into 5 so it would equal $\frac{1}{3}$ so if ~~you~~ ~~now~~ $\frac{1}{3}$=5, then ~~then~~
$\frac{3}{3}$=15

Figure 3. Student 6 response to Fraction Task 2.

For the following two fraction questions, Student 6 creates the same initial equivalence by writing $\frac{4}{7}$ = 12 and $\frac{7}{6}$ = 14. In both cases Student 6 finds the unit fraction and scales up to find the whole. Recording is again idiosyncratic in both cases, for example $\frac{1}{6}$ = 2 × 6 = 12 where Student 6 compresses two operations into one symbolic statement.

Other students’ written responses varied considerably as evident in the following examples of students’ successful working for Fraction Task 2. Student 2 (Figure 4) uses no verbal elaborations. The fraction four-sevenths is not stated explicitly but is implied in the division by four. In the second line the student states the equivalence between the number of objects and the fraction one-seventh. The third line shows the transformation needed to go from one-seventh to a whole without needing to refer to the fraction. Student 2 uses an identical method in the other two fraction problems which has been generalised regardless of the particular fraction and the number of objects represented by the fraction.



Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.

How many CDs does Kay have? 21

Show all your working.

$12 \div 4 = 3$

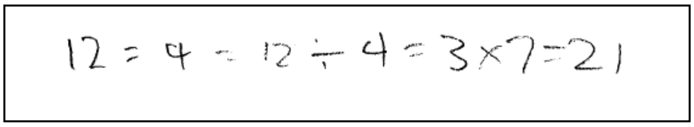
$3 = \frac{1}{7}$

$3 \times 7 = 21$

5.

Figure 4. Student 2 response to Fraction Task 2.

Student 1 (Figure 5) uses the equal sign in a way that some would deem incorrect. The 4 that is written after the 12 refers to the numerator of the fraction, hence the division by 4 to obtain 3. Multiplying by 7 is needed to transform one-seventh to a whole. The size of the fraction, although not stated, guides the reasoning to a correct solution in a way that is generalised by examining her responses to the other two fractional tasks.

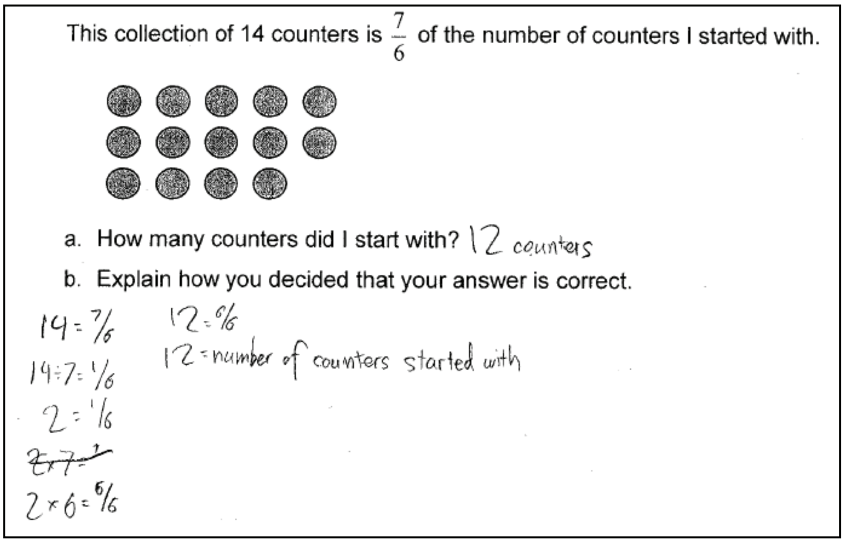


$12 = 4 = 12 \div 4 = 3 \times 7 = 21$

Figure 5. Student 1 response to Fraction Task 2.

In Fraction Task 1 Student 1 says: 10 is 2 parts = 5 each × 3 = 15. In Fraction Task 3 a similar abbreviated chain of reasoning is written without words as $14 \div 7 = 2 \times 6 = 12$

Two other students show their response to Fraction Task 3. Student 4 whose work is shown in Figure 6 consistently used the same method for all three fractional tasks.



This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.

a. How many counters did I start with? 12 counters

b. Explain how you decided that your answer is correct.

$14 = \frac{7}{6}$

$14 \div 7 = \frac{1}{6}$

$2 = \frac{1}{6}$

~~$2 \times 7 = 2$~~

$2 \times 6 = \frac{6}{6}$

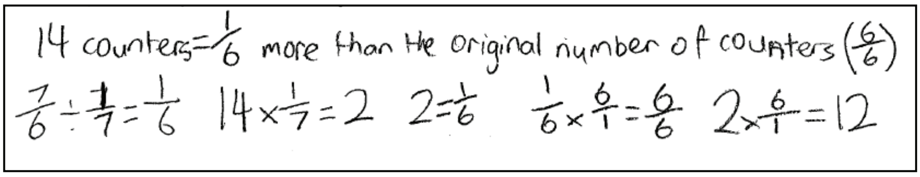
$12 = \frac{6}{6}$

12 = number of counters started with

Figure 6. Student 4 response to Fraction Task 3.

Like the previous students, Student 4 starts with the symbolic statement equating the 14 counters with the fraction seven-sixths. She uses idiosyncratic symbolism but the mathematical meaning is perfectly clear. This student moves symbolically and systematically from seven-sixths to one-sixth to six-sixths, at each step referring to the equivalent number of counters. Student 4 states the answer in words e.g. "12 = number of counters started with".

In the first fractional question (Figure 2), Student 5 uses an elaborated verbal description: "When I divided 2 parts by two I will get one part (10 ÷ 2 = 5) 1 part = 5) 1 part × (multiplied) by 3 = 3 parts (5 × 3 = 15). You started with 15 counters."



14 counters $= \frac{1}{6}$ more than the original number of counters $(\frac{6}{6})$

$\frac{7}{6} \div \frac{1}{7} = \frac{1}{6}$ $14 \times \frac{1}{7} = 2$ $2 = \frac{1}{6}$ $\frac{1}{6} \times \frac{6}{1} = \frac{6}{6}$ $2 \times \frac{6}{1} = 12$

Figure 7. Student 5.

For Fraction Task 3 Student 5 (Figure 7) uses incorrect recording to reduce seven-sixths to one sixth. Correct recording and execution is shown in: 14 × $\frac{1}{7}$ = 2. The student's next step 2 = $\frac{1}{6}$ shows equivalence between two objects and one-sixth. The remaining two steps are then self-explanatory. In the second fractional question Student 5 repeats the incorrect recording shown in Figure 7 by writing $\frac{4}{7} \div \frac{1}{4} = \frac{1}{7}$ = 3 CDs. Subsequent working correctly records and replicates the last two steps shown in Figure 7 demonstrating that Student 5 has a generalised method.

From the above responses it is evident that these Year 6 students know that they must move from any given fraction to its unit fraction and then scale up to one-whole. At the same time they keep track of the same operations on the number of objects represented by the given fraction. How they express their chains of reasoning vary from elaborated verbal descriptions to abbreviated mathematical expressions, sometimes not needing to refer explicitly to the fraction. They use "best available" symbols, especially involving the equals sign, to create and record their innovative chains of reasoning. We see these chains of reasoning as algebraic through students' use of equivalence, compensation and generalisation.

Conclusion

Researchers such as Jacobs et al. (2007) have emphasised that the foundations for algebraic thinking are laid in the primary school years through deepening students' understanding of number operations and relationships, especially through equivalence and compensation. The focus of their research has almost exclusively been on whole number operations. As a result, many teachers may think that students' thinking about fractions is not connected to the development of algebraic thinking. This is despite the fact that researchers such as Siegler et al. (2012) have demonstrated that competence with fractions is a unique predictor of students' subsequent attainment in algebra. This study has shown why that might be so.

It is important for teachers to move students in the upper primary years beyond more traditional types of fraction problems which limit students to calculating fractional parts of a known whole. The fraction tasks presented in this paper all require students to find an "unknown" whole when presented with a given collection which is a subset representing a given fraction of the whole. Having to use their rational number knowledge to find an unknown whole provides opportunities for students in the upper primary years to carry out chains of thinking that very closely mimic or resemble the kind of thinking that they will need later in the solution of simple algebraic equations. The same rules of equivalence and compensation apply. In both cases, students need to draw confidently on mathematical procedures that are generalisable. While later algebraic tasks will be explicitly symbolic, the fractional tasks discussed in this paper require students to develop their own repertoires of mathematical reasoning and to appreciate that these forms of reasoning can be expressed successfully in different ways. For this reason, we believe that the kind of fractional tasks discussed in this paper have high potential to provide a bridge to algebraic reasoning for all students.

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USING MULTIPLE CHOICE QUESTIONS TO IDENTIFY AND ADDRESS MISCONCEPTIONS IN THE MATHEMATICS CLASSROOM

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The *Progressive Achievement Tests in Mathematics* (PAT Mathematics) were published by the New Zealand Council for Educational Research (NZCER) in 1972. Revised editions were published in 2005. During the 1980s the Australian Council for Educational Research (ACER) also developed a range of PAT tests, including a number of versions of PAT Maths. These are continually being revised and developed. This paper describes the construction of the actual items of these tests which have a multiple choice format. It will be shown how developing, and then using multiple choice questions can inform teachers about students' misconceptions in mathematics. It will also explore the 'what next' once the misconceptions have been identified.

History of ACER PAT Mathematics

The first edition of the *Progressive Achievement Tests in Mathematics* (PAT Mathematics) was published by the New Zealand Council for Educational Research (NZCER) in 1972. These tests were designed to enable schools to record student progress and help teachers to identify the needs of their students.

The tests were based on classical test theory (Lord & Novick, 1968, cited in deKlerk, 2008) and achievement was reported in comparison with year level and age level norms (Darr & Stephanou, 2006). This method of reporting achievement was able to show how students compare to other students of equivalent age or year level. However, it could not be used to gauge the growth of an individual student over time, independent of the growth of other students.

The PAT Maths scale

In the 1980s, the Australian Council for Educational Research (ACER) developed a new range of PAT Tests based on *Rasch Measurement*. Rasch Measurement is a type of psychometric measurement, applied predominantly in the social, behavioural and health sciences, which enables a 'latent trait'—a characteristic that is not directly observable or measurable—to be measured indirectly by a well-designed instrument that provides indicators of this latent trait. In the case of PAT Maths, the latent trait being measured is mathematical knowledge and skill. The degree of mathematical knowledge and skill can be deduced indirectly from the responses to the test items (Wu & Adams, 2007; Lindsey et al., 2005).

The probability of a student getting a particular item correct is a function of that student's ability (mathematical knowledge and skill) and the question's difficulty. Using the mathematical model proposed by George Rasch, one can construct a measurement scale on which both item difficulties and student abilities can be displayed. Located higher on this scale are students with higher ability and more difficult test items. Located lower on this scale are students of lower ability and easier test items. Items from multiple test forms and year levels can be calibrated onto a single scale. Rasch Measurement can be used to track student progress over time, independent of which test form the student sat and the progress of other students.

Subsequent editions of the PAT Tests in Australia and New Zealand have continued to use Rasch measurement to produce and refine the PAT Maths scale.

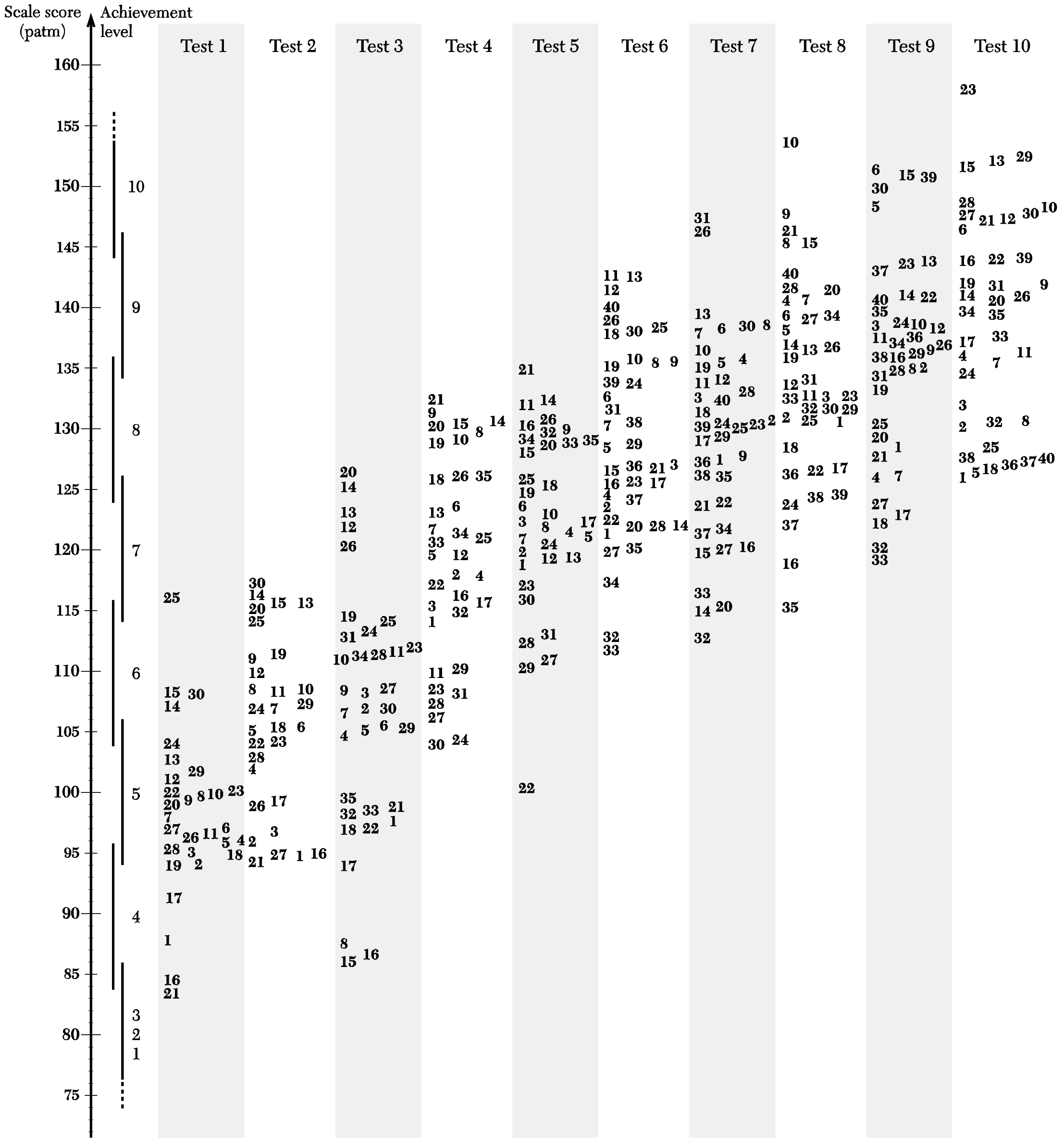


Figure 1. PAT Maths item distribution by year level on the single patm scale.

Structure of the tests

The PAT Maths tests consist of a bank of multiple choice items. These are developed around the main content areas of mathematics: Number and Algebra; Measurement and Geometry; Statistics and Probability. Test items have been developed and trialled with students at different levels and are mapped to a single scale using a national norming study (Darr & Stephanou, 2006).

During a norming study, trialled test items are arranged into test forms. Test forms are linked by including common items in different test forms and also by administering two different test forms to the same student. The data are then analysed with the Rasch model, and the relative location of items on the scale is determined. This process is known as item calibration. Figure 1 shows the result of the item calibration for the PAT Maths 4th edition tests. Once the scale is finalised, it becomes possible to estimate student locations on the scale and obtain the distribution of student achievement by year level. Each PAT norming study involves a representative sample of schools across Australia, from all systems and states.

In the fourth edition, the PAT Maths scale has been divided into bands or achievement levels, each ten scale scores wide. The achievement levels are a convenient way to describe the development of mathematical concepts and skills as students make progress up the PAT scale. The achievement levels may also provide the basis for grouping students with similar mathematical skills and knowledge and similar learning needs in a particular content area, in order to provide them with more targeted learning activities. The achievement levels are arbitrary. They have been defined using intervals of ten scale scores simply to make them easy to remember. There are no distinct boundaries that separate one level from another. Teachers can form groups based on different scale score intervals if they prefer.

Using assessment effectively to inform and improve teaching and learning

Figure 2 shows the results of the PAT norming study. It can be seen that there was a wide spread of achievement within any one year level and a large overlap in achievement between year levels. It also shows that on average, growth in learning is more pronounced in the early years and smaller growth is observed in the middle and later years of schooling. A similar pattern of achievement is observed in other large scale standardised assessments both within Australia and internationally (ACARA, 2011; Dadey & Briggs, 2012; Welch & Dunbar, 2014). For example, the national report for Australia's National Assessment Program in 2011 states that:

> Findings similar to this (less gain over later year levels) have been reported for other assessments in Reading and Numeracy that report data on a scale that has been vertically equated. Gains are smaller in later years of school than in earlier years of school. (ACARA, 2011, p.342)

The PAT Tests have been designed to reflect this. That is, adjacent test forms contain many items of similar difficulties and there is a spread of difficulty from very easy items at the beginning of the test form, to quite difficult items towards the end of the test form.

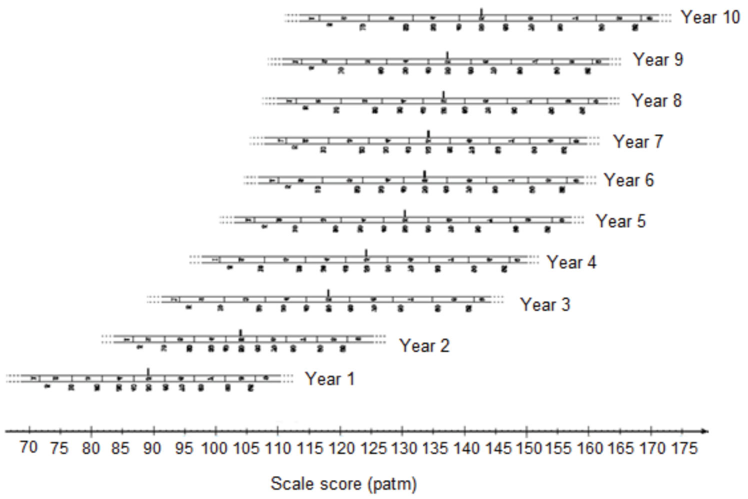


Figure 2. Distribution of student achievement on the pat scale by year level of students in the norming study.

By using PAT scale scores rather than percentage correct, students' achievement can be compared even when they have completed different test forms. Their degree of progress can also be tracked more easily over their years of schooling by tracking their growth in scale score.

A key element for the effective use of PAT Tests is selecting a test form that is appropriate for the particular student—neither too easy nor too hard. An appropriate test for a student is one in which the teacher can reasonably expect that student to get about half of the questions correct. This would result in a scale score with the least amount of measurement error. Tests that are either too easy (the student gets all or almost all the questions correct) or too hard (the student answers most of the questions incorrectly) would have very high degrees of measurement error and could not be used to place a student accurately on the scale.

In the case of standardised tests such as PAT Maths, this would have implications for tracking of growth over time. However, more broadly, for all types of assessment, an assessment that is too easy or too hard would mean that one cannot easily locate that particular student's level of development and would therefore have difficulty obtaining accurate information about what the student can and cannot do in order to design appropriate learning tasks for that student.

Having a means of assessing and tracking student progress in mathematics learning has the potential to significantly impact the effectiveness of teaching and learning within the classroom. However, it is of little use if this data is only used for reporting purposes (assessment *of* learning) but is not used to inform teaching and learning (assessment *for* learning). When teachers use data from well-designed assessments to target instruction where students are most ready to learn, profound improvements in performance can occur (Griffin et al., 2013).

Multiple choice questions in the PAT Maths tests

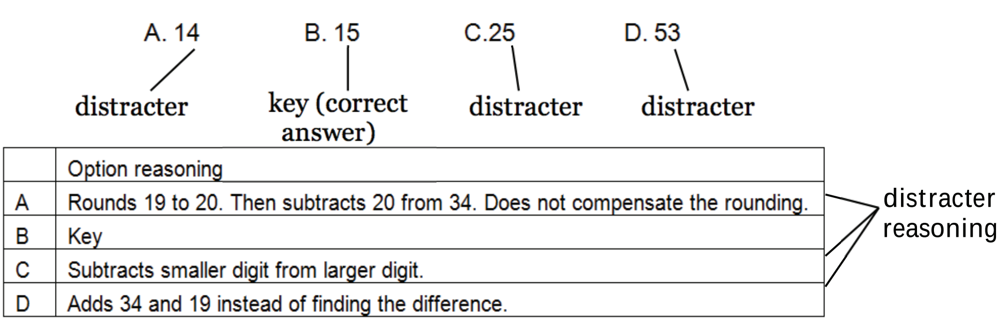
Analysis of the PAT Maths questions can be used to inform teachers about students' misconceptions in mathematics. Multiple choice questions generally consist of the question (stem), the choices provided (options), of which there is the correct answer (the key) and several alternatives (distracters). Literature suggests that a ‘good’ multiple choice question uses simple language, examines only the important facts, has brief and clear questions, avoids the use of the negative, avoids distracters such as ‘all of the answers are correct’ or ‘none of the above’ (Frary, 1997; Brame, 2015; Croation Academic and Research Network, 2008).

Mkrtchyan (2011) argues distracters are an important part in the development of multiple choice questions as they provide credibility to the test and provide important information about the test taker. Each distracter should be based on a common misconception about the correct answer and should be plausible (Mkrtchyan, 2011; Brame, 2015).

There is much debate around the number of distracters to include, and if there is indeed an optimal number. Rodriguez (2005) argues that three distractors are optimal for multiple choice questions and presents an in-depth study of the past 80 years of research illustrating this empirically. There are other arguments stating there is no psychometric advantage of having a uniform number, especially if the options become implausible (Frary, 1997; Haladyna & Downing, 1989; cited in Rodriguez, 2005).

There are many advantages for using multiple choice questions including that they are easier to mark, can be administered in an online form (Mkrtchyan, 2011) and can be machine marked if required. Objections to the use of multiple choice questions have included those that feel there are no tools or processes to analyse information or data about non-functioning distracters (i.e. distracters that only a very small percentage of students choose); there is always the possibility of guessing the answer (even with some statistical analysis designed to detect guessing); the options present a choice of answers rather than the test taker constructing the answer/response; and that they are simplistic and 'unable to test complex cognitive tasks’ (Athanasou & Lamprianou, 2002, p. 463, cited in Ljungdahl & Prescott, 2009).

However, many researchers and educators (Little et al., 2012; Mehrens, 2004) counter this last objection by showing that, when multiple choice questions are well constructed, with a clear stem and plausible distracters that target student misconceptions, they can be a very effective way of fostering higher order thinking and quality learning. Furthermore, for PAT Maths test questions, all questions are trialled and non-functioning distracters are removed before final test forms are constructed.
Figure 3 provides an example of a question. The different components of the question are labelled. The distracter reasoning is provided in the table below to illustrate the construction of the question.



	Option reasoning
A	Rounds 19 to 20. Then subtracts 20 from 34. Does not compensate the rounding.
B	Key
C	Subtracts smaller digit from larger digit.
D	Adds 34 and 19 instead of finding the difference.

Figure 3. Sample question with option reasoning included.

Using the PAT Maths questions to inform teaching and learning

PAT Maths 'allows teachers to gain an insight into their teaching and their students' learning' (Cooney, 2006, p. 463, cited in Ljungdahl & Prescott, 2009). By using the data and reports from the PAT Maths tests, patterns can be observed and information about a particular student's misconceptions can be determined. It is possible to analyse the data in terms of a whole class. For example, all students might appear to be struggling with a common question type, or they may all be having difficulties with a particular topic. Groups of students can be identified in a similar manner. This analysis can assist teachers in planning particular teaching content or organising students into particular groupings when addressing particular content or question types.

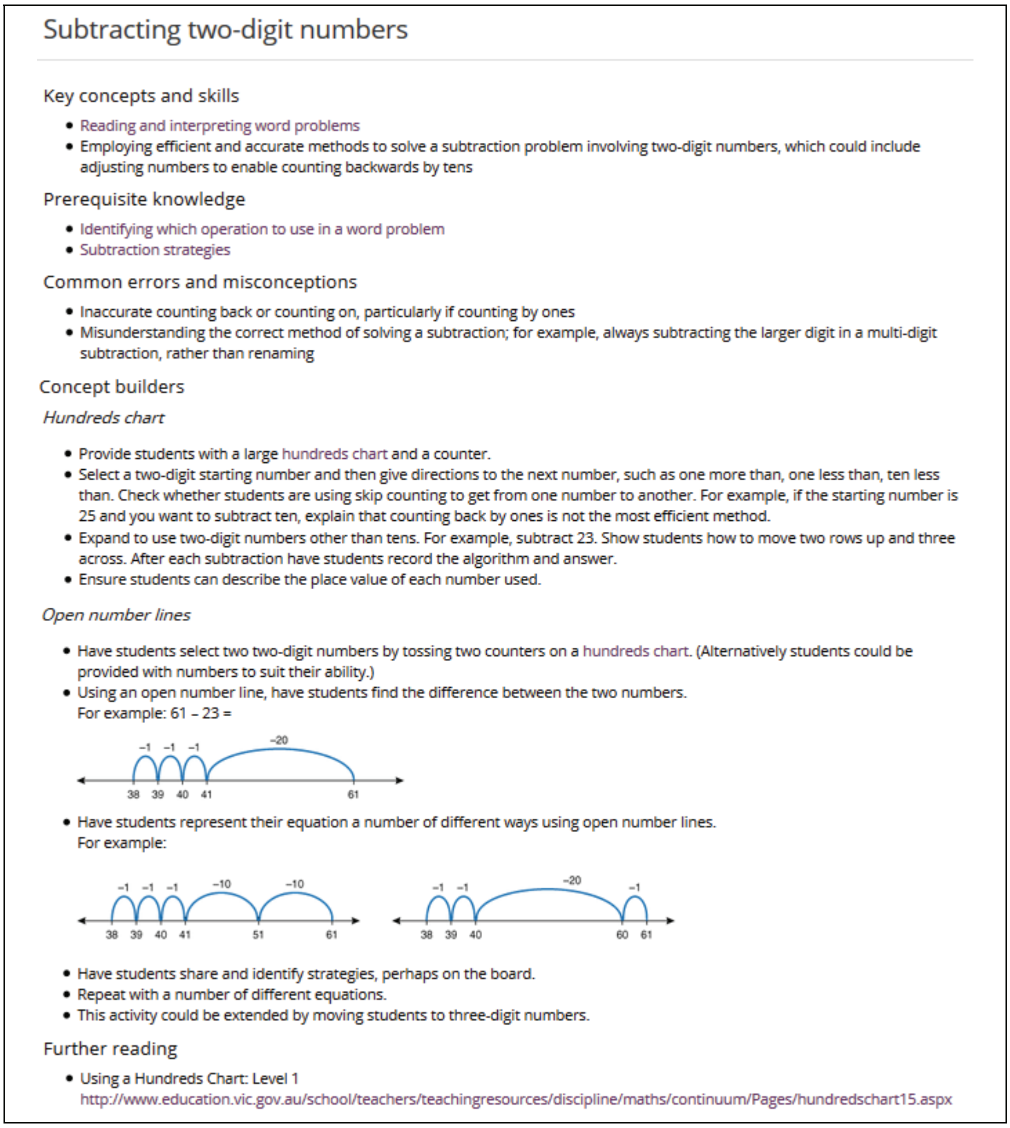
Analysis of the responses of a particular child to questions they have incorrect can provide teachers with 'valuable diagnostic information' (Ljungdahl & Prescott, 2009, p. 462) when the multiple choice questions are carefully constructed. It is identified as an important step in understanding the assessment tool as well as improving teaching and learning (Crisp & Palmer, 2007). Examining a particular question and identifying the distracter the student has selected can provide teachers with an insight into the student's misconceptions about a particular concept. This can be particularly powerful if a pattern emerges across a number of questions, allowing a common misconception to be identified and addressed. While an informative process, it is acknowledged that it is time consuming; hence to accompany the PAT tests, a PAT Resources Centre has been developed which includes Mathematics resources.

Maths section of the PAT Resources Centre

Hipkins (2006) states multiple choice questions 'may be quite complex and challenging in the thinking sequences required' (para 1). It is this distracter reasoning and interpretation that is used as the basis of development of the Mathematics component of the PAT Resources Centre. The PAT Resources Centre is an online subscription-based portal, designed to meet the needs of teachers who have conducted PAT assessments. The mathematics area of the PAT Resources Centre consists of a large set of PAT Maths items that have been selected, and deconstructed.

For each of the selected items the key has been identified and distracter reasoning has been provided. This has been presented in the form of a table for readability. Naturally teachers could perform this task, and Hipkins (2006) argues the value in identifying and even discussing distracters in developing understanding. However, it is acknowledged that developing multiple choice questions, including distracters that reflect common misconceptions, is time-consuming and difficult (Mkrtchyan, 2011). Thus there is value in the test developers (the people writing the test questions) providing the distracter reasoning, as this may provide the teachers with an understanding of how and why questions were developed in a particular ways.

It is all very well to identify student misconceptions, but the 'what next' is just as important. To assist teachers in addressing the identified misconceptions, accompanying each of the questions and their option reasoning are the following elements in the Maths component of the PAT Resources Centre: the identification of key concepts and skills; the identification of prerequisite knowledge; and suggestions for moving forward. These are presented in the form of what are known as *concept* *builders*–teaching activities and ideas that are provided around key concepts, often addressing common misconceptions and errors. These are not lesson plans, but are activities which could be used in whole class, small group and even in one-to-one situations. The identification that students may need to 'take a step back' to reinforce learning and understanding is supported with links to prerequisite knowledge of these concepts, acknowledging that many concepts form part of a learning continuum. Suggestions for moving forward are made for students who need the next challenge, but it is the further reading section sitting within the concept builders that supports the teacher's own personal learning on different mathematics concepts. An example of a concept builder from the Resource Centre is provided in Figure 4 below.



Subtracting two-digit numbers

Key concepts and skills

- Reading and interpreting word problems
- Employing efficient and accurate methods to solve a subtraction problem involving two-digit numbers, which could include adjusting numbers to enable counting backwards by tens

Prerequisite knowledge

- Identifying which operation to use in a word problem
- Subtraction strategies

Common errors and misconceptions

- Inaccurate counting back or counting on, particularly if counting by ones
- Misunderstanding the correct method of solving a subtraction; for example, always subtracting the larger digit in a multi-digit subtraction, rather than renaming

Concept builders

Hundreds chart

- Provide students with a large hundreds chart and a counter.
- Select a two-digit starting number and then give directions to the next number, such as one more than, one less than, ten less than. Check whether students are using skip counting to get from one number to another. For example, if the starting number is 25 and you want to subtract ten, explain that counting back by ones is not the most efficient method.
- Expand to use two-digit numbers other than tens. For example, subtract 23. Show students how to move two rows up and three across. After each subtraction have students record the algorithm and answer.
- Ensure students can describe the place value of each number used.

Open number lines

- Have students select two two-digit numbers by tossing two counters on a hundreds chart. (Alternatively students could be provided with numbers to suit their ability.)
- Using an open number line, have students find the difference between the two numbers.
  For example: 61 – 23 =
- Have students represent their equation a number of different ways using open number lines.
  For example:
- Have students share and identify strategies, perhaps on the board.
- Repeat with a number of different equations.
- This activity could be extended by moving students to three-digit numbers.

Further reading

- Using a Hundreds Chart: Level 1
  http://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/continuum/Pages/hundredschart15.aspx

Figure 4. Sample concept builder.

Sometimes talking through the question and the options with the student is enough for them to understand or identify areas of difficulty, particularly with word-based problems. To assist teachers with the process, links have been made to Newman's Error Analysis (White, 1999), identifying which of the five stages: Reading/Decoding, Comprehending, Transforming, Processing or Encoding (White, 2005) is illustrated by a particular distracter. Support material around an interviewing process for teachers and question prompts are also used.

The future of the PAT Resources Centre

At the time of this paper only the Number and Algebra part of the portal is live. Already basic Google Analytic data is showing in excess of 26 000 page views of the site in the period of one month. There are two phases of development to now take place. The first is to continue developing each of the content areas (Measurement, Geometry, Statistics and Probability) and have them available on the site. The second is to perform an in-depth study of the data from the pages viewed of the site, which may provide an indication of where students and/or teachers need support, the content areas for which they are seeking more information and questions that are proving to be a challenge.

Conclusion

Ljungdahl and Prescott (2009) argue that using distracter information from multiple choice questions can lead to assessment *for* learning, and aid teachers in developing students' numeracy skills. The ACER Progressive Achievement Tests (PAT) are used in over 4000 schools across Australia to monitor progress in key skill areas. The tests for the content area of mathematics consist entirely of multiple choice questions, and although these items are placed on a scale providing a score for students, there is value in taking the time to examine the carefully constructed questions in terms of the key and the distracter reasoning in order to identify misconceptions and adjust teaching and learning where necessary to challenge these misconceptions.

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WHITEBOARDING IN SENIOR MATHEMATICS CLASSROOMS

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Whiteboarding describes a distinctive method of teaching and learning where students work at large whiteboards that line the walls of a learning space. In this paper we report on the findings of a school-based research project into the benefits of whiteboarding in senior mathematics classrooms and the connections this project has prompted with the university where this particular method originated.

Introduction

Asked during a school-based professional learning workshop to list our most significant learning experiences, we nominated the unique tutorials we had experienced as undergraduate students at La Trobe University in the early 1980s. The tutorials were unique because they were conducted in rooms without chairs or tables and with all four walls lined with large blackboards. At the start of the tutorial the class would be given a worksheet of problems and we would proceed, in pairs, to try to solve them on the blackboards. To students used to a regular dose of lectures followed by relatively large practical classes, where it was easy (and sometimes convenient) to be overlooked, this was initially a daunting experience. This was particularly so because our problem-solving techniques, or lack thereof, were always on public display.

It did not take long, however, for us to realise that everyone else’s work was on public display as well. This brought about a small revolution in the way we learned: we began to discuss, share, and even passionately argue about the mathematics we encountered in our tutorials. We were also utterly engaged; it was impossible not to participate actively.

Fast forward 30 years to a school-based research project at Penleigh and Essendon Grammar School in Melbourne’s inner western suburbs, which focussed on replicating the ‘La Trobe Method’ in senior secondary mathematics classrooms. PEGS had recently built a senior secondary building that included rooms with entire walls made of whiteboard material, so we developed a series of lessons that used the 'La Trobe Method' of whiteboarding and recorded our observations. The first series of lessons in 2013 were limited in their structure and were designed simply to determine whether working at whiteboards, as opposed to working on similar problems in a traditional sit-down, pen-and-paper lesson, made any difference to student learning. In the second series in 2014 we structured the whiteboarding lessons carefully to determine how 'flipping' and the type of problems affected student learning.

Our observations over the two years suggest that whiteboarding does indeed make a significant beneficial difference to student learning and that those benefits are enhanced somewhat by paying close attention to the type of problems used in these lessons, by giving students additional instructional videos for homework, and by instructing students on how to collaborate.

The benefits

Deborah King, a student contemporary from our university days who went on to use whiteboarding in her own teaching at the University of Melbourne, suggests that the "simple activity"

> has a variety of benefits ranging from making friends to helping students to learn by explaining to others or learning from other students, exposing students to many ways of thinking about a problem. It developed a sense of identity within the group and with the tutor; we were all on a path together (Seaton, King & Sandison, 2014, p. 102).

Our school-based research project observations support King's finding and suggest that, broadly speaking, there are five key benefits to whiteboarding:

- whiteboarding makes student thinking visible
- whiteboarding provides immediate and effective feedback
- whiteboarding encourages mathematical discourse
- whiteboarding develops mathematical resilience
- whiteboarding demands participation

Whiteboarding makes student thinking visible

Perhaps the most striking benefit of whiteboarding is that every student's thinking is made publicly visible. When thinking is made visible students benefit firstly by being able to collaborate and share ideas and secondly, and perhaps more powerfully, by reflecting on their own thinking and looking through a 'window into the learning process itself' (Ritchhart & Perkins, 2011, p. 272). This combination of social collaboration and private metacognition are the hallmarks of the now widely accepted sociocultural learning theories espoused by Piaget and Vygotsky (Bush & Kelly, 2004, p. 9).

Teachers also benefit from thinking that is visible by being able to assess, especially in a whiteboarding lesson, every student's understanding quickly and effectively. Teachers are able to immediately "check student understanding, and to identify, confront, and resolve student misconceptions" (Wenning, 2005, p. 9) and inappropriate mathematical working. This cycle of early diagnosis and correction with students who need it is the foundation for effective feedback.

Figure 1. Year 11 students at Penleigh and Essendon Grammar School participating in a whiteboarding lesson. A lack of whiteboards is clearly not a problem; dry-erase markers work just as well on windows.

Whiteboarding provides immediate and effective feedback

It is probably safe to say that Black and Wiliam's (1998) focus on formative assessment (Black & Wiliam, 1998), and Hattie's (2013) finding that feedback is one of the most significant influences on student achievement (Hattie, 2013), have resulted in something of a paradigm shift in education. It is now widely understood and accepted that feedback is critically important for effective learning.

The visible, public nature of whiteboarding allows it to provide a mechanism for feedback not available in sit-down, pen-and-paper lessons. Not only can students in pairs give each other feedback, particularly if a Think Aloud Paired Problem Solving protocol is used (Flaherty, 1975; Lochhead & Whimbey, 1987; Van Someren, Barnard & Sandberg, 1994), but quick glances around the room allow students to learn from a wide variety of problem solving approaches. As Seaton, King and Sandison (2014) suggest, "it is amazing how many approaches to a problem you see . . . many you wouldn't have thought of yourself!" (p. 104). More importantly, teachers can also quickly diagnose any individual or whole-class misconceptions at a glance and provide feedback in a variety of forms from small individual corrections in reasoning to whole-class instruction of a significant concept. Even feedback to other teachers is possible: "if the whole class just didn't get something, I could feed this back to the lecturer, a really important link in the feedback chain" (Seaton et al., 2014, p. 104).

The kind of feedback that whiteboarding facilitates is effective because it focuses not on the student, but on the task and the processes required to complete the task and because it is immediate (Hattie & Timperley, 2007).

Whiteboarding encourages mathematical discourse

Another striking benefit to whiteboarding is the extent to which students engage in mathematical discourse. In our observations, not only did students engage in dialogue with their partners, but also with other students around the room and with their teachers. The discussions were not forced or manufactured but seemed to evolve organically from a shared desire to solve difficult, but interesting, problems. The discourse in the whiteboarding lessons contrasted intriguingly with sit-down pen-and-paper lessons based on similar problems. The discourse in the sit-down lessons focussed partially on the problems and substantially on students' social concerns, whereas the discourse in the whiteboarding lessons was demonstrably mathematical and focussed almost exclusively on the problems at hand.

The kind of discourse and interactions we observed suggested that the class was beginning to behave very much like the "community of mathematical inquiry" described by Goos (2004) where "students learn to speak and act mathematically by participating in mathematical discussion" (p. 259) in "a classroom that enables the practices, values, conventions and beliefs characteristic of the wider communities of mathematicians to be progressively enacted and gradually appropriated by students" (Goos, Galbraith & Renshaw, 2004, p. 36).

Our observations that whiteboarding encourages discourse were not made in isolation. Almost all discussions in the literature highlight the active and interactive nature of discourse in whiteboarding lessons (Bush & Kelly, 2004, p. 9; Henry, Henry & Riddoch, 2006; MacIsaac & Falconer, 2004; Seaton et al., 2014; Wenning, 2005, p. 5; Yost, 2003).

The discourse that students engage in during a whiteboarding lesson also has the added advantage of providing an insight into the world of professional mathematicians who often use blackboards themselves to visualise and experiment with their own thinking or to collaborate with colleagues, a practice best seen in films such as *Good Will Hunting* and *A Beautiful Mind* (Polster & Ross, 2012). Realising that they are working like real mathematicians brings an authenticity to the work that students are doing (Seaton et al., 2014, p. 111).

Figure 2. Students at La Trobe University engaged in discussion in a typical board tutorial.

Whiteboarding develops mathematical resilience

An observation that surprised teachers at PEGS was that when students solved problems in a whiteboarding lesson they seemed to be less dependent on teacher assistance and more inclined to persevere with problems. The students appeared far more resilient. The observation was surprising because the teachers used the same introduction and the same problems in a whiteboarding lesson as they did with a sit-down, pen-and-paper lesson and yet the student approaches in the two lessons were distinctly different.

There are two suggested explanations for this observation. The first is that students who engage in mathematical discourse, as they invariably do in a whiteboarding lesson, tend to be more resilient problem solvers. Johnston-Wilder and Lee (2008), for example, believe that "articulating mathematical ideas contributes to building what we will call 'mathematical resilience'" (p. 54).

The second is that students recognise that the thinking they are making so visible during a whiteboarding lesson is *not permanent*. Any error can easily be erased and a problem can be started from fresh. This recognition seems to give students the confidence to take risks and not worry about making errors. As Henry, Henry and Riddoch (2006) suggest,

> the ease of modification means that students may be willing to write or draw something that they are not quite sure of and expose it to group discussion and possible revision. Ideas are more easily exposed, discussed, accepted, discarded, and modified with the flexibility of a whiteboard (p. 1).

Henry, Henry and Riddoch (2006) were reflecting on the motivations of primary school children, but as Seaton, King and Sandison (2014) attest, the impermanence of whiteboarding even encourages the resilience of university students: "using blackboards (or whiteboards) invites the possibility for students to accept changes suggested either by their partners or by the tutor, fixing errors and false starts without the crossing out that comes when work is 'corrected' on paper, hence producing a polished final product" (Seaton et al., 2014, p. 104).

Whiteboarding demands participation

The public nature of whiteboarding means that students are compelled to be active participants in the lesson. In our observations no student, regardless of ability, was disengaged. Some students had difficulty but they got help from more able students who, in turn, consolidated their understanding by having to explain their thinking. Some students had little difficulty and others needed only to cast a glance around the room to get the tip or trick they need to proceed. The student's engagement in the task seemed to reflect a shift in the management of the class. Rather than rely on the teacher to manage and motivate the class from the board, each student became publicly and visibly responsible for their own learning. As Seaton, King and Sandison (2014) suggest, "moving the responsibility of driving the class from the tutor to the students opens up opportunities for students to discuss mathematics and engage with the content in an environment that is non-threatening and inclusive. Everyone is compelled to be active in class" (p. 112).

The ideal whiteboarding lesson: Homework or no homework?

By their very nature whiteboarding lessons are inverted or ‘flipped’ lessons because they allow teachers to “deliver targeted instruction to students one-on-one or in small groups, help those who struggle, and challenge those who have mastered the content” (Sams & Bergmann, 2013, p. 16) rather than rely on lectures or whole-class instruction.

Having established the benefits of whiteboarding in 2013 we set out in 2014 to determine whether whiteboarding lessons are improved by adopting a key component of flipping as defined by Bishop and Verleger (2013): “direct computer-based individual instruction outside the classroom” (p. 5).

We compared two classes with an identical set of problems. Class A was simply given the set of problems at the beginning of a lesson and asked to solve them using whiteboarding techniques. Class B was given some advance notice of the whiteboarding lesson and the goals of that lesson. The students in this class were also given some preparatory homework to do: watch a video we had prepared instructing them on the techniques for solving problems similar to those they would encounter in the whiteboarding lesson.

Interestingly, the only difference between the classes seemed to be that the students who watched the video arrived at their solutions more efficiently. Class A were eventually able to solve the problems successfully, but it took them a little longer to get there. It seemed that the only difference between the approaches was that Class B was easier to manage.

The ideal whiteboarding lesson: Pairs or no pairs?

Class B was also instructed to use the Think Aloud Paired Problem Solving (TAPPS) protocol. In this protocol one of a pair of students solves a problem while explaining their thinking out loud to the other student. The role of the other student is to act as a sounding board. The roles are then reversed for the next problem.

Class A, on the other hand, was given no instruction on how to collaborate. Some students in Class A began to solve the problems individually, some in groups and some in pairs. Eventually, however, students throughout the class began to collaborate in a unique and organic way. Individuals who were struggling to cope on their own began to discuss their difficulties with other individuals or initiated conversations with groups who seemed to be making progress.

Students in Class B, on the other hand, collaborated almost immediately, solved the problems more efficiently and were less reliant on teacher assistance. When they did engage the teacher in discussion it was more for clarification than for assistance in making a start in the problem solving process.

The ideal whiteboarding lesson: Problem types

We also compared the types of problems posed in 2013 and 2014 and analysed their merits. In 2013 we simply used problems that had previously been used for sit-down pen-and-paper lessons. Although the problems we used were not consistent across the classes that trialled the whiteboarding technique, some key common observations did emerge. It seemed that students lost motivation early in the lesson with problems that varied in difficulty but were essentially skills-practice problems. These sorts of problems also seemed to expose fewer misconceptions and encourage less interaction between students and their peers and between students and teachers. A similar observation was made of problems that required approaches limited to one form of representation: algebraic, graphical or diagrammatic. Finally, we observed that the skills-practice problems were limited in their scope. Students found the homogeneity of the questions repetitive and tedious. In response to these observations we developed problems in 2014 that

- exposed student misconceptions by creating cognitive dissonance, something that Mason (1999) suggests is the “fundamental experience that plays a crucial role in the construction, negotiation and reconstruction of meaning, both for individuals and for a social community” (p. 189);
- required a multi-representational approach where students were compelled to use a combination of algebraic, graphical and diagrammatic techniques;
- were differentiated from basic questions on fundamental concepts to complex questions requiring generalisations and proofs.

By making these changes to the types of problems posed in whiteboarding lessons we noticed an increase in engagement, resilience and mathematical discourse. Interestingly, the creation of cognitive dissonance and subsequent exposure of misconceptions added to the quality and quantity of mathematical discourse and seemed to motivate students rather than disengage and frustrate them.

It should be pointed out, however, that these preliminary observations often raised more questions than they answered. This suggests that further trials across a variety of topic areas are required to add to our understanding of the complex role that problem types have in the effectiveness of any whiteboarding lesson. Suffice to say that we recognise that writing rich and meaningful tasks is difficult and complex in the development of any lesson, let alone one that involves whiteboarding.

Coda

The findings in this paper were presented as a workshop at the 51st Annual Conference of The Mathematical Association of Victoria in 2014. Knowing that the conference was being held at La Trobe University we immediately sought out our old tutorial rooms and were pleasantly surprised that they still existed and were still being used in much the same way they had been since their experiences of the early 1980s. Unknown to us, the ‘La Trobe method’ of running university tutorials had eventually made its way to the Universities of Melbourne, Wollongong and Newcastle where they became an integral part of teaching units in all year levels (Seaton et al., 2014, p. 102).

La Trobe University staff, intrigued that the La Trobe method was being used in secondary schools, attended our workshop. Subsequent discussions developed a partnership that resulted in this paper and one that holds promise not just for the promotion of whiteboarding as a teaching and learning tool, but also for meaningful links between secondary and tertiary mathematics teaching.

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USING MATLAB IN SECONDARY SCHOOL MATHEMATICS INVESTIGATIONS

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Two mathematics investigations for secondary school students are presented in this paper. The investigations are intended to nurture complex reasoning processes and ways of working that are appropriate for students of this age group. The design of each investigation is analysed in relation to the Dimensions of Learning pedagogical framework, as well as the SAMR and TPACK models for technology integration. The tasks involve the use of MATLAB, which is software commonly used in industry and tertiary institutions. The analysis illustrates, however, that this technology is very suitable for use in secondary school mathematics investigations.

Introduction

Two investigative tasks that were written for secondary school mathematics students are referred to in this paper. The first task is an investigation into codebreaking, and the second task is a mathematical modelling task involving shooting in basketball. Both tasks were written in alignment with the Dimensions of Learning pedagogical framework (Marzano, 1992), and designed to make use of MATLAB software (The MathWorks, 2014). The connections between the tasks and the Dimensions of Learning framework are outlined and the suitability of MATLAB software for use in secondary school mathematics is examined with reference to the SAMR (Puentedura, 2006) and TPACK (Niess, 2009) models for technology integration.

Method

Analysing the design of the investigations with reference to the Dimensions of Learning framework

Both tasks are examined through the lens of the Dimensions of Learning framework. As illustrated in Figure 1, knowledge is central to the Dimensions of Learning framework.

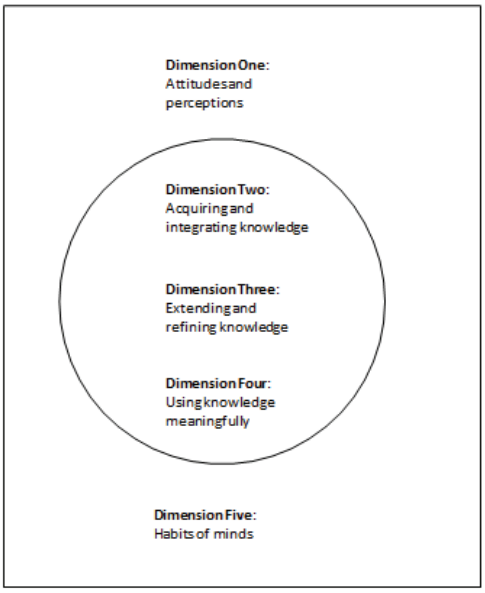


Figure 1. The Dimensions of Learning Framework.

The knowledge is described in two forms, namely *declarative* ('know what') knowledge and *procedural* ('know how') knowledge. The importance of extending and refining knowledge, and using knowledge meaningfully are emphasised in the framework. The framework extends beyond the acquisition of knowledge, however. It also addresses affective aspects of learning, including the attitudes and perceptions of learners and their ways of working. The Dimensions of Learning form a suitable frame of reference, therefore, for an analysis of investigations that are intended for adolescent learners.

Assessing the use of MATLAB with reference to the SAMR and TPACK models of technology integration

The benefit of using MATLAB in the investigations is considered with reference to the SAMR and TPACK models of technology integration. The SAMR model has four levels which can be used to assess technology integration. The first level is 'substitution', and in this level the new technology replaces a form of technology that was used previously. The second level is 'augmentation'. In this level the new technology is still essentially replacing the old method, but there are some added functionalities that enhance the learning process. The third level is 'modification'. At this level the educational process is transformed, and the technology facilitates the redesign of the task. Finally, the fourth level is 'redefinition'. In this level, new tasks can be designed that would not be possible without the new technology.

The TPACK model for technology integration provides a framework to describe an intricate blend of teacher knowledge. It spawned from a conceptual framework outlined by Mishra & Koehler (2006). The TPACK model describes a 'total package' of teacher knowledge which encompasses technological, pedagogical and content knowledge, and, importantly, the connections between these forms of knowledge. In their study of teachers in technology-rich environments, Goos and Bennison (2007) identified the need for such a framework. This need arises from the fact that the teacher has to cope with increased levels of complexity when new forms of technology are introduced (Hoyles & Lagrange, 2010).

Analysis

Connections between the codebreaking investigation and the Dimensions of Learning framework

One of the main objectives associated with Dimension One of the Dimensions of Learning framework is to help students develop positive attitudes and perceptions to classroom tasks. There are several methods that can be used to make tasks engaging for students. These methods include creating tasks that are relevant to life outside the classroom; ensuring that tasks provide an intellectual challenge; and stimulating interest in the task through the use of anecdotes related to the content of the task.

The codebreaking investigation is designed to adopt these methods. In the introduction to the investigation the relevance of codes throughout history and in many areas of real life is highlighted. Closely following the introduction, an intellectual challenge is presented to the students and they are asked if they can crack the code shown below

sspaoitini oietundnom mtonleogoa eipodotsng

tsloifhtei itenmtehcn mhteahtaae eehwgahtn

This codebreaking challenge is intended to initially arouse curiosity before it is solved using mathematical techniques later in the investigation. To raise interest levels further, a story is introduced in the form of a video clip from the movie “The Imitation Game” which is about codebreaking in World War II, and which contains a clue to cracking the code.

Dimension Five of the Dimensions of Learning framework outlines the productive mental habits of mind that help students become successful learners. Persistence is identified to be one of these habits of mind. Persistence is a quality that is associated with creative thinking. Codebreakers need to use their imagination and persevere. A link can therefore be drawn between this habit of mind and the codebreaking investigation.

The codebreaking investigation includes three main tasks. The first is to decipher a code which contains a quotation from Julius Caesar. The second task is to use the technique of frequency analysis to decipher a coded message. The third task is to crack a code that has been constructed using the transposition of letters.

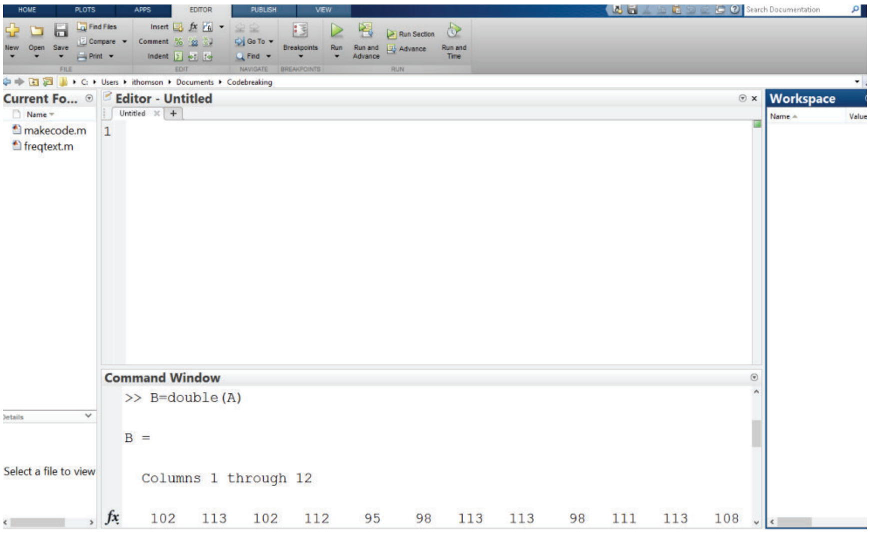
The tasks are all associated with stories that are designed to arouse curiosity and interest. The way that Julius Caesar coded his correspondence is described by explaining his use of the so called ‘Caesar Shift’ in which he shifted each letter three places to the left in the alphabet. FRIENDS ROMANS COUNTRYMEN LEND ME YOUR EARS becomes COFBKAP OLJXKP ZLRKQOVJBK IBKA JB VLRO BXOP.

Frequency analysis is connected to the true story of the plot to assassinate Queen Elizabeth I. The contents of coded letters between Mary Queen of Scots and her followers were revealed using frequency analysis and Mary was subsequently beheaded in 1587. In the third task, the students use the technique of transposition to crack the code that was set as a challenge in the introduction and which forms the last sentence spoken in the video clip from ‘The Imitation Game’, i.e. “Sometimes it is the people who no one imagines anything of who do the things that no one can imagine.”

Using SAMR and TPACK to assess the use of MATLAB in the codebreaking investigation

MATLAB is used in the codebreaking investigation in ways that can be related to the SAMR model. In the first task there is evidence of ‘augmentation’ when the Caesar Shift method is automated through the use of MATLAB. This is then transformed to the higher level of ‘modification’ when the task is redesigned to use ASCII code rather than just the letters of the alphabet. In the second task, there is evidence of ‘augmentation’ in that the frequency analysis can be performed faster and more efficiently using a MATLAB program than it could be done using pencil and paper. Arguably, the sheer power of MATLAB to deal with copious amounts of data increases the level to ‘modification’ since the students are able to analyse thousands of words rapidly from a variety of sources or languages. The third task also show evidence of ‘augmentation’ when it provides an automated method of transposing text to form a code. There is scope for ‘modification’ since the students can use this method to create their own codes on a large scale.

The codebreaking investigation can be assessed from a teacher’s perspective using the TPACK model. In this investigation the technological demands on the teacher in terms of the use of MATLAB are not high. The use of MATLAB in the investigation can often be restricted to the use of single line commands including, for example, the commands ‘char’ and ‘double’. An example of this is given in Figure 2 below which shows a screenshot of the MATLAB desktop.



```
>> B=double(A)

B =

  Columns 1 through 12

    102    113    102    112     95     98    113    113     98    111    113    108
```

Figure 2. The MATLAB desktop.

Some program scripts are used but they are provided and the teacher only has to cope with adapting them in minor ways. The mathematical content knowledge is not overly sophisticated either and the only concept of note is the transposition of a matrix. Overall, the investigation is designed in such a way that it would not stretch too much the technological, pedagogical and content knowledge of a secondary school mathematics teacher.

Connections between the basketball shooting investigation and the Dimensions of Learning framework

The basketball shooting investigation demands a level of declarative and procedural knowledge that is appropriate for senior secondary school students. The students are expected to know what the important features of a quadratic function are and know how these features can be used to construct a graphical model of the quadratic. Concepts and facts related to trigonometry and some elementary vector calculus are also required along with a reasonable degree of procedural skill in algebraic substitution and rearranging.

The basketball shooting investigation is also designed to be engaging. Although the context is very different from the codebreaking investigation, the methods used to make the investigation engaging are similar. The investigation is associated with life outside the classroom. It could even be introduced to the students in a basketball court. The investigation does present intellectual challenges in that it requires mathematical modelling and algebraic manipulation. The true story that the investigation is based on is perhaps the most engaging component of the design. In fact, the story is so powerful that the teacher can use it to help the students make connections with many features of the Dimensions of Learning framework. It is a tale of a young man at the end of high school who wishes to become a professional basketballer. He is not particularly tall but through a combination of dedicated practice and ingenuity he sets on a path to become the world's greatest basketball shooter.

A teacher who uses this story to introduce the investigation can help the students see the connections between the investigation and productive habits of mind that form part of Dimension Five of the Dimensions of Learning framework. 'Striving for accuracy' and 'restraining impulsivity' are habits of mind that are associated with critical thinking. 'Generating new ways of viewing a situation that are outside the boundaries of standard convention' is associated with creative thinking. These habits of mind are all depicted in the true inspiring story in which the basketballer perfects his shot and ingeniously improves it using unconventional techniques. Introducing the investigation in this way makes it relevant to students' interests and goals in a general sense. In the video clip that is incorporated in the investigation, the basketballer models self-efficacy when he writes on his shoes "I can do all things..."

Using SAMR and TPACK to assess the use of MATLAB in the basketball shooting investigation

With reference to the SAMR model, the use of MATLAB to perform curve fitting in the basketball shooting investigation can be viewed as a substitute for other methods. MATLAB performs the same operation that could be carried out on a spreadsheet or a graphics calculator. There is some evidence of augmentation in that the graph that is produced by MATLAB can easily be labelled and annotated. This can be done interactively using drop-down menus. Alternatively this can be performed through the creation of a script which could be adapted and reused in other contexts. The use of MATLAB to animate the basketball shots provides some evidence of modification. The visualisation of the shot is transformed from 2 dimensions to 3 dimensions. This helps the students make a connection between the mathematical model and the real world activity of shooting a basketball.

From the teacher's perspective the technological skills required to perform the curve-fitting and the three-dimensional animation are very manageable. This is because the curve-fitting can be carried out by entering coordinates into row matrices using single line commands followed by selections from an interactive drop-down menu. The program that is required for the three-dimensional animation is provided and only needs to be adapted by inserting a quadratic expression in a syntax similar to that used in a spreadsheet formula.

Conclusion

The mathematical investigations referred to in this paper have been analysed with reference to the Dimensions of Learning framework and shown to be suitable for secondary school mathematics students. The tasks have also been examined in relation to the SAMR and TPACK models for technology integration. This examination has shown that the tasks have been enhanced and transformed by the use of MATLAB without placing any great demands on the technological, pedagogical and content knowledge of secondary school mathematics teachers. From all of this it can be concluded that MATLAB can be used effectively in secondary school mathematics investigations.

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TEACHING MATHEMATICS TO YEAR 11 INDIGENOUS STUDENTS: MOVING FROM CONCRETE TO ABSTRACT MATHEMATICAL ACTIVITIES

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During Semester 2 2014, I taught a basic mathematics course to a group of Year 11 Indigenous students in a large remote Kimberley school. I will show how I attempted to take students from basic concrete spatial knowledge to the use of scale factors and similar triangles. The presentation highlights the importance of both starting from the students' current knowledge base and adjusting the teaching style to accommodate the learning strengths of Indigenous students. It is possible to rapidly move high-attending students from manipulating physical materials to the meaningful use of abstract representations and calculations in a 'high expectations' environment.

Introduction

Over many years much funding and effort has been directed towards ‘closing the gap’ between the literacy and numeracy achievement of Indigenous and non-Indigenous Australian students (Australian Government Department of Education and Training, 1989; Council of Australian Governments, 2009) with little evidence of success.

> Between 2008 and 2014, the proportion of Aboriginal and Torres Strait Islander students at or above the National Minimum Standards in reading and numeracy has shown no statistically significant improvement nationally (Australian Government Department of the Prime Minister and Cabinet, 2015, p. 14).

Although ‘numeracy’ has typically been included in reform agendas (MCEECDYA, 2010), the emphasis has mainly been on literacy improvement when reforms have been implemented. It is only relatively recently that a direct focus on improving mathematics for Indigenous learners (rather than the more nebulous ‘numeracy’) has been explicitly fostered (Grootenboer et al., 2009; Morris & Matthews, 2011).

Diagnostic assessment of mathematics understanding of Year 11 students at the school in 2014 revealed that they had learned very little mathematics beyond basic whole number during their previous ten years of schooling. This was in spite of the school’s engagement in national and state government literacy and numeracy reform programs over many years. Low attendance rates could explain some students’ low performance. The high-attending students, however, also exhibited extensive misconceptions, and mathematical knowledge and skills at least three years behind their current year level. The reasons for the continuing low performance of Indigenous students in mathematics are complex and partially involve aspects of culture, history and politics (Jorgensen & Sullivan, 2010) which I will not be considering in this presentation. Rather I will focus specifically on the role of curriculum and pedagogy in facilitating and accelerating the mathematics learning of the group of Year 11 students I had the opportunity to teach in the latter months of 2014.

Indigenous learners of mathematics: What do we know?

A four year project that supported schools across Australia to improve mathematics education for Indigenous students, *Make it count: Numeracy, mathematics and Indigenous learners*, was conducted by the Australian Association of Mathematics Teachers (AAMT) and culminated in an AAMT special interest conference in 2012.

The conference summary listed a number of pedagogical practices that could favour Indigenous learners and build on their strengths. The practices that particularly influenced my thinking and informed my planning include:

- making the learning goals—for the lesson, for the unit of learning etc.—intentional, explicit and understood by the students;
- giving clear and well-known processes for scaffolding students' learning in mathematics;
- students watching as teachers (or others) model the mathematics i.e. doing the mathematics, articulating it, applying it; and
- students interacting with mathematics through 'body–hand–mind'.

(Australian Association of Mathematics Teachers, 2012, p. 10)

Learning the mathematics

The stage one unit I was required to teach consisted of relatively low level content that would normally be introduced at around years 6 or 7. Even so, it was clear from previous assessments that the Year 11 students had achieved very little of the pre-requisite knowledge assumed in the unit description. The starting point for building the students' understanding therefore needed to be a lot lower than the knowledge and understanding required to directly access the unit's content. Conversely, the expected degree of mathematical complexity did not seem sufficient to match the normal developmental maturity of Year 11 students. Consequently, students who had missed large chunks of mathematical content (for whatever reason) had little chance of catching up if teachers restricted their planning to the limits of the unit content.

My approach was to implement pedagogical practices that favoured Indigenous students, while attempting to rapidly accelerate the students' understanding of concepts listed in the unit. If possible I intended to extend the complexity of the mathematics beyond that described in the unit.

The content and sequence of learning

The mathematics unit students were studying required that they were able to:

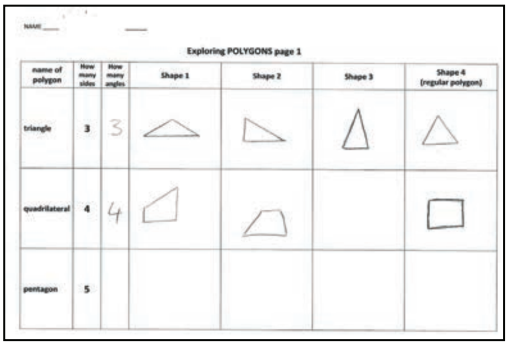
- measure, draw and estimate angles in degrees;
- classify angles as acute, right, obtuse, straight and reflex;
- name polygons including types of triangles and quadrilaterals; and
- draw polygons that meet criteria for angles, sides and vertices.

Triangles and angles

Previous testing had shown that students had little geometric knowledge, other than to name simple figures such as a triangle, a circle, a rectangle or a square. They thought 'angle' had something to do with corners but they had no knowledge of or skill in the measurement or classification of angles or the properties of polygons.

Exploring and categorising triangles

Student were given straws and wool (with a pipe cleaner 'needle') and asked to make different polygons, cutting the straws to provide a variety of different length sides for their shapes. During this process students discovered that no matter what the length of the sides, triangles were always rigid, while other polygons were flexible. This enabled me to draw attention to angles by asking the question, "So what changes when you change the shape of a quadrilateral?" Students completed the recording sheet shown in Figure 1 to reinforce the terminology. The action of physically making the polygons with straws interested students and even previously reluctant students attempted the recording sheet.



NAME

Exploring POLYGONS page 1

name of polygon	How many sides	How many angles	Shape 1	Shape 2	Shape 3	Shape 4 (regular polygon)
triangle	3	3				
quadrilateral	4	4				
pentagon	5					

Figure 1. Recording sheet used by students when creating different polygons with straws and wool yarn. This student usually refused to complete any written mathematical task.

By the end of this first lesson students understood that what makes triangles 'special' polygons is that the relative lengths of the sides of a triangle constrain the shape of its corners.[15] Talking with individual students and the class reinforced my perception that the students did not understand the geometric meaning of an 'angle' and had no knowledge of the way in which angles are measured mathematically. While they could make a square by manipulating straw sides of equal length, no one could tell me that the resulting corners were called 'right angles' or that they measured 90°. Some of the boys were familiar with the meaning of "I did a one-eighty," or "I did a three sixty," in relation to doing 'wheelies' when riding their bikes, but they did not know where those numbers came from or that they had a mathematical meaning.

During the next lesson, students used the straws and wool to construct a range of triangles, guided by a sheet illustrating standard categories of triangles and the angles they contain. Students worked together in pairs or triads to produce posters showing the different triangle types and their properties (Figure 2), necessitating the use of the target language in context. While some students made posters with the straws and

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[15] Further lessons explored the properties of other polygons including tessellations and perimeter/area activities, but this paper only focuses on developing understanding of angles and triangles.

wool, one student wanted to make his poster from paper triangles he had drawn and cut out. Figure 3 shows the result of his efforts.

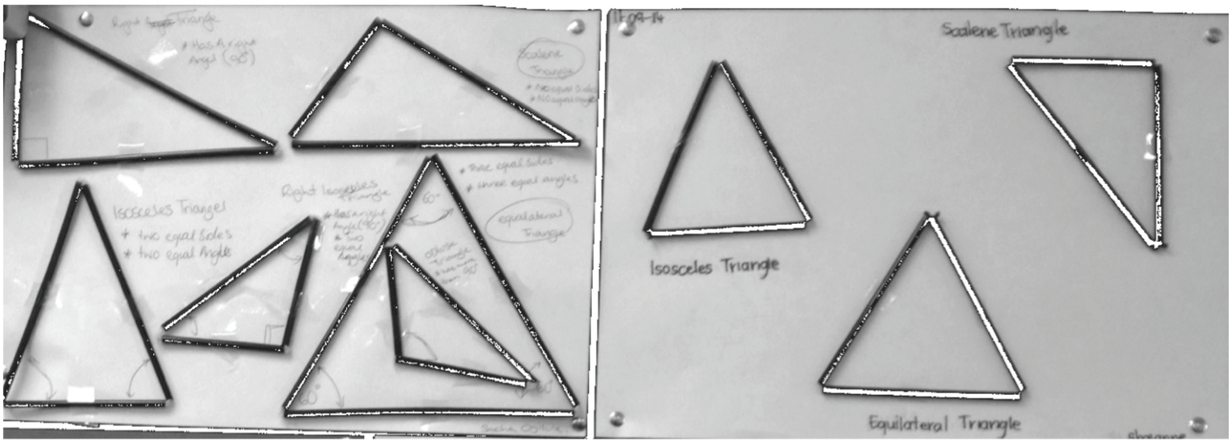


Figure 2. Students cut up straws and threaded them through with wool yarn to explore the lengths of sides and angles required to model the different categories of triangles.

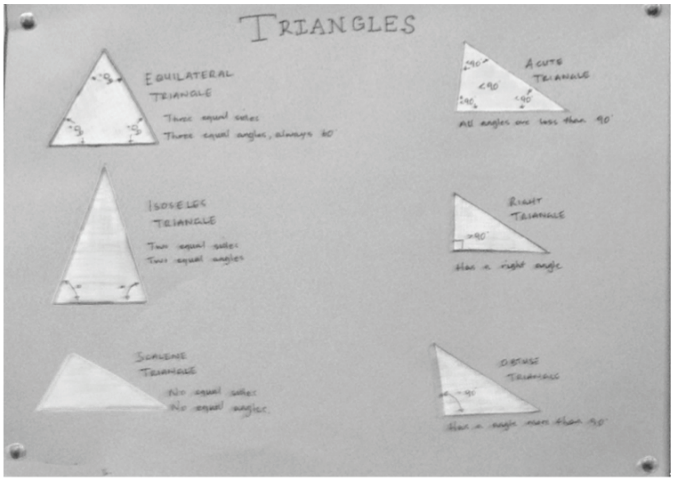


Figure 3. A student’s poster with annotated cut-out examples of the triangle types.

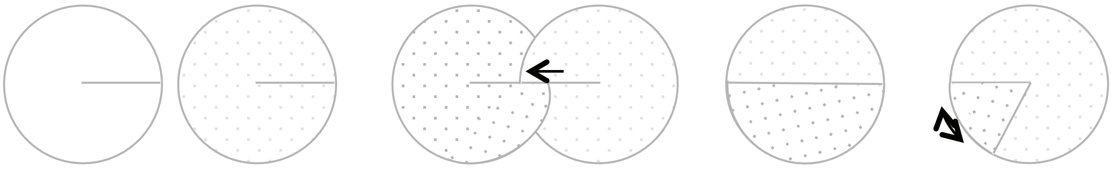
Explicit input

The sheet provided clear examples of triangle and angle types and the terminology that described them. In addition students were told that a ‘right angle’ matched the shape of the corners of their sheet of paper, the desk and the room, and also that it matched a quarter turn and measured 90°. I invited students to stand up and physically turn to match a 90° turn (quarter turn), and also 180° (half turn) and 360° (full turn, using four quarter turns), linking the language to bike riding slang.

The physical activity of creating the straw triangles taught students some of the properties of triangles and their angles. For example, they discovered that they could not make a triangle with two right angles, or with two corners greater than a right angle, and that if one corner was equal to or greater than a right angle (obtuse angle), the other two corners had to be less than right angles (acute angles). Physical actions were linked to the formal and informal language that describes angles and triangles.

Measuring angles

Students now knew a right angle measured 90°, but did not know why. To begin to establish this understanding, students were given two different coloured card circles, slit to the centre, intertwined and then rotated to illustrate the continuity of angles from zero through to a full circle. Figure 4 shows how the two circles are put together and rotated, one against the other to reveal angles in the coloured card.[16] Students were able to see how creating an acute or an obtuse angle with one colour created a reflex angle with the other colour, and how a straight line could still be considered an angle.



1) Cut out two card circles with slits to the centres.

2) Line up the slits and slide the two circles together. Now the circles can be rotated to create the full range of angles.

Figure 4. Creating circles to illustrate the turning actions and the relative size of angles.

The use of the circles led to the introduction of the units of angle measurement that are created around the 'turn' of the circle up to 360°. The use of a protractor as the implement or 'ruler' for measuring angle was introduced with the help of animated PowerPoint slides that I created for this purpose (Figure 5).

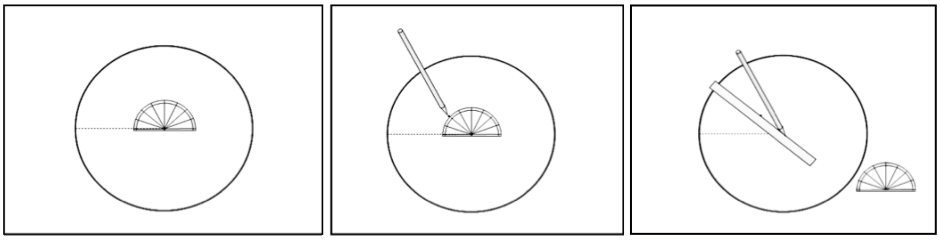


Figure 5. Slides that were animated to show the use of a protractor to draw an angle. The instruments physically move to demonstrate the exact actions involved.

After reshowing the animated sequence several times, students were confident enough to use a protractor to measure and then draw some of the different angles they were able to create with their intertwining circles. With a little more practice most students could successfully draw, measure and categorise angles without assistance.

Angles within triangles

At this point I returned to the exploration of triangles and their properties. Figure 6 shows an activity during which students established for themselves that the corners of any triangle sum to a straight angle, or 180°.

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[16] This activity was originally devised by Pamela Sherrard, WA Department of Education.

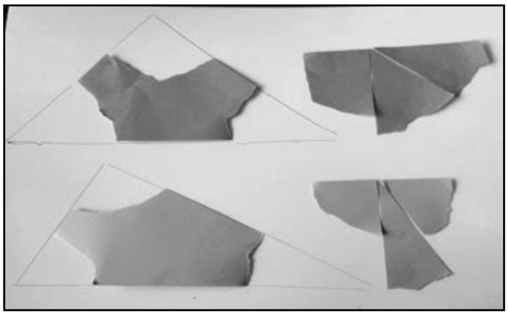
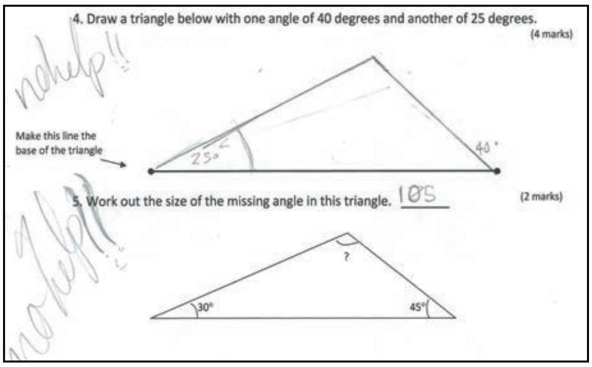


Figure 6. Students cut out different shaped triangles, drew around them, then tore off the corners and combined them to prove the angles of a triangle add up to a straight angle.

This and other angle and triangle information was reinforced and memorised by light-hearted incidental activities. For example, “Each of you must answer a question before you get to go out to recess.” “What do the angles of a triangle add up to?” “What is the size of the angles in an equilateral triangle?” “If two angles add up to 100°, what must the third angle be?” “What can you tell me about an Isosceles triangle?”

It was important that memorisation of the standard angle measurements and triangle types was only expected after students had developed an understanding of the concepts through scaffolded physical manipulation of materials and the development of strong mental images of the categories of angles and triangles. Figure 7 shows the kind of pen and paper assessment items that students could then complete with ease.



4. Draw a triangle below with one angle of 40 degrees and another of 25 degrees. (4 marks)

no help!!

Make this line the base of the triangle

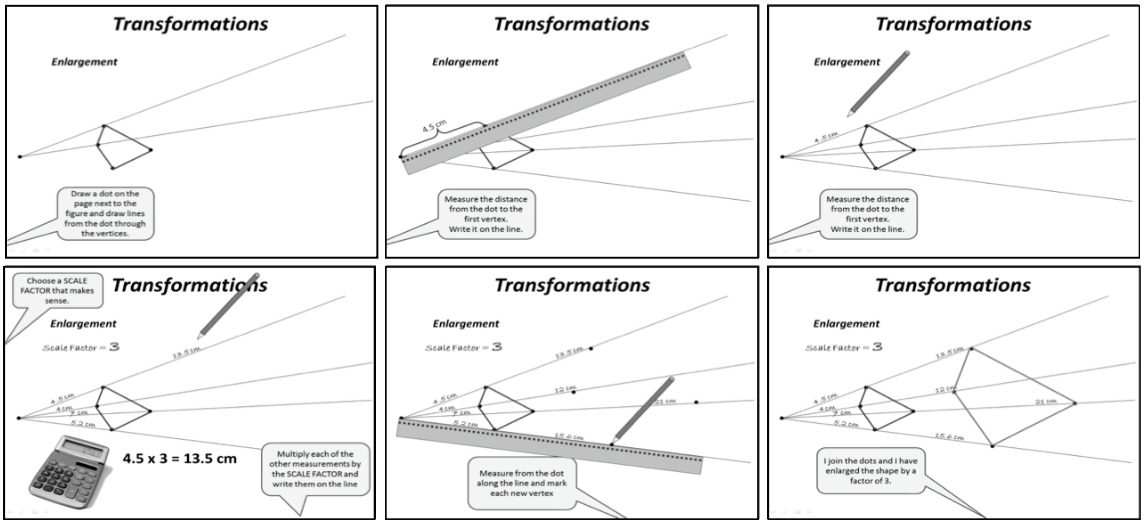
5. Work out the size of the missing angle in this triangle. 105 (2 marks)

no help!!

Figure 7. Example of assessment items. This work sample reveals the growing confidence of one student who usually avoided tests—she triumphantly wrote ‘no help!!’ next to each of the items.

Transformations and enlargements

Another content requirement of this unit included the introduction of transformations. I use animated slides to illustrate translation as the movement of points in a figure a fixed distance along parallel lines running through the points. Students found the scaffolded visual sequence easy to follow so I also created similar animated demonstrations for reflection and rotation. I then decided to extend the technique to create enlargements using rays and scale factors (see Figure 8).



Transformations

Enlargement

Draw a dot on the page next to the figure and draw lines from the dot through the vertices.

Transformations

Enlargement

Measure the distance from the dot to the first vertex. Write it on the line.

Transformations

Enlargement

Measure the distance from the dot to the first vertex. Write it on the line.

Choose a SCALE FACTOR that makes sense.

Transformations

Enlargement

Scale Factor = 3

4.5 x 3 = 13.5 cm

Multiply each of the other measurements by the SCALE FACTOR and write them on the line

Transformations

Enlargement

Scale Factor = 3

Measure from the dot along the line and mark each new vertex

Transformations

Enlargement

Scale Factor = 3

I join the dots and I have enlarged the shape by a factor of 3.

Figure 8. Animated slides stepping students through enlargement using rays and scale factors.

Students found the animations easy to follow. They were able to observe and then attempt to mimic the procedure shown in the animations (Figure 9). They could also view the steps over and over, and saw exactly what was expected of them.

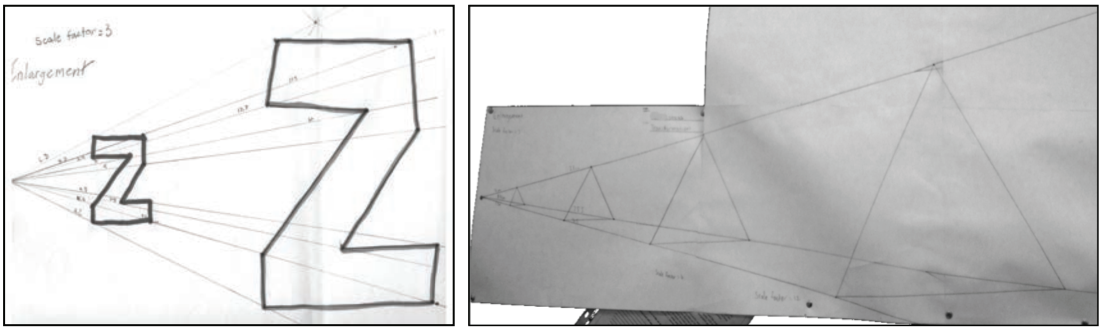
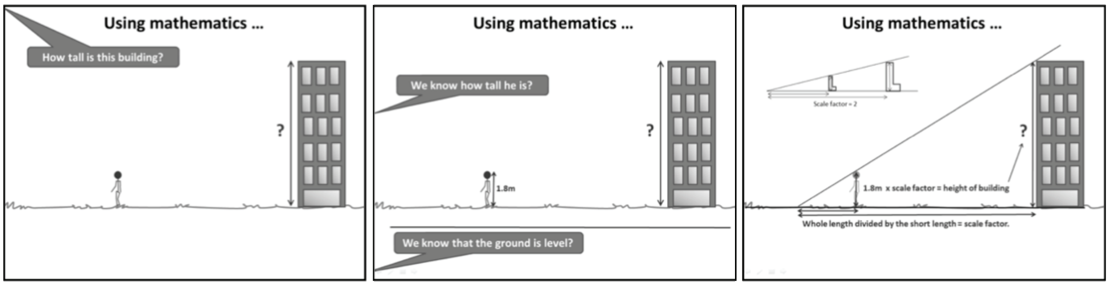


Figure 9. On the left is one student's enlargement after seeing the slide show, and on the right another student's enlargement of a triangle using scale factors of 3, then 6, and then 12.

The student's construction of the similar triangles using scale factors led me to believe that the students were capable of going well beyond the rather simple applications suggested in the unit content. Using animated slides to support their learning I introduced the idea that scale factors and proportional reasoning can be used to calculate unknown heights, and that the properties of similar triangles enables this.

Using similar triangles to calculate unknown heights

During the enlargement activity students explored how the scale relationship held true for every length within the enlarged figure. Students measured and used their calculators to prove this was true. I used animated slides to demonstrate how this knowledge could be used to calculate the unknown height of a building (Figure 10).



Using mathematics ...

How tall is this building?

Using mathematics ...

We know how tall he is?

We know that the ground is level?

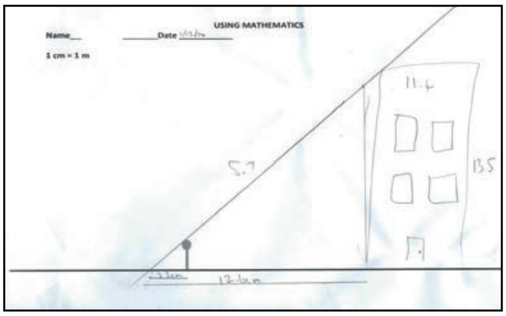
Using mathematics ...

1.8m x scale factor = height of building

Whole length divided by the short length = scale factor.

Figure 10. Slides from the animated sequence showing how an unknown height is calculated.

Students then drew their own tall object on a worksheet which was set up to scale with a baseline and a person measuring 2 metres tall. Students were asked to take the measurements on their page at 'ground level' in centimetres and use what they knew to work out the height of their building (or tree if they chose to draw a tree). I had not yet talked about the similar right triangles that were created in the diagram. Students did not, therefore, realise the necessity to keep their building or tree parallel to the 'person'. The work sample shown in Figure 11 reveals the effect of this. The student had measured the appropriate lengths to obtain an accurate scale factor, but because he had sloped his building, the actual height did not match his calculated height.



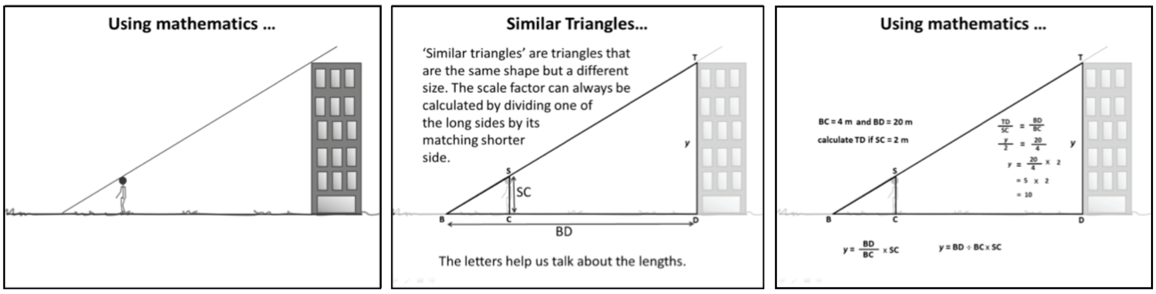
USING MATHEMATICS

Name__ ______Date 1/12/14

1 cm = 1 m

Figure 11. This student calculated a height of 11.4 cm, but his sloping building measured 13.5 cm. The problem was solved when the extra line drawn at right angles did measure 11.4 cm.

It was time to become more explicit about the role of similar triangles in making sense of why this process works. The animations in the next slides drew attention to the similar triangles in the diagram and the resulting relationships, as shown in Figure 12.



Using mathematics ...

Similar Triangles...

'Similar triangles' are triangles that are the same shape but a different size. The scale factor can always be calculated by dividing one of the long sides by its matching shorter side.

The letters help us talk about the lengths.

Using mathematics ...

BC = 4 m and BD = 20 m

calculate TD if SC = 2 m

$\frac{TD}{SC} = \frac{BD}{BC}$

$\frac{y}{2} = \frac{20}{4}$

$y = \frac{20}{4} \times 2$

$= 5 \times 2$

$= 10$

$y = \frac{BD}{BC} \times SC$ $y = BD \div BC \times SC$

Figure 12. Slides from animations introducing similar triangles, standard labelling of vertices and symbolic representation of the calculations.

At this point students were ready for a practical application of the method. We decided to measure the goal post at one end of the oval. We first measured the height of a student who volunteered to be the vertical side of the small triangle. We then used a long skipping rope and our eyes to line up the top of the goal post with the top of the student's head and a point on the ground behind the student. Students then measured the required ground level distances with a trundle wheel. We recorded the measurements, and then students drew a diagram and calculated the height of the goal post. Figure 13 shows one student's diagram and calculations using abstract symbols. The important point about this diagram and the student's calculations is that he understood every part of his representation and the meaning of the symbols he used.

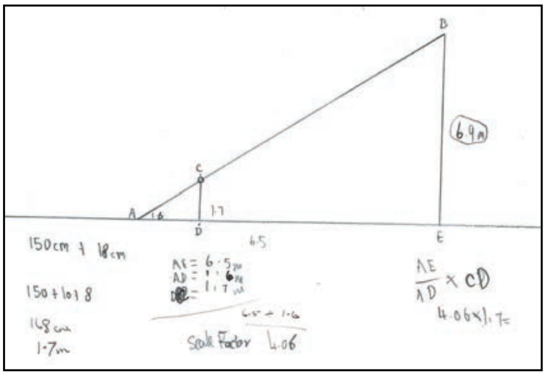


Figure 13. One student's representation of the measuring activity carried out on the oval to determine the height of the goal post, showing his calculations using abstract symbols.

Further activities and animated slides consolidated students' understanding and ensured that they could vary their diagrams of the triangles to suit different problems. A range of easily understood situations were presented to give students the opportunity to show that they could generalise the relationships between similar triangles in different contexts and had not just 'rote-learned' the procedure.

Reflections

The outcome of this sequence of lessons was extremely satisfying for me, the teacher, as well as for the students. They were very proud of the fact that they had used and understood some relatively advanced mathematics involving proportional reasoning. I believe that the successful acceleration of the students' learning can primarily be attributed to the students' exposure to explicit visual representations of the key mathematical concepts in a safe 'high expectations' learning environment.

At each stage students understood exactly what was expected of them and had opportunity to interact with the mathematics through 'body–hand–mind'. They were able to watch the modelling of the mathematical concepts and skills through the animated PowerPoint slides, which could be replayed until they were satisfied that they understood what was expected. The initial hands-on activities engaged all students by incorporating choices and opportunity to explore. Mathematical terminology and concepts were represented physically, diagrammatically, orally, and using symbols.

It seems that Indigenous students are particularly powerful visual learners. The challenge is to make the mathematics they see as explicit and meaningful as is possible.

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ONE SON, FIVE BIG IDEAS, AND TINKERPLOTS

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The aim of this workshop is to introduce a probability-based statistical investigation that can be carried out using the features of the software *TinkerPlots*. We will answer the question asked by Cliff Konold in 1994: Will changing China's one-child policy to a one-son policy cause a population explosion and/or result in an oversupply of girls? Following the four stages of a statistical investigation (pose question, collect data, analyse data, make decision) the process illustrates the five Big Ideas that form the conceptual foundation for statistics at the school level. The activity also illustrates some of the many learning affordances of *TinkerPlots*.

Introduction

An historic problem introduced by Konold (1994) is revisited for three reasons. Firstly it provides a still-current meaningful context within which to carry out a statistical investigation. Secondly it illustrates not only the four stages in carrying out an investigation but also the five Big Ideas that are the foundation for the study of statistics at school. Thirdly it employs the software *TinkerPlots*, developed by Konold and Miller (2005; 2011) more than a decade later to make large-scale interactive simulations possible in a hands-on fashion. This facility is one of the major affordances of the software for learning.

The one-son problem

The pure mathematics involved in this problem is from probability theory and can be stated and solved with a binomial model in a way leading to the sum of an infinite series. Adding a context, as Rao (1975) claimed is essential for any application of statistics, provides a meeting place for probability and statistics in the school curriculum. Later in the curriculum, probability provides the basis for the level of confidence one has in making an inference in statistics. Here it provides the link from a genuine context for a model that answers a problem in that context.

The Chinese government's one-child policy is well-known and was implemented to reduce a burgeoning population that was 1.1 billion when Kristof (1990) wrote about it. At that time Kristof reported that many couples were happy to accept the policy as it enhanced their economic prospects. This would have been more likely to be the case if the first child had been a boy. As well, however, there was a view of rural peasants that the policy should not be implemented until the first son was born. This would have amounted to a one-son policy. One of the consequences of the one-child policy in recent years has been an over-supply of men who now are often having difficulties finding wives (e.g., Brooks, 2013). China's attempt to alleviate this and other difficulties resulted in a policy change in 2013 to allow a couple to have two children if one of the parents is an only child (Burkitt, 2014).

Changing from a one-child policy to a one-son policy leads to the hypotheses that there may be a population explosion and that the balance of boys and girls would change to favour girls rather than boys. Konold (1994) expressed these hypotheses as questions based on the mathematics involved:

1. What would the average number of children be in a family?
2. What would be the ratio of births of girls to births of boys? (p. 232)

To address the problem with a probability model requires the assumption that the gender of a baby is a random phenomenon and that only single births are considered. This is the type of assumption that is made when students use two coins to model the likelihood of having two boys, two girls, or one of each in a two-child family. The situation here is that a single coin would be tossed until the side assigned to boy occurred. The number of tosses to reach a boy would be recorded as the number of children in the family.

The statistical investigation

Rao (1975) would now be happy that there is a context for a statistical investigation to take place. The four steps of an investigation advocated by the GAISE report of the American Statistical Association (Franklin et al., 2007) are the basis for the investigation, as shown in Figure 1.

GAISE Framework
I. Formulate Questions → clarify the problem at hand → formulate one (or more) questions that can be answered with data → anticipate variability
II. Collect Data → design a plan to collect appropriate data → employ the plan to collect the data → designing for variability
III. Analyse Data → select appropriate graphical and numerical methods → use these methods to analyze the data → accounting for variability in distributions
IV. Interpret Results → interpret the analysis → relate the interpretation to the original question → allowing for variability beyond the data

Figure 1. The GAISE Framework of the statistical investigative process (Franklin et al., 2007, p. 15–16).

1. Formulate questions

This has been done, with a model that anticipates the variability inherent in the random birth of a boy or girl for each addition to the family and the consequent variability in family size.

2. Collect data

The design to collect data uses the Sampler in *TinkerPlots* to simulate births based on the binomial model with $p = \frac{1}{2}$. The pseudo-random design of the Sampler provides the variation required for answering each of the questions. Figure 2 shows the Sampler set up with labels “B” and “G” in the Mixer (deleting other values), with Draw set at 1, and with the Attribute labelled “Child”. The Replacement Option should be set at “With Replacement”; this can be checked using the small arrow below the Mixer, as shown in Figure 3.

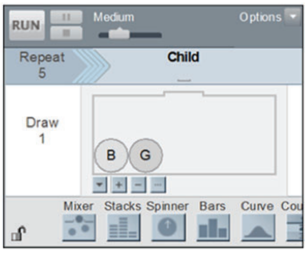


Figure 2. Setting up the Mixer in the Sampler.

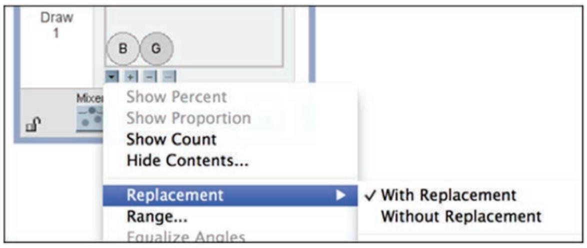
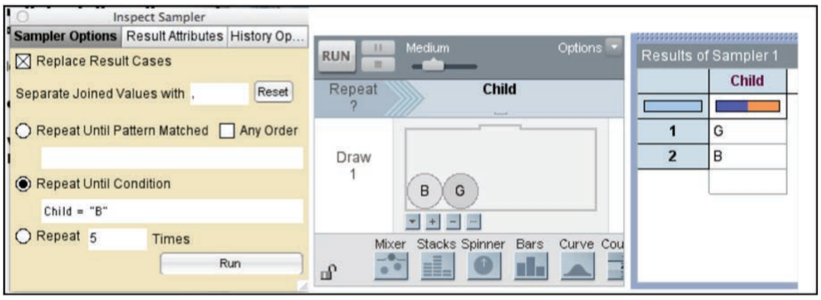


Figure 3. Making sure “B” and “G” are replaced after each “birth”.

Under Sampler Options in the Options menu (upper right corner of the Sampler), the “Repeat Until Condition” is chosen and Child = “B” is inserted in the box below, as seen in Figure 4. Then Run is clicked once to produce a Result as seen on the right of the figure. [The first family has two children with the boy born second.]



☒ Replace Result Cases

● Repeat Until Condition

Child = "B"

○ Repeat 5 Times

	Child
1	G
2	B

Figure 4. Setting the condition to draw from the Mixer until a “B” is drawn and recording each draw in the Results of Sampler 1 Table.

To keep track of the families and their sizes, first a Plot is dragged down for the Results of Sampler 1 Table. This displays the number of icons reflecting the number of births to get a boy. Clicking on N for the Plot (in the tool bar along the top of the window) shows the family size and clicking on Child in the Results of Sampler 1 Table colours the icons by gender (see Figure 5). To keep track of repeated families simulated, the History tool is used from the menu below the Plot. When it is clicked a grey box appears around the number displayed. Double clicking on the box creates a History of Results of Sampler 1 Table recording the count from the Plot (see Figure 6).

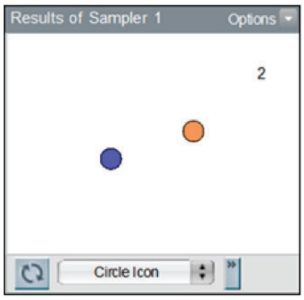
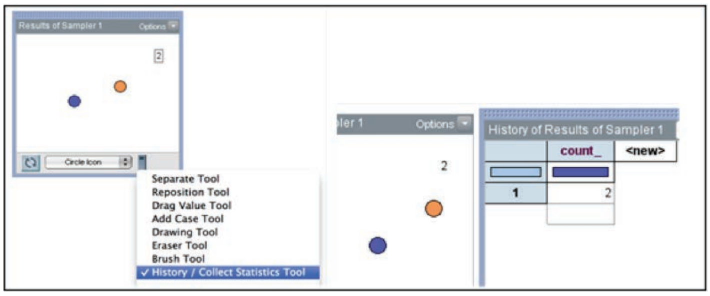


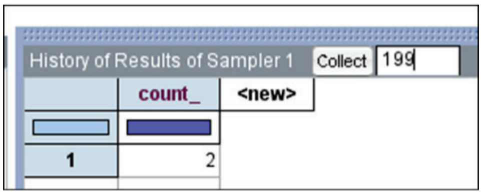
Figure 5. The plot of the first family with N counting the number of children.



	count_	<new>
1	2	

Figure 6. Setting up History to count the children in repeated families.

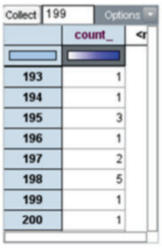
The Collect field in the History of Results of Sampler 1 Table can be changed to conduct more than one trial (creating more families) at one time. This is shown in Figure 7 to complete a total of 200 families and Figure 8 shows the number of children in the final families simulated. The data are now collected!



Collect 199

	count_	<new>
1	2	

Figure 7. Setting the Sampler to collect 199 more samples.



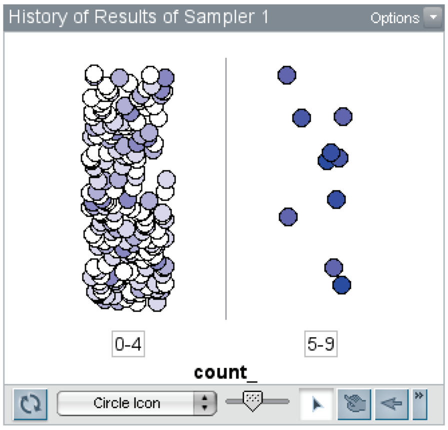
Collect 199

	count_	<r
193	1	
194	1	
195	3	
196	1	
197	2	
198	5	
199	1	
200	1	

Figure 8. The record of the 200 samples (sizes of families).

3. Analyse data

There are several ways to account for the variability in the data. To see how many families in the sample of 200 have each number of children, a Plot is used, dragging the **count_** attribute from the History of Results of Sampler 1 to the horizontal axis of the Plot. The data appear in two bins as seen in Figure 9, and dragging an icon to the right until each value is in a separate bin and clicking on the N tool shows how many families are in each bin as in Figure 10. The binomial distribution would predict a reduction of half in each bin to the right but there is variation resulting from the simulation process.



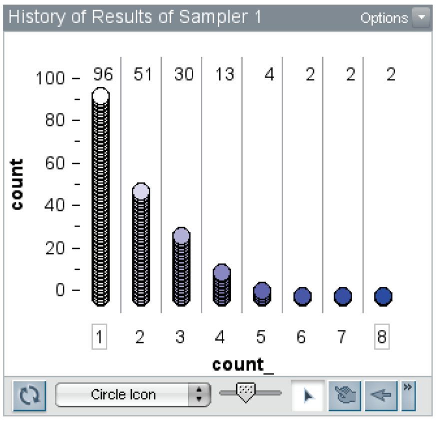


Figure 9. Plot showing **count_** data in two bins.

Figure 10. Plot with data separated into individual family-sized bins.

To find the average family size and the ratio of boys to girls, an icon is dragged to the right until the data are fully separated on the axis and the Average option is chosen from the tool bar (along the top of the window) as well as "Show Numerical Value". As seen in Figure 11, the average family size is very close to two and because every family has exactly one boy the average number of girls is 1.01 and the ratio of boys to girls is very close to 1:1.

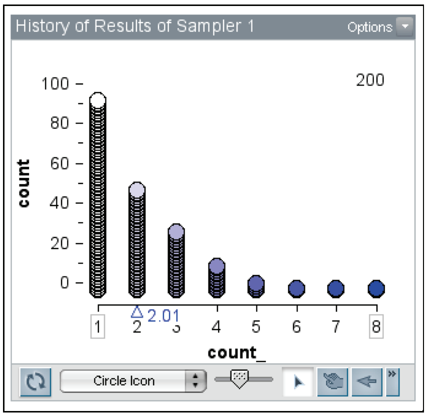


Figure 11. Plot with separated data showing the mean family size.

4. Interpret results

In the context of the problem (China's one-child policy) a change from a strict one-child to a strict one-son policy would double the number of births but there would not be a population explosion and the balance of boys and girls would return. Another, simulation of 200 families, would produce a slightly different distribution and value for the mean as seen in Figure 12. The confidence in the result is high but it would be possible to do 10 000 trials for an even greater degree of confidence.

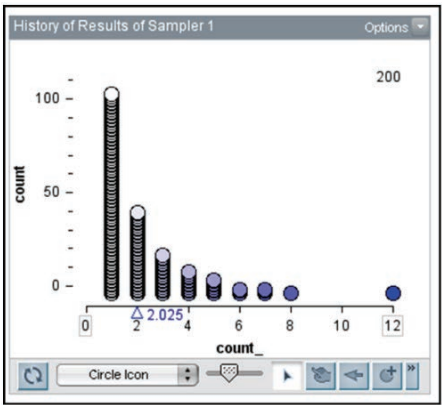


Figure 12. Another sample of 200 simulated families.

Five big ideas

The five big ideas for Statistics from the AAMT *Top Drawer Teachers* site (http://topdrawer.aamt.edu.au/Statistics/Big-ideas) are shown in Figure 13: Variation, Expectation, Distribution, Randomness and Informal Inference. They are described in detail on the Top Drawer site but Figure 13 is amplified to show how the one-son problem illustrates each of them. As demonstrated in the GAISE framework (Figure 1), variation is the cornerstone of every statistical investigation. Without variation there would be no statistics (Moore, 1990). Expectation arises from the question being investigated and the purpose of the investigation is to confirm it or otherwise. Here, two expectations were being explored within the variation of the data collected, related to family size and the balance of boys and girls. Distribution is the lens through which expectations are confirmed, here plotting the data to display the variation and find the average, and hence the ratio of boys to girls. Randomness provided the simulation of the births of boys and girls (draws from the Mixer) and hence the family sizes. Randomness and distribution provide the evidence to support the informal inference that answers the questions, in this problem with a high degree of confidence.

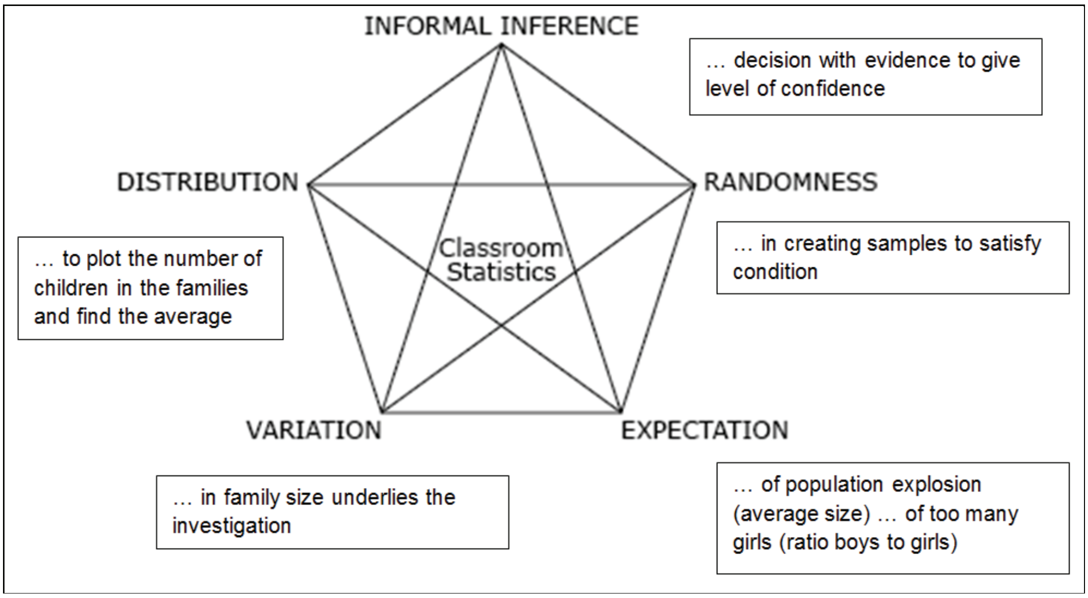


Figure 13. The big ideas of statistics underpinning the one-son problem.

Affordances of TinkerPlots

Much of the literature on the benefits of software applications in mathematics and statistics focuses on what the software can *do* and certainly as shown in this paper, there are many exciting and powerful things that *TinkerPlots* can do. Watson and Fitzallen (in press), however, review the affordances of *TinkerPlots* for *learning*. Among these affordances is the ability to analyse distributions in different and creative ways. Figure 14 shows the same data set displayed in bins and with a divider. Rubin and Hammerman (2006) found that the top configuration allowed teachers to discuss particular sections of the data and be aware of what their students would do. Using the bottom separated plot with a divider helped consider whole distributions and compare them meaningfully.

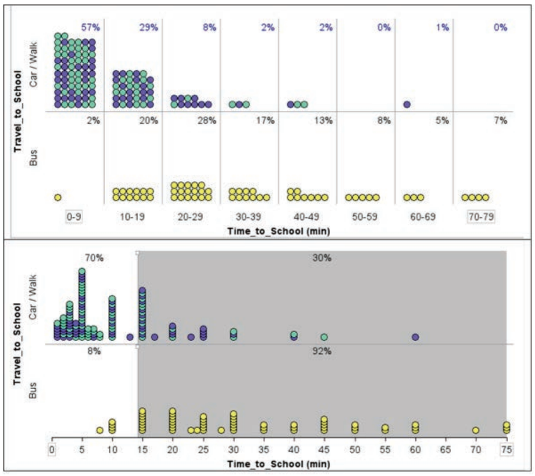
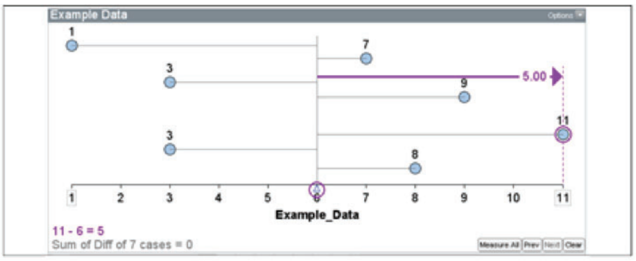


Figure 14. Using bins and a divider to analyse data in *TinkerPlots*.

Figure 15 uses the Ruler tool demonstrating that the sum of distances from data values to their mean is zero (Hudson, 2012). This is seen by imagining to balance the values on either side of six. Hudson also used the Drag Value Tool to move a data value to show the effect on the value of the mean.



11 - 6 = 5
Sum of Diff of 7 cases = 0

Figure 15. Using the Ruler to demonstrate the properties of the mean in *TinkerPlots*.

For the one-son problem not only are there several affordances related to visualising the process of creating the model for the simulation, but also another affordance is the ability to set up the Plot for the **count_** attribute and the mean at the beginning of the collection of the 200 (or more) samples. Using the History tool again, on the Plot of History of Results for Sampler 1, allows for the mean of **count_** to be calculated as each trial is added. Using the Formula option to define a new attribute **case** with the

Special Function caseIndex, it is then possible to plot the History of the mean as it approaches 2. This is seen for the end of another set of 200 trials in Figure 16. This enables students to *watch* the distribution of family sizes grow and the mean change over time and approach 2.0. Students not only enjoy this (and squeal when a large family occurs) but also observe how a distribution actually grows and changes as the samples accumulate. This is an important learning benefit of the software.

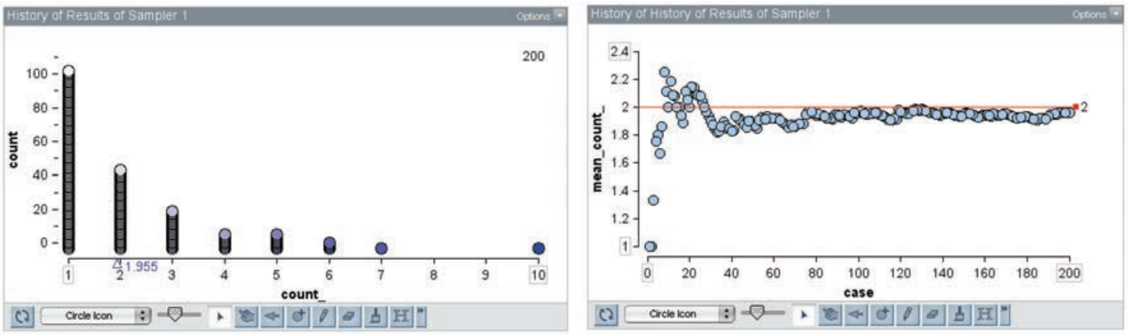


Figure 16. The final plots for an animation of the one-son simulation.

Conclusion

Without computers and software it is of course possible to simulate the one-son problem in the classroom. Each student is given a coin to toss (with Head for boy) and counts the number of tosses until a boy occurs. The teacher (or another student) is at the whiteboard with an axis drawn recording the number of children in families as they are shouted out by the students. This can be quite exciting when a very large family occurs. As described here using *TinkerPlots* this activity has been used as part of an extended unit for Year 10 leading to resampling (Watson, 2014) and as part of professional learning in the Reframing Mathematical Futures II project (Siemon, 2014).

Konold (1994) based his analysis of the one-son problem on an earlier probability simulation software before *TinkerPlots*, which could not keep track of repeated samples as the History tool does in *TinkerPlots*. This feature of the recent software (Konold & Miller, 2011) is an affordance for both doing statistics and learning how repeated sampling works to help reach decisions. One-son is a wonderful real-world problem, linking probability and statistics, illustrating the Big Ideas underpinning statistical investigations, and exhibiting the learning affordances of *TinkerPlots*. Thank you Cliff Konold!

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RICH TASKS TO BUILD STATISTICAL REASONING

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One of the foci of the Reframing Mathematical Futures II project (RMF2) is the development of statistical reasoning. The aim is to develop and trial rich tasks of an intermediate length at the middle school level that will assist teachers to make formative judgements on students' progress. Recognising the importance of all stages of a statistical investigation, as well as the big ideas underlying them, suggestions are made for an innovative approach asking students to reason statistically and provide arguments in support of their responses.

Introduction

Among the aims of the Reframing Mathematical Futures II project (RMF2, Siemon, 2014) is the development of classroom assessment tools that can be used to identify student learning needs in relation to statistical reasoning in Years 7 to 10. Although acknowledging the existence of items for surveys completed in testing contexts, and suggestions for extended statistical investigations where student progress is followed through written responses in workbooks, the approach taken in this paper is to suggest a middle ground incorporating a flexibility that hopefully can be adapted for classroom needs. These could then be incorporated in task-based professional learning modules aimed at deepening teachers' understanding. The ultimate aim is to provide rubrics for the multi-step tasks that allow the placement of responses on a hierarchical learning trajectory that can assist teachers in providing activities to move students to higher levels.

In this paper two multi-part tasks are presented with rubrics that have been trialled with students and levels of response are suggested in relation to previous research of the authors (Callingham & Watson, 2005; Watson & Callingham, 2003). As well, four new items are suggested for consideration by teachers and trialling in high school (Year 7 to Year 10) classrooms.

Statistical background

To ensure that tasks cover the concepts underlying statistical investigations and the steps involved in a complete statistical investigation, we first introduce five big ideas of statistics. These ideas are those that students in school need to develop. These ideas can be developed through conducting complete statistical investigations, and the steps in an investigation are described.

In the AAMT's *Top Drawer Teachers* website (http://topdrawer.aamt.edu.au), each drawer provides the big ideas that underpin the mathematics covered. These are the concepts that provide the foundation for the presentation of descriptions and activities associated with Misunderstandings, Good Teaching, and Assessment. For the Statistics drawer, there were five big ideas, as reproduced in Figure 1. Details from *Top Drawer Teachers* are presented in Appendix 1. As seen in the centre of the figure these ideas are intended to be the basis of classroom statistics. Planning of classroom activities, however, does not systematically trace a pathway around the corners of the pentagon in the figure because, as indicated, they are all interrelated. Classroom activities using the big ideas illustrate the practice of statistics, which is a step by step procedure for finding a decision for a statistical question posed in a context.

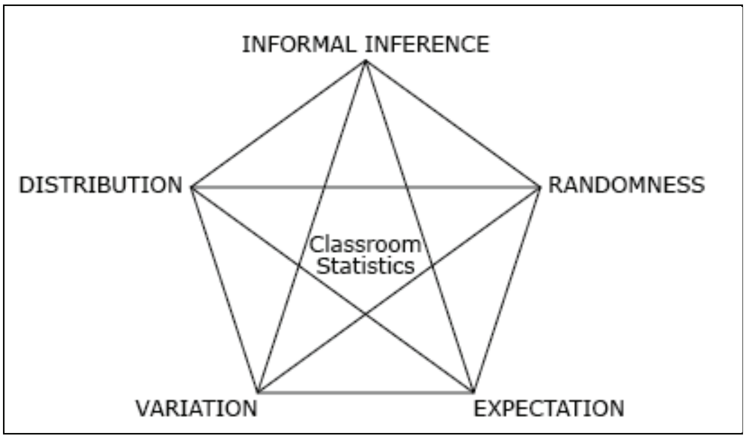


Figure 1. Interrelated big ideas underlying statistics (Top Drawer Teachers: http://topdrawer.aamt.edu.au/Statistics/Big-ideas).

Several models have been suggested for carrying out a statistical investigation. The PPDAC model of Wild and Pfannkuch (1999) has five steps: Problem, Plan, Data, Analysis, and Conclusion, with the possibility of cycling through the steps repeatedly. Watson (2009), building on the work of Holmes (1980), suggested a visual model for students including the importance of context for the statistical question posed and explicit links to variation for every ordered step: collect data, represent data, reduce data, level of certainty, informal inference. The *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2013) does not offer a specific step by step model for carrying out a statistical investigation. The US GAISE report (Franklin et al., 2007) is explicit in suggesting a four-stage model that reflects the work of Friel and Bright (1998) that can be represented as a poster in the classroom as shown in Figure 2.

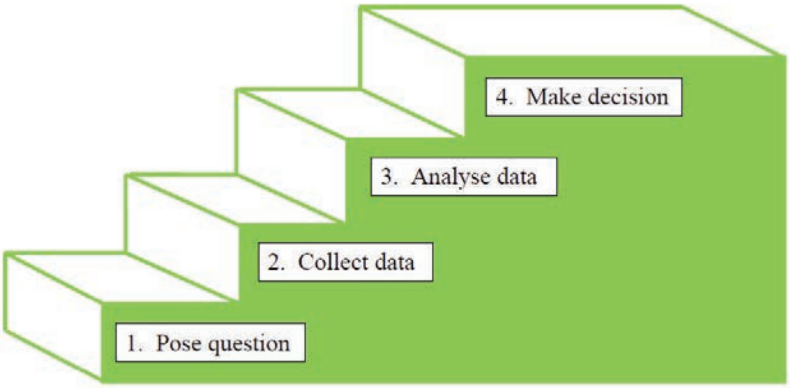


Figure 2. Four steps to making decisions with data.

It is important to appreciate the relationship of the big ideas to the stages of an investigation. Variation and expectation underpin all stages of a statistical investigation. If it were not for variation, questions would not be worth posing, but equally a question is sensible only if there is an expectation of some answer/s existing in the variation. A hypothesis or question is a way of expressing an expectation to be tested. Data collection is planned to allow for both variation and expectation and can at times be very complex. The big idea of randomness underlies the highest quality of data collection and influences the confidence with which a decision can be made at the end of the investigation. The purpose of analysing data is then to find or isolate the expectation within the variation. Here is where the big idea of distribution comes to the fore, as it provides a method of organising the variation in a way that others can understand the message in the data, and the evidence to support (or otherwise) the existence of the expectation. Distributions are usually displayed in some graphical form but they may also appear as tables, perhaps showing the association of two categorical variables. The purpose of a statistical investigation is to make a decision in relation to the expectation expressed in the question posed. At the school level the decision is likely to be an informal inference, recognising that at higher levels theoretical models allow the decisions to be formal inferences. All inferences, however, have some level of confidence related to the decision being made. The methods of data collection and analysis contribute to the confidence in the decision and evidence from these stages of the investigation are presented in support of the decision.

Based on the components of these big ideas for understanding statistics the framework for statistical investigations, the authors suggested a statistical literacy hierarchy (Callingham & Watson, 2005; Watson & Callingham, 2003) that can form the starting point for a proposed learning trajectory for statistics in the RMF2 project. A qualitative description of the six levels of the construct is given in Table 1. This hierarchy is used to map the suggested levels associated with the rubrics given for the tasks to follow, and to define the standard of the learning outcomes in statistics for students.

Table 1. Statistical literacy (SL) hierarchy (Watson & Callingham, 2003, p. 14).

Level	**Brief characterisation of step levels of tasks**
6. Critical Mathematical	Task-steps at this level demand critical, questioning engagement with context, using proportional reasoning particularly in media or chance contexts, showing appreciation of the need for uncertainty in making predictions, and interpreting subtle aspects of language.
5. Critical	Task-steps require critical, questioning engagement in familiar and unfamiliar contexts that do not involve proportional reasoning, but which do involve appropriate use of terminology, qualitative interpretation of chance, and appreciation of variation.
4. Consistent Non-critical	Task-steps require appropriate but non-critical engagement with context, multiple aspects of terminology usage, appreciation of variation in chance settings only, and statistical skills associated with the mean, simple probabilities, and graph characteristics.
3. Inconsistent	Task-steps at this level, often in supportive formats, expect selective engagement with context, appropriate recognition of conclusions but without justification, and qualitative rather than quantitative use of statistical ideas.
2. Informal	Task-steps require only colloquial or informal engagement with context often reflecting intuitive non-statistical beliefs, single elements of complex terminology and settings, and basic one-step straightforward table, graph, and chance calculations.
1. Idiosyncratic	Task-steps at this level suggest idiosyncratic engagement with context, tautological use of terminology, and basic mathematical skills associated with one-to-one counting and reading cell values in tables.

Possible tasks

The tasks that are suggested here are intended to be flexible in terms of problem context and number of parts, to fit the classroom situation (year level, time available, method of administration). When statistical literacy (SL) levels are given they refer to Table 1.

Tasks used previously

Task set 1: Coins

This set of related tasks is built on the big ideas of variation, expectation, and randomness.

(a) Imagine you are playing a game where you throw a coin 4 times. How many tails do you think might come up?

Code	**Description**	**SL Level**
2	2 tails or 50%	**3**
1	Any other number , “you don’t know, could be any of them”	**1**
0	No response	

(b) Explain why?

Code	Description	Examples	SL Level
3	2 but also recognising variations and / or attempts to quantify the highest likelihood	“2, because that has a 37.5% chance, but the others could happen, they are just less chance.” “Most times a tail should normally appear at least once in those four throws but there are a whole lot of possibilities as shown above” (student lists possibilities)	**6**
2	2 because there is a 50% chance, 50–50 of throwing a head or a tail, probability of a tail 1 in 2.	“2 tails because a coin has only 2 sides so we can only assume that the results will be 50–50” “2. Because the probability of a tails being shown is ½ .”	**4**
	Recognises variation by stating “you can’t really tell how many tails might come up”	“you can’t tell because you can’t control it, it falls the way it wants to fall” “you won’t know”	
1	Idiosyncratic reasoning or possible misinterpretation	“Because 8 possibilities and 4 successes” “Well I’m not sure there is a 20% chance of all 5 combinations”	**1**
0	No response, no explanation	“I don’t know”	

(c) Imagine you are playing a game where you throw a coin 4 times. Imagine that 100 people played the game. In the table below, fill in how many people you think will get each number of tails.

Number of tails	Number of people getting the number of tails
0	
1	
2	
3	
4	
Total	**100**

Code	Description	Examples	SL Level
3	Appropriate variability displayed incorporating probability and distribution	Does not give a probabilistic example like Code 2 All predictions made within the desired ranges below Prediction for 2 tails=30–45; Prediction for 1 and 3 tails=18–33; Prediction for 0 and 4 tails=3–10 “10, 20, 35, 25, 10”	6
2	Too narrow or no variation – extreme probabilistic outcome	Prediction for 2 tails = 37 or 38 (37.5) Prediction for 1 and 3 tails = 25 Prediction for 0 and 4 tails = 6 or 7 (6.25)	5
	Primitive understanding of proportion – 50% chance for 2 tails	“0, 25, 50, 25, 0” or “5, 20, 50, 20, 5” “10, 20, 40, 20 , 10” or “6, 10, 60, 20, 4” or “15, 15, 50, 10, 10”	
1	Assumes equality for all options	“20, 20, 20, 20, 20, 20”	3
	Seemingly random prediction Gets proportion in the wrong spot	“10, 30, 40, 1, 19” “0, 25, 25, 25, 25” “5, 40, 30, 20, 5” “1, 3, 89, 5, 2”	
0	Does not add to 100 – Possible misinterpretation	“10, 10, 10, 10, 10” “0, 0, 100, 0, 0”	
	No response		

(d) Explain why you think these numbers are reasonable.

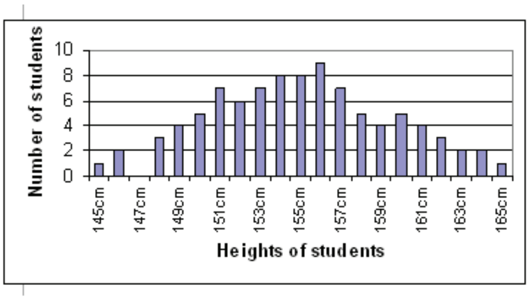
Code	Description	Examples	SL Level
4	Reasoning reflecting aspects of chance and probability (some kind of variation)	“Because out of 100 people not very many are going to get 0 tails or 4 out of 4 tails, it is most likely they’ll get 2 tails and 2 heads because there’s 2 different faces and 4 chances”	6
3	Implicit understanding of chance and probability, sometimes mentioning 50% or 1/2 chance (or answer reflecting a ‘4’ but not as clear)	“I think it would most likely be even because there’s a 50% chance it will come up” “They are reasonable because it is most likely that more people will get 2 out of 4”	5
2	Reasoning reflecting an even or equal chance for all numbers	“Because they all have equal opportunity”	4
	Anything can happen, chance and luck	“It’s random so no one knows what will come up”	
1	Idiosyncratic reasoning and personal beliefs	“Because tails never fails” “because it adds up to 100”	3
0	No reason		

Task set 2: Heights

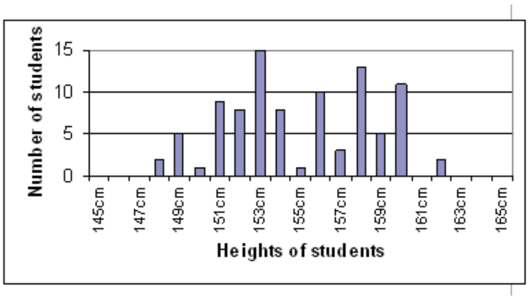
This set of related tasks is built on the big ideas of variation, expectation, and distribution.

The following graphs describe some data collected about Grade 7 students' heights in two different schools.

School A



School B



(a) How many students are 156 cm tall in each school?

School A ______________________ School B _______________________

Code	Description	SL Level
1	Reads correct values from graph, A=9 and B=10	**1**
0	Any other value/s or written responses	
	No response	

(b) Which graph shows more variability in students' heights?

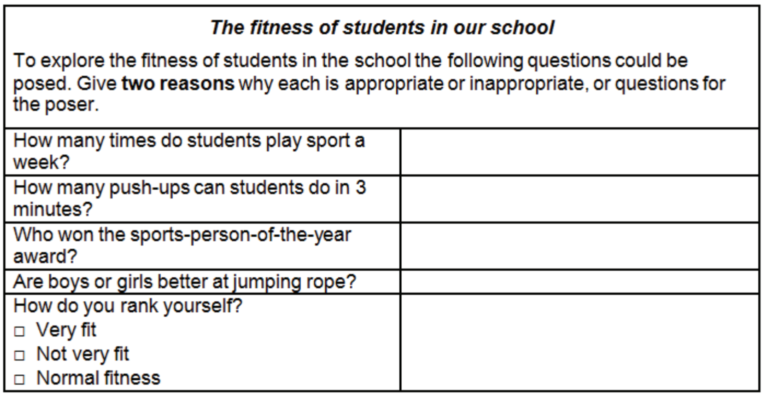
Code	Description	SL Level
2	School A	**3**
1	School B	**1**
	The Same	
0	Any other written response	
	No response	

(c) Explain why you think this.

Code	Description	Examples	SL Level
4	Mentions explicitly the wide range/spread and/or variety of arm spans	"b/c one of the students has an armspan of 165 cm and another has one of 145 cm and school B just has a range b/n 149 cm and 162 cm"	**Top 5**
3	Mentions implicitly the wide range/difference of arm spans	"School A takes up the whole graph and B doesn't"	**4**
2	Focuses on the size of the individual bars without regard to what they represent	"B – because it goes up and down and varies more" "B – they are nearly all different"	**4**
	Focuses on the number of individual bars without regard to what they represent (need to state some feature of the graph)	"A, because the graph in School A shows more lines" A – has more lengths, more numbers. "There are more numbers of arm spans to read"	
1	Misapplies notion of variability and focuses on an average height	"A, a lot of people are around about the same arm span in that school" B – it goes up higher	**Low 4**
0	Focuses on the content of data	"A, more people are bigger"	
	Aesthetic appearance and personal preference and graph lay out	"A, easier to see which is taller" "A, it has more detail" "School B goes up in 5's"	
	Misreads data or misinterprets question, unjustified statements	"There are more people in school A" "A – because it looked like they interviewed more people"	
	No reason		

Suggested new tasks

The next four task-sets are suggested specifically to address the four stages of a statistical investigation in Figure 2. As shown in Figures 3 to 6, students do not create their own answers but critique the answers of others. Students could also be asked to supply their own response first before responding to the others. This type of task requires students to provide arguments either for or against what is provided in the cells on the left. There could be more than one appropriate response in the list or there could be none. In the latter case students could be asked to provide their own answer at the bottom of the sheet. There could also be responses that have some positive and some negative attributes. The tasks can be as easy or difficult as required. The motivation for this type of task came from Jacobs (1999) who provided an example similar to Step 2 (Figure 4) in what follows. Given examples like the following as starting points, teachers in a school at a particular grade level can work together on adaptations and the development of rubrics. Possible rubrics and hypothesised levels of the statistical literacy hierarchy are provided in Appendix 2 for Step 2 as models for the other three steps, although these levels have not, as yet, been verified and are open to interpretation.

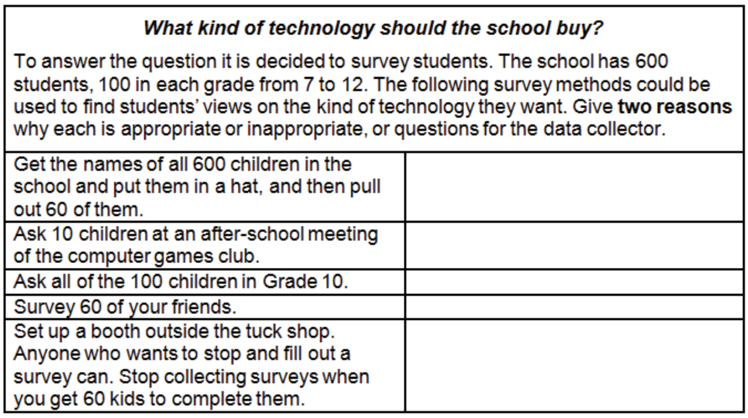


The fitness of students in our school

To explore the fitness of students in the school the following questions could be posed. Give **two reasons** why each is appropriate or inappropriate, or questions for the poser.

How many times do students play sport a week?	
How many push-ups can students do in 3 minutes?	
Who won the sports-person-of-the-year award?	
Are boys or girls better at jumping rope?	
How do you rank yourself? ☐ Very fit ☐ Not very fit ☐ Normal fitness	

Figure 3. Step 1 in a statistical investigation: Once the context for the investigation has been set, the *question is posed*.

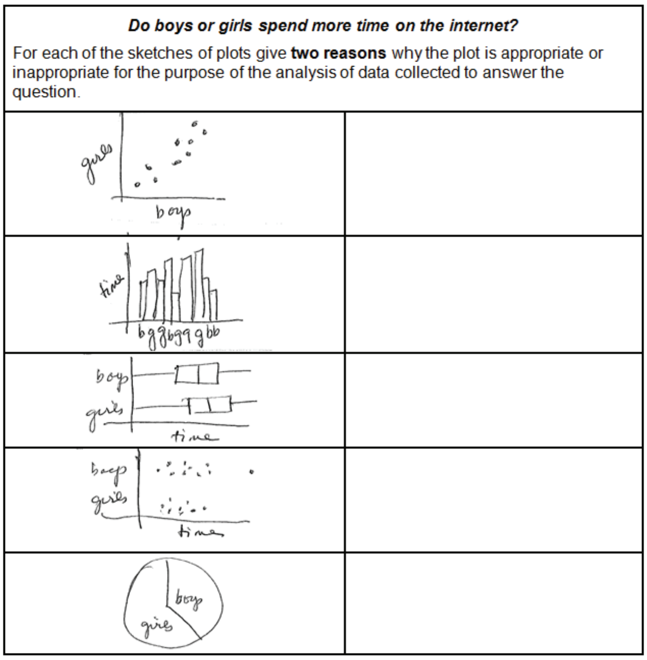


What kind of technology should the school buy?

To answer the question it is decided to survey students. The school has 600 students, 100 in each grade from 7 to 12. The following survey methods could be used to find students' views on the kind of technology they want. Give **two reasons** why each is appropriate or inappropriate, or questions for the data collector.

Get the names of all 600 children in the school and put them in a hat, and then pull out 60 of them.	
Ask 10 children at an after-school meeting of the computer games club.	
Ask all of the 100 children in Grade 10.	
Survey 60 of your friends.	
Set up a booth outside the tuck shop. Anyone who wants to stop and fill out a survey can. Stop collecting surveys when you get 60 kids to complete them.	

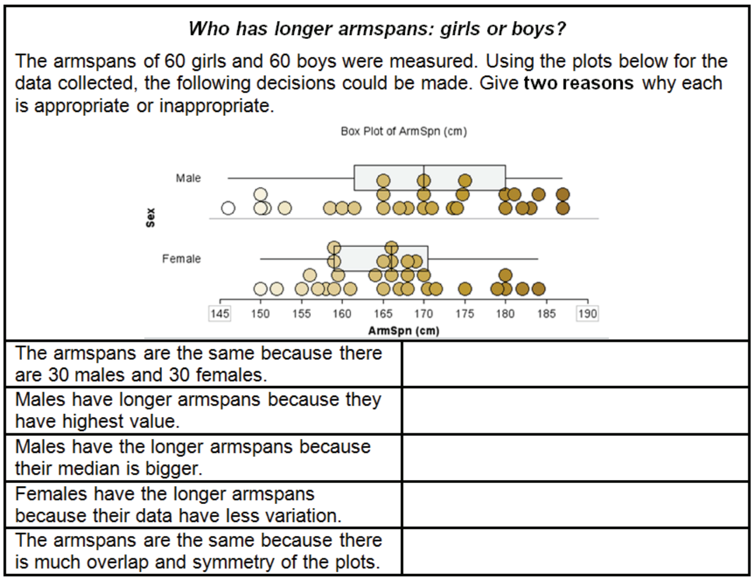
Figure 4. Step 2 in a statistical investigation: Once the context has been set and the question posed, *data are collected*.



Do boys or girls spend more time on the internet?

For each of the sketches of plots give **two reasons** why the plot is appropriate or inappropriate for the purpose of the analysis of data collected to answer the question.

Figure 5. Step 3 in a statistical investigation: Once the data have been collected to answer the question, the *data are analysed* to find evidence to answer the question.



Who has longer armspans: girls or boys?

The armspans of 60 girls and 60 boys were measured. Using the plots below for the data collected, the following decisions could be made. Give **two reasons** why each is appropriate or inappropriate.

The armspans are the same because there are 30 males and 30 females.	
Males have longer armspans because they have highest value.	
Males have the longer armspans because their median is bigger.	
Females have the longer armspans because their data have less variation.	
The armspans are the same because there is much overlap and symmetry of the plots.	

Figure 6. Step 4 in a statistical investigation: Once the representation of the evidence from the data is presented a *decision is made* on the question posed.

Conclusion

The suggestions for rich tasks for building statistical reasoning have been linked to the six levels of the Statistical Literacy hierarchy of Watson and Callingham (2003). The Scaffolding Numeracy of Siemon, Izard, Breed, and Virgona (2006) has eight levels for multiplicative reasoning. Further research is needed to link the two hierarchies but this paper presents some tasks that might be used as a starting point.

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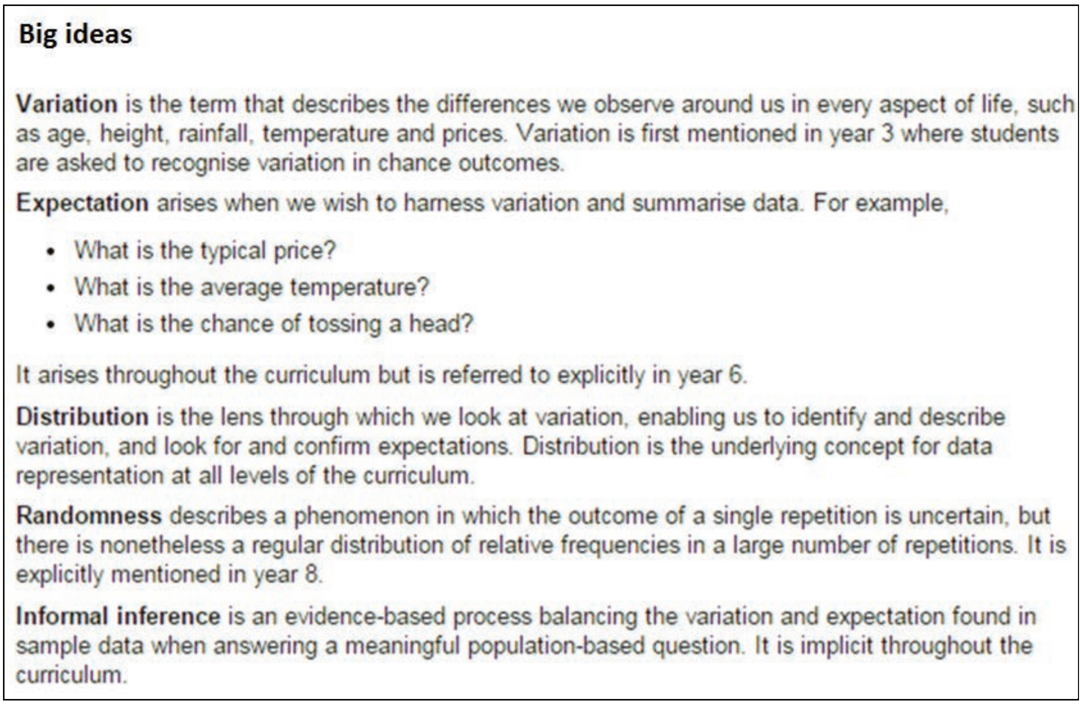
Appendix 1



Big ideas

Variation is the term that describes the differences we observe around us in every aspect of life, such as age, height, rainfall, temperature and prices. Variation is first mentioned in year 3 where students are asked to recognise variation in chance outcomes.

Expectation arises when we wish to harness variation and summarise data. For example,

- What is the typical price?
- What is the average temperature?
- What is the chance of tossing a head?

It arises throughout the curriculum but is referred to explicitly in year 6.

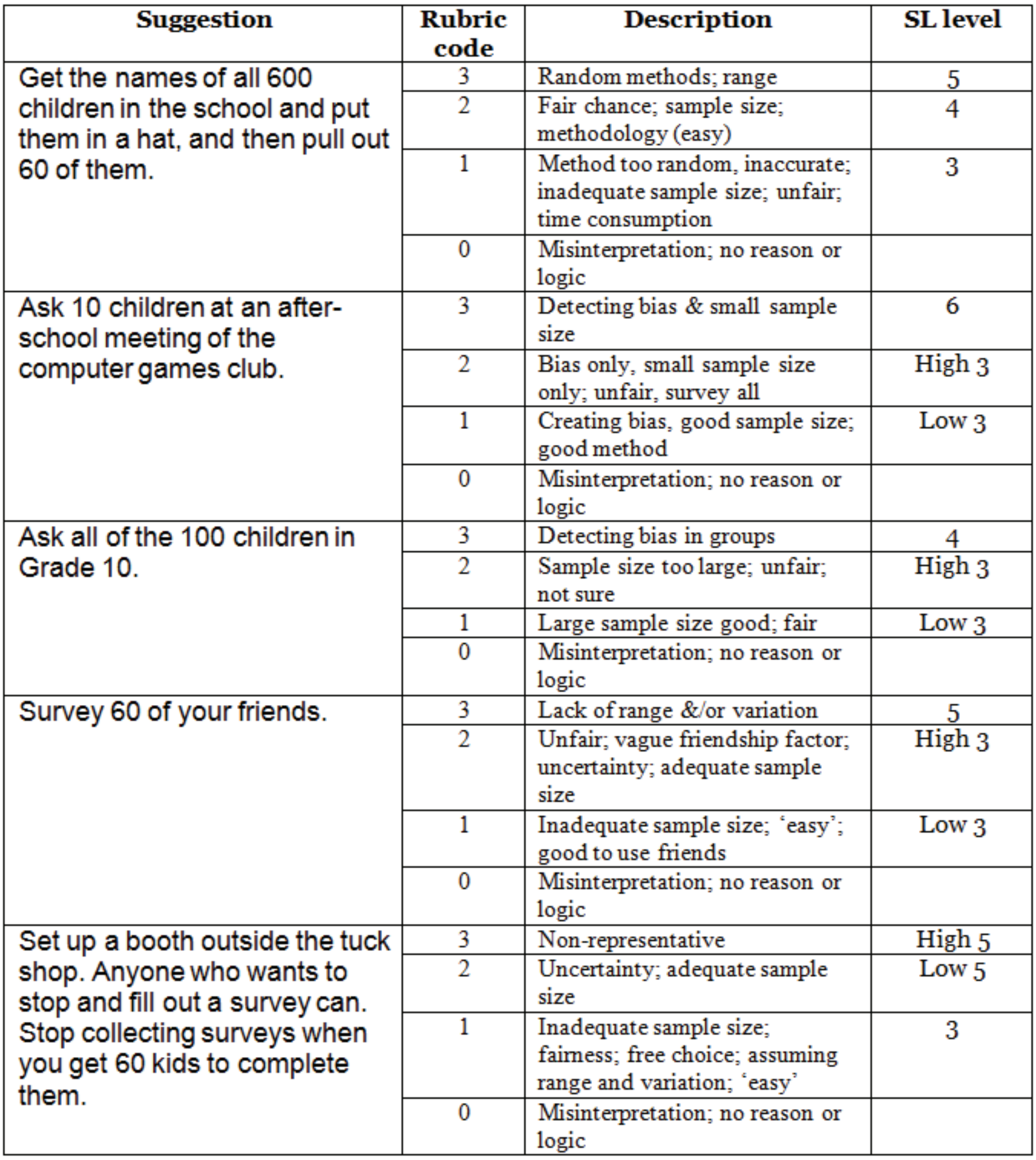
Distribution is the lens through which we look at variation, enabling us to identify and describe variation, and look for and confirm expectations. Distribution is the underlying concept for data representation at all levels of the curriculum.

Randomness describes a phenomenon in which the outcome of a single repetition is uncertain, but there is nonetheless a regular distribution of relative frequencies in a large number of repetitions. It is explicitly mentioned in year 8.

Informal inference is an evidence-based process balancing the variation and expectation found in sample data when answering a meaningful population-based question. It is implicit throughout the curriculum.

Appendix 2

Possible rubrics and statistical literacy hierarchy levels for tasks at Step 2 in a statistical investigation: Data collection(Figure 4).



Suggestion	Rubric code	Description	SL level
Get the names of all 600 children in the school and put them in a hat, and then pull out 60 of them.	3	Random methods; range	5
	2	Fair chance; sample size; methodology (easy)	4
	1	Method too random, inaccurate; inadequate sample size; unfair; time consumption	3
	0	Misinterpretation; no reason or logic	
Ask 10 children at an after-school meeting of the computer games club.	3	Detecting bias & small sample size	6
	2	Bias only, small sample size only; unfair, survey all	High 3
	1	Creating bias, good sample size; good method	Low 3
	0	Misinterpretation; no reason or logic	
Ask all of the 100 children in Grade 10.	3	Detecting bias in groups	4
	2	Sample size too large; unfair; not sure	High 3
	1	Large sample size good; fair	Low 3
	0	Misinterpretation; no reason or logic	
Survey 60 of your friends.	3	Lack of range &/or variation	5
	2	Unfair; vague friendship factor; uncertainty; adequate sample size	High 3
	1	Inadequate sample size; 'easy'; good to use friends	Low 3
	0	Misinterpretation; no reason or logic	
Set up a booth outside the tuck shop. Anyone who wants to stop and fill out a survey can. Stop collecting surveys when you get 60 kids to complete them.	3	Non-representative	High 5
	2	Uncertainty; adequate sample size	Low 5
	1	Inadequate sample size; fairness; free choice; assuming range and variation; 'easy'	3
	0	Misinterpretation; no reason or logic	