Mathematics: Traditions and (New) Practices

Edited by Julie Clark, Barry Kissane, Judith Mousley, Toby Spencer & Steve Thornton

Proceedings of the AAMT–MERGA conference

held in Alice Springs, 3–7 July 2011,
incorporating the 23rd Biennial Conference of
The Australian Association of Mathematics Teachers Inc.
and the 34th Annual Conference of the
Mathematics Education Research Group of Australasia Inc.

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PREFACE

This is a record of the proceedings of a joint conference incorporating the 23rd biennial conference of the Australian Association of Mathematics Teachers (AAMT) together with the 34th annual conference of the Mathematics Education Research Group of Australasia (MERGA). This is the first fully joint conference of the two associations in the history of mathematics education in Australasia. It brings together practitioners and researchers to discuss key issues and themes in mathematics education, so that all can benefit from the knowledge gained through rigorous research and the wisdom of practice.

The conference venue is Alice Springs in central Australia, a place rich in history and culture. This is reflected in the theme of the conference: *Mathematics: Traditions and (new) practices*. This theme highlights the importance of respecting traditional knowledge, including that of the first peoples of Australasia, and of forging new practices that promote rich and relevant mathematical experiences for students of all ages.

We are pleased to welcome any conference participants who are attending a MERGA or AAMT conference for the first time. We hope you will make yourselves known so you can be made welcome and introduced to others who share your research and teaching interests. Authors and presenters from many countries (e.g., Australia, New Zealand, Singapore, the United States of America, Papua New Guinea, the United Kingdom) are represented in these proceedings. There are participants from almost every university in Australia and New Zealand, teachers from government and non-government school systems throughout Australia, and officers from government Ministries of Education. Our collective interests span the entire range from pre-school to postgraduate teaching of mathematics, mathematics teacher education and research in mathematics education. We particularly welcome local teachers from the central regions of Australia, and thank you for your generous hospitality. We look forward to the dialogue that will emerge from the varying perspectives brought by these participants.

Reviewing procedures

All papers in these proceedings were submitted as either *research papers* or *professional papers*. Research papers were reviewed according to established MERGA processes, while professional papers were reviewed according to established AAMT processes. These two sets of papers are published as separate sections of the proceedings. They are preceded by a section that includes invited keynote papers, including the winner of the Beth Southwell Practical Implications Award, which was reviewed according to MERGA processes as part of its eligibility for the award.

MERGA reviewing process

All research papers submitted were blind peer-reviewed (without author(s) being identified) by one of ten review panels comprising mathematics education researchers with appropriate expertise in the field. Review panels were convened throughout Australia and New Zealand by experienced researchers who identified colleagues in their geographic region to join the panel. Panel convenors attended a training day where they reviewed 'early bird' papers according to clear reviewing guidelines that have been refined over a number of years. They then led their panels through the reviewing of a fixed number of conference papers. Each paper was independently reviewed by two panel members, who then discussed their assessments and produced a single consensus report that provided the author(s) with detailed feedback. For consistency, all reviews recommending that a paper not be accepted were reconsidered by two members of a small panel of highly experienced reviewers. Only those research papers accepted by two reviewers have been included in these conference proceedings.

The abstracts for round table discussions were also blind peer reviewed (without the authors being identified) by two experienced mathematics education researchers.

AAMT reviewing process

Professional seminars and workshops were selected as suitable for the conference based on presenters' submissions of a formal abstract and further explanation of the proposed presentation.

Authors of professional seminar and workshop proposals that were approved for presentation at the conference were also invited to submit a written paper to be included in these proceedings. These papers were then scrutinised blind by two reviewers (without the author(s) being identified) and rated on categories developed for use in previous AAMT conferences and AAMT professional peer-reviewed journals; contradictory reviewer reports were resolved by the editor after obtaining a third independent peer review (without the author(s) being identified). Reviewers were chosen by the editor to reflect a range of professional settings as well as expertise in the substantive area addressed by a paper. Only those professional papers accepted by two reviewers have been included in these conference proceedings.

List of reviewers

The panel of people to whom papers were sent for peer review consisted of:

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Caroline Yoon
Jenny Young-Loveridge

The editors of both the MERGA research papers and the AAMT professional papers would like to thank all those who assisted with the process for their professionalism and expertise. We particularly wish to thank Melinda Pearson and Kate Manuel in the AAMT office for coordinating the collection and dissemination of papers for reviewing and reports from reviewers, as well as providing professional editorial support. We also owe a great debt of gratitude to Judy Mousley for coordinating the MERGA reviewing process. Without her generous offering of time and expertise, the process would not have been possible. Toby Spencer's contributions to editing the AAMT professional papers and expertise in producing a high quality professional joint conference

proceedings publication under considerable time pressures have been invaluable, and we are grateful for his outstanding work.

The spirit of cooperation between MERGA and AAMT has been a wonderful advertisement for the unity with which mathematics educators in all sectors view the world. We all value equity and excellence in mathematics education for all learners at all levels, and strive to learn more through research and practice. As Kurt Lewin once wrote, “There is nothing so practical as a good theory.” Equally we could say, “There is nothing so theoretical as good practice.” This conference is a shining example of how theory and practice in education can come together to enrich and inform each other.

We thank the AAMT office which has provided the infrastructure and organisation to make this conference possible, and the Local Organising Committee for enabling things to run smoothly at the conference itself. We trust that all participants will enjoy both the professional dialogue and the social interchange throughout the conference.

Julie Clark, Judy Mousley and Steve Thornton (editors of research papers)
Barry Kissane (editor of professional papers)
Toby Spencer (editor of proceedings volume)

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MATHEMATICS ASSESSMENT: EVERYTHING OLD IS NEW AGAIN?

THE ANNUAL CLEMENTS / FOYSTER ORATION

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Over the past decade or so there has been much rhetoric about assessment. There are assessment websites replete with “rich tasks”, work samples, standards, and definitions. The MySchool website reports data from large scale assessments. Teachers are exhorted to use assessment as a tool for learning. What has all of this activity achieved? Research evidence is scant and conflicting. It is time to assess mathematics assessment and to reconsider the purpose, nature and use of assessment information.

This is the first Clements-Foyster lecture to be delivered to a combined audience of practitioner and academic researchers. When MERGA was established in 1976 by John Foyster and Ken Clements, the AAMT already existed. At that time it had a research committee, which suggests that mathematics teachers recognised the importance of research. With the growth of MERGA, the AAMT research committee ceased to exist but over the years the two organisations have developed a strong mutual respect and have worked together productively to address a range of issues in mathematics education. With the introduction of the Australian curriculum, assessment of mathematics is a re-emerging focus and the topic of this address.

I aim first to briefly outline the history of assessment with a focus on mathematical knowledge. I will then examine a number of influential developments in more recent times, and consider current practices, before proposing some new ways of thinking about the purpose, nature and use of mathematics assessment information.

Assessing mathematics

Assessment of mathematical understanding goes back to ancient times. The traditional owners of the land we call Australia had a complex mathematics to describe kinship groups, arrangements for sharing food and other resources, and for navigation and describing position. This mathematical knowledge was passed to the youth of the group in a variety of ways: modelling, practice, direct instruction and story-telling. How was this assessed? Some of the knowledge would not have been formally assessed. Some may have been part of secret initiation ceremonies and some may well have been assessed in a public display of knowledge (Peterson, 2008). The key point is that the “teachers” were also the assessors and getting the assessment right was fundamental to the survival of the group—very high stakes assessment.

Moving on to the ancient Greeks, Pythagoras is an important historical figure. In the Pythagorean brotherhood, whole numbers had religious status and formed the basis of secret rites. When one of their members, Hippasus, made the shocking discovery that the diagonal of a unit square could not be expressed as a whole number ratio, legend has it that he was drowned by the brotherhood members. This unhappy outcome was a consequence of challenging the teacher's assessment and knowledge base.

Imperial China used a complex system of examinations for admittance to the public service, the earliest system of standardised tests. Examinations took place at designated centres, and candidates were literally locked in for up to a week. Examinations were written, and all responses were copied by a scribe prior to assessment to prevent identification of the candidate. These examinations were very high stakes: success would guarantee a comfortable life not only for the examinee but for family and village as well. One of the “Six Arts” examined was mathematics, both applied, as in taxation, and pure problems being given. Successful candidates were “called to the bar” which separated them from the unsuccessful—similar language is used by lawyers to this day. Assessing mathematical knowledge has been an important element of education from the earliest days.

More recent developments

As schooling developed and became more formal, so did mathematics assessment processes. Teachers remained the principal assessors. Oral questioning of students, sometimes in public, was a recognised and respected approach to assessing students' knowledge for the purposes of determining attainment, and this tradition is continued in the oral defence of PhD theses. Such oral examinations “… allowed teachers to ask probing questions or even to help pupils by providing permissible hints” (Lewy, 1996, p. 225).

As educational opportunities expanded, formal examinations became more widespread in the west. Printed examinations were first used at Harrow School, one of the great public schools of England, in 1830. During the twentieth century the assessment emphasis moved to standardised tests and objective measurement that focussed on aspects such as identical conditions of testing, and statistical measures such as those relating to reliability. External tests at key points in schooling became widespread in some western countries, although not all. Bodin (1993), for example, described the French system, where students did not automatically move upwards from year to year, as one where assessment was carried out continuously by the teacher who did not have to account to anyone. The teacher awarded marks, calculated averages and these were assumed to be a measure of the achievement of the learner. Examinations were unknown.

The rise in external examinations, often presented at key points in schooling, in effect, separated testing and test development from the process of teaching (Grouws & Meier, 1992; Lewy, 1996). This separation was not unnoticed. Dennis (1926), for example, wrote

> In mathematics … the methods of testing have a strong effect upon the teaching. … For years we have been discussing and revising the teaching of mathematics, its aims, its

> curriculum, the materials to be used, and the methods to be employed. But we have not given equal attention to the ways of testing the results (p. 58).

Today, 85 years later, the same comments could be made about the new Australian Curriculum.

In Britain, by the early 1980s testing was widespread with over three-quarters of the responsible authorities using some form of testing, of which mathematics was usually a part. Much of this testing was driven by debate about standards of education (Gipps, 1988). The concerns about standards were not new. Early in the 20th century high failure rates in tests were accepted as a way of maintaining standards – only the brightest and best survived the process. As pressure grew for a better educated workforce, compulsory schooling was extended, and it became the norm for children to move through the years of schooling with their age peers. New arguments for testing developed, based on equity, but still rooted in standards (Resnick, 1980). Tests were used to set standards and test results were assumed to be a measure of the success of the system. Large-scale testing programs were used as part of a “carrot-and-stick” approach to improving the quality of education and teachers were expected to change their practice in response to this external pressure to raise standards of education (Darling-Hammond, 1990). The question of the use of tests not only to describe standards but also to raise them continues today.

In 1998, Black and Wiliam's seminal work changed the face of assessment. Their meta-analysis of a variety of research studies led to a series of influential publications about the use of feedback in classroom assessment (Black & Wiliam, 1998a, 1998b). Again the emphasis was on raising standards but this time through improving the classroom assessment process. Hattie (2009) reinforced the effectiveness of feedback, and reasserted the importance of teachers. When assessment and teaching are seamless, useful feedback is provided to students, and both teacher and students change what they do as a result, classroom assessment is a powerful tool.

Towards the end of the twentieth century, there were calls to build closer links between teaching, learning and assessment (e.g., Pellegrino, Chudowsky & Glaser, 2001; Shepard, 2000), and to involve teachers more closely in the assessment process. There was an expectation that teachers would not only test knowledge recall, but instead would use complex tasks intended to provide an intellectual challenge (Lewy, 1996). One approach to this matter was termed “authentic” assessment (Archibald & Newmann, 1988). Authentic assessment aimed to provide assessment tasks for students that were meaningful outside the school context, and which expected students to communicate their ideas through coherent writing, rather than through multiple choice responses. These ideas underpinned Queensland's Rich Tasks as part of the New Basics project (Education Queensland, 2004), although there were also other theoretical considerations around intellectual quality.

During the late 1980s and early 1990s, in various parts of the world attempts were made to return assessment to the classroom. In Britain, common assessment tasks were used at Key Stages in education. California had a large-scale program of teacher-judged assessment, as did Ontario in Canada. In Australia, the idea of a student “profile” took hold and this was seen as one approach to improved accountability in which teachers played a major role, culminating in the publication of *Mathematics: A curriculum profile for Australian schools* (Curriculum Corporation, 1994). These attempts to develop large-scale teacher-judged assessment processes failed on two counts. The first was political. Authorities did not believe the evidence that teacher judgement was as reliable as a multiple choice test. The second was industrial. Teachers refused to accept the additional responsibility and workload. Perhaps this was an opportunity lost.

The situation today

Today, the situation in Australia is bewildering. NAPLAN provides an external measure of numeracy but teachers are urged to use formative assessment. External testing has become high stakes, with schools compared to other like schools using widely available, complex statistical information on the MySchool website. At the same time, systems advocate use of assessment *for* learning or assessment *as* learning and provide examples of open tasks, rubrics, descriptions of expected standards and many other resources aiming to lift the quality of teaching. Wiliam and Black (1996) used the ideas of meaning and consequences as one way of distinguishing formative and summative assessment. Formative assessment, they suggested, is characterised by some action as a result—the consequences of the assessment—whereas summative assessment has a focus on maintaining the same meanings across individuals and groups, as well as across time.

Despite the stress on assessment for learning, the emphasis on feedback and the plethora of advice to teachers, and external testing, there is little evidence that overall this activity has created improvement in students' learning outcomes (Stiggins, 2007). Over a twenty-year period mathematics performance on statewide tests in Tasmania remained stable, although the tests themselves became harder, leading to a perception of falling standards (Griffin & Callingham, 2006). Initial comparisons on NAPLAN numeracy from 2008 to 2010 do not indicate any significant change across time for any grade group (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010). Burgess, Wilson and Worth (2010), writing from an economics perspective, claimed that "league tables" reporting assessment results for schools in England led to improved performance in contrast to Wales where league tables are not used. They quoted effect sizes of around 0.2 which is below Hattie's (2009) suggestion that an effect size of 0.3 represents what would happen through a process of natural development. Internationally, Australia has slipped somewhat in rankings, and in PISA 2009 its performance also declined significantly. In addition, a significant difference between male and female performance first seen in 2006 was confirmed, suggesting that gender equity issues are still of importance (Thomson, de Bortoli, Nicholas, Hillman, & Buckley, 2011). This decline happened despite the increased emphasis on statewide testing that grew throughout the 1990s and became NAPLAN in 2008. The evidence about improved performance from competitive assessment results is limited.

The situation in mathematics assessment in Australia today is thus somewhat confused. All states and territories undertake NAPLAN and these results are used for accountability at the local level. Australia participates in various international studies which are used as measures of the success of government policies. At the same time, teachers are bombarded with advice and resources about formative classroom assessment. There is an emphasis on giving feedback, improving teaching and providing detailed information to parents. Media and systems decry falling standards in numeracy, and parents are advised to consider assessment outcomes reported through MySchool when they choose a school for their child. In summary, teachers and schools are getting mixed messages about assessment. On the one hand they are urged to bring assessment closer to teaching, on the other the assessment that counts is externally imposed testing. Confusion reigns.

It seems that the education community has not clearly communicated to those outside it what assessment is about, and what inferences can be validly drawn from the information presented. In part this is an issue of numeracy—it is, after all, a data interpretation exercise. There are also, however, issues around the use of assessment information that have remained unquestioned. Messick (1989) coined the term “consequential validity” to describe the use of assessment information. He stated

> Validity is an integrated evaluative judgement of the degree to which empirical evidence and theoretical rationales support the *adequacy* and *appropriateness* of *inferences* and *actions* based on test scores or other modes of assessment. (p. 13, emphases in original).

The question that always needs to be asked is “does this assessment provide suitable information on which to base future actions” about whatever claim is being made, whether that claim is about an individual student, a school or a system.

Productive assessment

Assessment is arguably the most powerful element in teaching and learning. Quality assessment can provide information to students, teachers, parents and systems in effective and useful ways. To be helpful, however, it must be broad ranging, collecting a variety of information using a range of tasks before, during and after a teaching sequence. At present there is a lack of consistency—in terminology, in approach and in use of assessment information.

One resource that may provide some direction is the AAMT position paper on the “Practice of Assessing Mathematics” (Australian Association of Mathematics Teachers [AAMT], 2008). Taking account of both classroom and external assessment, this document clearly makes the call that

> Students’ learning of mathematics should be assessed in ways that:
> - are appropriate;
> - are fair and inclusive; and
> - inform learning and action (p.1).

This statement is consistent with Messick’s (1989) view of validity, and also recognises the reality of modern education. Large-scale external testing is here to stay, but does not have to have a negative impact on learning if it is used appropriately.

Assessment that provides useful, timely and appropriate information in fair and equitable ways is productive assessment. It may address the mathematical understanding of a child, the achievement of a class or the performance of a system, and can take place at any point in the learning and teaching cycle. Productive assessment includes productive tasks, productive dialogue, productive teaching practices and productive reporting. To illustrate these points, some examples of productive assessment are described here.

There are numerous wonderful tasks that promote and develop mathematical understanding in children. The key to making these tasks productive is to trust the students and allow them some freedom and control over what they choose to do. For example, a Year 7 class started exploring the Task Centre activity called “Sphinx” (Martin, 2000). They became very engaged with the problem and asked the teacher whether they could make a video about their investigation. Ultimately, the class produced a video that showed their learning about geometry, algebra, problem solving and many other incidental aspects of mathematics. This was an unintended assessment activity but one that produced very rich results for all concerned.

Productive dialogue can be any discussion that improves understanding. Take this example from a Year 8 classroom in a disadvantaged school during a learning sequence addressing 2D and 3D shapes:

> Student: We live on a circle, don’t we?
> Teacher: Are you sure? If we cut the earth in half we’d see a circle… Do we live on a circle?
> Student: Hang on, no, it’s a [long pause] It’s a cubic circle.

The student successfully demonstrated his understanding of the difference between a circle and a sphere without having the technical language to describe this. The teacher was able to build on this understanding and to develop the appropriate mathematical language—a productive episode for both parties.

Quality mathematics teachers can turn almost any activity into a productive teaching event. A Year 1 teacher decided to use her students’ birthdays as a starting point for what she intended to be a unit on time addressing the months of the year, and so on. When trying to sequence the birthdays in the class by hanging cards on a line, the children were very insistent that the sequence should begin in the current month, rather than January which the teacher had anticipated. The teacher decided to throw the challenge to the class to represent the birthdays in ways that could be understood by other people. The representations produced gave some deep, and surprising, insights into the children’s understanding of data representation.

Productive reporting can be at any level. This scenario was observed in a Tasmanian primary school (Callingham, 2010).

> The teachers are meeting in grade teams. They are sharing the “big books” about mathematics that the children in their class have produced. The discussion centres on what the books demonstrate about the children’s understanding, and what the teachers need to do to move that forward. In the discussion, teachers compare the work samples and make judgements about their own and other teachers’ students. They refer frequently to the state curriculum documents, NAPLAN results, the school policies and “throughlines” that have been developed collaboratively to ensure a common language and focus across the school. By the end of the meeting, all teachers have a commitment to some action for their class, and to increase the school focus on specific aspects of mathematics at which the students appeared to do less well on the NAPLAN.

The teachers were reporting to each other, using data from various sources and committing to action as a result.

Teachers make a difference. They assess continuously in a variety of ways. It is time for a return to the traditions of assessment practice by recognising teachers’ authority in the [new] practice of mathematics assessment.

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LESSONS LEARNED FROM THE CENTER FOR THE MATHEMATICS EDUCATION OF LATINOS/AS[1]: IMPLICATIONS FOR RESEARCH WITH NON-DOMINANT, MARGINALISED COMMUNITIES

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This paper centres on research on equity and mathematics education in Mexican American communities in the United States. This research is grounded on a socio-cultural perspective and encompasses work with teachers, students, and parents. We address questions such as: What are Latino/a immigrant parents' perceptions of mathematics instruction? What do teachers see as obstacles and advantages in the mathematics education of non-dominant students? How does language policy affect students' participation in the mathematics classroom? The findings are likely to be relevant to other settings with immigrant students and non-dominant students.

Some context

My entry into mathematics education was in great part through my experience as a Teaching Assistant for mathematics content courses for preservice elementary teachers. I became fascinated by how students made sense out of mathematics, and in particular those students who are not considered "successful" in mathematics by the traditional measures of success. As I write in Civil (2002):

> I became intrigued by the fact that the 'more successful' were less likely to make use of 'informal' methods, everyday type reasoning, and would rather use a formula, algebra, school-like methods. The 'less successful' were often trying to make sense of the problems, making connections to everyday life. (p. 135)

One of those "less successful" was Vicky, a preservice elementary teacher, who wrote in her journal, "There is hope yet when I can legally use my methods to solve a problem." Her words are a constant reminder of how crucial it is that we listen to our students' ideas and emotions about mathematics. Vicky's approach to problems showed great insight and deep understanding, yet she did not seem to value it as much as her peers' algebraic approaches. Preservice elementary teachers appeared to have rigid beliefs about what and how to teach in mathematics, beliefs largely grounded on their unsuccessful encounters with this subject (Civil, 1993). I could not help but wonder how they were going to respond to children's ideas about mathematics. When I shared

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alternative algorithms that children may have used, the prospective teachers would often attribute them to "gifted children." Years later, when I shared algorithms that immigrant children might bring with them, prospective teachers' comments such as, "how can we be expected to know all these different ways?" or "This is nice but they need to learn to do things the U.S. way," underscored for me the urgency to prioritize issues of equity in mathematics teacher preparation. Now I continue to wonder about the question of how teachers will respond to children's ideas about mathematics and I am particularly interested in how the sociocultural context of students may play a role in teachers' perceptions of students as doers of mathematics. In the last few years several projects across the world seem to be concerned with the need to make equity issues prominent in mathematics teacher education. But, as Gates and Jorgensen (Zevenbergen) (2009) write, in reference to two special issues of the Journal of Mathematics Teacher Education (JMTE) on social justice and mathematics teacher education:

> The publication of these two Special Issues is testimony to the continued concern in the mathematics education community over the problems of social justice, and the real need to bring it to the attention of mathematics teachers. However, we do need to ask—why has it taken so long? Why isn't everybody—or at least more people—concerned about social justice? … Surely few would claim that the social conditions of our pupils were not our concern. Yet, we claim that is exactly what does happen in the field of mathematics teacher education—to a great extent. (p. 165)

As I write this, another special issue from that journal on equity is almost ready. We could also raise the question of why "special issues" on these topics? What does that say about this topic in relation to what some call "mainstream" mathematics education research? As Martin, Gholson, and Leonard (2010) write in reference to the Journal for Research in Mathematics Education (JRME) special issues on equity, "In many ways this practice has helped to relegate these issues and the authors of such scholarship to the margins" (p. 15). This separation between "mainstream" mathematics education research and equity research issues in mathematics education is quite problematic. I have argued before (Civil, 2006) for the need to bring together cognitive and sociocultural approaches to address the complexity of doing research in mathematics education, particularly when working in minoritised communities. In the next section I give some insights from a line of work in which we sought to develop mathematics learning environments that built on children's and their families' backgrounds and experiences. My focus will be on the importance of listening to students' ideas about mathematics and of paying attention to the students' context.

Listening to students

I have written elsewhere about the challenges and the affordances in developing learning experiences that build on community knowledge (Civil, 2002; 2007). Some of the issues encountered have to do with our valorisation of knowledge (Abreu, 1995), what we count as being "valid" mathematics in a school context, for example; other related issues have to do with our own limitation to see mathematics in everyday practices, due in part to our background in "academic" mathematics (González, Andrade, Civil, & Moll, 2001). Over the years, when reflecting on our work in connecting home and school mathematics, I have often raised the question of "where is the mathematics?" (Civil, 2007). The garden project (Civil, 2007; Kahn & Civil, 2001) was one example of an experience in which we explicitly blended sociocultural and cognitive approaches to address this question of “where is the math?”. This project grew out of a teacher’s noticing that several of her students’ families had extensive knowledge and experience with gardening. The teacher decided to create small container gardens just outside her classroom and invited parents to come and help out with their knowledge as well with resources.

One of the mathematics topics explored through the garden project was that of area and perimeter. These concepts were grounded on the children’s experiences with the gardens that they tended to as part of this project. These gardens were enclosed in chicken wire (to protect them from desert creatures) and had to be covered with plastic during the winter nights. The gardens consisted of plants in pots since the ground outside the classroom was not conducive to planting. One real problem that the students (9-and-10-year-olds) encountered was how to maximize the space inside the enclosed gardens with the limited (and fixed) amount of chicken wire they had. From a practical point of view, we could argue that they did not need much mathematics to solve it. Students pulled here and there on the enclosure and crammed as many pots as they could. From a mathematical point of view, it is an optimization problem. We explored this problem in the classroom context, with artificial “gardens” made by a 3 feet string that they glued to poster board to make “gardens” for which they had to find the area (Civil & Kahn, 2001). Then, after the garden module was over and towards the end of the school year, I conducted task-based interviews with four students. Here is where I learned about Vickie’s thinking of linear units and square units. While she seemed to indicate that it did not make a difference whether one used centimetres or square centimetres for area, when talking about the plastic to cover she said square feet:

> Marta: OK, what about the feet? Because in the feet you told me very confident you said, “square feet” Could I have said that the area of this is 15 feet?
> Vickie: You should say square feet.
> Marta: OK, why do you think that?
> Vickie: Because they wouldn’t know what you mean you might say 15, 15 triangle feet or something.

The “they” in the last line was in reference to the people at the store where they sell the plastic to cover the gardens, a connection to outside school knowledge. Feet and inches are units that were familiar to them in their everyday context, while centimetres were more tied to the school context. Vickie may have known that in everyday life one uses “square feet” when talking about covering, but what this term meant to her remains to be seen, given her reference to triangle feet. Another student, Nathan, seemed confident that the shape with largest area (for a fixed perimeter) would be a circle, but when I probed him further, he added that for this particular task (where we were using square tiles to cover the area), a square would work best:

> Nathan: Well, our plant things were squares… I mean were circles, but I think that it would have to be like a square this way, to hold more because these are square units. Because, I mean you can’t cut a plant holder in half…. Umm, well because circles will fit into circles right? … I mean you can fit circles into squares, but it is hard to fit a square into a circle…. I mean it’s like if you wanted to fill up the edges you would have to cut it in half.

Once again, we see a connection to the real life experience, here with the fact that the pots in the real garden were circular but in the task-based interview we are using squares to find the area (but for Nathan, these become also the "pots"). And finally, there is Jimmy, who was very patient with me as he tried to help me understand his reasoning for why the perimeter of a 9 cm by 2 cm rectangle was 8 cm. His explanation was based on a kinaesthetic experience that they had had in which they had created rectangles by lying down on the floor and counted from head to waist as 1 unit, and from waist to feet as another unit. So he drew a rectangle as if composed of 4 people, one per side and thus came up with an answer of eight.

These are very brief snippets on three students' thinking about area and perimeter in the context of a project that was meaningful to them, judging by how eager students were to work in their gardens, talk about them, and engage in mathematical activities that were somewhat grounded in their experiences with the gardens. We made it clear to the students that we took their ideas seriously and in turn they took our questions and requests for explanations seriously. These experiences trying to connect home and school mathematics were fundamental to my most recent work, as they reinforced my view of the need for a holistic approach to the mathematics education of non-dominant students. Such an approach involves researchers listening to all interested parties, in particular parents, students, and teachers. I turn next to some lessons learned from this listening.

Listening to parents

Parental involvement in my context is typically associated with physical presence of parents in the schools. Thus, if "parents don't come to school" it often contributes to schools' (teachers', administrators', even other parents') deficit views of parents, particularly in working-class, non-dominant communities (Civil &Andrade, 2003). Our work with Latino/a parents is based on a redefinition of parental involvement. It is grounded on the literature on parental involvement from a critical perspective (Calabrese Barton, Drake, Perez, St. Louis, & George 2004; Delgado-Gaitán, 2001; Valdés, 1996) and draws on the concept of cultural and social capital applied to parental involvement (Lareau and Horvat, 1999). A key concept in our work is that of parents as intellectual resources (Civil & Andrade, 2003), which implies a need to learn about parents' views and understandings of mathematics to engage them in an authentic partnership with schools. I concur with Valdés (1996) when she expresses her concern for any effort at parental involvement that "is not based on sound knowledge about the characteristics of the families with which it is concerned" (p. 31).

As I point out in Civil (2008; in press), immigrant parents in different parts of the world share several perceptions about the teaching and learning of mathematics. For example, one such perception is that schools in the receiving country are less demanding in both discipline and content than the schools in their country of origin. How do we learn from parents about their perceptions on the teaching and learning of mathematics? We have taken several avenues to do this: ethnographic household visits; mathematics workshops with parents; mathematical "tertulias"; and classroom visits (Civil & Quintos, 2009). I describe each of them briefly here, but my focus will be on the classroom visits. The ethnographic household visits are grounded on the concept of Funds of Knowledge, which are "these historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being" (Moll, Amanti, Neff, & González, 2005, p. 72). These household visits were my entry into the world of working with parents and most specifically trying to see schools from their point of view.

The idea of mathematics workshops with parents originated through some of these household visits, as parents (mothers mostly) expressed an interest in knowing more about the mathematics their children were learning in school. This led to a large parental involvement project in mathematics where among other activities parents participated in short courses (about eight two-hour sessions) on a variety of mathematical topics (numbers and operations; algebra; geometry; data analysis, etc). These courses allowed us to establish rapport with the parents and in time learn about their ideas about the teaching and learning of mathematics as they engaged as learners of mathematics themselves. The "tertulias" emerged from this work. We borrowed this term from Spanish, where it is related to the idea of gatherings in cafes or people's homes to discuss literature, poetry, or art. Our mathematical "tertulias" (mathematical circles) are arenas for doing mathematics but also for engaging in a critical dialogue about issues related to mathematics education. For example, through these tertulias we learn about parents' concerns that their children are not being taught basic skills such as the multiplication facts; or we learn about their views on the algorithm for division in the U.S. as compared to the one in Mexico (Civil & Planas, 2010); or about their expectations for more homework and a stricter approach to schooling. Some of these issues also come up in the debriefing of classroom visits. This last approach has proven to be particularly useful towards engaging with parents in a dialogue about teaching and learning mathematics. The visit to a mathematics class provides a shared experience that we believe facilitates this dialogue while allowing for our beliefs and values to emerge.

A visit to a mathematics classroom

To illustrate aspects of the process and some findings I have chosen a visit to a 7^{th} grade classroom (age 12) in which 5 mothers, a parent liaison, and a graduate student participated. The topic of the lesson was order of operations. All the observers had a sheet with some questions to guide the visit (e.g., "what does the classroom look like?"; how would you describe the participation in this classroom?" "What kinds of problems / questions were posed in this class?"). During the debriefing we use these questions as starting points for the conversation. For instance, for the question "what does the classroom look like?", all the mothers were in agreement that there were too many distractions and non-mathematics related objects (posters, family pictures, etc.). Unpacking these observations led to several comments pointing to a shared preference for a more teacher-centred classroom. For example, the mothers were concerned about the way the desks were arranged in the classroom. Students were sitting in groups and their concern was that no matter how, some of them were always with their back to the board. The mothers questioned the value of being in groups:

> Berenice: Well, I think there is more distraction by being in a group all the time… rather than being individually. You're there by yourself attentive to what, to what the teacher is going to say…
> Dolores: Or many times, if you're in a group, the other one is going, is going to copy the one who, who…

Berenice: Yes.
Dolores: Who is doing it right. So he is going to depend on the one on the side.
Berenice: On the neighbour.
Dolores: I think that by being individually they learn better. They work harder.

Along the lines of a teacher-centred approach, some of the mothers wondered about the role of one female student who seemed to be mostly helping other students. What was she learning? the mothers wondered. This concern was reinforced because one of the mothers noticed that this girl had used a calculator to help one of the groups but had come up with the wrong answer. The mother wondered whether the teacher was going to notice that or whether she trusted this girl as her assistant.

To be expected the mothers use their own experience from when they were in school as one of the lenses through which they view this classroom experience:

Carlota: the tables were not like that [when she went to school]. We would sit normal…. There were no tables; everybody always facing the front. The board was always in the front. There were no boards around. Only in the front and they would just write down what they were going to teach you in the class, and here I saw a lot of things written, which I don't know if they're going to go over them or if they went over in another class, I don't know. That's how I was taught. Facing the front and the board and that was it.
Reina: The same for me, and when we finished the board would be erased and we would start over the next day, and now when we got in there, there were already some calculations on the board.
Carlota: It was written already. … But, but not in ours, because in our case children wouldn't leave the classroom. I mean, you stay in your classroom and the teachers would come in… but not here, you have to leave running, you go to your next class and you go to the next one, but not over there.

Notice how in the first line Carlota says, "we would sit normal" to describe her experience in school in contrast with what she saw in this classroom. They wonder about boards with parts of mathematics lessons already written on them, as opposed to seeing the lesson unfold in the class. They also noticed that teachers stay in the classroom and that students are the ones who switch. These observations point to aspects of the cultural script that is associated with teaching in different countries (Stigler & Hiebert, 1999). Discussing the implications for the teaching and learning of mathematics of these different cultural scripts is something that could be pursued with parents (and teachers).

The mothers were quite engaged in this conversation as they expressed their opinions about pedagogical issues such as group work and the flow of instruction. A concern for distraction was quite prominent in this debriefing. Students should be attentive, facing the board, working individually. Marianela, another mother says, "It's real casual… like one table, a boy finished but the other two didn't, then he started to talk, but it wasn't about math. So they should only talk about the math problems, not about other things."

Much of the mothers' talk could be associated with the tension "reform-traditional." Whether it is about the pedagogical approach (e.g., group work) or what to learn, as I illustrate next, we could argue that tensions are normal, generational, etc. But it has to be seen also through the power lens, by which I mean that low-income, immigrant parents' voices are often not heard in the school setting.

In terms of what to learn, two recurrent topics are the division algorithm, in which Mexican parents comment that their method is more efficient and requires mental arithmetic as the subtraction is done in the head (see Civil & Planas, 2010), and the fact that their children are not being asked to know their multiplication facts, while in Mexico they would. The mothers noticed that among the many things posted on the classroom walls were the times tables. One of them commented:

> Carlota: They [teachers in Mexico] never posted the tables on the walls for us…. They would tell you "This table is for tomorrow and learn it," and there we are "Taca, taca, taca," like a little machine, and backward and forward… I struggled a lot with my daughter. She's in sixth grade and I can't tell you she knows the multiplication facts. I can't tell you, and I struggled a lot with her. I tried every possible way, but what happened? That in Mexico the teacher demands for you to know the facts, but here the teacher doesn't. I mean, he gives you that little table [in reference to printed multiplication tables] and, and the child doesn't learn it. I'm like a crazy woman over here "Honey, learn it," and if the teacher doesn't demand it, I'm just like a crazy woman.

Carlota is not alone is expressing her frustration as to what many parents see as a disconnect between what they expect from schools in regards to their children's education and what schools do. Recurrent issues that parents bring up in the debriefings and focus groups are: lack of homework; teachers do not demand enough; not enough emphasis on learning the facts (e.g., multiplication); loose sheets of paper instead of note taking and a notebook; lots of distractions; approaches to doing mathematics that are seen as inefficient and do not stress mental arithmetic (e.g., division). Some parents, however, do comment on the fact that the approaches that they (and their children) are currently learning put more emphasis on understanding the why behind the procedures instead of rote memorisation. I have often argued for the need to have spaces in which parents can get experiences with the mathematics their children are learning (and the pedagogical approaches) and most importantly, opportunities to engage in dialogue about these experiences. In particular, it is important to develop a dialogue between parents and teachers / school personnel. In a large parental engagement project we had teams of teachers and parents facilitating mathematics workshops in the community. This allowed us to explore power issues as some teachers saw themselves as the experts in the teaching of mathematics and were quite critical of having parents teaching these workshops because they did not have the "proper preparation." A few parents asserted their right to be facilitating these workshops since they were participating in the same leadership development program along with the teachers. Furthermore, they saw themselves as better positioned to reach out to other parents since they were also parents in the community (see Civil & Bernier, 2006, for more details on these power issues parents-teachers).

In our current work, consistent with the need for a holistic approach to the mathematics education of non-dominant students, we have been working with teachers using a Teacher Study Group (TSG) format in which teachers and university researchers meet regularly to reflect on the teaching and learning of mathematics with an emphasis on equity. We worked with 26 teachers teaching grades 2 – 8 (ages 7-13) in two Teacher Study Groups (TSG). The overarching goals for the two TSGs were to enhance teachers' mathematical understanding, primarily through focusing on students' thinking, and to engage in conversations about the role of language and culture in the teaching and learning of mathematics. I turn next to some of our findings from listening to teachers in the TSG sessions.

Listening to teachers

At MERGA in 2009 I described some of the characteristics of the work with the two groups of teachers (Civil, 2009) and shared some of the findings with respect to what teachers see as obstacles and advantages in the mathematics education of non-dominant students. In particular I pointed out that engaging teachers to talk about the role of language and culture in the teaching and learning of mathematics was not easy. With the second group of teachers a prominent theme was that of teachers invoking the culture of poverty as an "explanation" for the students' performance in mathematics. However, as Gorski (2008) points out,

> The myth of a "culture of poverty" distracts us from a dangerous culture that does exist—the culture of classism. This culture [of classism leads]…into low expectations for low-income students…. The most destructive tool of the culture of classism is deficit theory…[which suggests] that poor people are poor because of their own moral and intellectual deficiencies. (p. 34)

In this section I focus on listening to the teachers as they talk about their students' families. I do that to further stress the need for dialogue parents-teachers. When we asked teachers what they perceived as being obstacles to the mathematics education of non-dominant students, many mentioned the home environment. Teachers' concerns ranged from parents' low levels of schooling, therefore not being able to help their children with homework to "parents not caring about their children's education" because they were not seen in the school. As I mentioned earlier, in my context, physical presence of parents at school is often used as an indicator of parental involvement. This is a narrow and not culturally responsive view of parental involvement. The topic of parents not caring / not valuing education came up several times during our TSG sessions. There are several avenues to challenge these deficit views, and a particularly powerful approach is through the concept of household visits in the Funds of Knowledge project (González, Moll, & Amanti, 2005). Another approach is through discussion of readings and experiences and having teachers offer alternative explanations when this issue is brought up. For example, in the exchange below, Olivia expresses her frustration at what she views as a lack of responsibility in her students (8-year-olds) and she attributes it to their parents. Michael offers a counter explanation.

> Olivia: When we hear about parents who are home, who stay at home and don't do anything, and you try to contact them, you try to have them coming to the classroom, and they make no response. That's frustrating. Like, I always tell my kids, "You have to be responsible. Your parents go to work. That's their responsibility. You're responsible for coming here and learn", and they all say to me, "My mom doesn't work. My dad doesn't work", blah, blah, blah, so they don't see that responsibility. So it's really, I mean, I think we're all sympathetic when we see that someone is struggling, and we do whatever we can to support them, but, on the other hand, when we see parents who are just, "Here, take my kid," then that becomes difficult.
>
> Michael: I had a parent who was unresponsive and her kid, who was really bright, was coming in late a lot and missing school a lot and it was really getting to be worrisome, and I could not get her to come in for a conference, and then finally, when she did, what I found out is that she had her sister's kids in the house with her because her sister is being deported, and one of these kids has a lot of mental health issues and it's just disrupting their entire home, and so it's really difficult to get all these kids ready for school in the morning, whether she works or not and that's the reason why the boy wasn't getting to

> school on time, because his cousin is disrupting their home life, and so I didn't understand that until I finally got to sit down with her. So, the point about being persistent with people and not making assumptions is really important, because once I got to talk to her, I could see she was definitely committed to her son's education, but she's just facing a lot of real challenges and getting it there. [TSG, May 2009]

Olivia's talk captures many of the stereotypes that I have heard in my work with teachers in non-dominant communities: "they stay at home and don't do anything" (how do we know they "don't do anything"? They could be having a situation similar to the one in Michael's case; or they may not be able to work for a variety or reasons, including immigration status); they do not come to school (how welcoming are schools? How is the visit to the school presented to them?); "when we see parents who are just 'here take my kid'" (parents, and this is particularly the case with many Mexican immigrant parents, trust the school with their child's schooling, they are not just handing him/her off).

I do not want to imply that teachers had deficit views about their students' families. The situation is more complex and we could see teachers going back and forth between discourses. For example, later in that same session Olivia said:

> Olivia: I think our parents truly want the best for our kids. I mean, for parent-teacher conferences, they do show up, I think they just don't know what to do. I have had parents telling me, "I can't help him with this work in third grade. I don't know this." They can't do the work. They weren't educated…. I think they are so overwhelmed with life and situations that happened, that they're not really, they don't have the tools…. I don't feel like they have the tools to help their child. It's not from lack of desire. They just don't know what to do. They're overwhelmed by everything else that happens in life, and sometimes we lose sight of that because we're so frustrated in the classroom, and we think, it they would just help him with his homework, if they just didn't help him read, but the reality is that we do what we can with what we have.

Of course there is a lot that we could question about Olivia's talk in this excerpt, such as her notion that parents do not have the tools. But we can also sense her not wanting to just dismiss the parents as uninterested in their children's education.

Although the teachers' talk about their students' families may be seen as "interesting but unrelated to mathematics education," I argue that whether teachers view parents as resources or as obstacles or as the reason for their students' behaviour and performance should be of concern to mathematics educators. As Jorgensen (Zevenbergen) (2010) writes, "the solution cannot be found in looking from a mathematical lens but must be much broader if increased access to mathematics education is to be a reality of the future" (p. 26). Her study is centred in an extremely disadvantaged context in a remote Aboriginal community; arguably it is a context that at many levels is quite different from the one in my work. However, the low-income Latino/a communities where my work is located are marginalised not only because of poverty but also in terms of language and culture, particularly in the current anti-immigration movement in my local context. I do believe that we need to get a better understanding of how teachers view their students' families and work towards moving away from deficit views to approaches that seek to understand parents' circumstances and experiences. The fact that teachers who have been teaching in schools with a large number of students of Mexican origin seemed unaware of the difference in the representation for the division algorithm in the two countries (Mexico and United States), points once again to a lack of communication between home and school. Some of the teachers in our work feel comfortable trying to make connections to their students' home experiences in other subjects but not in mathematics:

> Penny: Some kids do come in knowing a lot because they maybe have worked at home with their family in building things or working on a car or anything at home where they've had like pre-measurement experiences.… But you know when I mostly bring it out is in reading or writing time. In math I haven't seen it come out as much but then again maybe I am not being more aware of trying to ask those kinds of questions; maybe it's just I am faulting on my part of not asking those kinds of questions in math, because I do do it in other areas. [Interview, September 2006]

Making connections to students' cultural and language backgrounds is a complex endeavour. In the first part of this paper I gave a glimpse of some of our work connecting home and school mathematics (for more on this see Civil, 2002; 2007). In this last section I return to the students focusing now on a different topic, the issue of how language policy in schools affects participation in the mathematics classroom.

Listening to students, again

Since 2000 bilingual education in the State where my research is located has been severely restricted. Furthermore, since 2006, English Language Learners (ELLs) are to receive 4 hours a day of English language instruction, raising serious concerns about their opportunities to learn other content areas and the segregation that this approach promotes. It is a clear example of subtractive schooling (Valenzuela, 1999), in which the culture and language backgrounds of ELLs are not used as resources but instead seen as a problem that needs to "fixed." Many of the teachers in our work have had to switch from bilingual, additive approaches to Structured English Immersion (SEI) programs, where by law all instruction is in English with occasional clarification in the students' home language. Teachers have shared their frustration at the language policy and some of them have commented on the effect it has on students' participation in the classroom. Matilde, a middle school teacher at a school where ELLs were segregated for most of the school day, shares her perceptions of these students when they are approaching the time in which they may be switched to a "regular" classroom:

> Matilde: I work only with ELL students … Our kids feel afraid to be in the regular classroom because they feel the other kids have the power. So, even if I have a very brilliant a kid, he goes to a nor- class, a regular classroom, and he is going to be one X student [meaning anonymous]. Because he is not going to be that brilliant because they're going to ask them questions in English so they don't know how to explain themselves and they're going to be quiet. So they're going to be, relegated to the back of the class. So they are afraid to go to a regular class. [TSG, March 12, 2009]

I have discussed elsewhere several issues related to a restrictive language policy and students' participation in the mathematics classroom (Civil, 2009; 2011; Planas & Civil, 2010). Thus here, what I will do is provide a summary of some findings from this research. As Clarkson (2009) points out, we need to be aware of the specificity of different multilingual contexts as, "there is a danger that a model for teaching that may be useful in one such context can be assumed to be applicable in all multilingual contexts" (p. 151). My context is essentially a bilingual English-Spanish one, in schools in predominantly Latino/a communities with a strong affiliation with the Spanish language, with several teachers and school personnel being bilingual themselves. Thus it is not unusual to hear both English and Spanish in the school grounds, and for students to use Spanish in their small group discussions. The language of instruction, however, is English. This is the result of a language policy that cannot be separated from the political environment of anti-immigration in my local context (this is also the case in other countries, as I discuss in Civil, 2008; in press).

I will centre my observations on a study with a small group of 8 ELLs, 7 of which were recent immigrant students from Mexico (had arrived within two years prior to the study). They were in 7th grade (12 year-olds) and were with other ELLs for most of the day (except for one elective, in which they were mixed with non ELL students). Very likely due to the small size of the mathematics class, the atmosphere in the classroom was very relaxed and family-like; students knew each other well and there was a lot of teasing going on. The teacher was Spanish dominant herself but taught in English most of the time. Students were expected to write in English and there seemed to be an implicit expectation that they would communicate in English when talking to the whole group. Probably not surprisingly, when presenting in English, their verbal and non-verbal expressions were stilted and seemed tentative. Working in groups and then presenting a group's approach to the rest of the class was not the norm in this school. Thus, as we encouraged students to do this, we also gave them the freedom to use Spanish to explain their thinking when they so wished. Allowing for that to happen gave us access to very rich and lively mathematical discussions, which in turn gave us a window into their thinking about mathematics. We would have missed this, had they (and we) not been able to access their home language. I want to stress not only the cognitive aspect but also the affective one. By having access to their first language students could use humour (which is culturally situated) and metaphors when solving problems. Such is the case of Carlos in solving a probability problem that had two spinners, one split in fourths with Yellow, Green, Red and Blue; the other one split in thirds with Red, Green and Blue; one of the tasks involved finding the theoretical probability of getting a match. Carlos right away said 3 out of 12:

> Porque, mira, aquí no están hablando del yellow. … Nomás el yellow está de metiche ahí, porque, mire, nomás está... Sale green, red, and blue. Todo sale green, red, and blue, y el yellow también participa, pero haga de cuenta que el yellow no cuenta, pues. [*Because, look, here they are not talking about yellow…. Yellow is just a busybody here, because, look, it's just… We get* green, red, and blue. *In everything we get green, red, and blue, and yellow also participates, but just suppose yellow doesn't count*.]

Carlos' use of the word "metiche" (busybody) is a cultural referent, which combines both humour and metaphor. He knew that there were 12 possible outcomes, and only 3 of them were matches. His explanation combines mathematical talk with more of an everyday language by referring to Yellow as a "metiche."

Encouraging the use of the two languages also gave us a window on the different strengths that students bring, such as a student, Octavio, who enjoyed engaging in mathematical arguments but did not show that inclination till the use of Spanish with the whole class became more visible. When presenting with his group on a somewhat difficult probability problem where they had to determine whether a game was fair, he relied on another student to translate into English and write on the white board his explanation. When another student challenged their work, Octavio argued with him in

Spanish. Yet, later on as he tried to explain in English why he thought the game was fair, his explanation was much harder to follow. Was it because of the language or because of the mathematical content?

Our research shows similar findings to those of Clarkson (2006) and Planas and Setati (2009) on the reasons and situations in which students switch languages (e.g., perceived difficulty). We also conducted task-based interviews in which similar to Clarkson’s study, we asked the students about their language use to solve problems. Besides the cognitive implications of this type of research, I want to emphasize the affective component. Although this classroom provided a safe environment in which students could use both languages to do mathematics, it was after all a case of segregation and the students were very aware of this. They knew that they were not in the “regular” classroom and several of them shared with me that they would have liked to be in an environment that was more English dominant and with non ELL peers (see Civil, 2011; Civil & Menéndez, 2011; Planas & Civil, 2010). Thus, I wonder, were we “right” in encouraging them to discuss mathematics in Spanish? Or was doing this contributing to their perception that they were not having enough opportunities to learn English? This raises many questions in my mind around the effect of language policy on students’ language identity in the mathematics classroom (and in school in general).

Closing thoughts

Like Jorgensen (Zevenbergen) (2010) and Martin et al. (2010), I argue for the need to find other approaches to address the mathematics education of non-dominant students. As Martin et al. write,

> Rather than generating concern about studies that do not give priority to mathematics content, it may be more informative to understand why studies that have continued to do so have offered so little in the way of progress for students who remain the most underserved. Minimal progress for these students would seem to demand that we pursue *all* promising areas of inquiry informing us about how to help them experience mathematics in ways that allow them to change the conditions of their lives. (pp. 16-17)

One promising area of inquiry is one in which all interested parties really listen to each other and work on making “difference” a resource rather than an obstacle towards the teaching and learning of mathematics. We should examine how our values and beliefs about what counts as mathematics and who can learn it and how, support or interfere with the development of learning experiences that are culturally responsive to the students we have (we all bring values and beliefs, including students, parents, teachers, and university researchers). Teacher education programs need to engage teachers and preservice teachers in experiences with parents and children that allow them to examine the complexities of different perceptions and valorisations of knowledge as well as the role that multiple languages and language policies play in children’s learning of mathematics. To pretend that the cultural, social, language, and political contexts of non-dominant students can be put aside when teaching mathematics is educationally irresponsible.

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BETH SOUTHWELL PRACTICAL IMPLICATIONS AWARD

The Beth Southwell Practical Implications Award was initiated and sponsored by the National Key Centre for Teaching and Research in School Science and Mathematics, Curtin University, Perth, Western Australia. Curtin sponsored the "Practical Implications Award", as it was then known, for the first ten years. The Award is now sponsored by the Australian Association of Mathematics Teachers (AAMT). In 2008, MERGA was honoured to be able to rename the PIA as the Beth Southwell Practical Implications Award, in honour of MERGA's and AAMT's esteemed late member, Beth Southwell.

The award is designed to stimulate the writing of papers on research related to mathematics teaching or learning or mathematics curricula. Application for the award is open to all members of MERGA who are registered for the conference.

Applications for the PIA are judged against specific criteria by a panel consisting of two members of MERGA, two from AAMT, and chaired by the MERGA Vice President (Development).

LEARNING OVER TIME: PEDAGOGICAL CHANGE IN TEACHING MATHEMATICAL INQUIRY

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Inquiry pedagogies are often advocated for equipping students with 21st century skills, but teaching mathematics through inquiry is difficult. A longitudinal study investigated teachers' experiences of learning to teach mathematical inquiry over time. Using the Productive Pedagogies framework, this paper reports on aspects of practice that evolved for twelve primary teachers as they gained experience with inquiry over three years.

School mathematics is criticised for emphasising closed problems with set answers (Hollingsworth, Lokan & MacCrae, 2003). Many students find mathematics boring and lacking relevance (McPhan, Morony, Pegg, Cooksey, & Lynch, 2008). Declines in students studying advanced mathematics has prompted recommendations to build capacity and interest in mathematics by improving mathematics teaching and promoting inquiry (Australian Academy of Science, 2006; Council for the Mathematical Sciences, 2004). Inquiry addresses ill-structured problems, where the problem statement, goals, or solution paths contain ambiguities that require negotiation (Reitman, 1965). Most everyday problems are ill-structured; evidence is often conflicting, requiring one to seek potential causes of the problem and generate a range of possible solutions (Walker & Leary, 2008). Through mathematical argumentation, justification, and hypothesis, mathematical inquiry generates fresh understandings, appreciation of complexities in problem contexts and new questions to explore (Magnusson & Palincsar, 2005).

A major issue in mathematics education is to find ways to support teachers to develop inquiry pedagogies in mathematics. Researchers have identified challenges that teachers face when teaching inquiry (mostly in science): envisioning inquiry processes, managing uncertainties that arise, and creating a culture of inquiry (R. Anderson, 2002; Crawford, Krajcik, & Marx, 1998). Little is known about teachers' experiences as they move from these challenges towards expertise.

A longitudinal study was designed to understand teachers' experiences as they developed proficiency teaching mathematical inquiry. This paper presents findings from analyses of classroom lessons of twelve primary teachers' over three years using the Productive Pedagogies framework (State of Queensland, 2002). Areas of their pedagogies that shifted are discussed.

Literature

Inquiry is relatively uncommon in mathematics classrooms where the focus is on problems that are well-structured—that is, problems in which there are no ambiguities (context-free), or where the problem is embedded in context but decisions have already been made to address the ambiguities. Because of this, learners and teachers in mathematics may lack confidence to contend with uncertainties that arise or manage the deliberation needed to wrestle with complexities in the problem. Initial experiences can be especially daunting, as teachers are often disappointed when lessons do not run as expected (R. Anderson, 2002; Makar, 2010). "There is a danger that … initial difficulties with implementation and disappointment with student performance can lead to a premature rejection of [these] new pedagogies" (Krajcik et al., 1998, p. 341).

In a large scale review of literature on mathematics professional development, Doerr, Goldsmith, and Lewis (2010) conclude that "repeated cycles of experimentation, reflection, and revision [are] required to change elements of instruction" (p. 4), particularly in areas such as inquiry which are strongly connected to teachers' beliefs. They suggest that key features of professional development that do make a difference—substantial time investment, systemic support, and opportunities for active learning—are rare in programs involving more than a few teachers. In evaluating sustained professional development projects, Heck, Banilower, Weiss, and Rosenberg (2008) report that teachers' use of innovation was greatest in the first 80 hours of interaction and then leveled off, but after 160 hours, innovation increased again. This suggests that innovation is sustained in the long term, but only if teachers are supported over time, remembering that change is non-linear and idiosyncratic (Clarke & Hollingsworth, 2002; S. Anderson, 2010).

The Productive Pedagogies framework

In order to understand teachers' changing experiences of teaching inquiry, there is a need to document classroom observations of inquiry practices both within a single classroom over multiple years and collectively as teachers gain experience. Finding a framework without shortcomings was unlikely, particularly since characteristics that make up quality classroom pedagogies are contested. Productive Pedagogies was developed for the Queensland School Reform Longitudinal Study (QSRLS, 2001a) as a way to observe and document pedagogical practices across Queensland.

Table 1. Productive Pedagogies (QSRLS, 2001a).

Intellectual Quality Knowledge presented as problematic Higher order thinking Depth of knowledge Depth of understanding Substantive conversation Meta-language	*Supportive Classroom Environment* Students' direction of activities Social support for student achievement Academic engagement Explicit quality performance criteria Student self regulation Narrative
Connectedness School subject knowledge is integrated Link to background knowledge Connectedness to world beyond classroom Problem-based curriculum	*Recognition of Difference* Knowledge explicitly values all cultures Representation of non-dominant groups Group identities in a learning community Active citizenship

The opportunities provided by the Productive Pedagogies seemed positive in their emphasis on many of the same qualities valued in inquiry. Productive Pedagogies consists of 20 pedagogical practices organised into four main clusters: Intellectual Quality, Connectedness, Social Support and Recognition of Difference (Table 1). Although they have been critiqued even by their authors (e.g., Ladwig, 2007), the framework has been used extensively. Researchers have refined and updated the framework (Mills & Goos, 2007), but it has remained substantially unchanged since its publication in 2001.

Method

The research question was, "Which aspects of teachers' practice change as they gain experience in teaching mathematical inquiry?" The data reported in this paper come from teachers who completed at least three years in an ongoing longitudinal design-based research study. In design-based research, the researcher focuses on simultaneously studying and improving the research context through a number of reflective and retrospective cycles. The benefit of design-based research is that

> … in contrast to most research methodologies, the theoretical products of design experiments have the potential for rapid pay-off because they are filtered in advance for instrumental effect. They also speak directly to the types of problems that practitioners address in the course of their work. (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 11)

Figure 1 represents the model used in the project to understand teachers' changing experiences as articulated by the teachers in the study (Makar, 2008).

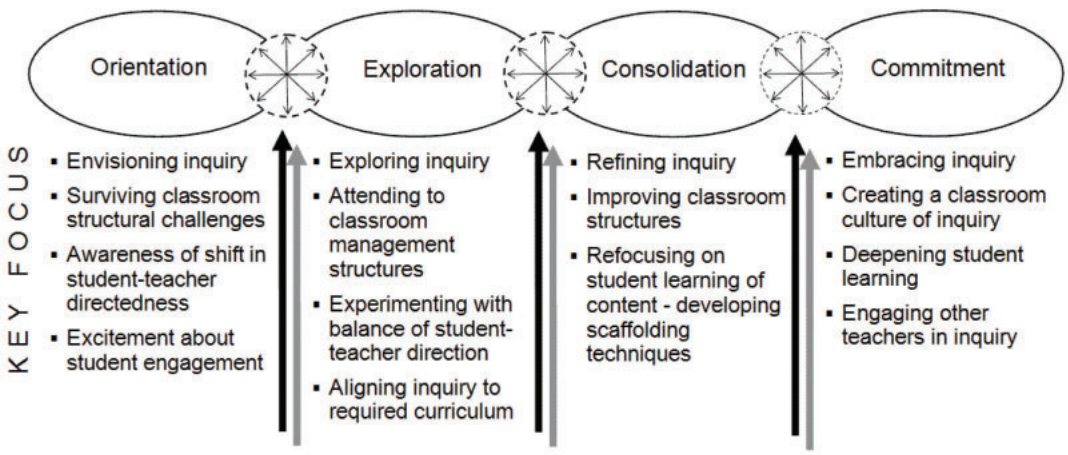


Figure 1. Model of teachers' changing experiences in learning to teach mathematics through inquiry.

The first phase (2006–2007) included five teachers from a large primary school in a middle class suburb. The next phase (2007–2009) expanded to twenty teachers—six from the original school and fourteen teachers (the entire school) from a rural school—in a low socio-economic area in the same region. As is common in longitudinal research, the project experienced attrition due to transfer. Data were collected from 23 teachers, with new teachers recruited as teachers left. This paper limits its focus to the 12 teachers in the study for at least three years. Five teachers (all female) were from the suburban school and seven (six female, one male) from the rural school.

Teachers participated in three to four days of professional development per year and taught a mathematical inquiry unit each term (a term lasts 10 weeks). Professional development seminars gave teachers time to collectively reflect, share experiences and obtain peer feedback. During the seminars, teachers also engaged in learning experiences that highlighted particular aspects of inquiry (e.g., ill-structured problems, assessment, emphasising concepts), built understandings of inquiry processes (e.g., working with ambiguity, understanding the role of evidence) and developed a learners' perspective of inquiry (e.g., experiencing frustration, breakthroughs, cognitive drivers).

The teachers developed their own units or modified published units; a unit lasted anywhere from two lessons to several weeks. During or after lessons, the researcher and teacher engaged in informal conversation to offer individualised support, query experiences, validate uncertainties and offer advice if requested. Advice was used sparingly to understand teachers' experiences with limited support (the current phase includes more explicit and systematic feedback and targeted skills in teaching inquiry).

Data collection and analysis

Classroom lessons were videotaped; it was not possible to videotape every lesson, but in most cases at least two lessons from every teacher were taped each term. Five hundred and sixty-five lessons were videotaped in the first two phases (2006–2009). This paper presents analyses of these videos, limited to teachers in the project for at least three years. To gauge teachers' pedagogies over time, a stratified random sample of lessons was selected to analyse, with lessons from each teacher randomly sampled according to the criteria in Table 2 to align with the model used in the project (Figure 1).

Table 2. Categories of lessons in the sample coded.

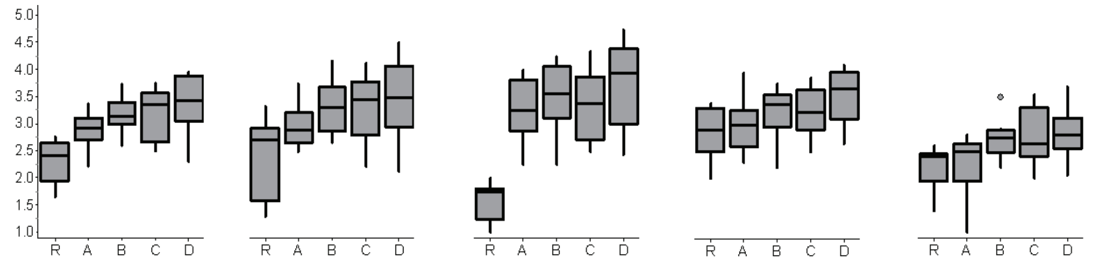
Category	Cumulative terms teaching inquiry	Random sample of lessons coded
R	Regular (non-inquiry) maths lesson (any term)	1 per teacher
A	First Inquiry (term 1 of their participation)	2 per teacher
B	Remainder of first year (terms 2-4)	2 per teacher
C	Second year (terms 5 – 8)	2 per teacher
D	Third year (terms 9 – 12)	2 per teacher

Lessons were identified by a code to mask their category and analysed with the Productive Pedagogies Classroom Observation Scheme (Queensland School Reform Longitudinal Study [QSRLS], 2001b). The Scheme describes qualities of practice in each Productive Pedagogy on a scale of 1 to 5 (with 5 high). A team of researchers led by the author scored the sample after a period of moderation (and interim cross checks). One hundred and one lessons were analysed in all (in a few cases, only one taped lesson was available), with scores averaged across teachers' two lessons.

Teachers were also assigned a difference score for each Productive Pedagogy and pedagogy cluster based on shifts during the first year (AB, the difference in their average A score and average B score), the first three years (AD) and comparisons between initial inquiry lessons and non-inquiry lesson (RA). Distributions of RA, AB and AD were tested with a two-tailed test of a single mean (e.g., H_a: RA≠0).

Results

Figure 2 provides a snapshot of the distributions of the twelve teachers' scores overall (average across Productive Pedagogies) and for each pedagogy cluster in the five categories—R (regular lesson), A (first inquiry), B (first year), C (second year) and D (third year). The graphs suggest that the teachers' pedagogical practice generally improved over time. Some patterns are more complex, however, than the graphs reveal.



Overall	Intellectual Quality	Connectedness	Supportive Environment	Recognition of Difference
R: 2.3 (0.42)	R: 2.4 (0.73)	R: 1.6 (0.37)	R: 2.9 (0.49)	R: 2.2 (0.37)
A: 2.9 (0.37)	A: 3.0 (0.42)	A: 3.3 (0.57)	A: 3.0 (0.47)	A: 2.3 (0.57)
B: 3.2 (0.32)	B: 3.3 (0.47)	B: 3.5 (0.64)	B: 3.2 (0.51)	B: 2.7 (0.35)
C: 3.2 (0.49)	C: 3.3 (0.61)	C: 3.3 (0.63)	C: 3.3 (0.44)	C: 2.8 (0.52)
D: 3.4 (0.55)	D: 3.4 (0.73)	D: 3.7 (0.84)	D: 3.5 (0.53)	D: 2.9 (0.49)

Figure 2. Distributions of scores in each pedagogy cluster and overall for a regular lesson (R), first inquiry (A) and in inquiries in the first (B), second (C) and third years (D). The table shows means (sd) of each pedagogical cluster in each category of inquiry experience (R, A, B, C, D).

Such broad comparisons offer only a vague impression of the teachers' changes in pedagogical practices in their first three years. Of interest was whether patterns emerged within Productive Pedagogies over time. For example, did different pedagogical practices evolve at different times? Table 3 details breakdowns of change scores (RA, AB, AD) for each Productive Pedagogy and pedagogy clusters, discussed below.

Intellectual quality

Although it had a plateau in the second year, Intellectual Quality improved by about one point on average on the five point Observation Scheme from the regular maths lesson to the third year of inquiry. Three Productive Pedagogies showed significant improvement from the regular maths lesson to the first inquiry lesson (RA), particularly *Problematic Knowledge* and *Higher Order Thinking*. *Metalanguage* improved gradually but was consistently higher (low standard deviation) by the teachers' third year. *Substantive Conversation* and *Depth of Understanding* appeared difficult areas to change.

Connectedness

Connectedness was the lowest cluster in the regular maths lessons ($\bar{x}$ = 1.6) yet increased to the highest ($\bar{x}$ = 3.7) by the third year. The improvement from a regular maths lesson to the inquiry lessons in all pedagogies in this cluster was significantly higher almost immediately. *Knowledge Integration* and *Problem-based Curriculum* increased quickly then plateaued or slightly declined. *Link to Background Knowledge* and *Connectedness to the World* increased slowly, ending strong by year three.

*Table 3. Productive Pedagogies - shifts over the first three years. Mean (sd), p-value by category, * p<0.05, **p < 0.01.*

Productive Pedagogies	RA (Regular lesson to first inquiry)	AB (across first year)	AD (across first three years)
Overall	**0.56 (0.43), p = 0.00088****	**0.30 (0.37), p = 0.016***	**0.50 (0.47), p = 0.0035****
Intellectual Quality	**0.62 (0.74), p = 0.015***	0.31 (0.57), p = 0.085	0.44 (0.80), p = 0.085
• Knowledge presented as problematic	**0.77 (0.85), p = 0.0093****	0.47 (1.29), p = 0.24	0.69 (1.29), p = 0.092
• Higher order thinking	**0.88 (0.84), p = 0.0040****	0.24 (0.55), p = 0.16	0.40 (0.73), p = 0.089
• Depth of knowledge	**0.71 (0.80), p = 0.011***	0.39 (0.85), p = 0.14	0.46 (1.03), p = 0.15
• Depth of understanding	0.17 (1.09), p = 0.61	0.15 (0.88), p = 0.58	0.35 (0.93), p = 0.21
• Substantive conversation	0.81 (1.44), p = 0.077	0.22 (1.05), p = 0.48	0.35 (1.28), p = 0.36
• Metalanguage	0.38 (0.93), p = 0.19	0.41 (0.93), p = 0.15	**0.38 (0.58), p = 0.046***
Connectedness	**1.69 (0.63), p < 0.0001****	0.21 (0.59), p = 0.23	0.42 (0.77), p = 0.085
• School subject knowledge is integrated	**2.0 (1.04), p < 0.0001****	-0.10 (1.09), p = 0.75	-0.35 (1.83), p = 0.52
• Link to background knowledge	**0.85 (1.04), p = 0.016***	0.60 (1.05), p = 0.074	**0.77 (0.92), p = 0.014***
• Connectedness to world beyond classroom	**0.96 (1.01), p = 0.0073****	0.42 (0.97), p = 0.16	**1.17 (1.23), p = 0.0071****
• Problem-based curriculum	**2.96 (1.08), p < 0.0001****	-0.063 (0.69), p = 0.76	0.10 (0.69), p = 0.61
Supportive Classroom Environment	0.12 (0.39), p = 0.32	0.22 (0.41), p = 0.09	**0.54 (0.40), p = 0.00074**
• Students' direction of activities	0.67 (1.40), p = 0.13	0.34 (1.34), p = 0.40	0.75 (1.23), p = 0.058
• Social support for student achievement	-0.15 (0.89), p = 0.58	0.25 (0.91), p = 0.36	**0.69 (0.85), p = 0.018***
• Academic engagement	**0.56 (0.89), p = 0.050***	-0.22 (0.92), p = 0.43	0.13 (0.92), p = 0.65
• Explicit quality performance criteria	0.06 (1.09), p = 0.85	0.29 (0.73), p = 0.19	**0.63 (0.99), p = 0.050***
• Student self regulation	-0.56 (1.17), p = 0.12	0.42 (1.17), p = 0.24	0.5 (1.01), p = 0.11
• Narrative	0.58 (1.16), p = 0.11	**0.88 (0.88), p = 0.0054****	0.67 (1.32), p = 0.11
Recognition of Difference	0.02 (0.55), p = 0.90	0.46 (0.72), p = 0.052	**0.60 (0.59), p = 0.0044****
• Knowledge explicitly values all cultures	**0.75 (0.54), p = 0.00057****	-0.10, (0.81), p = 0.69	-0.46 (0.81), p = 076
• Representation of non-dominant groups	0.04 (1.33), p = 0.92	0.57 (1.27), p = 0.15	0.50 (1.33), p = 0.22
• Group identities in a learning community	-0.13 (0.93), p = 0.65	0.38 (1.38), p = 0.36	**0.63 (0.77), p = 0.017***
• Active citizenship	**-1.15 (0.46), p < 0.0001****	**0.55 (0.59), p = 0.0089****	**1.69 (0.86), p < 0.0001****

Supportive classroom environment

Teachers noticed at once that students were engaged in inquiry lessons (similar reports are made by Kennedy, 2005), an observation supported by the data (t_{11}=2.2, p=0.050). Some areas declined initially; this was most evident in *Student Self-Regulation*, related to classroom management, where teachers may have felt uncomfortable with less control and higher noise levels. *Student Self-Regulation* was generally high in regular lessons ($\bar{x}$ = 3.9, s=0.43) and never reached this level in inquiry. These pedagogies typically took longer to improve, with only two significantly higher by year three (*Explicit Quality Criteria*, *Social Support for Achievement*). This suggests that developing a classroom culture of inquiry may be one of the most challenging aspects of teaching mathematical inquiry.

Recognition of difference

This category did not demonstrate strong growth in teachers' first year of teaching inquiry, particularly *Active Citizenship* which dropped dramatically. This may be a reflection of teachers' initial classroom management concerns (Makar, 2010). *Active Citizenship* improved substantially by year three, ending strong ($\bar{x}$ =3.5, s=0.75).

Discussion

This paper examined evidence of teachers' pedagogical shifts over time as they gained experience in teaching mathematics through inquiry with support. In regular mathematics lessons, every pedagogical cluster scored on average below mid-level (score of 3 on a scale of 1 to 5) and most ended well above mid-level by the third year of inquiry teaching (Figure 2). In some pedagogies, such as *Connectedness to the World*, the average for the regular mathematics lessons was disappointingly low ($\bar{x}$ =1.3). This may say as much about many regular maths lessons as it does about inquiry. If students are not made aware of the way that mathematics is used in the world, it is of no surprise that many students believe mathematics lacks relevance and choose to discontinue studying it (Australian Academy of Science, 2006; McPhan et al., 2008).

Many pedagogies improved in the first year, declined slightly in the second year, and improved again in the third year. This pattern suggests the importance of supporting teachers in the first year (Makar, 2010), throughout the second and into at least the third year where they are gaining confidence. Innovative pedagogies place significant demands on teachers, and targeted, timely support appears to be vital. Recognising that the primary goal of professional development is the *long term improvement of student learning*, Doerr et al. (2010) counsel that professional development must likewise focus on *sustained, long term change of teacher practice*. In particular, their review of the literature suggests that (1) extended time investment, (2) sustained support and (3) repeated opportunities for teacher learning over time are required if there is an expectation for teachers to demonstrate shifts in practice.

It is well established in the literature on teacher education that pedagogical change is difficult. The research reported here provides preliminary insights into the potential for teachers' pedagogical change when these three features—extended time investment by teachers and a teacher educator, sustained support and repeated learning opportunities—

are in place. The teachers in this study committed substantial amounts of time and energy in developing, teaching and reflecting on mathematical inquiry units for their students, the researcher invested hundreds of hours in classrooms observing individual teachers' lessons, and three to four days per year of professional development provided multiple opportunities for learning and reflection. Such a commitment from all parties questions whether this type of research can be scaled up. The next phase of the study (2009-2012) with over 40 teachers is currently underway, focusing on investigating and building foundations for a scalable model.

Practical implications

Teachers as well as those involved in teacher education and professional development must understand the nature of challenges and shifts associated with mathematical inquiry. For teachers, acknowledging that learning to teach mathematics through inquiry takes time can assist them in persisting through periods of frustration. Having a better understanding of the difficulties of learning to teach mathematics through inquiry may assist teacher educators in better supporting and validating teachers' experiences with inquiry pedagogies. The study reported in this paper suggests several practical implications for teachers, schools and teacher educators, including areas of greatest challenge, improved awareness and attention to the "implementation dip" of new pedagogical practices and the value of longitudinal professional development.

Patterns of pedagogical change and the "implementation dip"

Although the combined average of teachers' overall productive pedagogy score tended to rise as they gained experience (Figure 2), the changes did not happen in a linear, predictable fashion. In some areas, the improved practice was evident almost immediately. Even among those areas which improved in the first inquiry unit, the progression of pedagogical change in the following again was unpredictable. The pedagogical clusters of *Supportive Classroom Environment* and *Recognition of Difference* eventually rose significantly above that of a traditional mathematics lesson, but this took substantially more time.

Teacher educators, principals and policymakers need to expect rather than eschew the non-linear nature of teachers' adoption and adaptation of new pedagogical practices. This study is a reminder that new practices not only take time, but improvement pathways shift and turn in unexpected ways. For example in this study, dips and plateaus were evident in pedagogies from every pedagogical cluster. Although implementation dips have been reported in the literature (see for example, work by Fullan (2007) and Pendergast (2005) on implementing whole school pedagogical innovation), they are typically met with surprise and disappointment. In some cases, a judgement is made hastily that the dip indicates the new pedagogical practice has gone into disuse. Instead, implementation dips need to be acknowledged as a normal part of the process so that teachers are supported and encouraged to persist through them rather than left feeling guilty.

Sustained professional development

The data suggest that the teachers' overall pedagogical practices in the study improved within the first year of their engagement with inquiry (agreeing with research by Heck et al., 2008). It is important to note, however, that *sustained* engagement with professional development was likely needed beyond this first year to maintain and further improve practices. The plateau or "implementation dip" that appeared for many of the teachers after the first year strongly suggests the importance of this ongoing support through this period when teachers' initial engagement with inquiry may be starting to wane. The pressures of "performativity" (Ball, 2003) may have also amplified the dip as inquiry is sometimes considered to be at odds with accountability.

The design of this study ensured that the teachers received regular classroom support and professional development throughout the study. While it points to some positive outcomes of school-university partnerships, there are questions about whether this type of professional development can be applied more broadly. This and several other questions are raised by this study requiring further investigation.

- What models of "scaling up" improve inquiry-based pedagogies in mathematics more broadly (e.g., peer coaching, whole school adoption)?
- What supports can target an "implementation dip" to lessen its impact or duration?
- How can more long-term classroom-based professional development be encouraged? What aspects (e.g., classroom feedback, reflection, collaboration) are most critical?
- What other frameworks are effective for evaluating and self-assessing inquiry-based teaching practices in mathematics?
- How do teachers' experiences with inquiry-based teaching affect their teaching of regular maths lessons?

This study has implications as well for both teachers and teacher educators about the pedagogies of regular mathematics lessons. In particular, the pedagogy of *Connectedness with the World* was unexpectedly low. School mathematics is often poor at being explicit about connections between content being taught and the world beyond school walls. This study is a reminder of the importance in regular mathematics lessons of making the relevance and interconnectedness of mathematics explicit.

Conclusion

The findings of the longitudinal study previously published had been based on teachers' self-reports of challenges and opportunities in teaching inquiry (Makar, 2007, 2010) and case studies of exemplars of teaching and learning (Allmond & Makar, 2010; Makar & McPhee, 2009; Fielding-Wells, 2010). This paper extends the evidence base of this longitudinal study by presenting analyses of quantitative data from repeated observations of teachers' classroom practices over three years. As a body of work, the interweaving of multiple research approaches strengthens the overall message that implementing mathematical inquiry, while highly promising as a pedagogical practice, is challenging for teachers and requires substantial time and resources to operationalise.

In this study the changes to teachers' pedagogies were non-linear and did not follow a predictable curve. This is a timely reminder that facilitating teacher change is complex, even when teacher development strategies tick all of the "effective professional development boxes". Perhaps this is because innovation is not a process of adoption, but rather a process of implementation involving progress and outcomes that are, necessarily, highly reliant on interaction with the particularities of the local context (S. Anderson, 2010).

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RESEARCH PAPERS

INSTRUCTIONAL COHERENCE: A CASE STUDY OF LESSSONS ON LINEAR EQUATIONS

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In this paper we examine the nature of the instructional coherence across a series of lessons on linear equations. Using video and interview data from a Year 9 class in the New Zealand component of the *Learner's Perspective Study* (LPS) we explore how the teacher's pedagogical strategies associated with the selection and enactment of tasks and the action of 'sowing seeds' were key factors in establishing instructional coherence. We provide excerpts from classroom episodes to illustrate how instructional coherence supported students' learning of mathematics.

Introduction

It is widely agreed that raising achievement, especially for those groups of students who are currently underserved in our classrooms, is a priority focus for educational reforms. In New Zealand we have an array of policy initiatives, supported by a 'standards agenda', that are designed to identify and address underachievement. Underlying these policy initiatives is the belief that teacher quality—and thus classroom instruction—is a major determinant of student progress in schools.

We know that effective pedagogy can take many forms. Anthony and Walshaw (2009) in their research review of practices relevant to New Zealand education context offered ten pedagogical principles. However they note that "any practice must be understood as nested within a larger network that includes the school, home, community, and wider education system" (p. 6). In arguing that teaching is a holistic and complex endeavour, it is clear that other synthesis, especially those related to East Asian classrooms, may offer different combinations of key principles that define effective pedagogical approaches.

'Coherence' is one such factor that features in cross-national comparative studies. As reviewed by Chen and Li (2009), coherence is promoted as an important characteristic of mathematics classroom instruction in Asian countries. A review of earlier studies led these researchers to conclude that "coherent mathematics lessons can help lead to students' better mathematics learning with connected and coherent conceptual understanding" (pp. 711-2). But what does a coherent mathematics lesson—or unit of lessons—look like, and how might coherent instruction support student learning?

Instructional coherence

Instructional coherence is not a descriptor that is specific to the mathematics classroom. Indeed, Finley, Marble, Copeland, Ferguson and Alderete (2000) proposed that any teacher who brings “the components of the system—curriculum, instruction, assessment, external mandates, and community context—together intentionally with a focus on student learning” (p. 4) creates instructional coherence. Coherent instruction, they claim, supports teachers to make instructional decisions by using both the information collected in the classroom and information from external sources about what is important for students to learn.

Aligned with this perspective, existing studies on instructional coherence in mathematics classes have tended to explore the connectedness or integration of instructional elements. For example, Wang and Murphy (2004) defined instructional coherence as activities or events that are casually linked in terms of the structure of instructional content and the meaningful discourse reflecting the connectedness of topics. Schmidt (2008) argues that topics in mathematics “need to flow in a certain logical sequence in order to have coherent instruction” (p. 23)—a characteristic of mathematics curricula of top-achieving countries. As described by Fernandez, Yoshida, and Stigle (1992), lesson events that are coherent relate to each other in ways that allow students to infer relationships among events

Existing studies on coherence are largely sourced from cross-cultural comparative studies or Asian countries (e.g., Cai & Wang, 2010; Chen & Li, 2009; Shimizu, 2009). An impetus for studies in Asian countries originated in the widely disseminated finding by Stigler and Perry (1998) that found that both Japanese and Chinese mathematics lessons were structured more coherently than American lessons. It was noted that students in Japan would frequently spend an entire lesson studying one or two problems, a feature that was different to classes in American schools. Additionally, Hiebert et al.’s (2003) analysis of lesson in the TIMSS 1999 video study highlighted the explicit linking within the Japanese classroom that formed an identifiable lesson pattern. The lesson pattern was typically organised around (i) review of the previous lesson; (ii) presenting the problem(s) for the day; (iii) students working individually or in groups; (iv) discussion of solution methods; and (v) highlighting and summarising the main learning. Shimizu (2009) claimed that the ‘pulling together’ of the main points of the lesson, the ‘Motome’, was a key instructional factor to the effectiveness of the lesson.

Some studies suggest that coherency involves more than lesson structure as represented by sequencing of lesson events—it also involves the coherency of discourse that frames these events. For example, discourse associated with learning objectives may well serve to guide students learning in productive ways (Chen & Li, 2009). Conversely, when learning objectives override opportunities for students to build on their own thinking and reasoning, such discourse may limit opportunities for sense making (Askew, 2004). Discourse when viewed as a pedagogical tool also enables teachers to provide opportunities for students to participate in mathematical practices of argumentation. Such practices can help students build on their former mathematical knowledge, connect with their new knowledge, and comprehend their mathematical knowledge more deeply (Walshaw & Anthony, 2008).

Utilising a coherence lens, Sekiguchi (2006) characterised the effectiveness of a Japanese-style lesson by four aspects of the teacher’s classroom discourse management.

Rhetorical' management, or coherence between goals and discourse production, is organised by the lesson's "four-phase script" (p. 84) comprising the introduction, student working independently and in groups, student explanations, and teacher summary. Thematic' management involves the coordination of the related topics within and across lessons that comprise a particular mathematical theme. Referential management refers to the ways that the discourse participants—the students and teacher—keep track of referents during discussions. Strategies include the deliberate use of processes such as "naming, symbolizing, drawing, reviewing, summarizing, and using textbooks, blackboards, worksheets, notebooks, and projectors" (p. 86). "Focus' Management deals with strategies that direct students' attention to see the 'point' of the lesson, including the use of comparison and contrast, discussion, and summary.

Teacher knowledge is also a key factor in developing coherence across linked lesson events or activities. Ma's (1999) comparative study of teacher knowledge has been influential in highlighting the knowledge teachers were able to draw on as related to particular content topics, and how such knowledges influenced their instructional sequences for developing ideas and their access to students' thinking.

We see from the literature that whilst there is agreement that instructional coherence is desirable, there is also evidence that what defines coherence, or is key to obtaining coherence, may in some instances be culturally prescribed. While educational systems and curricula may support coherence (Leung, 2005), it is clear that individual teacher's enactment of pedagogical strategies in relation perceived students needs, mediated by individual teacher knowledge in its many forms, may serve to influence levels and qualities of instructional coherence.

The case study

As part of our participation in the New Zealand component of the *Learners Perspective Study* (LPS) the authors had sustained access to three different secondary classrooms. For one classroom our research team was particularly struck by the apparently seamless flow of the lessons across the unit of 10 lessons—creating a sense of coherency. It is difficult to describe the feeling of watching these lessons—for us as observers there was sense that learning was happening in a continuous fashion—that is, the learning trajectory seemed to evolve continuously rather than in discrete units defined by specific lessons objectives. The mathematics lessons of this classroom, we felt, could usefully be analysed using an instructional coherence lens. We were concerned to discern those key elements of the teacher's pedagogical practice and knowledge base that determined the observed instructional coherence across the sequence of lessons.

Case study context

The case comprises a unit of lessons from a New Zealand Year 9 (Grade 8) classroom in a large coeducational urban school, catering for students from, in the main, the middle socioeconomic sector. The classroom teacher, Dave, with 4 years experience, was identified by the local mathematics community as an effective practitioner. His class of 30 students was one of two extension classes at the Year 9 level in the school.

In an interview following the sequence of lessons, Dave explained his teaching goals for the unit as twofold: (i) for students to be able to solve and understand linear equations of the form ax ± b = cx ± d; and (ii) for students to develop an understanding of the meaning of equality (=). The sequence of lessons is summarised as follows:

- L1: Revisions of order of operations and algebraic notation and manipulation.
- L2: Solving 1-step linear equations of the form x ± b = c [using a working backwards model].
- L3: Solving linear equations of form ax ± b = c.
- L4: Solving linear equations of form ax ± b = c using function boxes.
- L5: Solving linear equations of form ax ± b = c [introduced fractions, decimals].
- L6: Real world applications of solving linear equations.
- L7: Review of definition of equations and refocus on the meaning of the equal sign, introduction of balance model to solve 3x + 4 = 2x + 9.
- L8: Use of balance model to solve ax ± b = cx ± d.
- L9: Solving equations of form ax ± b = cx ± d. Introduction to systems of equations with infinite or null solution sets.
- L10: Real world applications of forming and solving equations.

Data collection and analysis

The teacher and his students agreed that our team could collect video, interview, and observe across a sequence of 10 lessons that represented a unit on algebra—focused on solving linear equations. Video capture involved three cameras: focused on the teacher, a group of students, and the whole class. Post lesson video-stimulated recall interviews involving the teacher and students generated further data. Triangulation of the video and interview data was enhanced by reference to researcher classroom observation notes, photocopies of written work by the focus students, photocopies of class activities, and teacher questionnaire data (for a description of LPS research design see Clarke, 2006).

We adopted a three-pronged analysis of instructional coherence. Firstly, we examined coherence across the intended and enacted curriculum across the lesson sequence. The analysis of lesson content and specified teacher objectives for lessons was the main data source. Then we tracked the connections of mathematical knowledge within and across the lessons, looking closely at the nature of tasks and at links between previous learning/knowledges and new knowledge construction. Lastly, we used the post-lesson student interviews (two for each lesson)—focused on students' perspectives of their learning and the teaching process—to examine coherence in terms of zones of proximal development (Vygotsky, 1986).

Establishing instructional coherence

As we have seen in the literature review instructional coherence involves a combination of factors related to curriculum sequence, making connections within and between topics, creating clear organisational patterns and establishing social and mathematical norms, attending to and building on students' existing understandings and knowledge, to name a few. But what is not so clear is what specific pedagogical actions a teacher might take within a lesson sequence to ensure that coherence is developed and maintained. In this section we discuss two distinct pedagogical strategies that we claim significantly contributed to the observed instructional coherence: (1) springboard tasks, and (2) sowing the seeds. We provide excerpts from three lessons to demonstrate how these instructional approaches helped lead students to develop effective mathematical practices and sound mathematical understandings.

Episode 1: Introducing methods for solving linear equations

As was typical, Dave started lesson one (L01) with a set of student problems (see Figure 1). Some problems required students to access previous knowledge in the form of consolidation/practice tasks; others required students to use existing knowledge to move towards new knowledge. The intention of the latter task was that these would act as a springboard for new learning.

1. 5 x 14 - 9 2. 13 + 2 x 17 3. 3 + 8 x 7 - 10	4. $5 + 3^2$ 5. 111 - (17 x 3) 6. $(9 + 2)^2 + (3 + 2^2)$	7. 120 ÷ 12 x 5 8. 3 + (21 - 6 x 3) 9. 2.4 + _ x 2 = 5.8

Figure 1. L01 task.

Dave's instruction to the students that they should show their working reaffirmed the shared mathematical obligation (Cobb, Gresalfi, & Hodge, 2009) that was evident within the classroom environment:

> So all I want to do with this lesson is do a little bit of revision of the work that is going to be crucial to your understanding of the next topic. … the answers aren't necessarily the most important thing, but the process of how we get the answer is really important.

While students were engaged in this activity, Dave walked around the class providing individual assistance. Updating on progress, Dave remarked to the whole class: "I don't expect everyone to finish number nine. ...But if you have had a go at number nine we will get some answers of those."

In the whole-class discussion requests for solutions to problem 9 resulted in two alternative approaches. The first approach offered was: "I minused 2.4 from 5.8 and then what was left I divided by 2. Another student demonstrated his solution as follows: "2.4 plus 0.5 because 2.9 is half of 5.8." Dave's invitation for students to offer reasons why they might have two solutions, and whether or not both were correct, prompted students to refer back to the BEDMAS rules of operations. In summarizing their contribution Dave remarked:

> Okay, so another reason to be careful of BEDMAS is even if we don't know what this is, it is a number that I have smudged. But it is still a number and normal rules work—times before plus.

Utilising the student contributions, Dave drew attention to the new idea of 'working backwards' remarking that, "We are working backwards. Why did he undo the plus first when BEDMAS says do times first?" He provided justification for the method that would be revisited in Lesson 2 in the context of solving one-step linear equations:

In the post-lesson interviews one student confirmed the expectation they have a go at the springboard problems:

> Pat: Question 9 was easy to understand. I just subtracted the 2.4 and went on from there.
> I: What made you subtract?
> Pat: I reversed what I would usually do and it just worked out.

I: So is the reversing thing not a foreign thing for you to do in maths even though you haven't really been formally taught?

Pat: It was quite unusual but I just automatically tried to reverse it and see how it would work out, and it worked out.

However, the second student, Vanessa, reported that she was unable to work out how to do number nine in the first instance. In the post-lesson teacher interview it was apparent that the choice of task was planned to support on-going learning linked to previous learning. Dave offered a metaphor of "sowing the seed" as follows:

> I just really wanted to make sure they were comfortable with that before the second day which was the start of solving equations. … so I sowed the seed in the first lesson with the idea of working backwards of solving an equation.

Episode Two: Introducing a balance model

The second lesson (L02) began with a discussion of the strategies for solving a set of puzzles that had been set as homework in L01. One of the puzzles is shown in Figure 2.

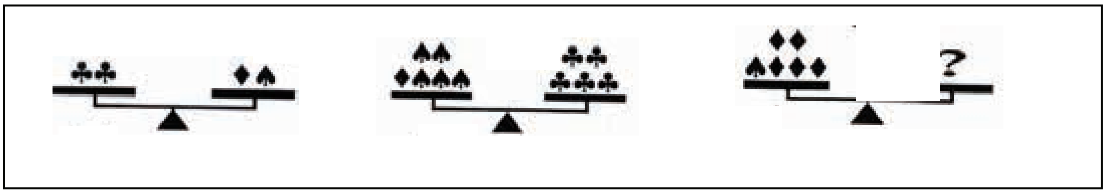


Figure 2. The starter learning task in L02.

Early in the discussion it became apparent that some students had used a trial and error method to balance the scales. After further discussion the teacher drew the students' attention to the strategy of keeping the scales balanced:

> What I want you to be thinking about is the strategy of keeping the scales balanced. That is how Henry and Jack and some of you others managed to work it out. And that is a theme that we are going to revisit over the next few

In the post-lesson interview Dave indicated that the use of the balanced scale puzzles was a precursor to extending their understanding of the meaning of the equal sign:

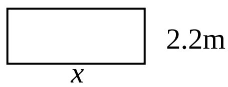
> Later on they are going to start doing complicated equations and they are going to need to understand that the equal sign doesn't just mean calculate and up until now most of them think the equal sign means 'works out to be' or 'I get this'. .. I am going to have to adjust their view of what the equal sign means and think of it in terms of balance scale.

The post-lesson interview with the two students Pat and Ruth confirmed their struggle with the puzzle activity. Sowing the seeds early in the sequence of lessons possibly was an indication that the teacher had a strong sense of his students' need to revisit these ideas over an extended sequence of lessons. The idea of the balance model to solve equations was not formally introduced until lesson 7.

Episode three: Using equations in solving practical problems

The fifth lesson began with students working on a problem set that included two challenge problems:

> Challenge 1: The perimeter of this rectangle is 15m.
> Form and solve and equation to calculate the length of this rectangle.



Challenge 2: A rectangle has an area of $72m^2$. Its length is twice its width. Calculate the perimeter.

After a brief review of the first four problems, Dave invited one student Charley to present his thoughts about the fifth problem. While Dave accepted Charley's solution method, he pressed his students to apply their current learning to the challenge problem:

Charley: I did 2.2 times 2 is 4.4 and then I did 15 minus 4.4 which is equal to 10.6 so then I divided it by 2 and got 5.3.
Dave: What does 5.3 represent?
Charley: The length.
Dave: The length. Who agrees that that is the length? Anyone like to suggest another method that uses that x? Henry.
Henry: I did 2x plus 2.2 times 2. Like the way it is up there and so I worked backwards.

Even though Henry tried to use x to solve the unknown Dave engages in further probing aimed to highlight the formation of an equation:

Dave: Where did you get your 2x from?
Henry: The fact that x equals the two lengths.
Dave: Right, opposite sides of a rectangle have the same length. If that is x, that's x, and that's 2.2, and this might be 2.2. What does perimeter mean?
Henry: All of the sides added together.
Dave: And what does all of that [2x + 2.2 × 2] equal if we add them all together?
Henry: I don't know.
Dave: Have we been given more information in that question?

The students' preference to use pre or partial algebraic process to solve the unknown was again observed in the discussion of the sixth problem. Here, Dave required a student to explain his solution of 6 + 6 + 12 + 12 = 36 to the sixth problem as follows:

Dave: So how did you get that one must be twelve and one must be six?
S: Because it said that the length must have been twice the width.
Dave: Good so did you use the x at all?
S: No, I didn't.
Dave: Okay, that is a really good method the guess and check … I would like if you have had a go at this to try and form an equation like we did for the last one and solve that equation to get this answer which is the perimeter.

The intent to introduce a new method with this springboard task was confirmed by the teacher in the post-lesson interview:

> The main goal was partly to cement their ideas of how to solve equations really and also to introduce the idea of using equations to solve problems. …I deliberately try and push them as far as I could today. …But I don't mind doing that because when I go back on Monday …I am going to give them exercises out of [text] and after what I have done with them today they are hopefully going to find it straightforward.

Dave's pedagogical decision to return to the problem was informed by his knowledge about his students' learning potential and current understandings:

> I felt at this stage if I take it any further I was going to lose some of them, I was hoping someone would come with ... we choose 72 so that when they divide by 2 they are going to get 36 and they would be able to spot that if 'x square' is 36 they will spot 6 they wouldn't have necessarily gone onto explanation on how we got that 6. So I was hoping someone would have come up with that equation in which case I would have followed it through. But I felt at this stage the only solution I got were the guess and check points, so

> I thought I would sow the seed that there's an algebra method there but I felt if I was going to carry on too far I was going to lose too many of them.

Discussion and implications

In the foregoing section we provided three episodes to elucidate the coherence of Dave's instruction. These episodes were selected to exemplify two instructional approaches that were regularly observed in the 10 lesson sequence: one concerns the nature and enactment of the tasks; the other is the teacher's action of "sowing seeds".

The teacher regularly posed a set of tasks for students to work on prior to new instruction. Typically, the first a few problems acted as revision or consolidation activities, while the last few problems were challenging. These 'springboard' tasks generated new ideas that were central to learning goals that were more fully developed later in the lesson, and or revisited in subsequent lessons. We claim that this instructional approach supported coherence as characterised by the connections of students' existing (and prior) knowledge to new knowledge. The success of the linking was supported by the opportunities for student to work independently on the problem set prior to the more formal introduction of new knowledge and the subsequent obligation for students to explain and justify their thinking through class discussion.

In our exploration of Dave's metaphor 'sowing the seeds', the 'seeds' correspond to the multiple layers of new knowledge and methods embedded in the intended curriculum. It appears that Dave sowed seeds with a careful consideration of the distance between the actual and potential development of individual students. These seeds assisted Dave to plan a logical sequence of knowledge construction that builds and links to students' existing and emergent ideas. Such orientation to and anticipate of further learning, we claim, is a hallmark of coherence.

In combination, sowing seeds and the use of springboard tasks supported the provision of appropriate challenge for students and affirmed the expectation that struggling with the task was the norm and that not all students would be immediately successful. We hypothesize that Dave's expectation that students needed to experience struggle as a natural way of learning was indicative of his awareness of the need to establish appropriate zones of proximal development for his students. As Dave reflected in a post-lesson interview: "I deliberately try and push them as far as I could today (to introduce the idea of using equations to solve problems)". The press for students to engage and resolve springboard tasks, combined with the sowing of seeds, appeared to be a conscious way the teacher scaffolded students within their zone of proximal development. However, an overriding question remains concerning the reasons for the implementation and effectiveness of the two instructional approaches: How effective would these strategies be in other classes, especially those classes which contain mixed or predominantly low-achievement levels?

This analysis is limited in that it offers an insight in to one classroom. But as noted earlier, this classroom stood out for us in terms of the seamless, almost invisible, way that learning and teaching appeared to be structured. Attending to features of coherence, a concept that is more typically applied to Asian classrooms studies, has provided a useful lens to look at how this teacher promoted mathematical learning.

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TEACHER AND PRESERVICE TEACHER BELIEFS ABOUT MATHEMATICS TEACHER EDUCATION

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This paper reports on the perceptions of mathematics education of in-service and preservice primary school teachers involved in an innovative model trialled in the final mathematics curriculum unit of a B.Ed. program. Questionnaire items asked about the value of time spent in classrooms, the importance of theoretical understandings, and of linkages between theory and practice. Both groups reported valuing time in schools, understanding the theories that underpin practice, and lecturers with recent classroom experience, but there were also interesting differences between the groups at the beginning and at the end of the unit, and some change for each group.

Introduction

Concerns about best practice and pedagogy for mathematics teacher education and the perceived theory-practice divide have been raised by researchers, teacher educators, school educators, and the public (Kolthargen, Loughran, & Russell, 2006). There is a perceived gulf between the pedagogies that preservice teachers are introduced to and encouraged to adopt through their education courses, and the practices they encounter in classrooms (Kolthargen et al., 2006; Taylor, 2002). Preservice teachers report dissatisfaction with what they have learned in their teacher education programs (Australian Secondary Principals' Association, 2007) considering some of it irrelevant (Kolthagen, 2010; Shuck, 1996). Consistent with this, there is evidence that they consider the most valuable aspects of their university courses to be those which have the most apparent relationship to classroom practice (Beswick, 2006; Shuck, 1996). Klein (2006, p. 335) suggested that "preservice teachers' ways of being a teacher of mathematics has less to do with theory and policy than their previous (and current) experiences of institutionalised teaching and learning."

The study reported here was designed to examine the potential, in terms of closing the theory-practice divide, of closely linking university learning experiences with classroom practice in the context of the final mathematics curriculum unit of a primary (Grades 3-6) and early childhood (Grades K-2) Bachelor of Education program. The specific research questions addressed were;

1. To what extent do preservice and practising teachers share beliefs about mathematics teacher education?

2. How might these beliefs be influenced by a mathematics curriculum designed to link university and school contexts?

Relevant literature related to nature and origins of the perceived gap and to influencing belief systems is reviewed in the sections that follow.

Bridging the gap

What preservice teachers view in schools during their practicums has a profound effect on their view of what is best practice in mathematics teaching (Beswick, 2006; Kolthagen, 2010; Shuck, 1996) and often this reinforces preconceived ideas of teaching pedagogy which were formed during their own schooling, and that are contrary to understandings that their teacher education courses are designed to develop (Beswick, 2006, Klein, 2006). Consistent with this, Calderhead and Robson (1991) found that preservice teachers' experiences and beliefs held from their own education influenced the pedagogy they used in classroom teaching and their ability to make the transition to new ideas presented to them. Kolthagen et al. (2006) suggested that programs need to focus on the preservice teacher as a learner, able to reflect on experiences and practice and to be able to analyse and make meaning from them.

The perception of a theory-practice divide is shared by many practising teachers and may be linked to distorted recollections. The ability to recall events accurately naturally declines over time (Basden, Reysen, & Basden, 2002). This fact and the propensity for people to form false memories, perhaps influenced by recollections of others shared and reinforced in social contexts such as school settings, can result in a lack of realisation by teachers that many of the practices they use are in fact linked to their university teacher education courses (Basden et al., 2002; Beswick & Dole, 2008).

Allen, Butler-Mader, and Smith (2010) argued that the theory-practice gap can be bridged by forging university school partnerships. Their study involved the recruitment of practising teachers as secondees and sessional tutors to a university as part of a university school partnership. Such an approach was supported by Nelson (2005, cited in Allen et al., 2010 p. 623), in his role as Australian Minister for Education, who commented that, "Many who train teachers do not see themselves as members of the teaching profession itself. Perhaps we need more teachers in universities with teaching appointments". However, according to Allen et al. (2010, p. 622), school personnel working in universities still "saw the work as separate and distinct from their work in schools" and the study identified the need for ongoing communication and the sharing of ideas between the university and the schools involved.

An important role of mathematics educators is to influence preservice teachers to teach differently from the ways in which they were taught (Goos, 2009). Because preservice teachers value lecturers who are enthusiastic and passionate about mathematics, and know their subject (Beswick & Dole, 2008; Hill, Lomas, and McGregor, 2003), the credibility of lecturers and tutors themselves may have an impact on helping preservice teachers to embrace new ideas. In fact Hill et al. (2003) found that the quality of lecturers was one of the two most influential factors in determining the quality of a preservice teacher education program, and that this was influenced by the lecturer's expertise in school classroom contexts. Programs such as that which formed the context of this study have the potential to strengthen lecturers' knowledge of school classrooms and build their connections with the contexts that preservice teachers value, thereby enhancing their credibility and influence and hence the value attached to university aspects of mathematics teacher education.

Beliefs systems and change

A further aspect of the theoretical underpinnings of the study lies in understandings of beliefs and the conditions under which they are most likely to change. Beliefs are understood as anything that a person regards as true (Beswick, 2007) and, consistent with a constructivist view, as distinguishable from knowledge only in terms of the degree of consensus that they attract (Beswick, 2011; Guba & Lincoln, 1989). Greens' (1971) widely accepted description of belief systems in which beliefs are characterised by varying degrees of centrality (a function of the number and intensity of connections with other beliefs), and subject to clustering, whereby parts of an individual's belief system can be held in isolation from other beliefs, is foundational. Belief systems are also dynamic with the relative centrality and influence of beliefs shifting according to the context (Beswick, 2003).

Clustering can result when beliefs arise in differing contexts. For example, beliefs about teaching that originate in an individual's experience of teaching as a school student and beliefs about teaching that are formed in the context of university based teacher education may be held in distinct clusters. Belief clustering provides an explanation for the ability of teachers to endorse the aims of teacher education programs whilst simultaneously agreeing with apparently contrary practices in a school context.

Together belief clustering and the dynamic nature of their interconnections explain why preservice teachers so often revert to teaching in the ways that they were taught (Ball, 1990). Classroom contexts evoke beliefs formed in similar contexts as students, and these may not have been reconciled with contradictory beliefs formed subsequently. It is these classroom connected beliefs that exert the dominant influence on practice in that context. Awareness of a disjunction between beliefs about teaching that underpin practice and those that are promoted in teacher education programs may lead teachers to rationalise the difference by rejecting those perceived as less relevant and adopting the notion of a theory-practice divide. Bridging the gap can therefore entail substantial and onerous intellectual work and requires that teachers have the opportunity, time and support to work through the process to arrive at an integrated system of beliefs about teaching and hence more balanced views of the benefits of the university and school based aspects of their courses. Experiences that closely link university and school based learning might provide such an opportunity.

The study

Consistent with the literature, the project that formed the context of this study was embedded in a partnership between schools and a university in which connections and communication were forged. Importantly, the partnership was initiated by a school principal who saw mutual benefits for school and preservice teachers. Opportunities were provided for students to apply their theoretical understandings and knowledge of mathematics teaching to a classroom situation in partnership with classroom teachers and university personnel. In this way preservice teachers and the practising teachers who acted as their mentors were assisted to marry new ideas with their own.

The final mathematics curriculum component of the B.Ed. (Primary and Early childhood) program in within which the study was conducted aimed to bring together aspects of the preservice teachers' knowledge described by Shulman (1987), namely their knowledge of mathematics content, general pedagogy, mathematics curricula, students as learners of mathematics, and pedagogical content knowledge for mathematics teaching, applying them to the classroom context. The previous units in the mathematics curriculum sequence had been half-units comprising weekly 1-hour lectures and 1-hour tutorials over a 13-week semester. The students had also had the opportunity to study some mathematics education elective modules.

Participants

Ninety six of the 106 preservice teachers enrolled in the fourth and final half unit of mathematics curriculum in the B. Ed (early childhood and primary) course at the University of Tasmania participated in the study, along with 32 teachers (referred to as mentor teachers) and school leaders from three primary schools. The preservice teachers who chose not to participate in the study were involved in the unit in exactly the same ways as those who did but simply opted not to submit data.

One of the schools involved was a small (enrolment of approximately 160) city school in a socio-economically disadvantaged area and the other two schools had approximate enrolments of 260 and 380 and were in moderately socio-economically disadvantaged areas. The smaller of these schools was an outer city suburb with an intake from some country areas as well as adjoining suburbs. The other was an inner suburban school in a smaller regional city.

Questionnaires

Data were collected in a range of ways including interviews, field notes, and classroom observations but only the questionnaire data are relevant to the current study. Participating mentor teachers, principals and preservice teachers were invited to complete pre- and post- questionnaires. The initial questionnaires were identical for all groups and comprised six sections that asked about: (1) expectations of the project; (2) confidence to teach mathematics; (3) beliefs about mathematics and numeracy in everyday life; (4) beliefs about mathematics in the classroom; (5) beliefs about mathematics teacher education; and (6) the respondent's role, gender, school or campus. Sections 2, 3, 4, and 5 comprised items requiring responses on 5-point Likert types scales such that 5 represented the highest level of agreement or confidence and 1 the lowest. Section 5 on Mathematics teacher education is relevant to the current study and its 16 items are listed in Table 1.

The final preservice teacher questionnaire repeated all of the sections from the initial questionnaire that required responses on Likert type scales whereas the final teacher questionnaire (also completed by principals) repeated only the section on mathematics teacher education. Both final questionnaires contained additional open-response items focussed on evaluation of the unit. In all cases responses were anonymous with respondent devised codes used to match responses across the two surveys.

Procedure

Prior to the start of the unit the preservice teachers were randomly placed in groups of four with a mentor teacher from one of the three schools involved in the project. This meant that students may not have been working in their chosen specialisation (early childhood (Grades K-2) or primary (Grades 3-6)). This was appropriate because the degree towards which the preservice teachers were working qualified them to teach from K-6. The randomised allocation to groups was also designed to mirror the realities of working with unfamiliar colleagues in school settings.

The initial questionnaires were distributed and completed in meetings that involved preservice teachers and mentor teachers on each of the two campuses where the program ran. The main purpose of these meetings was to introduce the unit structure and provide opportunities for the principals to address the preservice teacher cohorts and for initial meetings of preservice and mentor teachers to occur.

Mentor teachers identified the school students with whom the preservice teacher groups would work, and the first 6 weeks of the semester were used for collaborative planning by preservice and mentor teachers, administration of agreed pre-assessment tasks to the small groups of students with whom the preservice teachers would be working and for preservice teachers to become familiar with the classroom environment and particular students to which they had been assigned. Preservice teachers also attended mathematics education workshop/tutorials at the university. These 2-hour sessions focused on assessing and planning models, mathematics curricula, creating a positive classroom climate, use of ICT, and the mathematics knowledge required for teaching. Individual groups met with their university lecturers for further pedagogical and content support as they were planning and designing assessment tasks and analysing student responses to these.

In the following weeks the preservice teachers worked in the schools for six weekly sessions and had ongoing meetings with their mentor teachers. University staff maintained contact with the preservice teachers and school personnel and visited the schools several times. At the end of the semester the preservice teachers, teachers and school leaders participated in a meeting and celebratory afternoon tea to share experiences and highlights of the project. The final questionnaires were administered at these sessions.

Assessment of the unit was entirely separate from the research and required preservice teachers to submit reflective journals detailing their learning from the experience and philosophical statements relating to their beliefs about mathematics teaching and learning. Pre- and post-project comparisons of questionnaire responses were made using t-tests and effect sizes, d, calculated as described by Burns (2000).

Results and discussion

Table 1 shows the means and standard deviations for the teacher and preservice teacher responses to each of the 16 items about mathematics teacher education on the initial and final questionnaires. There were many fewer responses to the final questionnaire, particularly from preservice teachers. This reflects the much lower attendance at the final meeting as a result of the competing priorities for preservice teachers’ time at the ends of semesters.

Table 1. Teachers' and preservice teachers' beliefs about mathematics teacher education.

Item	Teachers				Preservice teachers			
	Initial mean	Initial SD	Final mean	Final SD	Initial mean	Initial SD	Final mean	Final SD
	(n=32)		*(n=20)*		*(n=96)*		*(n=27)*	
1. The more time preservice teachers spend in schools and classrooms the better.	4.77	0.43	4.75	0.44	4.62	0.63	4.63	0.63
2. It is important to understand the theories on which teaching practices are based.	4.20	0.71	4.35	0.59	4.24	0.75	4.27	0.67
3. All aspects of teaching can be learned in schools and classrooms.	2.93	1.02	3.10	0.97	3.20	1.16	3.23	1.14
4. What is taught at university about maths teaching is useful in the classroom.	**3.47**	0.78	**4.05**	0.51	3.71	0.74	4.00	0.69
5. The classroom teacher is the most important influence on school students' mathematics learning.	3.60	0.81	3.80	0.89	3.88	0.75	3.84	0.85
6. All aspects of teaching can be learned at university.	1.80	0.81	1.60	0.75	1.98	1.03	1.85	1.12
7. Working with individual students is a useful part of teacher education.	**4.10**	0.80	**4.55**	0.61	4.20	0.71	4.48	0.65
8. Teachers can easily describe the reasons for their teaching decisions.	3.70	0.79	**4.00**	0.86	3.45	0.77	**3.19**	0.85
9. Regular time in school classrooms throughout the semester is more effective than blocks of time.	3.97	0.81	3.85	0.93	3.88	0.90	4.04	0.87
10. The university teacher is an important influence on preservice teachers' learning about mathematics teaching.	3.83	0.75	4.10	0.64	4.04	0.75	3.88	0.82
11. It is important that teachers can articulate the theory that informs their teaching decisions.	3.90	0.80	4.20	0.70	3.96	0.75	4.12	0.77
12. I can see connections between what I have learned about teaching maths at university and working in school settings.	3.55	1.15	**3.65**	0.67	3.93	0.70	**4.12**	0.77
13. It is important that lecturers have recent classroom teaching experience.	4.37	0.67	4.50	0.61	**4.33**	0.72	**4.77**	0.43
14. Spending time in schools and classrooms is not necessarily beneficial.	**1.77**	0.73	***1.30***	0.57	1.83	1.10	*2.23*	1.42
15. University and school based learning experiences are equally important.	3.63	0.89	3.80	0.83	3.73	0.86	3.69	1.09
16. Analysing the work of individual students can provide important insights into mathematics teaching.	4.23	0.82	4.50	0.51	4.19	0.67	4.23	0.65

On average, the preservice teachers and their mentor teachers agreed at both the start and end of the project with Items 1, 2, 7, 13, and 16. These concerned the value for preservice teachers of spending time in classrooms, working with individual students, and analysing students' work, as well as the importance of understanding the theoretical bases of teaching practices, and having lecturers with recent classroom experience. Both groups of participants at both survey administrations disagreed on average with statements that "All aspects of teaching can be learned at university" (Item 6) and that, "Spending time in schools and classrooms is not necessarily beneficial" (Item 14). Although consistent with the literature documenting preservice teachers' valuing of classroom based learning (Beswick, 2006; Schuck, 1996), these results suggest that both preservice and inservice teachers also regard theoretical understandings of their work as important.

Pairs of significantly different means are in bold or, in the case of Item 14 for which there was a significant difference between the initial and final means of the teachers' responses, and also between the final mean for teachers and the final mean for preservice teachers, one significantly different pair is bold and the other italicised. There were no significant differences between the overall views of the preservice teachers and their mentor teachers at the start of the semester, but there were differences for three items at the end. Preservice teachers finished the unit less inclined than their mentor teachers to agree that teachers can easily give reasons for their teaching decisions (Item 8, $t(44)=3.18$, $p=0.003$, $d=0.94$) and more likely to agree that they could see connections between their university learning about mathematics teaching and their work in school settings (Item 12, $t(44)=-2.06$, $p=0.046$, $d=0.54$). Their disagreement with Item 14, that time in schools is not necessarily beneficial, was on average less strong than their mentor teachers' at the end of the semester ($t(44)=-2.75$, $p=0.009$, $d=0.65$). In each case the effect sizes were moderate to large (Burns, 2000).

The direction of change of the means for Item 8 for mentor teachers and preservice teacher indicate that the difference between their responses at the end of project resulted from a combination of the changed levels of agreement of the two groups. Increased agreement on the part of the mentor teachers that they could articulate reasons for their teaching decisions, perhaps as a consequence of needing to do so in their work with the preservice teachers, made a contribution. In addition, preservice teachers adopting a more critical stance in relation to teachers' knowledge and decision making also contributed to the significant difference for Item 8. Both changes can be regarded as positive outcomes of the approach. There is also evidence from Item 12 that the project assisted preservice teachers to connect their learning in the two contexts. The difference for Item 14 is a consequence of stronger disagreement on the part of mentor teachers and weaker disagreement on the part of preservice teachers that time in schools is not necessarily beneficial. This suggests that the mentor teachers may on average have viewed preservice teachers' involvement in their classrooms more positively than the preservice teachers themselves did.

The mentor teachers' views differed from the start to the end of semester for three items. They were more inclined at the end to agree that both university learning about mathematics teaching (Item 4, $t(48.0)=-3.21$, $p=0.002$, $d=0.75$), and work with individual students were valuable (Item 7, $t(48)=-2.13$, $p=0.038$, $d=0.56$). They disagreed more strongly than before that time in schools was not necessarily valuable

(Item 14, $t(46.6)=2.53$, $p=0.015$, $d=0.64$). The project thus appears to have influenced mentor teachers to value university learning more highly while at the same time reinforcing the value they attach to preservice teachers spending time in schools. The only change for preservice teachers was towards stronger agreement that their lecturers should have recent classroom experience (Item 13, $t(69.1)=-3.85$, $p=0.000$, $d=0.61$). Given the other changes noted this difference may reflect the preservice teachers' appreciation of the way that their lecturers were able to work with the schools and to mediate their involvement in the school context. Although beyond the scope of this project it is possible that, consistent with Hill et al.'s (2003) reasoning, the lecturers' ability to perform this role enhanced their status and influence with the preservice teachers.

Conclusion

In terms of the research questions, this study provides evidence that inservice and preservice teachers share similar beliefs about mathematics education. School placements are therefore likely to reinforce preservice teachers' beliefs in the value of classroom experience in learning to teach (Beswick, 2006; Schuck, 1996). However, the data also show the potential of integrated school and university programs such as that described here have the potential to influence the beliefs of both inservice and preservice teachers towards a more balanced view of the worth of university and classroom based learning.

The data suggest that inservice and preservice teachers ended the program valuing both classroom practice and the theories on which it is based. There was a significant increase in beliefs that what is taught at university is useful in classrooms. This suggests that working with university courses and preservice teachers may help practising teachers to see the connection between theory and practice and may counteract some of their distorted recollections of their own teacher education courses (Basden et al., 2002; Beswick & Dole, 2008). The study also suggests that strong links made between university courses and practice, and strong communication pathways between school personnel, university lecturers and preservice teachers as in this study may assist preservice teachers to make connections between their learning in the two contexts.

Preservice and inservice teachers agreed that working with individual students can provide important insights into mathematics teaching. There is likely, therefore, to be value in strengthening this element of preservice mathematics education courses even in more traditional contexts where work samples and video excerpts can be used.

The study also raises questions about the implications of preservice teachers' valuing of lecturers with recent classroom experience. Specifically, what qualities of these lecturers are considered important by preservice teachers? And to what extent does the status that experience affords them affect their ability to influence students' beliefs?

Although ideas from the beliefs literature constituted part of the theoretical framework of the program and have explanatory power in terms of teachers' practices, little is understood of the ways in which particular beliefs interact and are influenced by the myriad factors involved in learning to teach mathematics. There is a need for fine-grained in-depth studies using mixed methods to chart the changes in individual's beliefs, including about mathematics education, and the factors that trigger them.

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UNSCRIPTED MATHS: EMERGENCE AND IMPROVISATION

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It may seem that improvisational drama and primary mathematics are two diametrically opposed disciplines, the former being based around emergence and uncertainty and the latter based around predictability and certainty. In this paper I argue that creative mathematics teaching and learning requires a certain amount of unpredictability and that, particularly with regard to problem solving, learners' solutions have a certain quality of emergence that is not dissimilar to improvisational drama. By examining children's solutions to non-routine problems I consider what mathematics education might gain from attending to the discipline of improvisation.

Theoretical background

Collaborative emergence

Sawyer (2001) traces the origins of the concept of emergence to work in 1875 by the philosopher George Henry Lewes and Lewes' distinction between two types of effects: resultants and emergents. The main qualities of emergent effects, Sawyer argues, are that outcomes cannot be fully understood or predicted by studying the constituent parts, as illustrated by Lewes' example of the effect of water emerging from the combination of oxygen and hydrogen. Understanding the properties of water cannot fully be achieved by reduction to the study of the properties of oxygen and hydrogen (although quantum physics now overturns this claim). This non-reductionist aspect of emergent phenomena means that they are multiplicative rather then additive in their nature (Davis & Simmt, 2003). Sawyer does not define resultant effects but I take these to be those effects that are predictable through the study of their component parts, typified by the behaviour of billiard balls.

Although the concept of emergence has been developed since Lewes' time, particularly in the physical sciences, it probably began to have most impact on educational research with the development of artificial intelligence systems that displayed intelligent behaviour based on simple, local rules of interaction and without the need for a central leader. Thus models of how insect colonies create complex structures or birds fly in symmetrical flocks became canonical examples of emergent systems (Clark 1997). From these simple forms of emergence it has generally become accepted that group behaviour can be considered as emergent when there is no structured plan for the group to follow, and where there is no leader directing the group (Sawyer, 1999). Classrooms and students are, however, fundamentally different from anthills and ant, or flocks and seagulls, in the range of actions and agency available to the participants. To distinguish between systems where there is interaction but not agency, in the sense that individuals within the system can intentionally change the direction of what is emerging, I am using Sawyer‘s phrase of *collaborative emergence* to encompass phenomena ―that result from the collective activity of social groups" (Sawyer, 1999, p. 449).

Whilst not necessarily using the terminology of collaborative emergence, most teachers and researchers might consider group behaviour as emergent when there is no pre-determined plan or script that a leaderless group is following. In the context of this paper, the groups that I am considering to be engaged in collaborative emergent activity are pairs of children working on finding a solution to non-routine mathematics problems. As the children were not given any direction from the teacher as to how to solve the problems, nor were they assigned particular roles within their pairs (in particular, neither child was asked to act as ‗leader‘ of the pair) their problem solving activity fits with Sawyer‘s criteria to be classed as emergent.

A key theoretical and analytical shift in treating group activity as collaboratively emergent is that ―interaction among constituent components leads to overall system behaviour that could not be predicted from a full and complete analysis of the individual components of the system" (Sawyer, 2000, p. 183).

Performance and improvisation

Performance in some of the educational literature has perjorative overtones. For example, Dweck (2000) talks of ‗performance oriented‘ learners as learners who are keen to be seen to ‗performing‘ in correct and acceptable ways and that as such ‗performance‘ is not always linked to understandings. Similarly, there are overtones sometimes of being taught to ‗perform‘ in the ‗training‘ sense of the word.

In contrast to such views of performance as not creative or allowing for agency, I am using the term in the sense used by Holzman (2000), in that the majority of our activity could be thought of as having an element of performance, and that one reading of Vygotsky is that we learn and develop through performing.

> Performative pyschology is based in an understanding of human life as primarily performative, that is, we collectively create our lives through performing (simultaneously being who we are and who we are becoming) (Holzman, 2000, p. 88)

Although very young children learn to talk through joining in performances of conversations that are co-created and improvised between the child and more experienced others, as they grow older much of what children learn becomes routinized and rigidified into behavior (Holzman, 2000). An important distinction that Holzman makes here is between behaviour and activity: the former being a focus on the ‗self-contained individual‘ and activity as what people engage in together ―rather than as the external manifestation of an individualised, internal process" (Holzman, 2000).

One activity that adults engage in which is clearly performative, in the sense of collectively creative, is improvisational drama, in which actors create scenes without a pre-determined script. I explore here ways in which problem solving could be considered similar to improvisational drama. Of course much of what passes for problem solving in school mathematics would be better described as exercises in that the method of solution is, in a sense, scripted and all the performer (child) has to do is replace certain elements. But problems for which pupils do not have a ‗script' could, I argue, be understood as involving improvisation. Together, improvisation and emergence provide lens for examining problem solving activity and raise questions about assumptions and practices in teaching primary mathematics.

An example of collaborative emergent problem solving

The school context

This example comes from a two-year teaching experience in a primary school, Bow Bells, in the east end of London. The school is located in a traditionally working-class area but more recently there was also a high immigrant population, with many of the children starting school speaking almost no English. On measures of performance judged by national tests, only around 45% of pupils at age eleven were attaining the expected level 4 in the tests, compared with government targets of 80%. Inspection reports painted a picture of a school in difficulty, a consequence of which was that teachers were reluctant to apply to work there. The school was thus in a downward spiral. To counteract this, the local education authority had put in a new head-teacher, a specialist in literacy.

At that time I was looking to go back to do some school teaching. Several years of my own research had revealed little evidence of the sort of problem solving that was written about, and I had begun to wonder if teachers were right in sometimes thinking that academics in their ivory towers had got it wrong and that, given the constraints of schools, problem-solving based teaching was not possible. In approaching a local authority for a school to work in, Bow Bells was suggested.

At initial meetings with the teachers, two things were frequently commented on. First, teachers would talk about the limited language facility of the children (even for those children for whom English was their first language) and that consequently there was little point in asking the children to talk about mathematics. Second, and linked to the first point, there was a general sense that the children had little to contribute to mathematics lessons: it was important to equip children with the ‗basics' before they would be able to engage in any form of problem solving. This attention to the ‗basics' permeated throughout the school from the classes of five-year-olds to the eleven-year-olds and the predominant style of teaching across all the years was one of the teacher demonstrating a method on the board and the children subsequently completing practice worksheets.

The local authority was able to provide money for support in mathematics and so a colleague, Penny Latham, and I were able to work there more intensively than I originally anticipated: I was there one day a week for eight weeks each term and Penny there for two days a week, both of us over the course of two years. We agreed with the staff that our main focus would be on supporting the children in being able to talk about mathematics and to develop their mathematical understanding from problems and problem solving.

I have set out this context at some length as I want to make it clear that the children we were working with were not ‗privileged' and did not have the kinds of teaching that might pre-dispose them to finding solutions to problems without being shown a method for doing this. The example that follows is typical of the sort of work we did. It comes from a class of six- and seven-year olds, one term into our second year of work with them and their teachers.

Improvised solutions: Jelly beans

The lesson started with a discussion about the idea of equal. I put up on the board

$$25 + 10 = 15 + 10 + 10$$

and asked the children to decide in their pairs whether they thought this was true or not. The class were all agreed that it was not true: because 25 + 10 made 35 not 15. Asked about the +10 + 10 that followed after the 15 and the children were clear that these were not relevant (one child suggested that the board had not been cleaned properly). Like many children of this age they had appropriated the idea that the equals sign means ‗makes' and that what immediately followed it had to be the answer.

I talked about how the three numbers to the right could be added, representing this by adding them pairwise, 15 + 10 and 25 + 10, drawing lines down from the 15 and 10 and recording 25 below and then drawing a line down from the 25 and second 10 with the 35 below. Amid groans that I had (again) tried to ‗trick' them, there was a general agreement that the statement was true. My recording stayed on the board for the rest of the lesson.

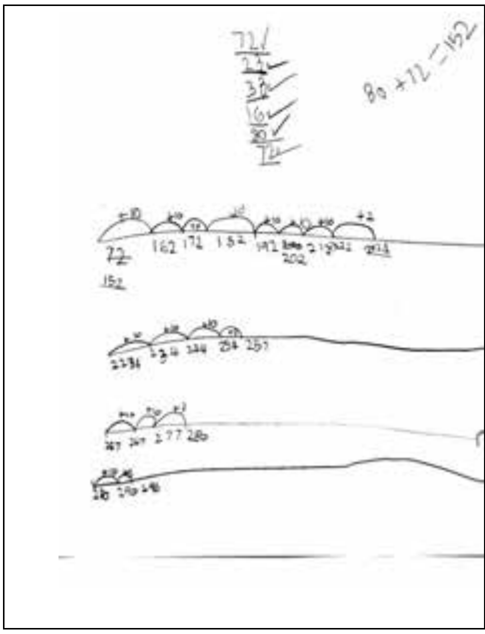
I then set up the main problem for the lesson, presenting it orally. I talked about visiting a friend, Richard, who ran a sweet shop and how he had posed a problem that he hoped the children could help with. He kept jars of different flavours of jellybeans from which he made up orders. I asked for suggestions as to the flavours of beans he might have, in the expectation that the children may have seen the Harry Potter movies and come up with some ‗exotic' flavours. But they stuck with traditional fruit flavours, so I added in fish and broccoli. The six flavours were listed on the board and how many beans there were of each flavour:

Strawberry	72
Orange	23
Apple	33
Cherry	16
Fish	80
Broccoli	72

The problem was that Richard had an order for 300 beans and did not know if he had enough beans in total. Were the children able to find out?

There was a general murmuring of this being hard, but this was the second year of working with the children and they had come to accept that we would give them challenging problems to work on but also trust that they would get there in the end. In particular, good habits for working in pairs had been established, including that when working in pairs the children would share one piece of paper. They also knew that they could use any of the practical materials in the room and record their working in whatever way they found helpful.

A few pairs got out base-ten blocks to model the problem practically but most used paper and pencil. I want to examine the solutions of two pairs of children that are typical of the sorts of approaches the children took.



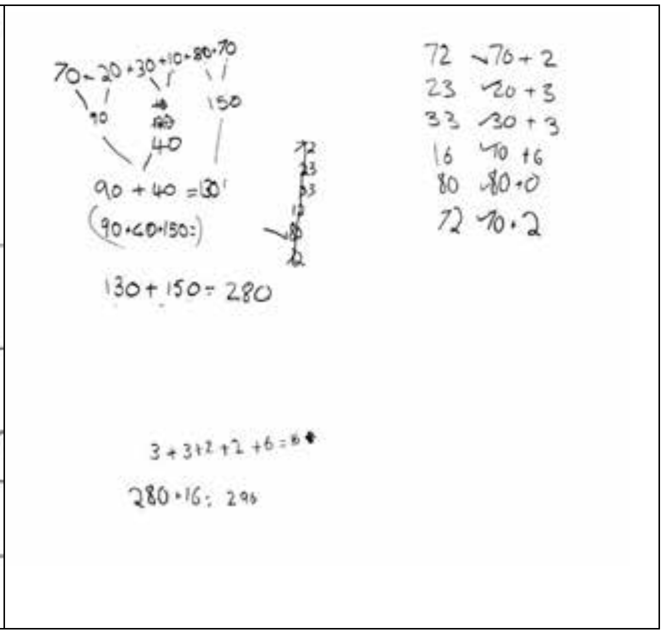


Figure 1. Amy and Ali's solution. *Figure 2. Ben and Beth's solution.*

Sawyer (2000) argues for the analysis of collaborative emergence through examining group interactions, texts produced (including spoken texts), and the historical development of the group. As data were not systematically collected on the group interactions, I focus here on the physical texts that the pairs of children produced and then turn attention to the historical context that I consider supported the emergence of these texts.

Texts

Figure 1 shows the work of Amy and Ali. They copied down the numbers in the order in which they were on the board, but then started adding them systematically. They began with the largest pair, 80 and 72, adding these through the co-ordinated actions of Amy counting on in tens from 72 while Ali kept track of the number of tens added on. Both children put out fingers to keep track of action and keep their counting in time. Hence when Ali reached 80 Amy simultaneously reached 152. Then to add on the second 72 they turned to using the empty number line, drawing other number lines to add on 33, 23, and 16 in that order.

Figure 2 shows the work of Beth and Ben. Like Amy and Ali they started by copying down the list of numbers, ticking off 80 but then could not decide what to do next. Ben suggested writing the numbers down as tens and ones and they wrote the tens out, in the same order but horizontally, ticking them off as they went. It was not clear who chose to record the pairwise addition of the tens by appropriating the ‗pull-down' notation that was on the board from the introductory activity.

Do these examples constitute examples of a collaborative emergent system?

Sawyer argues that a collaborative emergent system has the characteristics of:

1. unpredictability;
2. non-reducibility to models of participating agents;
3. processual intersubjectivity;
4. a communication system that can refer reflexively to itself, and within which the processes of communication themselves can be discussed; and
5. individual agency and creative potential on the part of individual agents

(Sawyer, 1999, p. 453)

These solutions, and those of other children, were unpredictable given the range of solution methods. I had not planned to use the notation that morning that Beth and Ben used, but in the language of improv drama, it proved to be a ‗good offering'. In improv drama scenes, good offerings are ‗lines' that open up possibilities for other players, as opposed to bad offerings that close things down. For example, in response to a simple opening of ‗Hi Mike' ‗Hey sis' would be a good offering, while ‗Who are you? I've never seen you before' is a bad offering.

We knew the children well enough to know that the difficulty of the problem meant that no-one in class would have been able to solve the problem alone and the origins of the solutions cannot be reduced to an account of the understandings of individual children. The whole lesson was based on the processes of intersubjectivity together with individual agency and creativity. Sawyer's fourth point is the least obvious, although the lesson finished with these pairs of children presenting their solutions to the class and a discussion of the clarity of each solution and which the children preferred.

Do these examples count as improvised?

It is easier to determine what is not improvisation than what is (Sawyer, 2000). Although we had worked with the children on using empty number lines, we had not used them for successive calculations as the children did here, and the use of the ‗pull-down' notation was certainly improvised as the children had never been exposed to this before. Similarly we had never taught or observed the co-ordinated counting in tens activity of Amy and Ali. While the popular impression of improvisation is that it all has to be made up, it is more a sense of coordinating previously known and used elements in new ways, and it is in that sense I argue these are improvised solutions.

> Improvisation, like composition, is the product of everything heard in past experience, plus the originality of the moment. The contents of even a very accomplished improviser's solos are not all fresh and original, but are a collection of clichés established patterns, and products of memory, rearranged in new sequences, along with *a few* new ideas. (Coker, 1964, p. 36, original emphasis).

Historical context

One aspect of the historical context is the attention to artefacts and tools that the children drew on. They were familiar with base-ten blocks. We had worked on fluency in adding multiples of ten, and emphasised the strategy of starting with the larger number when adding two numbers. We had introduced the children to the empty number line and had worked with it long enough for this to be a model for addition for many of the children (Gravemeijer, 1999).

But in addition to these ‗cognitive' supports I want to make links to play and performance and the history of this that we, the class, had established, as I consider these as central to the children ‗buying into' a problem that had a level of challenge beyond anything they had met before.

Becker (2000) in his analysis of jazz improvisation argues for the importance of having ―a real shared interest in getting the job done" (p. 175). Like other researches leading to rich pupil solutions (for example, Fosnot & Dolk, 2001) the considerable time spent at the beginning of the lesson setting the context for the problem was not simply window dressing or a device to make some unpalatable calculations acceptable. There was a general ‗suspension of disbelief' created by spending time setting up the scenario and in getting ‗buy-in' from the children. This is not mere speculation: in the early part of setting up the problem, one of the girls repeatedly whispered to her neighbour ―It's not true you know. He doesn't really have a friend with a shop." This increasingly became a stage whisper obviously intended to be overheard by everyone, so I stopped and we spent some time talking about whether it mattered if the ‗story' was real or not. Although some of the children were disappointed that I would not reveal the truth, they were generally content to ‗play along'. Such ‗playing along' helps, I suggest, in the children being willing to ‗play' with a problem. This is in contrast to some views that ‗artificial' problems do more harm than good. While I would agree that the ‗quick' word problem about shopping, followed by another about ‗cooking' does not encourage engagement, I think more use could be made of more extended narrative scenarios to hook children in.

Discussion

Persons in environment

In her interpretation of the work of Vygotsky as a performative psychology, Holzman (2000) argues for being clear about *distinguishing* learners from their context but not treating them as *separate* from the context.

> While we surely can be (and are, in Western cultures) distinguished from environment, this does not mean we are separate from it. Instead of two separate entities … there is but one, the unity ―persons-environment." In this unity, the relationship between persons and environment is complex and dialectical: environment ―determines" us and yet we can change it completely (changing ourselves in the process, since the ―it"—the unity ―persons-environment"— includes us, the changers). (p. 86–87)

This has echoes of the emergence concept of downward causation (Campbell, 1974). ―In downward causation, an emergent higher level property begins to cause effects in the lower level, either in the agents or in their patterns of interaction" (Sawyer, 1999, p. 455). Is it meaningful (or helpful) to talk of downward causation in the sense of the solutions that the children produce having some quasi-autonomous effect on the learners? In other words, is there a sense in which the solutions are manifesting themselves through the children, rather than the children are simply producing the solutions?

Experienced improvisers testify to downward causation. Although at the beginning of a scene, improvising actors have a whole range of options open to them (indeed, one of the disciplines of improv is to keep these options open for as long as possible), once the form and content of the scene starts to emerge, actors will talk afterwards of the scene ‗writing itself'. Similarly jazz musicians report a sense of the music playing the band:

> The players thus develop a collective direction that characteristically—as though the participants had all read Emile Durkheim—feels larger than any of them, as though it had a life of its own. It feels as though, instead of them playing the music, the music, Zen-like, is playing them. (Becker, 2000, p. 172)

Even if it is only metaphorical to talk of downward causation, engendering a sense of this plays, I suggest, an important role in moving from either a teacher-centred or a pupil-centred lesson to a mathematics-centred one. The (tacit) sense of the solutions having some agency rather than being the ‗property' of specific children may account for why that even at this young age the children were able to talk about the solutions without being defensive or possessive of them. Again there are resonances with jazz.

> Likewise, people must have a real shared interest in getting the job done, an interest powerful enough to overcome divisive selfish interests. In an improvising musical or theatrical group, for instance, no one must be interested in making a reputation or protecting one already made. (Becker, 2000, p. 175)

Conclusion

If it is the case that paired or group work that allows for collaborative emergence can result in more sophisticated, improvised, mathematical performance than could be achieved by the individual pupils then this has implications for the planning and implementation of lessons. First, most teachers base their planning on what they consider to be the needs of the individuals in their class which, as indicated here, are necessarily at a lower level of mathematics than could be achieved collectively. In the case of the children at Bow Bells school there was a clear pay-off from working at this more challenging level. These children were in Year 2, one of the years of primary schooling where the children have to take one of the externally set ‗National Tests'. Not only did over 90% of the children reach the expected level on these tests (substantially higher than in previous years), but the children themselves commented on how easy they had found the test. Second a focus on the collective outcome presents challenges to the discourse of the individual that currently structures assessment activity.

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THE INFLUENCE OF TEACHERS ON STUDENT ENGAGEMENT WITH MATHEMATICS DURING THE MIDDLE YEARS

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Recent decades have seen growing concern over the lowering levels of engagement with mathematics in Australia and internationally. This paper reports on a longitudinal study on engagement with mathematics during the middle years and explores the influences of teachers on the participants‟ engagement with mathematics. Findings reveal that development of positive pedagogical relationships forms a critical foundation from which positive engagement can be promoted.

Introduction

In recent decades there has been growing concern over the lowering levels of engagement with mathematics in Australia (Commonwealth of Australia, 2008; State of Victoria Department of Education and Training, 2004; Sullivan & McDonough, 2007; Sullivan, McDonough, & Harrison, 2004) and internationally (Boaler, 2009; Douglas Willms, Friesen, & Milton, 2009; McGee, Ward, Gibbons, & Harlow, 2003). The issue of lowered engagement levels in mathematics during the middle years (Years 5 to 8 in NSW) could cause wide-reaching consequences that have the potential to affect our communities beyond the obvious need to fill occupations that require the use of mathematics. Disengagement with mathematics leads to reducing the range of higher education courses available to students through exclusion from courses requiring specific levels of mathematics. In addition, students who discontinue studying mathematics limit their capacity to understand life experiences through a mathematical perspective (Sullivan, Mousley, & Zevenbergen, 2005). Arguably one of the most significant influences impacting on engagement in mathematics is the teacher and teaching practices, or pedagogy (Hayes, Mills, Christie, & Lingard, 2006; NSW Department of Education and Training, 2003).

This paper is derived from a longitudinal case study on engagement in mathematics during the middle years of schooling in which a group of 20 students experienced a range of mathematics teachers and pedagogical practices during their final year of primary school and the first two years of secondary school. Data were collected from the group across the three school years through individual interviews and focus group discussions. This paper is an exploration of the influences of teachers and their practices on the participants‟ engagement with mathematics. The theoretical framework underpinning this paper is based on current theories and definitions of engagement, and literature defining „good‟ teaching of mathematics. A brief overview of the literature is now provided.

Engagement

Recent research into student engagement, the *Fair Go Project* (Fair Go Team, NSW Department of Education and Training, 2006) focussed on understanding engagement "as a deeper student relationship with classroom work" (p. 9). The *Fair Go Project* found that students need to become „insiders‟ within their classroom, feeling they have a place and a say in the operation of their classroom and the learning they are involved with. They have a need to identify themselves as „insiders‟ as well as to be identified as „insiders‟ by their teachers, students and all stakeholders

In addition to the definition of engagement described above, there are others that should be considered. Some definitions view engagement only at a behavioural level (Hickey, 2003), where others view engagement as a multidimensional construct (Fredricks, Blumenfeld, & Paris, 2004). Fredricks et al. (2004) define engagement as a deeper student relationship with classroom work, multi-faceted and operating at operative, affective, and cognitive levels. Operative engagement encompasses the idea of active participation and involvement in academic and social activities, and is considered crucial for the achievement of positive academic outcomes. Affective engagement includes students‟ reactions to school, teachers, peers and academics, influencing willingness to become involved in school work. Cognitive engagement involves the idea of investment, recognition of the value of learning and a willingness to go beyond the minimum requirements. In terms of engagement with mathematics, engagement occurs when students are procedurally engaged within the classroom, participating in tasks and „doing‟ the mathematics, and hold the view that learning mathematics is worthwhile, valuable and useful both within and beyond the classroom.

Why is engagement with mathematics so crucial? In an investigation into the reasons students are choosing not to pursue higher-level mathematics courses, McPhan, Moroney, Pegg, Cooksey and Lynch (2008), claim "curriculum and teaching strategies in the early years which engage students in investigative activities and which provide them with a sense of competence are central to increasing participation rates in mathematics" (p. 22), yet many attempts to investigate the lack of engagement with mathematics have failed to find good reasons for students‟ difficulties. It is claimed students who are engaged with school are more likely to learn, find the experience rewarding and continue with higher education (Marks, 2000).

'Good' teaching and mathematics

The pedagogical practices employed within mathematics classrooms cover a broad spectrum from the „traditional‟, text book based lesson, to the contemporary or „reform‟ approaches of problem solving and investigation based lesson, or a combination of both. When asked to recall a typical mathematics lesson, many cite a traditional, teacher-centred approach in which a routine of teacher demonstration, student practice using multiple examples from a text book and then further multiple, text book generated questions are provided for homework (Even & Tirosh, 2008; Goos, 2004; Ricks, 2009).

An alternate approach to teaching mathematics reflects a constructivist perspective where students are provided with opportunities to construct their own knowledge with a focus on conceptual understanding rather than instrumental understanding. Such an approach fosters problem solving and reasoning and is consistent with frameworks for quality teaching (Newmann, Marks, & Gamoran, 1996; NSW Department of Education and Training, 2003).

Although there are arguments for using either or both approaches, there is strong support for an investigational, contemporary approach to teaching and learning mathematics (Anthony & Walshaw, 2009; Boaler, 2009; Clarke, 2003; Lovitt, 2000). Open-ended, rich tasks transform students‟ beliefs about problem solving and alter the culture of mathematical engagement. Evidence suggests that providing middle years students with engaging mathematical tasks supported by appropriate teaching strategies leads to sustained improvement in learning outcomes (Callingham, 2003).

Much research has been conducted on effective teaching of numeracy and mathematics, with a particular emphasis on the pedagogical content knowledge (PCK) required for effective teaching of mathematics (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997a; Delaney, Ball, Hill, Schilling, & Zopf, 2008; Hill, Ball, & Schilling, 2008; Shulman, 1986). In support of the need for strong PCK it can be argued that teachers with higher mathematical qualifications do not necessarily produce strong learning outcomes in their students as a result of weak understandings of how students learn and the pedagogies that are appropriate for particular mathematics content (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997b).

In recent years the Australian Association of Mathematics Teachers (AAMT) (2006), developed a set of standards that reflects current literature on effective teaching of mathematics and represents national agreement of teachers and stakeholders on the required knowledge, skills and attribute of quality teachers of mathematics. Data informing this paper were analysed against the backdrop of the above literature on engagement, effective teaching and current teaching standards. The following is a brief description of the methodology used in the study.

Methodology

The participants in this case study were derived from a Year 6 Cohort in a western Sydney catholic primary school and were identified through Martin‟s (2008) *Motivation and Engagement Scale (High School)*, as having strong levels of engagement with mathematics. The instrument consisted of a 44 item Likert scale requiring students to rate themselves on a scale of 1 (Strongly Disagree) to 7 (Strongly Agree) and was adapted to be specific to mathematics. The group of 20 participants made the transition together to the local catholic secondary college which had been in operation for only two years prior to the group‟s arrival. The participants represented a diverse range of mathematical abilities and cultural backgrounds, and most came from families with two working parents.

During the study the students participated in individual interviews during Year 6 and again in Year 8, and a series of focus group discussions at five points across the duration of the study. In addition, teachers identified by the students as „good‟ mathematics teachers were interviewed and observed during several mathematics lessons. The students formed three focus groups; a boys group, girls group and mixed gender group. Each interview and focus group discussion was loosely based on the following set of discussion points/questions: (a) Tell me about school; (b) Let‟s talk about maths; (c) Tell me about a fun maths lesson that you remember well; (d) When it was fun, what was the teacher doing?; and (e) What do people you know say about maths?

The data gathered were transcribed and coded into themes. In terms of the students‟ perceptions of mathematics teaching, two major themes emerged as being influential on their engagement with mathematics: teachers‟ pedagogical repertoires, those day-to-day practices that teachers implement in their teaching of mathematics, and the pedagogical relationships formed between teachers and students.

Results and discussion

During Year 6, the participants experienced pedagogies that included a significant focus on cooperative learning. The opportunities for interaction and dialogue that this provided had a positive influence on the students‟ engagement with mathematics, with one student saying: "You‟ve got like more options to choose from rather than if you‟re by yourself" and another: "working with partners is fun because you could find different strategies and you have fun and it‟s easier". It can be argued that the classroom practice of cooperative learning has positive results in terms of providing a safe environment in which the students are able to learn within a positive classroom culture. The ability to associate learning in mathematics as fun appeared to be a powerful influence on engagement, and the following quote summed up the collective feeling of the majority of participants: "The group can work it out together to try and solve the problem and you‟ve like learned something new or how to work out something".

One Year 6 teacher, Linda, who was identified by the students as the „best‟ mathematics teacher, was described by several students as someone who enjoys teaching and has a passion for mathematics. Alison attributed this quality to increasing her own engagement: "She just puts a lot of enthusiasm in maths and makes it really fun for us. She gets all these different maths activities. She just makes it really fun for us and I quite enjoy maths now because of that".

It appeared the teacher‟s enthusiasm for mathematics fostered positive attitudes and excitement towards mathematics, reflecting the findings from research (Askew et al., 1997b) and recommendations by the AAMT (2006). In addition to her passion for mathematics, the students witnessed Linda as having fun teaching. Tenille said: "It‟s fun when the teacher, like, while you‟re doing the work, she also has fun teaching the maths as well".

When the students moved on to Year 7, they were faced with a new set of pedagogies, and a new set of mathematics teachers. In contrast to the approaches used during their primary years, the students were expected to work on an individual basis, using computer-based interactive tutorials and mathematics textbooks. This significantly reduced classroom interaction and dialogue, and rather than having a single mathematics teacher, students were provided with a rotation of four different teachers.

Although the availability of computer technology provided the opportunity for teachers to deliver a new and relevant way of teaching and learning (Collins & Halverson, 2009), they instead appeared to be used as replacements for teachers. Alison picked up this emerging idea among the students:

> … it's probably not the best way of learning because last year at least if you missed the day that they taught you, you still had groups so your group could tell you what was happening. Where now, we‟ve got the computers and it‟s alright because there is, um, left side of the screen does give you examples and stuff, um, but if you don‟t understand it, it‟s really, hard to understand.

It is reasonable to suggest that the website and textbook were not necessarily bad resources. However, the data was showing that it was the way they were used in isolation that meant the students began to disengage from mathematics. During Term 2 of Year 7 the students were given the opportunity to engage in tasks that were more interactive and hands-on, consistent with recommendations from research (Boaler, 2002; Callingham, 2003; Lowrie, 2004). Several of the students commented on this change, with Fred saying: "We‟re doing more hands-on tasks than what we were used to, like what we used to do. It‟s more interesting". The students found the incorporation of concrete materials made their mathematics lessons more interesting, and the opportunity to work in groups during one particular activity made those lessons memorable, with Rhiannon giving this reason: "… because we got to create the shape by using straws, in groups. Not by ourselves". In addition to the benefits of being able to work collaboratively, George felt he and his group made more of an effort than usual: "It was good because we could make it ourselves and we could like put effort into it".

When the students reached Year 8, the school‟s structure had been reviewed and during Term 2, the students were provided one regular mathematics teacher per group. The newly formed mathematics classes appeared to increase the students‟ engagement, allowing stronger teacher/student and peer relationships to develop. In terms of the resources that were used in the Year 8 lessons, there was less reliance on the students‟ laptops and more emphasis on using text books. Kristie described a typical routine:

> Well, we just got our text book and the laptops don‟t come out in maths as much or at all, unless you‟ve forgotten your text book or something like that. And, um, maths is good, we separated into groups and the teacher‟s out the front and he‟ll tell us what to do and you pretty much put your hand up if you need help, and he‟ll help you and then you have the text book out and you answer the questions in your maths book.

Although it has been found that a traditional approach to teaching mathematics may have a negative influence on student engagement, in this case the students saw it as an improvement on previous pedagogies and appeared to experience higher levels of engagement. One aspect of the teachers‟ pedagogies that had a positive effect on the students‟ engagement was the students‟ perceptions of an improvement in teacher explanations. George made this comment which reflected the feelings of many of the students: "I think maths has improved because the teachers go through it with you more, whereas last year they would just set you a task and leave you with it". Billy, a student who struggled to maintain his engagement in mathematics, added: "Sir just writes stuff on the board and then he explains it really good and we learn about stem and leaf graphs. He teaches it really good and other teachers just write it down and say „go do that‟".

During the final data collection, Alison made a comment that was reflective of the group‟s feelings once they were assigned their regular teachers and were able to begin building positive pedagogical relationships: "The teachers know where we‟re coming from and what we need to learn and they learn, not what the group needs, but what we need". The data shows that the students appeared to re-engage with mathematics when they felt the teachers knew them in terms of their mathematics learning needs. The opportunity to establish positive pedagogical relationships with teachers appeared to provide students with a sense of belonging, an important aspect of an effective mathematics classroom (Boaler, 2009).

Implications and conclusion

The most powerful influence on engagement in mathematics for these students appeared to be that of their teachers. This influence can be viewed at two interconnected levels. The first level includes the pedagogical repertoires employed by the teacher, and the second, the pedagogical relationship that occurs between the teacher and students. That is, the connections made between the teacher and student, and the teacher‟s recognition of and response to the learning needs of his or her students. Although this study has limitations in terms of the selective nature of the sample, it is suggested that the development of positive pedagogical relationships forms a critical foundation from which positive engagement can be promoted and this may be generalised to a wider student population.

The findings discussed in this paper imply that many middle years students are still dependent on high levels of interaction within the mathematics classroom. Repetition of the current study within different school contexts would be of benefit in further exploring the concept of engagement with mathematics. Further studies on engagement with mathematics during the later years would be beneficial in terms of investigating whether pedagogical relationships remain as important for older students. Although student achievement and its relationship to engagement levels was not a focus of this study, such an exploration would also be worthwhile for future research.

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TEACHING PRACTICES FOR EFFECTIVE TEACHER-STUDENT RELATIONSHIPS IN MULTIETHNIC MATHEMATICS CLASSROOMS

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Teacher-student relationships can strongly influence academic achievement and motivation, particularly for minority group students. Teaching practices contributing to strong academic relationships are therefore vital to understand. This article describes such practices drawn from observations of 100 Year 10 mathematics lessons involving six teachers and their classes across three mid-low socio-economic schools. For many indigenous (Māori), New Zealand Pacific, and New Zealand European students, evidence emerged that essential caring teacher behaviours include and extend beyond traditional mathematics teaching practices. Findings are presented using an holistic model of health and well-being that encompasses cognitive, social, physical, and spiritual dimensions.

Introduction

Culturally-linked issues affecting academic achievement are essential for educators to address (e.g., Alton-Lee, 2003; Castagno & Brayboy, 2008; Ministry of Education, 2008; Pang, 2005; Tyler et al., 2008; Villegas & Lucas, 2002), particularly in mathematics, a gate-keeper subject, where differences in achievement by ethnicity are often found. The importance of effective teacher-student relationships for learning, particularly for indigenous and marginalised students, is well documented (e.g., Bishop, Berryman, Tiakiwai, & Richardson, 2003; Eccles, 2004; Gay, 2000; Gorinski, Ferguson, Wendt-Samu, & Mara, 2008; Ladson-Billings, 1994). Teachers‟ care for their students is seen by many as an essential component of learning-focussed teacher-student relationships (e.g., Bishop et al., 2003; Gay, 2000; Hackenberg, 2010; Hill & Hawk, 2000; Noddings, 1992). Students who see their teachers as caring are more likely to continue with mathematical study (Noblit, Rogers, & McCadden, 1995; Ocean, 2005), and have positive academic attitudes, motivation, and engagement (Gay, 2000; Hudley & Daoud, 2007; Wentzel, 1997).

New Zealand schools, like many internationally, are becoming increasingly ethnically and culturally diverse. This paper describes factors that contribute to teacher care found within one part of a mixed-method study carried out with six New Zealand multiethnic (indigenous Māori, New Zealand Pacific[1], and New Zealand European) mathematics classes and their teachers. Firstly, the theoretical and contextual background to the study is discussed. Next, Durie"s (1998) holistic model of health and well-being is described in relation to this study, and then the study, analysis, and findings are outlined.

Theoretical background

Teachers showing care for their students is advocated by many and a broad range of caring teacher behaviours are discussed in the literature (e.g., Hackenberg, 2010; Haynes, Ben-Avie, & Ensign, 2003; Noddings, 1992, 1995). Authors in the field of culturally responsive teaching also promote caring teacher practices (e.g., Bishop et al., 2003; Gay, 2000; Ladson-Billings, 1994; Wlodkowski & Ginsberg, 1995). Yet how caring relationships can be nurtured may vary across ethnicities (Thompson, 1998) and research is needed to illuminate such differences and teacher care in general within mathematics instruction (Hackenberg, 2010).

Particularly relevant to this study, *manaakitanga* (nurturing relationships) is a bedrock concept for all *tikanga* (Māori cultural practices) (Macfarlane, Glynn, Grace, Penetito, & Bateman, 2008). Caring for people is also fundamental for many Pacific groups and, using a Tongan example, involves developing three aspects of the "*tangata kakato*" (the total person) (Koloto, 2004, p. 61): "*mo'ui fakasino*" (physical well-being), "*mo'ui faka'atamai*" (intellectual well-being), and "*mo'ui fakalaumalie*" (spiritual well-being) (p. 62). Implications of such cultural perspectives of interpersonal care within classrooms include expectations by students and their families of the constant use of caring teacher practices.

Care can be shown in many ways: such as by showing respect, giving advice, or by acknowledging someone or their feelings (Noddings, 1992). Students" experiences of teacher care are affected by the classroom environment (e.g., Bishop et al., 2003) and teaching practices (e.g., Anthony & Walshaw, 2007; Bishop et al., 2003; Gay, 2000; Noddings, 1992; Pang, 2005; Wlodkowski & Ginsberg, 1995). Specific teacher practices found to help develop caring teacher-student relationships include: involving students in classroom decision-making (e.g., Alton-Lee, 2003); using „safe" questioning practices (e.g., Bills, 2000); creating a sense of shared endeavour; and incorporating particular pedagogies, for example, collaborative work (e.g., Hill & Hawk, 2000).

Understanding factors conducive to caring teacher-student relationships in mathematics learning is particularly important given persistent achievement gaps between those of dominant and marginalised ethnic groups, and the lack of representation of marginalised groups within mathematically-rich and mathematically-dependent disciplines. The literature provides a strong case for teachers to show care to their students. However, less focus has been given to how teacher care can be holistically shown within multi-ethnic classrooms, classrooms with indigenous students, and within *mathematics* learning. In this study, teacher care was explored by focussing on their care for students" mathematical progress and for their students as culturally located individuals.

[1] People living in New Zealand who have strong cultural, heritage, and family connections to their Pacific Island countries of origin.

The whare tapa wha model

Durie‟s (1998) *whare tapa wha* model of health and well-being (literally translated as „the four-sided house‟) was chosen to discuss factors contributing to teacher care found in the study. The model has four mutually supportive dimensions: *taha hinengaro* (representing people‟s cognitive, psychological, and emotional well-being), *taha whānau* (relating to interpersonal characteristics), *taha wairua* (representing spiritual elements), and *taha tinana* (relating to physical well-being). In Durie‟s model, balance across the four dimensions is important.

Mathematical thinking can contribute to the *taha hinengaro*, and elements contributing to „affect‟ such as mathematical self-concept to *taha hinengaro* and *taha wairua*. The suitability and timeliness exploring the model in this study is confirmed through the more recent use of the *whare tapa wha* model within mathematics educator professional development towards engaging Māori learners (Tertiary Education Commission, 2010).

The study

Participants included six mathematics teachers and their Year 10 classes from three urban mid-low socio-economic secondary schools with roughly equal proportions of Māori, New Zealand Pacific, and New Zealand European students. Teachers‟ ethnicities included four New Zealand European, one Māori/New Zealand European, and one New Zealand Asian. The 161 student participants included roughly even numbers of the target ethnicities[2] and of males and females. Students‟ self-reported ethnicities indicated many had multiple heritages, making reporting of results by ethnicity unsuitable.

Māori and New Zealand Pacific cultural advisors were consulted regarding all stages of the study (Averill, 2009). Reported here are the findings from the central data collection method of the larger investigation, classroom observation. An observation schedule was designed using ideas drawn from a wide range of literature, extensive consultation with cultural, teacher, and student advisors, and extensive piloting (Averill & Clark, 2007). Three data gathering periods were used - the initial four weeks of the school year, two-weeks after roughly 10 school weeks had passed, and another two-weeks 10 school weeks later. The complete data set comprised 100 lesson observations.

Analysis

Analysis is largely drawn from the data from the two study teachers whose practice most consistently displayed acts described in the literature as caring. Durie‟s (1998) *whare tapa wha* model was used as an interpretive typology and results will be discussed as they relate to the four sides of the *whare* in turn.

Taha hinengaro

Practices that assisted in building safe, purposeful, and engaging learning environments are pertinent to the *taha hinengaro*. These included teachers creating warm, caring classroom atmospheres with a clear focus on mathematics learning by consistently reinforcing firm boundaries, and setting high (yet attainable) expectations and ensuring students were aware of these. Teachers showed care for student learning by giving clear

[2] As recorded by schools for the Ministry of Education.

signals ("I‟ll know you are ready when your arms are folded and your mouths are closed."), being consistent in and explicit about their practice ("I‟m coming around to see your progress. You‟ve got five minutes and then we‟ll see what we think about one another‟s ideas"), capitalising on students‟ reactions and responses to promote learning, and by showing they liked and respected their students.

Caring teachers created a sense of urgency for completing tasks and maintained students‟ engagement by constantly challenging their thinking (e.g., by adding new ideas to discussion, helping students invent and incorporate new rules to a game, and varying activities or their styles of questioning). Involving students in lesson-related decision making (e.g., in selecting tasks or the level of difficulty of the tasks), varying lesson tasks, and sanctioning humour were further practices that showed care for students‟ mathematical progress.

Observed behaviours less caring of students‟ mathematical learning relating to *taha hinengaro* included teacher-directed rather than student-centred practices, lack of variety in lesson activities, setting uninspiring tasks, low teacher expectations (e.g., providing little work, accepting off-task behaviour), and reacting negatively to student humour or suggestions.

Taha whānau

Practices relating to *taha whānau*, those that appeared to help develop a sense of community and social responsibility, fell within four areas: nurturing class community; nurturing personal responsibility; care for learning needs; and care for students as individuals within their wider family and community contexts. Teachers nurtured students‟ sense of class community by letting students know something about them as individuals by telling students about themselves at the start of the year and sharing personal information relevant to students‟ learning. For example, one teacher discussed her child‟s collection of cereal box cards when introducing related probability simulation work.

Teachers nurtured students‟ sense of personal responsibility by showing interest in and concern for them within and outside mathematics learning (e.g., discussing health issues related to smoking with a student athlete smoker). Teachers used inclusive language (e.g., "let‟s see what happens when…"), prioritised students and their learning, and incorporated activities that encouraged a sense of community (e.g., mathematical games, group tasks, stories). They carefully selected learning activities that required or enabled students to share aspects of their own knowledge and personalities and used opportunities to acknowledge shared endeavours (e.g., a school athletics competition). Caring practices included addressing students‟ learning needs by attending to students‟ concerns, making calculators, equipment, and homework books available, and by showing care regarding *whakaiti* (humility) and *whakamā* (shyness, embarrassment).

Practices relating to *taha whānau* that undermined students‟ senses of community and community purpose included not knowing or making mistakes with students‟ names, disregarding students‟ concerns and interests, and public admonishment.

Taha wairua

Teacher practices relating to caring for the *taha wairua* included showing respect for students (e.g., by using praise, providing timely feedback about learning, and explaining their practice) and encouraging students to self-assess their progress. The teachers typically exhibiting most care for their students incorporated one-to-one interaction with all students every lesson, often multiple times. These teachers enhanced students‟ sense of personal mathematical identity through modelling their own, showing students they were aware of their progress, providing many suitable engaging tasks to fill the lesson, relating tasks to students‟ lives, and being positive and encouraging.

Teach care was less apparent in lessons with few opportunities for students to share responsibility for their learning and little provision for student enjoyment, interest, one-to-one teacher-student interactions, or mathematical success.

Taha tinana

Practices pertaining to students‟ physical well-being and movement and were interpreted as relating to the *taha tinana*. Examples included students writing mathematical questions and working on the board, moving as part of a game or to indicate their progress, carrying out mathematical tasks outside the classroom, and being able to move around the room (e.g., to obtain assistance from peers or open windows). Teachers showed care for students‟ physical well-being through ensuring the classroom environment was comfortable, acknowledging the effects of the environment (e.g., heat) on students‟ learning, and showing concern for students‟ physical well-being. In the most caring classrooms teachers worked close to students, placing the students as important participants, showing them that the teacher was ready to assist, and enabling privacy.

Teachers less caring of students‟ physical well-being included few opportunities for movement and at times gave instructions resulting in student discomfort (e.g., insisting on the removal of non-uniform jersey).

In summary, teacher practices relating to care for students as culturally located individuals fitting within one or more dimensions of Durie‟s (1998) model were present, with many discussed above. Other examples included teachers pronouncing students‟ names correctly, acknowledging students‟ culturally-based knowledge, and greeting or praising students using Māori or Pacific Nations‟ languages. However, very few instances of further use of Māori and Pacific languages or of mathematical contexts drawn from these cultures were observed.

Discussion and conclusion

Attaining equitable access to mathematical achievement has been a persistent challenge for many education communities. Caring teacher-student relationships focused on enhancing learning offer one pathway towards maximising motivation and achievement. This study adds to the literature on teacher care by illustrating how caring classroom practices can be linked to the interrelated cognitive, social, emotional, physical, and dispositional aspects of mathematics learning.

Few culturally-linked models for mathematics teaching exist. This study illustrates how a model drawn from indigenous perspectives can inform teaching of indigenous, minority, and dominant culture students. Whilst the framework is drawn from indigenous Māori perspectives, the dimensions are universal and transferable to other cultures as they relate directly to the human condition and interpersonal relationships. Similar models drawn from other communities are worthy of investigation to further add to our understanding of how best to enhance teacher-student relationships towards maximising mathematics learning.

Further exploration is possible into ways of using the model to develop teachers" practice in mathematics and other curriculum areas, and how caring teacher-student relationships can be enhanced through reflecting students" families, cultures, cultural identities, and lives in instruction.

This study indicates that a model recommended for teachers to improve their teaching of a group of students less well served by mathematics education can be relevant for teachers of students of many ethnicities. With increasingly diverse combinations of students" ethnic backgrounds and the challenges sometimes found within projects targeting specific ethnic groups (e.g., McKenzie & Scheurich, 2008; Theoharis, 2007), the generic nature of many aspects of caring teaching practice as suggested by this study provides a way towards enhancing equity of access to mathematics learning.

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PRESERVICE TEACHER PERCEPTIONS OF GOOD MATHEMATICS TEACHERS: WHAT MATTERS?

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That preservice teachers' understanding of what constitutes good teaching is partly shaped by their experiences as students in the classroom is well documented; but how they give shape to their recollections is underexplored. In this study, preservice teachers wrote about their perceptions of good teaching in narrative reflections of their experiences as school students. Almost 25% of the cohort chose to write about a mathematics teacher. Narrative analysis was used to investigate the content and the form of the 31 reflections to provide insight into how preservice teachers reconstruct their narratives of experience. Three distinctly different story types were found.

Preservice teachers come into teacher education programs with existing beliefs about what is good teaching mostly developed from their experience as a student; however, preservice teacher beliefs appear limited, underdeveloped, and particular to individual experience (Fajet, Bello, Leftwich, Mesler & Shaver, 2005; Lortie, 1975; Minor, Onwuegbuzie, Witcher & James, 2002; Prescott & Cavanagh, 2006). They have been shown to affect how preservice teachers respond to teacher education programs (Richardson, 2003). This paper contributes to a better understanding of preservice teacher beliefs by investigating the form as well as the content of narrative expressions of beliefs concerning good teaching.

The line of inquiry reported here emerged from a larger study that investigated how a second year subject in a four year teacher education program contributed to the development of professional teacher identity of preservice teachers (Balatti, Knight, Haase & Henderson, 2010). Prior to this subject, the preservice teachers (primary and secondary) have little or no experience teaching in a classroom. Also at this point, students have had no curriculum methodology subjects.

The learning identity framework (Falk & Balatti, 2003) underpinning the pedagogical approach used, views teacher identity in terms of "identity resources" that is, those behaviours, knowledges, beliefs and feelings that come from having a sense of belonging (or aspiring to belong) to a community of practice (Wenger, 1998) in this case, of teachers. According to this framework, identity resources come from the identity formation, re-formation and co-construction that occur through interacting and storying.

Polkinghorne (1988, p. 18) defines story or narrative as "a meaning structure that organizes events and human actions into a whole, thereby attributing significance to individual actions and events according to their effect on the whole." Narratives are open to contention and revision and can be reworked any number of times.

In the subject, preservice teachers were asked to write online weekly narratives that included reflections on their past school experiences and their teaching experiences during their school placements. One of the first tasks was to write about their experience of good teaching at school.

Approximately 25% of the responses concerned good teaching in the context of mathematics classrooms. Moreover, despite the prescriptive requirements of the task, the narratives displayed a range of content and structure that suggested further investigation was warranted.

This paper reports the insights that a narrative analysis of this set of experiential stories provides concerning preservice teacher understandings of what constitutes good teaching. In particular, it provides responses to the following questions:

- How do preservice teachers appear to make their judgements of what constitutes good teaching?
- What do preservice teachers consider to be good teaching or a good teacher in the mathematics context?

Method

The texts analysed were written in the first week of the subject which is delivered in the first semester. The non-assessable task read as follows:

> We'd like you to write about a memory you have of a good teacher or of good teaching that you experienced as a child. Start by giving a context (e.g., your age at the time, subject). Then tell us why you thought that person was a good teacher. (Did you think this at the time or is this what you think only in retrospect?) Follow this by telling the story of a particular incident (or practice) that supports your claim. Conclude with a sentence or two showing the links between what the teacher did and what you know so far from your readings about what constitutes good teaching.

Of the 145 preservice teachers enrolled at the time, 136 completed the writing task. Of these, 31 wrote narratives of good teaching in a school mathematics context.

Analysis of the data comprised narrative analysis (Clandinin & Connelly, 2000; Riessman, 2008) followed by inductive analysis (Patton, 1990). The narrative analysis entailed both structural analysis and thematic analysis (Riessman, 2008).

As a starting point, the structural analysis used Labov's six elements of narrative structure (as cited in Riessman, 2008) and the thematic analysis identified the content of the narratives that directly or indirectly revealed participants' perceptions of qualities of good teaching. This phase produced the aspects of the narratives that were further analysed inductively.

The coding categories emerged from the data and were readjusted through the constant comparative method (Patton, 2002) across the 31 narratives. Figure 1, below, summarises the analytic process. The Nvivo 9 qualitative data processing software was used to assist with the coding of the data and the cross-tabulating of codes.

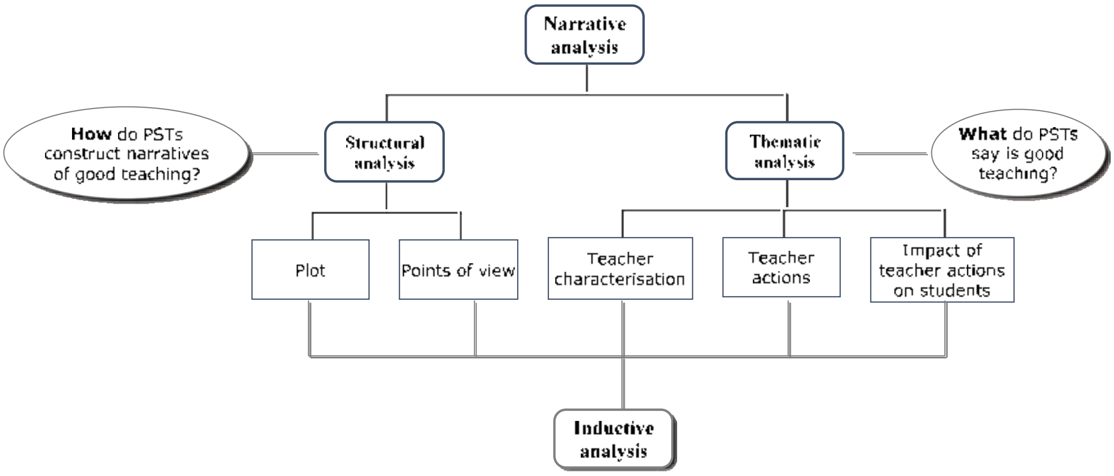

Figure 1. The analytic process.

Findings and discussion

This section is organised around the two research questions. The data selected to illustrate the findings are the three complete narratives reproduced below. The choice to use complete narratives rather than fragmented excerpts aligns with the preference in narrative analysis to retain stories "intact for interpretive purposes" (Riessman, 2008, p. 74). Examples were chosen for their brevity, to illustrate the three story types evident in the data, and to illustrate some of the indicators that preservice teachers use to determine what constitutes good teaching in the mathematics classroom.

How preservice teachers construct narratives of good teaching

The task elicited three different story forms which we called turning point stories (8), critical moment stories (7) and pattern of practice stories (16). All texts fell into only one of the three categories.

Turning point stories (see Tom's below) refer to the stories in which a new teacher changes the student's mathematics learning trajectory from a negative to a positive one in terms of engagement and/or performance. In these stories, the teacher is presented as the "saviour" to whom the student (the narrator) responds positively and becomes the "saved".

Story type one: Turning point stories: Tom's story

> I started high school at an all boys school in NSW. When males going through puberty are all together, rebelling and all of the things that we go through during that time, it can be tough for teachers. I can see that now. I had a mathematics teacher that would show no respect for his students and would actually insult us if we didn't understand a mathematics problem. I wasn't understanding mathematics at all and I thought it was just me and my family genes. I am the first in the family to finish high school and the younger brother.
>
> I had to change schools half way through year nine because I couldn't keep up with athletic commitments. I changed to a co-ed state school. One of my first classes was Maths and I dreaded it. I sat down to a female teacher and actually understood everything the teacher wrote on the board. By creating a safe and supportive learning environment my new teacher had me loving mathematics. I wasn't worried about being embarrassed to ask or even answer questions, because it was all right to get something wrong.

The second story type, the critical moment stories (see Cathy's below), is similar to the first story type in that the story narrates an event in which the main characters are the teacher and the student (the narrator). However, unlike the first story type, the student does not have a history of poor results and/or disengagement. Rather, they are stories of teachers who recognise that the student is in trouble and who intervene in a timely manner thus averting a negative outcome for the student. These are the "just in time" stories where through the vigilance and action of the teacher, students are spared a bad outcome. In these stories, the teacher is the watchful protector and the student is again the "saved".

Story type two: Critical moment stories: Cathy's story

> In primary school (grade 3) I could never grasp the concept of measurement. Measurement was one of the most difficult concepts for me to learn. I couldn't comprehend the difference in length between a millimetre, centimetre or a metre. My teacher noticed I was struggling in this area. Instead of making me feel stupid (like the other students did, when I answered a question wrong), she took me aside one afternoon and asked me to do a "special" task for her. Of course, being young and receiving personal attention from my teacher, I greatly accepted her "special" papers and completed them for her. As years went on, I now realise that her "special" papers were extra Math exercises that helped me understand.

The third story type is markedly different from the previous two. The patterns of practice stories (see Pat's below) as the name suggests describe patterns of behaviour of good teachers. Of the 16 stories of this type, 12 were of the one teacher and the other four described patterns across more than one teacher with one of the teachers being a mathematics teacher. These stories do not refer to a pivotal or critical event and the narrator is usually absent from the story other than as one of a number of students. In these stories, the students are content and even thrive. The teacher is the trustworthy and trusted shepherd or nurturer and the students are the flock.

Story type three: Pattern of practice stories: Pat's story

> Mr Jones, my Maths teacher in Year 10 who later became my Maths C and Physics teacher in Year 11 and 12 was the best teacher I ever had. He was always so enthusiastic about what he taught even when we were struggling to understand the concept of an 'imaginary' number or, in my case, graphs. He always listened to our questions, no matter how stupid they seemed later and encouraged us to explore anything that interested us in our classes. In my case, this meant that Mr Jones was always willing to talk over the possibility of aliens and the creation of the universe. He involved us in debates and discussions and always tried to make everything interesting and fun. Which was a hard thing to do when you're teaching complicated maths and science to a group of teenage girls! He was also a teacher who was unafraid to have a little fun in the classroom which made the whole learning process so much easier. He was most definitely the kind of teacher I want to be someday....only, I hope to have slightly better drawing and spelling skills!

Stories often conclude with a coda in which the readers are brought back to the present and this is sometimes done with changes in perspective. To encourage students to reconstruct their narrative from the viewpoint of a preservice teacher, the task had invited them to relook at a past experience either with "older eyes" or through the lens of the literature. A little over half the group included this aspect in their narrative (Table 1).

Table 1. Evaluation of significance of story from a new viewpoint.

Narrative type	No of narratives including new viewpoint		
	Revision of past experience	Reference to academic literature	Neither
Turning point stories (n=8)	1	2	5
Critical moment stories (n=7)	3	4(1)	1
Patterns of practice stories (n=16)	4	7(3)	8
Total	8	13(4)	14

Note: Numbers in brackets indicate stories that included both viewpoints.

Preservice teachers' understandings of good teaching

Within their narratives, the subjects described good teachers in terms of personal qualities they displayed, the behaviours they demonstrated, and/or the impact they had on their students. Table 2 summarises the attributes that "good teachers" possess. Although present in some narratives as contributing to good teaching, the least noted attributes were the mathematical content knowledge and behaviour management skills of the teacher. The most cited were the teachers' capacity to relate to their students and good teaching skills. A finer grained analysis revealed that good teaching involved organisational skills, communication skills, use of creative and enjoyable tasks, using real life applications, and learner centred strategies.

Table2. Narratives recording attributes.

Narrative type	No of narratives recording Attributes				
	Teaching skill	Content knowledge	Forging relationship	Behaviour management	Personal style
Turning point stories (n=8)	5	1	3	1	4
Critical moment stories (n=7)	4	0	4	1	0
Patterns of practice stories (n=16)	11	4	7	3	6
*Total	20	5	14	5	10

*Note: Some narratives had more than one category of attribute of good teaching evident.

Most preservice teachers also defined good teaching in terms of the teachers' impact on them personally or on others. Table 3 shows the categories of impacts that emerged from the data. Learning outcomes refer to improved performance in mathematics, the strongest impact for story types one and two; affective-self refers to outcomes to do with feeling valued or self esteem, the strongest impact for story types two and three; and affective–mathematics refers to students' feelings toward mathematics.

The final level of coding undertaken looked for patterns between teacher attributes and the student impacts for each of the story types. Because of the small size of the data set, no meaningful comments can be made other than to say the teaching skills was the category most cited in relation to impacts for all three story types. For both the turning point and critical moment stories, teaching skills were associated most with improved learning outcomes; for the pattern of practice stories, teaching skills were associated mainly with affective outcomes to do with the self.

Table3. Narratives recording impacts.

Narrative type	Impacts			
	Learning outcomes	Affective–Self	Affective–Maths	None
Turning point stories (n=8)	6	2	2	0
Critical moment stories (n=7)	7	5	1	0
Patterns of practice stories (n=16)	3	8	4	4
Total	16	15	7	4

Note: Some narratives described more than one kind of impact.

Conclusion and implications

This research studied snapshots, in the form of written narratives, of 31 reconstructed memories of preservice teacher experience of good teaching in a mathematics classroom. Using narrative and inductive analytic techniques, it explored how the narratives were constructed and what they said about good teaching. The three distinctly different story types that emerged and the different levels of engagement evident in adopting new viewpoints suggest that **how** preservice teachers think about their experiences of good teaching may be as relevant to teacher educators as the content of their narratives. Further research is required with larger data sets that investigates possible patterns involving preservice teacher story types and openness to learning the new identity resources (Falk & Balatti, 2003), including deeper knowledge of what constitutes good teaching, that come with becoming a teacher. Even without these larger studies, there may be merit in having preservice teachers use the methodology to study narratives of teacher experience. Understanding **how** they have constructed their narratives may help preservice teachers reflect more profoundly upon their own developing practice.

In the second set of findings concerning the characteristics of good teaching, the narratives looked to the attributes of the teacher and to the impact that the teaching had on the students. There was strong awareness that teaching skills and ability to relate well with students contribute to good teaching. In contrast, less awareness existed for the importance of content knowledge. Interestingly, while learning outcomes i.e., mathematical content knowledge, was the most cited form of impact that good teaching had on students, mathematical content knowledge was one of the least noted as an attribute of a good teacher. Further research that develops a better understanding of the reasons for preservice teachers not articulating it is recommended.

As teacher educators we can look at what preservice teachers bring in terms of their past experiences as a deficit or as resource to capitalise on and to further develop. A better understanding of their past experiences as students that the analytic approach used is this paper offers, may improve the likelihood of transforming past experiences into a resources for developing professional teacher identities.

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ANALYSING INTERVIEW DATA FOR CLUSTERS AND THEMES

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This paper outlines a qualitative approach for analysing interview transcripts. The approach involves identification of comments related to research interests, formation of clusters to group comments and then confirmation of themes in the data related to pre-determined research interests. Descriptions of each participant's views on each theme are summarised as case-studies. Illustrative examples from teacher interview data on teaching and learning will show how the approach can be implemented. The approach used is based on the work of Chesler who used grounded theory to explore views of a group of professionals.

Theme analysis is a common approach used to analyse interview transcripts (see for example Strauss & Corbin, 1990). This paper will outline an approach for theme analysis and for developing case studies related to themes. The approach will be illustrated using data from a study on teaching and learning mathematics with technology.

Given that themes are statements that encapsulate recurring ideas in interview transcripts and are likely to 'emerge from them on intensive analysis' (Tesch, 1990, p. 60) it is important to find a way to identify themes. In theme analysis there is the possibility to explore all themes emerging from data or else restrict the identification of themes to those which are related to specific research interests. The approach in this paper limits the themes by identifying only those related to specific research interests. Flick (2006) and Chesler (1987) also reported approaches where themes were focussed on specific areas of interest. This differs from some approaches to theme analysis where all emerging themes are identified in interview transcripts, often as an initial stage in the formation of theories (Strauss & Corbin, 1990).

The next section will outline one method for identifying themes based on an approach described by Chesler (1987). Chesler provided a seven step sequential analysis for development of theories and the first four steps in his approach can be used to identify emerging themes related to specific research questions. His approach, also reported in Miles and Huberman (1994), can be used to provide a description of phenomena under investigation. In this paper use of pen-and-paper or technology (in this case a computer algebra system, or CAS) for understanding in a secondary mathematics classroom will be used to provide illustrative examples of the approach used for theme analysis. The context of the study is discussed further in the section describing the data used to provide illustrative examples.

Chesler's approach for analysing interview transcripts

The first step of Chesler's analysis required researchers to "underline key terms in the text" (p. 9). This involved reading the transcript and underlining key terms deemed to be relevant and important to the research questions.

The second step, carried out concurrently with underlining of key terms, was to "restate key phrases in the margin of the text" (p. 10) using words as close as possible to the text in the interview transcript. Chesler stressed the need to be able to use the restatements of the key phrases to go back to comments in the transcript.

The third step was to "reduce the phrases and create cluster" (p.10) by placing phrases with the same focus together in a cluster. Chesler suggested checking the original transcript when assigning phrases to clusters as he believed it could be difficult to know the exact meaning of a phrase in the absence of the context of the original text. This highlights the importance of step two in Chesler's approach where the researcher ensures that the restated phrases are able to be used to return to the original transcript. Next, 'constant comparison' (see for example Bryman, 2004, p. 403) was used to reduce the number of clusters to form 'meta-clusters'. Strauss and Corbin (1988) acknowledge the role that the researcher's experience and knowledge plays in grouping of comments.

> Later, as we move along in our analysis, it is our knowledge and experience (professional, gender, cultural, etc) that enables us to recognize incidents as being conceptually similar or dissimilar and to give them conceptual names. It is by using what we bring to the data in a systematic and aware way that we become sensitive to meaning without forcing our explanations on data (Strauss & Corbin, 1988, p. 47)

Analysis of the meta-clusters provided the overriding themes emerging from the data. Finally, prior to developing a theory, Chesler's fourth step involved generalisation of the phrases within a cluster to provide an analysis of the meaning of the cluster.

Data to illustrate approaches

The data used to illustrate the process of theme analysis and development of case studies in this paper is from teacher interview transcripts. These interview transcripts are from a study that was part of a Victorian research project investigating the implementation of CAS in year 12 mathematics (see Stacey, McCrae, Chick, Asp, & Leigh-Lancaster, 2000). CAS is a technology which is able to automate many mathematics procedures and when this research project was undertaken it was the first time that CAS was allowed in year 12 examinations in Victoria. The teachers and students were the first to undertake a year 12 mathematics subject where CAS was allowed in all aspects of the course. Students were provided with a handheld CAS for use in class, at home and in examinations.

The teachers were experienced year 12 mathematics teachers who had taught year 11 maths with CAS, but this was the first time they had taught a Year 12 mathematics subject where CAS was assumed in the examinations. Semi-structured interviews were conducted with each teacher mid-year and also following the end-of-year examinations in November. The interviews were intended to cover a number of issues, as part of data collection for the research project, so there were numerous foci for questions. As a result, it was necessary to decide on an approach for analysing interview transcripts to investigate particular research interests for this study, which was part of the larger research project (for further information about the research project see Ball & Stacey, 2005a, 2005b, 2006; Flynn, 2003; Stacey, 2003).

The research area investigated in this paper is the use of pen-and-paper or CAS in senior secondary school mathematics. For the purposes of this paper it is useful to know that the teachers had to come to personal decisions about the extent and nature of pen-and-paper and CAS use in their classrooms, in the context that students were allowed to use CAS in examinations. In the following sections the examples used to illustrate the analysis are related to use of pen-and-paper or CAS.

Analysis of interview transcripts to determine themes

This section describes the nine-stage approach used for analysis of interview transcripts to determine themes in this paper. Figure 1 provides an overview of the analysis process, which comprises nine stages. The two terms 'theme' and 'emerging theme' indicate the results of two different aspects of the analysis of interview transcripts. *Emerging themes* are the product of stages 1–3 and *themes* will be confirmed following stage 6. There are two additional stages for the development of case studies which are shown in Figure 5 and which will be described later in this paper. Stages 1–2 will now be described.

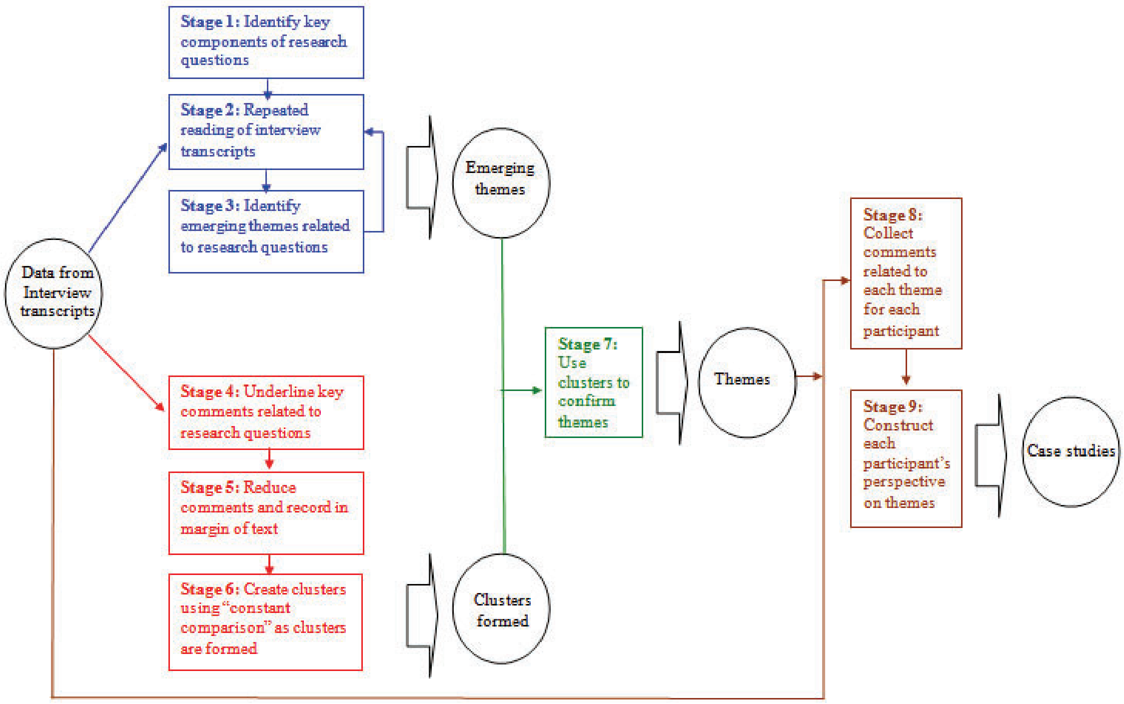

Figure 1. Cycle for analysis of interview transcripts to determine themes and produce case studies.

Identification of emerging themes

The product of stages 1–3 will be a list of emerging themes evident in the interview transcripts and related to specific research questions (refer to Figure 1). These three stages are different to the initial steps described by Chesler, who started his analysis by underlining key terms and then paraphrasing these key terms in the margin of the text.

Stages 1–3

In stage 1, the researcher identifies the key component of the research question. The reason that the term key component is used is that some research questions may have a number of components which need to be investigated. For this paper the key component is use of pen-and-paper or CAS in class.

Stages 2–3 are carried out concurrently. To identify emerging themes the interview transcripts need to be read a number of times. The process is commenced by reading the interview transcript and recording a draft list of emerging themes. This list is a first attempt at emerging themes and will be modified (either by adding, deleting or rewording) as interview transcripts are re-read a number of times. In this analysis the goal is not to find all emerging themes in the data but instead to find those specifically related to research areas of interest. Themes unrelated to the research questions are not considered.

As outlined previously, Strauss and Corbin (1988) stressed the importance of a researcher's experience and knowledge in recognising similarity and differences in comments and this will be important when identifying themes. To recognise recurring ideas in data requires the ability to recognise words and expressions that may be referring to a common idea, hence the need to have appropriate experience and knowledge of the research area. Themes are not going to be repeatedly stated in interview but instead will need to be in inferred by the researcher. As a result it is important, using the illustrative examples in this paper, that the researcher is able to recognise recurring ideas associated with pen-and-paper or CAS. The documentation following stages 1–3 is a list of emerging themes. The final wording of the confirmed themes will be decided in stage 6.

As there is no documentary evidence for stages 1–3, except for a list of emerging themes, it is only possible to provide a statement of a theme here to illustrate the stages. One emerging theme was "students' understanding" in the context of classes with access to both pen-and-paper and CAS.

Following are descriptions of the other stages (4–6) used to confirm themes with the illustrative examples related to the emerging theme "students' understanding".

Formation of clusters

The goal of stages 4–6 is to produce a list of clusters, each of which contains paraphrased comments with the same focus related to the research interest (see Figure 1). These three stages are independent of stages 1–3 and hence the emerging themes are not referred to for stages 4–6. To carry out stage 4 it is necessary to go back to the data and do a separate analysis of the interview transcripts.

Stage 4

First return to the interview data and research question/s and highlight key comments related to each research question. This process will most likely require re-reading of the interview transcripts a number of times, particularly if each research question is considered separately. Note that a sentence may contain more than one key comment, particularly if sentences are lengthy.

Figure 2 provides an illustrative example of the outcome of stage 4 using an excerpt from a teacher interview. This excerpt was chosen as there is a focus on *understanding*, which linked directly to the emerging theme being used as an illustration in this paper. In this case the research question was related to use of CAS or pen-and-paper in class and four comments are highlighted. It is important to note here that the researcher had observed many classes where CAS was available and this aided in the ability to recognise words that signalled pen-and-paper or CAS use. To conduct stage 4 the researcher recognised words or phrases that identified relevant comments to be highlighted. Some examples in Figure 2 are the terms "by-hand" and "manual" which referred to *pen-and-paper* in the context of the research question. Part of the third highlighted comment "if they didn't fiddle with it a bit themselves they didn't have a strong sense of ownership over it" may seem irrelevant at first. However, in the context of the fully highlighted comment this statement suggested that students needed to have an understanding of the mathematics by performing some pen-and-paper or CAS work (i.e. "fiddling with it") so that they would not be perturbed by unexpected technology displays or outputs. This stresses the need for the researcher to be able to interpret the interview comments in the context of the study and to recognise relevant comments, rather than only look for key words. If a researcher only highlights comments with the terms *by-hand* or *pen-and-paper* or similar then the third highlighted comment "there was an element of magic with a lot of the things ... they were less able to say well that's just a case of this or I know why that's doing it now" would be excluded. The second highlighted comment "A more likely sequence is for me to start with manual" where the term "manual" refers to pen-and-paper may also be missed. The ability to interpret the comments made by the teacher in the context of the research questions is essential here in order to be able to identify relevant phrases and comments to highlight.

Stage 5

The purpose of stage 5 is to reduce the data to assist in formation of clusters in stage 6. Highlighted comments will be paraphrased and the paraphrased comments will be recorded in the margin of the interview transcript. The intent here is not to focus on the precise wording of the paraphrased comments, as they are not used for reporting, but instead to ensure that each paraphrased comment accurately summarises the key idea highlighted in the interview transcript. This will assist in stage 6 when paraphrased comments with a common focus are collected to form a cluster. As stated previously some sentences may contain multiple foci and hence one sentence may result in more than one paraphrased comment.

Paraphrasing will enable teacher comments with a common meaning, but different wording, to be represented by the same or similar comments. Referring again to Figure 2 it can be noted that the last three paraphrased comments all refer to learning mathematics with pen-and-paper prior to use of CAS, even though the wording of each phrase is slightly different. The difference reflects the additional focus of the teacher comments, namely the ability to "deal with syntax and unexpected outputs" (i.e. for working with CAS) and "for simple cases to develop understanding of mathematics" (i.e. to perform simple cases using pen-and-paper to develop mathematical understanding). Figure 3 provides additional examples of paraphrased comments related to pen-and-paper or CAS. Implicit in the teacher's second highlighted comment in

Figure 3 is that pen-and-paper should be used to develop mathematical understanding ("Once they understand how they've got it") before a CAS can be used ("then use the calculator" implies following pen-and-paper). Again, the researcher needs a good understanding of the research context in order to recognise this.

Stage 4 – Highlighted interview comments related to research question	Stage 5 – Paraphrased comments
Sometimes, when I'm getting them to look for patterns and things and see if they can see anything that's there I might start with that and then use that as the basis for exploring why that might be the case and then often we would go back to then doing something by-hand. A more likely sequence is for me to start with manual because I kept getting the feed back in different ways from the kids and from their work that there was an element of magic with a lot of the things and if they didn't fiddle with it a bit themselves they didn't have a strong sense of ownership over it and once they got lost in the syntax or they got lost in the unusual nature of the output, an unfamiliar output, then they were less able to say well that's just a case of this or I know why that's doing it now because this is just a separated fraction or (...). So I think I kept feeling reinforced in the view that with all the new procedures you really needed to spend some time to make sure they could do the simple cases by-hand and if you didn't you were battling, you were battling with the majority of the kids (...) because they don't know (...), they don't know what the building block was. So I'm more reinforced about the view than I was even when I was doing graphical calculator stuff (...)	Sometimes CAS used to generate patterns for exploration and then pen-and-paper. Normally start with pen-and-paper Pen-and-paper first to understand mathematics and be able to deal with syntax and unexpected outputs Pen-and-paper first for simple cases to develop understanding of mathematics.

Figure 2. Interview excerpt illustrating stages 4-5 (Teacher 1 end-of-year).

Well I think that's when you have to do things by-hand. I feel you need to be able to show them how to come up with things by-hand so that they can understand where this part of the equation comes from, where the solution might come from, and how it all fits together. Once they understand how they've got it, they can then use the calculator (...)	Pen-and-paper for understanding Once students understand the maths (using pen-and-paper) students can use CAS – teacher legitimizing CAS use.

Figure 3. Interview excerpt illustrating stages 4-5 (Teacher 2 mid-year).

Stage 6

The purpose of stage 6 is to form a number of clusters, with each cluster containing a group of paraphrased comments with the same focus. This is the last stage where the paraphrased comments are used.

Two or more paraphrased comments will be required for formation of an initial cluster. "Constant comparison" (see for example Bryman, 2004, p. 403) is used to finalise the clusters and occurs as paraphrased comments are assigned to clusters. Constant comparison involves comparison of paraphrased comments within and across clusters as they are formed to ensure that each cluster accurately reflects the data within it. Where a paraphrased comment appears to belong to two clusters the comment will be compared to other comments in each of the two clusters, searching for similarities and differences. If it is still not evident which cluster the paraphrased comment belongs to then the two possible clusters will be reconsidered to determine if they are sufficiently different to warrant two clusters, or whether they should be consolidated to form one cluster. If the two clusters are combined then each paraphrased comment will be individually reconsidered to determine appropriateness for inclusion in the new combined cluster.

Each cluster will be labelled with a name to represent the main focus of the paraphrased comments within the cluster. An example of a cluster with some associated paraphrased comments is shown in Figure 4. Note here that each comment refers to the order in which pen-and-paper or CAS is used in class. Some comments specifically state that CAS or pen-and-paper should be first, while other paraphrased comments, for example, to use “CAS for checking pen-and-paper sketching of graphs”, suggested that pen-and-paper was first. One cluster, “students’ choice of CAS or pen-and-paper”, contains the paraphrased comment “Teacher encourages able students to use pen-and-paper first and then use CAS once they know how to do something”. This paraphrased comment was originally assigned to the cluster “pen-and-paper or CAS first” (Figure 4), but then on reconsideration was placed in the “students’ choice of CAS or pen-and-paper” as the focus was on the teacher encouraging able students to use pen-and-paper first. With a focus on mathematically-able students, there is a suggestion that there might be different decisions to be made, depending on students’ facility with pen-and-paper techniques.

Pen-and-paper or CAS first	• Teacher demonstration of pen-and-paper first then CAS for speed • Teach pen-and-paper first and then CAS • CAS for checking pen-and-paper sketching of graphs • Always do graphs with pen-and-paper first and then check results

Figure 4. Example of cluster with some paraphrased comments in cluster.

Confirmation of themes

The intention of stage 7 (in Figure 1) is to confirm and hence finalise the themes. This involves clarifying the wording of the themes and checking that the themes are supported by the interview data. Stage 7 uses the outputs of stages 1-3 and 4-6, making use of the emerging themes and clusters.

Stage 7

First allocate each cluster to an emerging theme. Where clusters appear to align to two emerging themes, the statements of the emerging themes will be reconsidered and reworded if necessary to clearly distinguish the two themes. Themes will be discussed with a second researcher, who will also have read the teacher interview transcripts, to clarify descriptions of themes. Themes will be confirmed when they have one or more clusters listed under them. If an emerging theme does not have an associated cluster then this will not be included as a final theme.

Development of case studies using themes and teacher comments

The final two stages of the process (stages 8 and 9 in Figure 1) are used to produce case studies for each theme.

Stage 8

In this stage participants' comments related to each theme are collected. Note that individual participants may or may not make comments relating to every theme.

For stage 8 the researcher returns to the original interview transcript, considers each highlighted comment and assigns each comment to a theme. The result of this process is a collection of interview comments for each theme.

Figure 5 illustrates two teacher comments related to the theme "students' understanding". The first comment suggests that the teacher believes that students need to see pen-and-paper work ("by-hand") to understand the mathematics prior to use of CAS ("understand where this part of the equation comes from, where the solution might come from, and how it all fits together"). The focus for this comment appears to be the teacher's desire for students to develop understanding before using the CAS calculator. The second comment in Figure 5 also focuses on the importance that the teacher places on students developing understanding before using CAS.

- I think that's when you have to do things by-hand. I feel you need to be able to show them how to come up with things by-hand so that they can understand where this part of the equation comes from, where the solution might come from, and how it all fits together. Once they understand how they've got it, they can then use the calculator.
- ... it helps them to see how the actual solution is developed and what different parts of the solution refer to, and where they fit in. But then once they know how to do it and once they understand how to do it then they don't need to do that over and over again, they can use the calculator. So I think they do need to do that in order to develop understanding but then once they know how to do it, they know how to do it, and they can use the calculator.

Figure 5. Collection of sample interview comments related to a theme - students' understanding.

Stage 9

In stage 9 the comments for each participant for each theme are summarised to produce case studies. The case studies provide a summary of each participant's position in relation to each theme for which comments are provided. There is no illustrative example provided for this stage in this paper.

Conclusion

The cycle for analysis requires identification of terms, comments and emerging themes related to a given research question, familiarity with the research area and the context in which the participant (in this case the teacher) works. The paraphrasing of comments is a practical way for the researcher to engage with the interview data and decide on the key focus for each interview comment. The formation of clusters by grouping paraphrased comments provides a second opportunity to consider key messages in the data. The use of clusters to confirm themes provides yet further consideration of the data. This structured cycle for analysis of interview transcripts has proven helpful for identifying emerging themes, confirming these themes and then producing case studies.

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CHILDREN SOLVING WORD PROBLEMS IN AN IMPORTED LANGUAGE: AN INTERVENTION STUDY

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This paper reports on one aspect of a two-year design study aimed to assist second-grade Filipino children solve additive word problems in English, a language they primarily encounter only in school. With Filipino as the medium of instruction, an out-of-school pedagogical intervention providing linguistic and representational scaffolds was implemented with 17 children. Pre-intervention, children experienced linguistic difficulties and were limited to conceptualising and solving simple additive structures. Post-intervention interviews revealed improved understanding of more complex structures, but only when linguistic difficulties were minimised.

Filipino children from disadvantaged families are expected to learn mathematics and solve word problems in English, a language they primarily encounter only in school (Young, 2002). Thus, it is not surprising that many Filipino students who have completed two or three years of schooling are unable to solve even simple addition and subtraction word problems (Bautista, Mitchelmore, & Mulligan, 2009; Bernardo, 1999). While language problems often arise as a cause for poor performance in mathematics (Philippine Executive Report on the TIMSS, cited by Carteciano, 2005), what is not clear is whether lack of English language proficiency is the main reason for Filipino children's poor problem-solving performance. This study attempts to provide insight into these issues by addressing the following research questions:

1. Is the failure to solve problems due to linguistic difficulties and/or to an inadequate understanding of the semantic structure and associated mathematical relationships in the given problem?
2. Is it possible to improve young Filipino children's strategies for solving addition and subtraction word problems presented in English?

Although the study was conducted in the Philippines, it has applications to similar contexts where children learn mathematics in a language not widely spoken in the community. Such is the case in remote Indigenous communities in Australia, as well as in several developing nations in Asia and Africa.

Theoretical background

The classification of addition and subtraction word problems according to their semantic structure (see Table 1) has formed the basis of a long tradition of research on addition and subtraction word problems (Carpenter & Moser, 1984).

Table 1. Some types of addition or subtraction word problems.

Problem Type	Problem
Join	Alvin had 3 coins. Then Jun gave him 8 more coins. How many coins does Alvin have now?
Separate	Dora had 11 mangoes. Then Dora gave 6 mangoes to Kevin. How many mangoes does Dora have now?
Combine	Tess has 5 hats. Rodel has 8 hats. How many hats do they have altogether?
Missing Addend	Jolina had 7 pencils. Then Alma gave her some more pencils. Now Jolina has 12 pencils. How many pencils did Alma give her?
Part Unknown	Jimmy and Mia have 11 marbles altogether. Jimmy has 4 marbles. How many marbles does Mia have?
Compare	Rica has 12 books. Luis has 7 books. How many more books does Rica have than Luis?
Equalise	Rica has 12 books. Luis has 7 books. How many books does Luis need to have the same number of books as Rica?

Recent theories on word problem solving processes have drawn on the text comprehension theories of van Dijk and Kintsch (1983). When solving problems, the solver first integrates the textual information into an appropriate *situation model*, or a mental representation of the situation being described in the problem, which then forms the basis for a solution strategy (Mayer, 2003; Thevenot, 2010). Because the construction of a coherent situation model depends on adequate proficiency in the language of the text (Zwaan & Brown, 1996), children solving problems in a language not widely spoken outside school are clearly disadvantaged. Unless children's proficiency in their second language allows them to use their bilingualism as a cognitive tool (Clarkson, 2007), they struggle with linguistic structures that would not be as problematic for native speakers (Martiniello, 2008).

This is not to say that linguistic factors are the only barriers to problem comprehension and solution. Strong part-whole knowledge and a flexible understanding of number meanings are seen as essential for recognising the structure of additive problems (Poirier & Bednarz, 1991; Zhou & Lin, 2001). For example, children may fail to solve the Missing Addend problem in Table 1 if they can reason about a set only if they know its cardinal measure. In Vergnaud's (2009) terms, they lack essential *concepts-in-action*. Interestingly, the advantage of expertise in the problem domain (in this case, part-whole knowledge) on the construction of situation models is widely recognised in text comprehension research (Hirsch, 2003).

Method

The intervention study reported here is part of a larger project aimed to improve word problem solving performance in the Philippine context. A design research methodology

(Lesh & Sriraman, 2010) was adopted, as it is particularly appropriate for identifying and responding to conditions for success (Dede, 2004). The study involved several iterations of assessments and interventions (Table 2).

Data reported in this paper refer to 17 children (11 girls, 6 boys; mean age: 7.8 years) from public schools in the Greater Manila area who voluntarily participated in a parish-based tutorial program from June to September 2009. They were taught in shifts of 4-8 students by the author and two volunteer tutors who were trained on the pedagogical approach.

Table 2. Design study process and timeline.

Oct-Nov 2008	Feb-Mar 2009	Apr-May 2009	Jun-Sep 2009	Oct 2010
Written test (*N* = 75) Interview (*N*=7)	Written test (*N* = 348) Interview (*N*=50)	Pilot intervention (*N* = 90)	Intervention (*N* = 17)	Community consultations (*N* = 23 teachers)

Consistent with features of a design study, pedagogy was informed by an integration of van Dijk and Kintsch's (1983) linguistic comprehension theory and Vergnaud's (2009) theory of mathematical learning, as well as by earlier stages of the study (Table 2). The following section briefly describes how the pedagogical approach was designed.

The decision to use Filipino as the medium of instruction during the intervention, to provide word lists of common English words, and to present text in simplified formats was based on several convergent findings. First, two written tests administered to two different samples of Grade 2 and Grade 3 students (Bautista, Mitchelmore et al., 2009; Bautista & Mulligan, 2010a) confirmed that Filipino students were more successful in solving word problems written in Filipino than equivalent problems written in English. Second, interviews with 57 children from 15 public schools (see Bautista, Mulligan, & Mitchelmore, 2009, for interviews with 7 of these children) showed that children could not use English even for social conversation, and a considerable number used Filipino rules to decode English text, making it very difficult to teach them in English.

Because it was hypothesised that word problem solving involves more than linguistic competence (Vergnaud, 2009), the intervention aimed to strengthen children's concepts-in-action by presenting each additive structure in Table 1 through a range of representations (Lesh, Post, & Behr, 1987). For example, a concrete representation for the Separate problem structure in Table 1 was to briefly display, then screen, 11 counters (Wright, Martland, & Stafford, 2000). Without allowing the child to see, 6 counters were then removed. The child was then asked in Filipino, ―There were 11 counters, but then I took away 6 counters. How many counters are there now?" These various representations were particularly helpful given that the children in this study struggled with textual representations (Bautista & Mulligan, 2010b).

The primary data source was the individual scaffolded pre- and post-intervention interviews illustrated in Figure 1 (see Bautista & Mulligan, 2010b for details). In essence, the interview schedule involved presenting the first six word problems in Table 1 for the child to read and solve in English. However, if the child reached an impasse, successive linguistic and mathematical scaffolds were provided. The mathematical scaffold was either a concrete representation of the task or a transformation of the

Compare problem to a mathematically simpler Equalise task (see Table 1). All number triples were in the range 1-20, and based on Carpenter and Moser's (1984) procedure. Pre- and post-intervention tasks differed only in their surface elements (e.g., using mangoes instead of coins) and in the number triples used. The interviews were conducted in Filipino by the author.

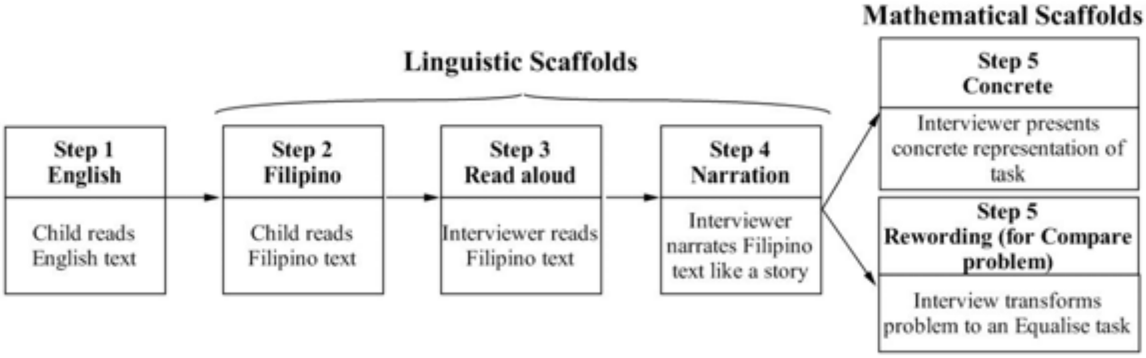

Figure 1. Structure and sequence of the interview protocol.

Results

The results are discussed in terms of the two research questions.

Linguistic and/or mathematics difficulty?

The scaffolding techniques in the pre-intervention interviews were used to investigate to what extent linguistic or mathematical factors impeded word problem solving. The Pre-intervention graph in Figure 2 shows the type of scaffold that facilitated correct solutions. Darker areas in the graph represent instances when linguistic scaffolds were necessary and sufficient for success. The extent of the dark regions shows that the children were dependent on linguistic scaffolds—very few of them could solve problems in English, without assistance. However, the linguistic scaffolds were primarily helpful for the Join, Separate and Combine problems. In contrast, the linguistic scaffolds facilitated correct solutions for less than a quarter of the children for the remaining problems, indicating underlying mathematical difficulties.

Linguistic difficulties were reflected in children's struggle to interpret the text. Thirteen children had difficulties in decoding text (7 in English, 6 in Filipino), and one could not read at all. Further, several children knew only a few basic English words. For example, 11 children did not understand the statement, ―Alvin had 3 coins." Difficulties in retrieving textual information also occurred for Filipino problems. For example, 4 children could not identify the giver from the Filipino translation of the statement, ―Then Alvin gave her 8 more mangoes."

Mathematical difficulties were observed in the Missing Addend, Part Unknown, and Compare problems. Some children were limited to conceptualising and reasoning about disjoint subsets with known quantities. For example, C7[1] constructed two disjoint sets, instead of one set having a subset for the Part Unknown problem, even when a corresponding concrete task had been provided, and even when smaller numbers were

[1] To preserve anonymity, codes were used in place of children's names. The coding conventions will be explained in a later section.

used (2 + □ = 6). Only 10 children correctly solved the Compare problem, pre-intervention, and 8 of these managed to solve only the corresponding Equalise task.

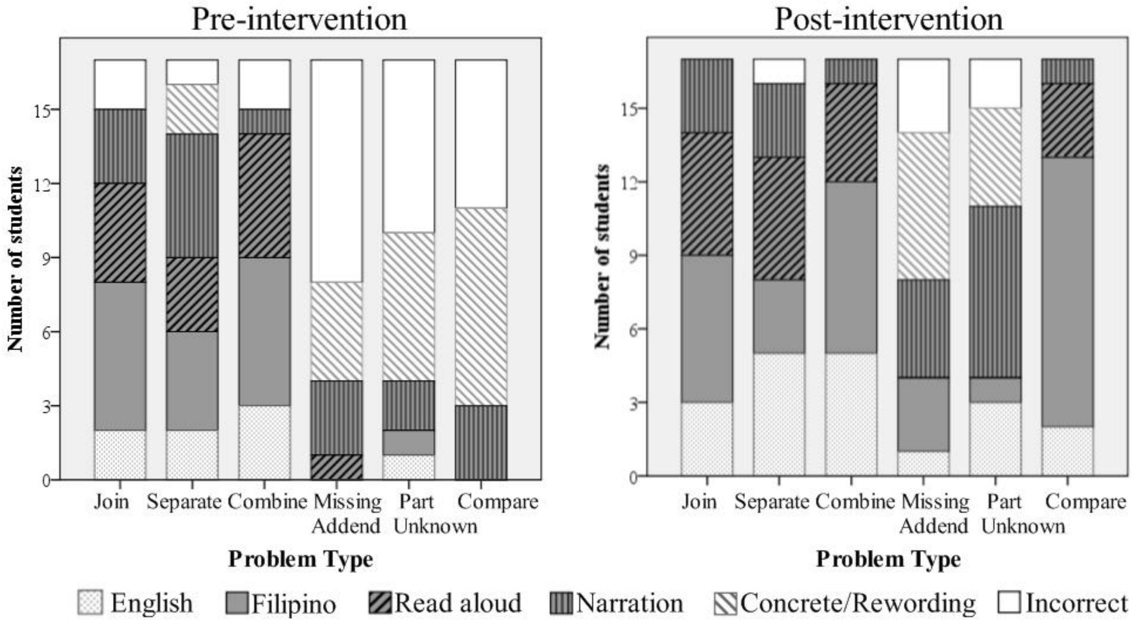

Figure 2. Text processing strategies before and after the intervention.

Intervention outcomes

The Post-intervention graph in Figure 2 presents the step in the interview, post-intervention, at which a correct solution was achieved. While performance on the Missing Addend, Part Unknown, and Compare problems improved post-intervention, the children's unfamiliarity with the language continued to prevent them from solving word problems presented in English. When A2 was asked if there was any word he did not understand, he looked at the text and said, ―*Lahat 'yan* [All of them].” The words directly taught during the intervention were largely just memorised. When C5 was asked what ―more” meant, she said, ―*Nakalimutan ko* [I forgot].”

Children were also found to construct a situation model based on isolated words from the text. For example, C2's understanding of ―Alvin had 3 coins” was reduced to one word: ―*Pera* [money]”. Having been exposed to various additive structures during the intervention, however, some children tried to determine which of these structures matched the problem text. For instance, after B2 read the English Missing Addend problem, she asked whether it was an ―*Ilan yung lagpas* [How many more]” task.

Individual student profiles

To further investigate the outcomes of the intervention, an analysis of each child's progress was made. An analysis of the interviews revealed that children could be classified into distinct categories according to their (1) level of mathematical strategies, and (2) level of text processing strategies. Table 3 describes children's increasing levels of mathematical strategies, from counting strategies to more advanced relational strategies (e.g., calculating 9 + 6 as 9 + 1 + 5). Similarly, Table 4 shows levels of text processing strategies, which are based on the interview structure in Figure 1.

Table 3. Most sophisticated strategy observed at each level.

Level	Addition Strategies	Subtraction Strategies
1	Erroneous Strategy/Count All	Erroneous Strategy/Separate
2	Count All	Separate
3	Count On	Count Up
4	Mental	Mental
5	Bridge-through-ten/Compensation	Bridge-through-ten

Table 4. Level of text processing strategies.

Level	Description
1	Needed to have the text elaborated or concretely presented to them for most problems
2	Could use Filipino text to solve word problems, albeit limited to Join, Separate, and Combine problems
3	Could use Filipino text to solve at least one of the Missing Addend, Part Unknown, and Compare problems
4	Could use English text to solve at least three problems

The matrices in Figure 3 display the levels for each child pre- and post-intervention along two dimensions: mathematical and text processing levels. The pre-intervention matrix was divided into four regions (A, B, C, and D), representing various combinations of high and low levels on each dimension. Children were then assigned codes based on the region where their results were located in the matrix. For example, children in the upper right region were all prefixed B. This system was developed in order to more easily compare pre- and post-intervention results. The numbers in parentheses represent the number of problems each child solved correctly.

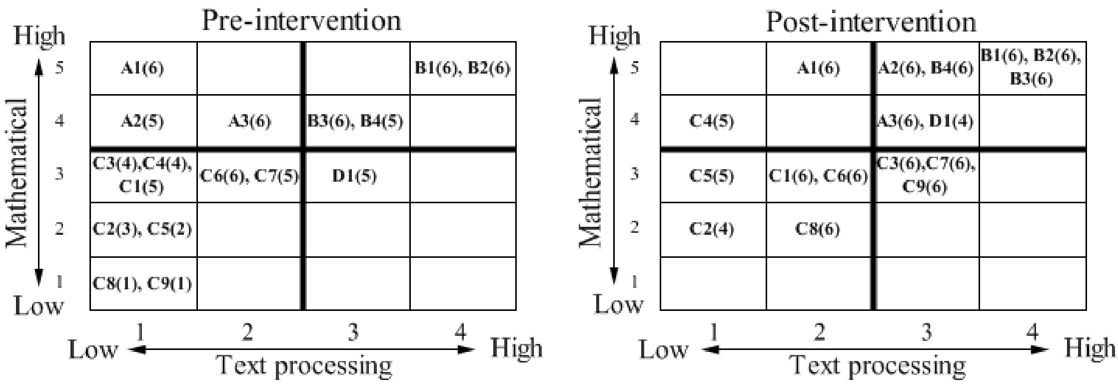

Figure 3. Student profiles before and after the intervention.

Prior to instruction, children's mathematical levels were associated with the number of word problems they solved correctly. Children at higher mathematical levels tended to solve more problems than those at lower mathematical levels. However, post-intervention, it became possible for children with low mathematical levels to solve five or six problems. Remarkably, all children who could solve problems in English (Level 4 in text processing) could also utilise sophisticated mathematical strategies (Level 5 in mathematics), both pre- and post-intervention.

A comparison of the matrices demonstrates how each child progressed during the intervention. However, the matrices also reveal conditions for success. For instance, the only children who reached Level 5 in mathematics were those prefixed A and B. Thus, these were children who already utilised a range of strategies before the intervention. The rest of the children continued to count by ones. Similarly, post-intervention, only three children could solve word problems in English (Level 4 in text processing), and they were all prefixed B. These were the children who, prior to the intervention, could solve some of the more difficult problems without the need for read-aloud or narration supports. In contrast, seven children continued to rely on substantial help from the interviewer. These seven included two children who, in spite of having low text processing levels, had high mathematical levels—A1 who was a non-reader and C4 who read one syllable at a time, often with errors.

Discussion

Although this part of the larger study involved a small sample, which does not permit generalisation, the results provide a rich description of how language proficiency and reading skill interact with word problem solving performance. There were apparent linguistic difficulties observed, as when children could not understand simple English statements, or when reading difficulties prevented them from retrieving information explicitly stated in the text. Indeed, these challenges were more pronounced than those commonly reported in the literature, which tends to relate to difficulties with academic, rather than conversational, language (Fillmore, 2007), and to comprehension difficulties associated with ambiguous text (Cummins, 1991).

This is not to say that linguistic difficulties were the only obstacles to solving word problems. Mathematical difficulties were uncovered, but only when linguistic difficulties were minimised through the provision of linguistic scaffolds. Consistent with findings from monolingual children (Carpenter & Moser, 1984), the data indicate difficulties in conceptualising certain semantic structures. Some children found it difficult to conceptualise relations involving comparisons and sets with unknown quantities. Thus, they failed to solve the Missing Addend, Part Unknown, and Compare problems even when linguistic scaffolds were available.

Concerning the attempt to help children solve word problems in English, the findings demonstrate that while it is possible to help children conceptualise a wider range of additive situations and advance their mathematical strategies, children's pervasive reliance on linguistic scaffolds suggests difficulties in mapping the text to mathematical knowledge. To compensate, some children constructed situation models based on a few words and the situations they encountered during the intervention. Although the data could not directly establish that children's weak linguistic skills encouraged such coping strategies, it remains clear that their linguistic difficulties inadvertently presented them with no other option.

The finding that all children who had advanced text processing strategies in English also utilised advanced mathematical strategies suggests possible connections between mathematical strategies and the ability to solve word problems in an imported language. Further research is needed to investigate this conjecture.

Implications

This study has a number of educational implications. First, it critically questions the use of an imported language for mathematics instruction. However, as there are pragmatic difficulties in changing the language policy in Philippine classrooms (Bernardo, 2008), other avenues for coping with language issues need to be explored. Recommendations include code-switching, the development of materials in the local language, and equipping teachers with tools for teaching in the imported language.

Second, as reading difficulties definitely limited children's text-processing strategies, reading comprehension strategies should be integrated into the mathematics classroom (Fogelberg et al., 2008), and reading instruction should be provided to non-readers.

Third, teachers should provide children with opportunities to develop their conceptions of relational structures by creating lessons that incorporate various representations. A range of representations is particularly helpful as children who struggle with one representation may be able to handle other forms of representation.

Fourth, children's continued reliance on unitary counting suggests that they may benefit from an intervention specifically focussed on developing relational strategies (Gersten et al., 2009). Left unattended, these unitary counting strategies may impede performance on multidigit addition and subtraction (Ellemor-Collins, Wright, & Lewis, 2007).

Finally, written tests should be supplemented with individual interviews or informal conversations because language issues may conceal underlying mathematical difficulties. However, considering the onerous time demands these may place on teachers with large classes, a whole-class assessment followed by individual interviews for a target group of low-attaining students may be feasible (White, 2008).

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A STUDY OF THE AUSTRALIAN TERTIARY SECTOR'S PORTRAYED VIEW OF THE RELEVANCE OF QUANTITATIVE SKILLS IN SCIENCE

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The ability to apply mathematical and statistical thinking within context is an essential skill for graduate competence in science. However, many students entering the tertiary sector demonstrate ambivalence toward mathematics. The challenge, then, is to determine how science curricula should evolve in order to illustrate the integrated nature of modern science and mathematics. This study uses a document analysis to examine degree structures within science programs at a selection of Australian tertiary institutions. Of particular interest are the signals these degree structures send in terms of the relevance of the study of mathematics prior to entry to university and the quantitative content within.

Introduction

The increasing dependence of modern science on data, algorithms and models has resulted in a greater need for science graduates to achieve competency in Quantitative Skills (QS)[1]. This fact is acknowledged through publications such as the *Bio 2010* report from the National Research Council of the USA (NRC, 2003). More recently, the Learning and Teaching Academic Standards—Draft Science Standards Statement Consultation Paper, published by the Australian Learning and Teaching Council (ALTC, 2010), provides clear statements on learning outcomes for Australian science graduates. The document represents the opinion of academic scientists in Australia and details *threshold learning outcomes* that all recent graduates of science are expected to demonstrate. These are "minimum standards" and many are explicit regarding the use of QS.

[1] In this article the adjective "quantitative" is used to describe the fundamental skills that allow a scientist to use mathematical and/or statistical thinking and reasoning to gain understanding of scientific processes. In the context of primary and secondary education, the term "numeracy" has frequently been used in place of quantitative skills. For example, in The Report of the Numeracy Strategy Education Development Conference, published by the Australian Association of Mathematics Teachers (AAMT, 1997), numeracy is described as the ability to use mathematics to achieve some purpose in a given context.

Meeting the need for increased proficiency with QS is a considerable challenge for tertiary educators when one considers the environment in which these advances need to occur. The report by Brown (2009) details the downwards trend in the mathematical preparedness of students entering the sector. With students displaying weaker skills and increasingly negative feelings towards quantitative tasks, tertiary science educators are struggling to understand how best to foster the development of QS in science students.

The challenge for science and mathematics educators of how best to demonstrate the intimate relationship between the disciplines is a source of continuing conjecture and robust debate. Wood and Solomonides (2008) argue that when teaching mathematics, a context-based approach produces graduates who are more workplace-ready. Thus, many academics seeking to engage students in learning how to use mathematics skills favour interdisciplinary or integrated approaches because they involve context (see for example, Matthews, Adams and Goos (2009)). Similarly, Venville, Wallace, Rennie and Malone (2002) report that secondary school teachers employ these approaches to enhance pupil engagement in learning. While placing material in context may be a useful motivator, Tariq (2008) is one of many who report that the contextual nature of the problems in science requiring QS poses additional challenges for many students.

Despite the large body of literature discussing the teaching of QS to science students, there is still substantial confusion and variation in opinion regarding its importance. The negative attitude students display towards applying QS is perhaps reflective of a larger body of opinion in society that expresses confusion, or worse mistrust, when quantitative arguments are used to discuss issues in science. Undoubtedly there are many factors leading to this negative view of the quantitative nature of modern science. This publication considers the influence the Australian tertiary sector has on the perception of the importance of QS in science. The analysis uses publically available documents (internet web pages) which the institutions either contribute to, or publish themselves, regarding academic preparation for, and content within, science degree programs. Accessible information includes (i) science degree entry requirements; (ii) prerequisite or assumed knowledge requirements for subjects within the degree program; and (iii) subject or unit descriptions. Through these documents, it is possible to gain some insight into the portrayed value of proficiency with QS in science. This information is of interest and importance, not only to prospective science students, but also to secondary educators who have to grapple with this issue and frequently look to the higher education sector for leadership in terms of a consistent message that can help motivate themselves and their students. Furthermore, secondary teachers and guidance officers often advise students on subject choice in the later years of secondary school, and their views are heavily influenced by the content of these documents. Therefore, we explore the following research question:

> What is the apparent relevance of QS in Science in tertiary education, as portrayed by publically available documents?

We also briefly comment on the literature to illustrate some approaches to the delivery of QS to science students. Whilst not within the direct scope of the above research question, this allows conclusions to be drawn regarding alignment between the importance of QS in science as portrayed beyond the tertiary education sector, and efforts within institutions to embed QS in science.

Method

The study involved documenting four characteristics of the Bachelor of Science degree programs at a selection of Australian universities. These particular degree programs were chosen over other degree programs that may be labelled as science degrees, in order to maintain consistency through the study. The only exceptions to this protocol were in instances where a Bachelor of Science was not offered by particular institutions, but a Bachelor of Applied Science was in existence.

The universities chosen for the study were those that were reported to have had enrolments of greater than 1800 students in the natural and physical sciences in the year 2005 in a study commissioned by the Australian Council of Deans of Science; see Dobson (2007). This results in a sample of 17 tertiary institutions representing over 73% of the total cohort of students across Australia studying the natural and physical sciences in 2005. It is anticipated that the messages these institutions transmit have the greatest impact on the public perception of importance of QS in science.

The following characteristics were chosen as indicators of the publicly portrayed importance of QS in science:

1. Entry requirements requiring prior study in mathematics; and
2. Compulsory requirements for mathematics, statistics or QS within the degree program.

In addition to investigating these two factors in the context of science in general, they are also applied to study within a specific science major[2], chemistry. The decision to investigate a particular major arose for two reasons. Firstly, the requirements for prerequisite study for some majors may be different to the requirements for the science degree as a whole.[3] In this study we measure the publicly portrayed importance of prior knowledge of mathematics for success in a chemistry major as part of a science degree. Secondly, through subject sequences and prerequisite and assumed knowledge requirements, we wish to determine whether it is possible to observe development of QS through the major. Specifically, we investigate how deeply is it possible to observe *subjects that develop QS*[4] within the chemistry major.

Chemistry was chosen as an appropriate representative major for two reasons. Firstly, it is recognised that the variety of programs represented by the Bachelor of Science is considerable, so using minimum standards to measure the articulation of the importance of QS may be similarly varied. By focusing on the sequence of subjects that defines the chemistry major in each program (note that most Bachelor of Science degrees have such a major), some of the variability is removed. Secondly, we note that the Draft Science Standards Statement Consultation Paper, published by the ALTC (2010), has resulted in a corresponding document for the discipline of chemistry, titled

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[2] In this publication a major refers to a sequence of subjects that a student must complete as a part of a science degree, in order to be deemed proficient in the discipline area named as the major. Typically, a major represents about one third of the total number of subjects required for graduation with a science degree.

[3] For example, this frequently occurs in physics majors where successful completion of secondary school physics may be a prerequisite for the major, despite the absence of a physics prerequisite for entry in to the degree program itself.

[4] *Subjects that develop QS* are defined to be discipline-specific subjects that have mathematics, statistics or quantitative skills subjects, as either prerequisite, or assumed knowledge.

the Chemistry Academic Standards Statement—Consultation Phase published by the ALTC (2011). Similar to the overarching document it states very clearly that QS are essential for graduating chemistry majors.

Prior mathematics study required for entry into science degree programs

The requirements for prior study in mathematics for entry to a science program were identified using information from tertiary admissions centres in each state of Australia. These centres were used because many students prefer to access information regarding tertiary entrance requirements in one place. For example, the University Admissions Centre (UAC) in New South Wales "processes applications for admission to most undergraduate courses at participating institutions" (UAC About us, 2011a). In their first quarterly newsletter for 2011 (UAC, 2011b) they comment on their publication *University Entry Requirements 2014 Year 10 Booklet*, stating that:

> The booklet is a valuable tool for Year 10 students choosing their subjects for years 11 and 12. It shows all the prerequisites, assumed knowledge and recommended studies for university courses starting in 2014. Each Year 12 school in NSW and the ACT will receive four complimentary copies of the booklet in mid-May.

For the purposes of this study we report the occurrence of a secondary level mathematics subject as a prerequisite or assumed knowledge for entry into a science degree.

Compulsory mathematics, statistics, or QS subjects

Some science degree programs have compulsory mathematics, statistics or QS subjects embedded within their structure. Although many majors require some such subjects, we identified the minimum requirements within *any* major of the degree program, and report the minimum number of these compulsory subjects that must be completed to be awarded the degree.

Prior mathematics study for entry into the chemistry major

We report the percentage of first year chemistry subjects that have an explicit mathematics prerequisite, statement of assumed knowledge, or recommendation of previous study.

QS requirements within a program to satisfy chemistry major

For the chemistry major in each Bachelor of Science degree, we determine the percentage of chemistry subjects beyond first year that develop QS.

Results

A summary of the data collected is presented in Table 1. The institutions in the table are ordered according to enrolments in the natural and physical sciences in 2005, as presented in the table by Dobson (2007, p. 23). The institutions are labelled according to their affiliations; "G8" represents membership of the Group of Eight, "ATN" represents membership of the Australian Technology Network and "IRU" represents membership of the Innovative Research Universities.

Table 1. Summary of results showing entry requirements in mathematics, and compulsory mathematics/statistics/QS subjects for Bachelor of Science degrees, and prerequisite requirements for chemistry majors and higher level chemistry subjects.

University	Mathematics background from secondary school: P=Prerequisite A=Assumed	Compulsory mathematics /statistics/QS subject	Percentage of 1st year chemistry subjects in the chemistry major with secondary level mathematics as prerequisite or assumed knowledge	Percentage of 2nd and 3rd year chemistry subjects in the chemistry major with 1st or 2nd year mathematics/statistics/ QS subjects as prerequisites or assumed knowledge
University of Melbourne (G8)	P	0	33%	0%
University of Sydney (G8)	A	2	50%	33%
Monash University (G8)	None	1	0%	0%
University of Queensland (G8)	P	1	0%	0%
University of New South Wales (G8)	A	0	0%	38%
University of Western Australia (G8)	P	0	0%	0%
University of Adelaide (G8)	A	0	0%	0%
RMIT University (ATN)	None	1	0%	0%
Australian National University (G8)	None	0	0%	15%
Murdoch University (IRU)	None	0	0%	25%
Queensland University of Technology (ATN)	A	1	0%	0%
University of Technology, Sydney (ATN)	A	1	0%	8%
La Trobe University (IRU)	P	0	0%	0%
Curtin University of Technology (ATN)	P	1	0%	0%
University of Western Sydney	None	0	0%	12%
Griffith University (IRU)	P	1	0%	0%
James Cook University (IRU)	P	1	50%	0%

One immediate observation from the table is the dominance of the Group of Eight in terms of enrolments in the natural or physical sciences. Dobson (2007, p. 23) reports that these institutions account for the preparation of nearly half of Australia's science graduates. We anticipate that this group sends strong signals regarding the importance of QS in science.

Mathematics preparation from secondary school

The strongest signal in any of the four data categories was in the required mathematics background from secondary school. Twelve of the seventeen institutions include mathematics from secondary school either as a prerequisite or as assumed knowledge. This signal was almost uniform across the Group of Eight institutions with only two of the eight not requiring or assuming mathematics from secondary school for students in their science degree program.

Anecdotally there is some discussion around what many regard as the weaker message associated with the word "assumed" when it is used in place of "prerequisite".

Compulsory mathematics/statistics or QS subjects in science degrees

Eight of the 17 institutions in the table have a compulsory mathematics, statistics or QS subject within the Bachelor of Science. It is difficult to draw a conclusion from this statistic, except perhaps to observe that a significant number of institutions appear to believe that students enter their science programs with adequate QS preparation from secondary school for the needs of the full degree program. Development of QS within these programs must be facilitated solely within discipline-specific subjects, relying at most on previous secondary-level mathematics study.

QS in chemistry majors

The table shows some uniformity in the portrayal of the importance of QS within chemistry majors: very few subjects comprising chemistry majors appear to develop QS. That is, there is very little reliance on building QS through links between secondary school mathematics and first year chemistry, or through links between tertiary mathematics/statistics/QS subjects and higher-year chemistry subjects. Anyone using these measures alone may conclude that the relevance of QS to becoming a capable chemical scientist is quite tenuous.

Discussion

The data presented in Table 1 are revealing in terms of measuring *external perception* of the value of QS in science. This type of data represents information that is accessed by practicing secondary school teachers and guidance officers, as well as by budding science students and their parents when choosing senior secondary school. Some students may enjoy science, but experience anxiety towards mathematics, so any hint that the study of mathematics is unnecessary, or can be postponed until later, may result in poor subject selection in the senior school.

Misalignment of external perceptions with efforts towards increased understanding of the need for QS

Whilst being a credible measure of the *portrayed* importance of QS in science, the data accessed in this study and summarised in Table 1 are a crude measure of the *actual* relevance of QS in science. These data hide significant efforts in both secondary and tertiary education to demonstrate the links between mathematics, statistics and science.

Huntley (1998) explains that approaches to curriculum organisation that foster understanding of the intertwined nature of mathematics and science almost certainly involve *interdisciplinary* or *integrated* approaches. It is certainly the case that such approaches may not be revealed by the documents analysed in this study.

At the secondary level, Venville et al (2002) conclude that the authenticity offered by integrated or interdisciplinary approaches provides an opportunity to enhance pupil engagement with school, and that process and higher-level cognitive skills may be increased. Goos and Askin (2005) report on a problem-based Year 10 course at a Brisbane school that integrated mathematics and science. The course showed success with students empowered to make more effective decisions about future careers through an understanding of how mathematics and science are used in “real life” situations.

In the Australian higher education system, Bridgeman and Schmid (2010) report on an interdisciplinary approach in laboratory exercises in first year chemistry subjects at The University of Sydney, which facilitate the development of skills in statistical analysis in a science context. Similarly an interdisciplinary teaching intervention at The University of Queensland highlighting the links between mathematics and science is discussed in Matthews *et al*. (2009). The intervention, in the form of a first year subject, was designed to demonstrate the need for QS in modern science and to improve mathematics skills of students when applied in the context of science.

The preceding two paragraphs briefly touch on the literature revealing efforts to foster an understanding of the value of QS in science. The approaches adopted in these examples are not widely recognised outside science education, and often struggle to gain acceptance amongst educators themselves as Goos and Askin (2005) and Henderson, Beach, Finklestein and Larson (2008) discuss.

Perhaps one of the greatest barriers to the tertiary sector transmitting uniform signals regarding the importance of QS in science is a lack of understanding within the sector as to the most effective way to demonstrate the links between mathematics and science. Without continued efforts in these areas, tertiary science educators are unlikely to be able to meet the ambitious goals they have set themselves through the standards reported in the Learning and Teaching Academic Standards draft consultation paper published by the ALTC (2010).

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INTERACTIVE WHITEBOARDS AS POTENTIAL CATALYSTS OF PEDAGOGIC CHANGE IN SECONDARY MATHEMATICS TEACHING

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It has been established that the use of interactive whiteboards (IWBs) does not of itself imply interactive pedagogy. Indeed it has been argued that precursors for a change from teacher-centred to interactive pedagogy include a high degree of technical IWB competence. Based on the responses of secondary mathematics teachers at one school to a brief professional learning program we suggest that awareness of the potential of IWBs to enhance student engagement and hence learning, and commitment to collaboration and improved teaching, can motivate experimentation with the technology such that technical competence and pedagogical change occur together.

Over the past decade interactive whiteboards (IWBs) have been embraced by school systems in the UK and more recently Australia and New Zealand. Initiatives such as the Schools Whiteboard Expansion project have provided funds for at least one IWB for each subject department in participating UK secondary schools (Moss, Jewitt, Levačić, Armstrong, Cardini & Castle, 2007), while in Australia the significant costs involved have not deterred their rollout in Victoria (Jones & Vincent, 2006) and other states. Lee (2010) used the term “digital take-off” to describe teachers’ rapid adoption of the technology in their classrooms, but others (e.g., John & La Velle, 2004; Serow & Callingham, in press) have reported that a minority of teachers resist IWBs, avoiding their use by citing technical and other difficulties.

In spite of the widespread enthusiasm for IWBs research that demonstrates impacts on students’ learning is scant (Jones & Vincent, 2006). Changes to teaching that have been associated with the use of IWBs include speeding the pace of lessons, providing access to a wider range of multimedia resources, and allowing for greater interaction in lessons (Moss et al., 2007). Although these changes can be positive they are not necessarily so. For example, Biggs (1987) suggested that increased speed of delivery can result in surface learning, while Moss et al. (2007) cautioned that greater access to resources can result in increased reliance on commercially prepared materials and observed that this appeared to be more likely for mathematics teachers than those in other areas. In addition, they noted that the interactive potential of IWBs required intentional planning in order to be realised, and Hodge and Anderson (2007) have suggested that an IWB can result in less interaction and a greater emphasis on whole class teaching.

Beauchamp (2004) proposed a five stage hierarchical model for the adoption of IWBs. The five stages, as described by Muir, Callingham and Beswick (2011, p. 2) are as follows:

1. Blackboard substitute: Turn on IWB, Find relevant files, Use the IWB pen, Students don't use the IWB
2. Apprentice: Use prepared files—predominantly presentation, Save new pages created during lesson, Students have some access planned by teacher, Sometimes use other programs (e.g., Powerpoint), Sometimes use material from Internet or elsewhere
3. Initiate: Have multiple windows open and available, Use "flip charts" created with IWB software, Save work systematically in "favourites" folder, Students have access to choice of IWB tools on teacher direction, Use of a wider range of programs including specialist software, Use of different Internet sites
4. Advanced: Use work from students (scanned or saved), Students have frequent access to the IWB, sometimes spontaneously, Use of media files (e.g., video, sound files) prepared by teachers, Use of hyper-links—non-linear thinking, Use of "improved" lessons with focus on student learning rather than technical capability
5. Synergistic: Teacher *and* students confident and competent with IWB, Teacher has technical and content competence so that lesson structure is fluid and responsive to students, IWB use embedded in lesson activities beyond presentation.

In the study reported here, Beauchamp's (2004) hierarchy was used both as a framework for teachers to reflect on their current and desired IWB use, and to analyse the use of IWBs by five secondary mathematics teacher participants. In light of the conflicting literature about teachers' willingness to embrace IWBs and the impacts of the technology on teaching, we were particularly interested in the extent to which IWBs might be a catalyst for pedagogic renewal in mathematics when they were the focus of shared professional learning. The specific research question addressed by the study was: How might the use of IWBs influence the mathematics teaching of a group of secondary mathematics teachers in the same school?

The study

The five teachers whose mathematics pedagogy was the focus of this paper were the secondary teachers in a group of eight teachers who participated in a small study of the potential pedagogical impacts of IWBs. The study was conducted over a period of approximately 12 weeks in the final school term (of three).

Participants

The five secondary teachers all taught at least one mathematics class at the same Grade 7–10 suburban government high school. Details of their teaching experience, qualifications, and current mathematics teaching responsibilities are shown in Table 1. Mathematics Extended is an elective subject chosen by students who enjoy the subject or want a firmer basis for subsequent study of the discipline. Maths Applied and Maths Methods Foundation are preparatory subjects for pre-tertiary subjects available in Grades 11 and 12.

The school, Queensbridge High, had very recently invested in IWB technology but funding had not extended to the provision of training for teachers in their use.

Table 1. Details of participating teachers.

Teacher	Qualifications	Teaching experience	Position in the school	Current mathematics teaching
Tammy	B.App.Sc, B.Ed	10 years	Teacher	8 Maths, 10 Maths Applied
Kylie	B. App.Sc.	12 years	Advanced Skills Teacher, Mathematics leader	10 Maths Methods Foundation, 9 Maths Extended, 7 Maths , 9 Maths
Louise	B.Ed., M.Ed.	24 years	Assistant principal	9 Maths
Steve	B.Ed. (Prim)	5 years	Teacher	7 Maths
Claire	B.Ed. (Prim)	10 years	Teacher	7 Maths, 8 Maths, 10 Maths

Instruments

Data were collected using a range of instruments including student surveys. Although relevant to the current study students survey data are not included here due to space constraints.

Lesson observation

Each teacher was observed teaching one mathematics lesson using the IWB as they normally would. The observer recorded as much detail of the lesson activity as possible focussing on the teacher’s actions (e.g., instructions, explanations, monitoring), student activity (e.g., groupings, extent of engagement and participation), and the use of the IWB (e.g., what was displayed on it, who used it). The times at which various episodes of classroom activity changed were also recorded.

Interview

Immediately after the lesson observation each teacher was asked about the degree to which the lesson was typical in terms of their IWB use, the extent to which they believed that the lesson could have been conducted without the IWB, student involvement in the lesson, and specific aspects of IWB use or related resources that had been observed. They were then asked to describe what they regarded as the main advantages of using IWBs in mathematics teaching, how they would like to use the IWB, and what supports they believed would be necessary to help them to achieve this.

Teaching journal

Each teacher was asked to document their IWB use for a period of 10 school days. To this end they were provided with 10 pages, each containing a table in which to record predefined codes which referred to the topics being taught, instructional objectives, the student grouping used (whole class, small group, or individual), role (integral or supplementary) and primary use (whiteboard, data display, IWB) of the IWB, resources used in conjunction with the IWB (internet, virtual manipulative, game, text book, other), IWB features used (e.g., cover and reveal, blinds, spotlight) and the approximate division of IWB use between teacher and students (teacher dominated, 50/50 teacher and students, student dominated), throughout the day.

Workshop notes

Three, approximately monthly workshops were held with the teachers, and notes were made during these by one of the researchers and a research assistant. In the final workshop teacher presentations and other evidence of their work with IWBs was also collected.

Procedure

The lesson observations and teacher surveys were conducted prior to or immediately after the first of the workshops. Teachers were asked to complete the journal for a 10-day period during the four weeks between the first and second workshops. After each of the first and second workshops the teachers were encouraged to try something of what they had learned and to report back on this in the next session.

The first workshop was a half-day event that introduced the teachers to the project and to Beauchamp's hierarchy of IWB use. Each of the researchers involved shared some IWB resources that would be potentially useful for the teachers. These included the Gapminder website, virtual manipulatives, GeoGebra, and Learning Feder@tion objects. The teachers also had an opportunity to share their favourite IWB resources and features.

The morning of the second session was devoted to teachers discussing their recent IWB use followed by a presentation from an external IWB professional learning provider that focussed on the features of IWBs, commercially available IWB software packages, and peripheral devices. In the afternoon each of the researchers presented a mini-lesson aimed at provoking discussion of the variety of ways in which an IWB might be integrated with traditional tools, used to enhance activities that could be done without the IWB, or could facilitate learning experiences that would be very difficult to provide in another way.

The final, half day, workshop was an opportunity for the teachers to showcase their IWB use and to report on their experiences of experimenting with new approaches to IWB use in their contexts over the course of the project.

Results and discussion

In the following sections results are presented and discussed chronologically. Due to the short time frames involved and the complexities of school life not all data were collected for all teachers but sufficient were gathered to chart the progress of the Queensbridge High teachers.

Lesson observations and interviews

Four teachers (all but Louise) were observed and interviewed although one teacher, Tammy, was in a room that had no IWB. Kylie, Claire, and Steve were all teaching Grade 7 classes and all used the IWB for an initial activity that involved a puzzle or game to be completed within a limited time. Claire and Steve made no use of the IWB during the main part of the lesson but in each case a group of two or three students who completed their work early were allowed to use the IWB for a further task or game. In Claire's lesson the task related to the lesson focus on expressing patterns algebraically but the number puzzle chosen by Steve was not connected with the lesson. Kylie did not use the IWB at the end of her lesson but did use it at various times throughout the lesson, primarily to display and record information. She also made use of an online dictionary to show the meaning of ‘round’ in the context of decimal numbers. Consistent with Moss et al.’s (2007) observation, Kylie’s lesson was noticeably fast-paced and although this may have been her practice regardless of whether the IWB was used or not, the IWB did appear to facilitate the pace to some extent.

All of the teachers indicated that the observed lesson involved fairly typical IWB use and all expressed a desire to use the IWB more, and more effectively. For Steve this meant going beyond just using it to start lessons and for early finishers, whereas Kylie was keen to explore how students could be more involved in using the IWB and Claire was interested in finding out what a “really good lesson” with the IWB looked like. They agreed that the activities for which they used the IWB could have been done without it but that it was easier with the technology. For Tammy, access to an IWB was the major issue cited as preventing her from using it regularly and hence developing her skills. Claire taught all of her maths lessons in rooms with an IWB but described being hindered by a lack of technical expertise. For example, she said that it was necessary to check and often to recalibrate the board in the break before lessons, and that she had learned what she had by trial and error in the absence of any professional learning. Steve had also experienced difficulty in his lesson as a result of the IWB needing to be calibrated. In keeping with her desire to have students make more use of the board, Kylie believed that a second IWB in each classroom would be useful.

Workshop 1

Having been introduced to Beauchamp’s hierarchy all of the teachers considered themselves to be at the Apprentice stage except for Kylie who viewed herself as being at the Initiate level. These judgements were broadly consistent with the lesson observations although it was not possible, on the basis of a single lesson, to determine the extent to which students had choices with respect to the IWB tools that they used or the range of software and internet sites that were used. All of the teachers cited new ideas and web-based resources from the workshop that they saw as potentially useful.

Teaching journal

Louise, Claire, and Steve completed teaching journals. Louise’s journal included just four lessons over a 3-week period including one lesson in which the IWB was “not working!” In the remaining three she reported using the IWB in whole class contexts except for part of the third lesson, in which a small group used it. Explicit instruction characterised her IWB use in all three lessons and was accompanied by revision in the second lesson and the introduction of new concepts in the third. In the first lesson the IWB was used to display a text book exercise and was operated solely by Louise. In the second, Louise indicated that some IWB feature was used, in addition to using it to display information, and in the third lesson she described its use as entirely with the IWB. Internet resources were used in lessons 2 and 3 along with a text book exercise display in the third. In both the second and third lessons Louise reported some use of the IWB by students.

Steve reported on six mathematics lessons over a 10-day period, with the IWB used in the first four of these, and always in whole class contexts. It was used for explicit instruction in the first and second lessons and for revision in the remaining two. Steve reported using the IWB as an IWB only in the third lesson but as a data projector on the other occasions. Resources used included internet sites in all four lessons, games in the first three, and as a text book in the first two. He reported approximately equal use by students and teacher in the first two lessons, student dominated use in the third and teacher dominated use in the fourth.

Claire reported on nine mathematics lessons in a 2-week period. Of these the IWB was used in seven. In each of these lessons small groups used the IWB, accompanied by whole class use in the first three lessons and individual use in the final three. Claire reported using the IWB for explicit instruction (3 lessons), revision (3 lessons) and introducing a new concept (2 lessons). All of the lessons made use of the IWB as an IWB with just one lesson (the third) in which it was also used as a whiteboard. Claire reported using internet sites in each of the first four lessons; virtual manipulatives in the first, third, and final three lessons; and games in the first two lessons and the final three. For the first three lessons Claire reported a balance of student and teacher use of the IWB, and student dominated use for the final four lessons.

The differing patterns of IWB use that were reported in the teachers' journals is likely to be related to their differing roles in the school—Louise's AP responsibilities meant that her teaching load was relatively light and frequently disrupted—and differing levels of commitment to the project and/or to changing their use of the IWB in their teaching. Nevertheless, there was some evidence of a willingness to experiment on the part of all three teachers. Overall the reported IWB use was consistent with the tendency reported in the literature (Hodge & Anderson, 2007; Muir, Callingham, & Beswick, 2011).

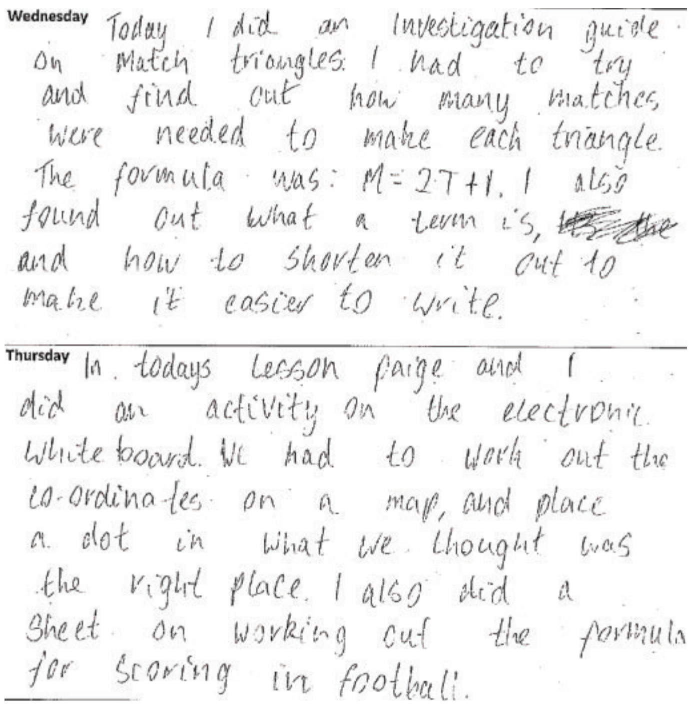
Claire's teaching journal was unique in both the number of lessons documented and the progression in her IWB use that was evident. Specifically there was a trend toward to less whole class use, greater use of manipulatives, and greater student access to and use of the IWB. Although there is insufficient evidence to conclude that Claire progressed from the Apprentice level (Beauchamp, 2004), the changes evidenced were in the direction of more sophisticated use.

Workshops 2 and 3

As described already the second workshop provided the major professional learning component of the program focussing on features of the IWB and ways in which its use could be incorporated into existing mathematics teaching approaches. The third workshop was primarily a forum in which the teachers could share their learning.

In that workshop the Queensbridge High School teachers chose to make a group presentation lead by Claire and assisted by Tammy and Steve. They described how they had collaboratively planned and implemented a Grade 8 algebra unit that incorporated the use of hands-on tasks, interactive tasks using the IWB, and traditional tasks. The unit was divided into weeks each with a list of objectives and pathways for working through the activities. No whole class teaching was used, but the weekly overviews were supplied to students at the start of each week and they were expected to be self-directed. Access to the IWB was rostered to ensure that all students had opportunities to complete the interactive tasks. An important feature of each lesson was a 10 minute reflection time at the end during which students wrote at least 50 words about what they had achieved that lesson. The teachers shared three reflections from each of two students, and two of these from one of the students are shown in Figure 1. The mention of the "electronic whiteboard" in the Thursday reflection was the only reference to IWBs in the examples they presented.

The teachers explained that IWB resources were easy to find but that the availability of technical support on just one day per week meant that technical issues with the IWBs remained a frustration and necessitated always having a back-up plan. In addition, they had found that access to several PCs as well as an IWB was very useful when this was possible. Overall they believed that the approach taken in the unit had resulted in improved student engagement and a more rewarding teaching experience.



Wednesday Today I did an Investigation guide on Match triangles. I had to try and find out how many matches were needed to make each triangle. The formula was: M=2T+1. I also found out what a term is, ~~its the~~ and how to shorten it out to make it easier to write.

Thursday In todays lesson Paige and I did an activity on the electronic whiteboard. We had to work out the co-ordinates on a map, and place a dot in what we thought was the right place. I also did a sheet on working out the formula for scoring in football.

Figure 1. Examples of student reflections.

Although the teachers appeared to rely on pre-prepared materials there was no evidence of negative effects of this as suggested by Moss et al. (2007). Rather, they integrated the online resources with others in a purposeful way as part of their planning. In contrast to Hodge and Anderson's (2007) warning, the use of the IWB, in the context of their overall revised approach, did not have the effect of increasing teachers' reliance on whole class teaching but had, on the contrary, reduced it. This was also a change from the lesson observations made at the start of the project.

Conclusion

The short duration of the study meant that large changes in teachers IWB use or mathematics pedagogy were unlikely to be observed. Nevertheless, Beauchamp's (2004) hierarchy was effective in facilitating teachers' reflection on their IWB use and in setting goals for its development. Apart from Kylie, all placed themselves at the Apprentice level at the start of the project and all, including Kylie, were able to identify specific ways in which they wanted to develop their IWB use.

The teachers' presentation at the final workshop showed evidence of development and suggests that one or more of the individuals involved in preparing the unit had moved to the Initiate stage (Beauchamp, 2004). Compared with the lesson observations at the start of the project, where the IWB was used primarily as something of a novelty to engage students at the start of the lesson and to reward early finishers, its use in the Grade 8 algebra unit was integral to the achievement of the unit's objectives. Importantly, these changes occurred without any change to the availability of the technology or technical assistance with its use. Improvements in these areas would undoubtedly be helpful and appreciated by the teachers but they did not present an insurmountable obstacle to teachers progressing their objectives in relation to IWB use.

Arguably the most positive aspects of their work were not directly related to IWB use. Rather, it appeared that planning collaboratively was a relatively new way of working, perhaps prompted by the need to "have something to share". In addition, the teachers had not set out to use the IWB at every possible opportunity but had incorporated it purposefully into their overall plan as a part of varied menu of tasks for students to undertake. The incorporation of reflection time for students was a successful innovation to assist students to be aware of and to take responsibility for their own learning and also constituted a lesson ending that focussed students on what they had achieved. The student reflections that the teachers selected to share provided further evidence that they were not preoccupied with using the IWB as an end itself but in improving their teaching and using the IWB as one of range of tools to this end.

In terms of the research question that was the focus of this study, the data point to the potential of a challenge to incorporate IWB use to be a catalyst for more fundamental pedagogical change. Of course, the study was small and the intervention brief but it raises questions that warrant further exploration. For example, what role is played by teachers' existing pedagogical repertoires in their uptake of new technologies and their ability to rethink their teaching approaches? What role might teachers' pedagogical content knowledge play? How and to what extent might other specific innovations be used as catalysts for pedagogical change?

The results also support the inclusion in professional learning around new technologies of a pedagogical focus from the outset. The Queensbridge High School teachers were novice IWB users but were able to change their pedagogy at the same time as developing their technical skills. Indeed it could be that the pedagogical possibilities presented in relation to IWB use may have motivated them to engage with the technology. This is yet another avenue for future research.

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PREPARING FOR SCHOOL TRANSITION: LISTENING TO THE STUDENT, TEACHER, AND PARENT VOICE

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Moving on from primary school provides many different challenges. This paper explores multiple perspectives about preparation for school transition in mathematics. As a qualitative case study it draws on student, teacher, and parent, voices through the use of questionnaires and interviews. Sixty seven students and six teachers from three different schools participated in the study. There were commonalities and differences in beliefs about mathematics learning and teaching that contribute towards successful transition. The results illustrated that facilitating successful transitions requires that attention be given to the perceptions and values of students, teachers, and parents.

Introduction

For all students, transition across educational sectors is an important event in their schooling lives, whether as an internal transfer (e.g., from junior school to senior school) or as an external transfer (e.g., from primary school to middle school). A body of literature (e.g., Anderson, Jacobs, Schramm, & Splittgerber, 2000; Demetriou, Goalen, & Rudduck, 2000; Galton, & Hargreaves, 2002) signals the many challenges which students may encounter at this important time. Broadly, these challenges include difficulties with continuities in learning mathematics, teaching styles, teacher expectations, friendships, and school systems. Recent New Zealand studies by the authors (Bicknell, 2009; Bicknell, Burgess, & Hunter, 2010; Bicknell & Hunter, 2009) explored different aspects of the transition process. The focus of these studies was on preparedness, support, and transitional success and failure across differing sectors for students in mathematics.

Preparedness includes academic preparedness, independence, and industriousness. Support may be provided by teachers, parents, and/or peers whilst transitional success or failure can be judged by grades and academic orientation. These were the three key elements for analysis provided by Anderson et al's (2000) conceptual framework. This paper provides an opportunity for us to address in more detail the issue of preparedness and to hear multiple voices of students, teachers, and parents. We want to understand how these three stakeholders view preparedness in mathematics for successful transition from primary school. Our research question asked: How do students, teachers, and parents view preparedness for a successful transition in mathematics?

Review of the literature

Successful transition is not only important for students' social and learning trajectories (Noyes, 2006) but also to maintain their motivation to continue to engage with mathematics (Athanasiou, & Philippou, 2006). One reason Akos, Shoffner, and Ellis (2007) suggest students lose interest in mathematics is due to the increased focus placed on performance-oriented teaching and learning as students move up the schooling system. This performance orientation emphasises student demonstration of mathematical skills and increased competition at higher levels of the education sector and contrasts with the more task-orientated focus of primary school classrooms. Within a task-orientated focus, emphasis is placed on students working to improve their competencies (Zanobini & Usai, 2002). In recent times, certainly in New Zealand, primary mathematics classrooms have been strongly orientated towards task-focused teaching and learning through the New Zealand Numeracy Development Projects (NDP) (Ministry of Education (MoE), 2008). However, as students move into higher sectors in the New Zealand school system the focus shifts towards more assessment-driven pedagogies.

Noyes (2004) describes this shifting emphasis in pedagogy as schools being responsive to political influences within "mathematics learning landscapes" (p. 28). While policy has a broad influence on what happens in mathematics classrooms, other more local factors impact on the transition process in individual schools and classrooms. The barriers and enablers to successful transitions vary depending on differing contexts and situations. They involve more than individual students. Teachers (Pietarinen, 2000), parents (Mizelle, 2005; Cox & Kennedy, 2008), and peers, (Wentzel & Caldwell, 1997) all play a key role in the transition process (Jindal-Snape & Foggie, 2008).

> It is widely accepted that fluency should be underpinned by the continuity and progression designed into the curriculum, by the efficient and purposeful transfer of information at the interface and by comprehensive liaison between the various parties involved: teachers, pupils and parents. (Nicholls & Gardner, 1999, pp. 1–2)

Of particular interest for this study is how continuity in the mathematics learning landscapes is enacted across mathematics classrooms at each sector level; that is how the cultures of classrooms at primary level (as the students and their teachers perceive it) links to how the students, their teachers, and their parents perceive how this supports their preparedness to transition to middle school. Such continuity of learning has been identified by researchers as an area of action that will improve learning across transitions.

In this paper we take the culture of the classroom enacted by teachers as a key factor which shapes students' relationships in and with mathematics. We explore how students view the culture of the mathematics classroom in preparedness for transition and acknowledge the influence of recent reforms in mathematics education (founded in constructivist learning theories in which students construct, explain, and justify their reasoning using multiple strategies). This led us to listen to student voices to see how they view their current classroom culture and how they see this as preparation for transitioning to middle school mathematics learning.

The theoretical perspective taken in this study adopts an ecological view suggested by Bronfenbrenner (1979). In this cultural frame the different social environments are recognised as directly impacting on students as they prepare to make an "ecological transition" (p. 26) across school sectors and make adaptations on multiple levels to the perceived changes in roles and settings they will encounter.

The study

This paper reports on data extracted from a larger study that investigated the different transitions of groups of students within centres and school sectors across a three-year period. In a previous paper (see Bicknell & Hunter, 2009) we reported on the systemic transition in mathematics of students in the second phase of the study (primary school year six to intermediate school year seven). In this paper we return to the findings from the second phase to investigate how the mathematics learning environments (classroom cultures) are directly responsive to the New Zealand Numeracy Project (MoE, 2008). Further, we are interested to see how how year six students, their teachers, and their parents view preparedness for mathematics in the next sector.

The sample for this study included 67 students (65 returned complete questionnaires) and their teachers (n=6) from six different schools. The six primary schools were from a decile[1] range of three to seven from two different geographical regions in New Zealand. The students completed a questionnaire that included both open-ended and likert scale questions. This was supplemented by group interviews. The teachers also completed an open-ended questionnaire and participated in semi-structured interviews. The parents (n=34) also completed a questionnaire. To determine the key themes and the commonalities and differences in perceptions about the transition process, responses were systematically coded initially based on Anderson and colleague's (2000) conceptual framework. This was followed by a second level of coding. Tables were then created for some of the pattern codes to give a quantitative view of the data from the multiple sources (Cohen, Manion, & Morrison, 2007).

Results

The classroom contexts

All teachers reported that they had recently participated in numeracy professional development provided for the New Zealand Numeracy Development Projects (MoE, 2008) and placed an emphasis on numeracy in their mathematics programme (between 60-100%). Key features of all of the teachers' lessons (as reported and documented) included the use of streamed groups (based on strategy levels). Their lessons featured an introductory basic facts whole class activity followed by group rotations where students developed solution strategies for problems in small groups and talked about their strategies with the teacher in larger sharing sessions. Follow-up work included activities to reinforce previous learning through the use of numeracy resource materials. These lessons reflect the model promoted by the New Zealand Numeracy Development Projects.

[1] Each school in New Zealand is assigned a decile ranking between 1 (low) and 10 (high) based on the latest census information about the education and income levels of the adults living in the households of students who attend that school.

The students concurred with their teachers and they almost uniformly described working in groups and working with the teacher as a key aspect of their mathematics lessons. The majority of the students noted that the learning of basic facts was consistently a focus of mathematical activity. Other activities the students listed as common practice in their mathematics lessons included working from textbooks and worksheets and explaining their strategy solutions. The least common occurring activities were writing their own word problems, participating in competitions, and convincing others about their mathematical thinking.

In the next section we present firstly the students' perceptions of their preparation to transition to middle school in mathematics. This is followed by the teachers' and parents' perceptions. Then we provide a synthesis of the three voices in which we highlight commonalities and differences among the stakeholders.

Student perceptions of their preparation for transition in mathematics

The students presented their ideas initially in a written questionnaire. Table 1 below provides a summary of student responses to the question: How important do you think each of the following are in preparing you to do well in mathematics?

Table 1. Student Responses to Question 4.

	Extremely important	Very important	Somewhat important	Not important
Working in a group with other students	8	37	18	2
Working alone	15	19	27	4
Working with the teacher	31	20	11	3
Sharing your ideas in a large group	23	26	13	3
Working from a textbook	13	20	27	5
Working from a worksheet	7	30	22	6
Learning using games and activities	20	19	21	5
Knowing your basic facts	47	12	4	6
Being able to use a calculator	21	27	10	7
Explaining your strategy solutions	30	25	5	5
Convincing others about your mathematical thinking	17	31	15	2
Writing your own word problems	11	28	24	2
Learning from your mistakes in mathematics	42	18	1	4
Learning from the mistakes of others	20	27	12	6
Being able to ask for help in mathematics	40	21	3	1
Taking part in competitions	16	24	19	6

Table 1 shows that the factors that related to students' attitudes towards, and ways of participating in, mathematics drew the most positive responses. The strongest placed factor ranked by the students as either extremely important or very important was 'being able to ask for help' (94%). This was followed closely by 'learning from your mistakes in mathematics' (92%) and 'knowing your basic facts' (91%). The selection of these factors suggests that the students had a sense of responsibility and autonomy towards themselves as mathematical learners. They also illustrate a task-focused orientation in which the students are doing what Zanobini and Usai (2002) describe as improving aspects of their mathematical competencies.

The second group of responses included: explaining your strategy solutions (85%); working with the teacher (78%); sharing your ideas in a large group (75%); convincing others about your mathematical thinking (74%); learning from the mistakes of others (72%); and working in a group with other students (69%). The selection of these factors by the students reflects a focus on them taking personal responsibility for their mathematical learning and at the same time illustrates the importance the students placed on ways of communicating about, and participating in, mathematics. However, only half of the students believed that working from a worksheet (51%), working alone (52%), and working from a text book (51%) were important in preparing them for mathematics in their next school setting. These three factors more closely represent mathematics learning within performance-orientated settings.

The findings in Table 1 were triangulated using additional data from an open-ended question in the questionnaire and focus group interviews. The open-ended question asked: What do you think are the most important things to do to be prepared for mathematics in year 7? The students' responses provided further evidence that they believed that they needed to improve their mathematical competencies. They were also aware of an attribute Anderson and his colleagues (2000) describe as essential for preparedness, which is industriousness. This is exemplified by a student who noted:

> Work very hard when it's getting closer to the end of the year; learn all the basic things you should know in year 6 so you're prepared for the next year. (Student D5)

A second student commented:

> ... make sure that I know my basic facts well, know how to do word problems and work well with new people. (Student C8)

This student perceived the importance of group work as equipping him to continue to work with others including a new teacher when he transitioned to his new school. Other students also noted that they viewed the ability to work in groups, work with other students, show their ways of thinking (publicly), know how to use different methods and strategies, and learn from mistakes as important aspects of preparedness for transition. For example one student stated:

> I think I have to ask the teacher for help a lot. And always give ideas to the group that I'm working with and speak up. (Student E5)

Other students illustrated that they recognised that their current ways of working would change in the next setting. These students identified that part of their preparation required that they worked independently, harder, and were prepared for challenge. For example one student recorded a need to:

> ... be prepared for a challenge, new types of working and working with your new teachers and classmates. (Student F1)

The students were asked how they thought their teachers were preparing them mathematically for transition to the next schooling sector. Across the sample they provided similar responses. Most often they stated that they were encouraged by their teachers to work alone. Many stated that their teachers now gave them worksheets and had them working from text books as a preparatory step. Accepting greater challenge, taking personal responsibility, and increased homework was also mentioned. As one student stated:

> Our teacher challenges us and gives us different work nearly every day and we either get a maths book like a text book or just a sheet and we work off those and each time there are different levels and challenging levels for your group and like for homework it will be hard for us and so we do quite hard questions now. (Student M1)

A large number of students also described how their teachers were focusing on teaching them more numeracy solution strategies (NDP mental strategies) in preparation for what the teachers perceived would be required in the new school. However, they themselves did not see the learning of multiple strategies as an important part of their preparation for transition. Similarly, the other factors which the students identified in this final section (textbooks, worksheets) are those which only 50% of the students thought were important to prepare them for the move to the next stage in their schooling. This however was the students' perceptions of what were important aspects of their preparation and as the next section will show differed somewhat from that of their teachers.

Teacher perceptions of preparing the students for transition in mathematics

The majority of the teachers described the importance of the students holding strong knowledge of their basic facts. Like their students, the teachers took a task-orientated focus (Zanobini & Usai, 2002) and described an emphasis placed on improving aspects of their students' competencies across mathematical skills and strands. For example, one teacher described a broad emphasis across the mathematics strands:

> Ensure that they have basic facts 'down pat'. Lots of exposure to a variety of strategies and problem solving skills. Experience in all maths strands. Above [all] give them the confidence to take risks with their thinking. (Teacher F)

Other teachers described a central focus on the teaching of numeracy strategies. They stated that they wanted to ensure that the students had a repertoire of strategies; a focus which fitted within the current politically focused NDP and connects to what Noyes (2004) suggests as the influence of wider policy. But, at the same time they outlined how they wanted the students to have had experience with the written standard algorithms for the four operations; a focus complicit with previous teaching methods prior to the introduction of the NDP. Other themes the teachers described included ensuring student knowledge of place value and developing a range of problem solving strategies. Some teachers also demonstrated that they were aware of a need to make mathematics relevant to their students' lives, to developing student confidence to take risks with their thinking, and have the skills and confidence to use textbooks. One teacher specifically focussed on her goal to increase her students' awareness of their own levels of achievement and weaknesses so they could take shared responsibility in identifying the next steps in their learning. She stated:

> My transition approach is the same with all areas [curriculum]. I ensure the child is aware of their level, what they can do, what their next steps are. Some children take this on board many don't. (Teacher E)

Like their students, the general focus of the teachers' preparatory steps was directed towards ensuring that gaps were filled for individual students within a task-orientated focus (Zanobini & Usai, 2002). However, they also addressed other aspects of preparedness that Anderson and his colleagues (2000) maintain are essential for successful transition. These focused on the need to develop independence, industriousness, and coping mechanisms.

Parent perceptions of preparing the students for transition in mathematics

Parents balanced their perceptions of what they wanted in preparedness for transition between wanting their children to be competent across mathematical dimensions and being able to perform competently at the next level. This included the mastery of basic facts which they saw as not only the responsibility of the school or teachers but also acknowledged their contribution towards their children's rapid recall of basic facts. They also placed importance on coverage of the curriculum; they wanted there to be no gaps in their child's mathematics education. They also wanted the mathematics lessons to be targeted at the student's level with clear progressions.

The parents, like the teachers, recognised a need to develop a range of coping skills as well as a sense of independence and industriousness. These included helping the children to work in a variety of ways: to work from worksheets and textbooks; to work independently; and to work under pressure. One parent stated:

> I feel he will either 'sink or swim' depending on how he starts the year (Year 7). If he finds it too difficult at beginning he will lose interest in doing well. I am hopeful he will do well and am preparing him and myself to get stuck into the new year's studies, as I think he might need help initially to settle into a work routine. (Parent C1)

Other aspects raised by the parents included the need for their children to have a positive attitude, self confidence, and a willingness to ask for help in mathematics. They also identified the importance of listening to the teacher, asking questions, risk-taking, and good work habits including accepting and working towards an increasing workload including homework.

The majority of the parents believed that the responsibility for the preparation rested predominantly with the school and teacher, although acknowledged that their support and encouragement would help with the transition. When mathematics was valued at home and links made to real life contexts, they believed, their children's preparedness for the transition was strengthened. However, not all parents felt that preparedness had been successful for their children; some had 'no idea' and one parent acknowledged concerns. Four parents stated that their children had not been prepared to succeed in mathematics in the following year but did not articulate reasons why.

Given that we received questionnaires from nearly 50% of the parents, we believe this could be viewed as reasonably strong parental interest in transition. It supports previous studies such as Mizelle's (2005) that parents are interested in the transition process although the level of commitment and sense of responsibility for preparedness, in this sample, was variable.

Conclusion and implications

Clearly, there were commonalities and differences between the groups of stakeholders. However, a common theme of preparedness was the importance of learning and mastery of basic facts. This theme extended beyond basic facts to include coverage of the mathematics curriculum and for all stakeholders preparedness also meant ensuring competency across all mathematics strands. Voices from the three stakeholders all described the importance of improving competency in a variety of different ways. As other researchers (Akos, et al., 2007; Zanobini & Usai, 2002) previously noted, a task-orientated focus is consistent with students in lower levels of the education sector. Another theme common to the group, focused on the importance of preparedness to work from worksheets and textbooks within a more individualised setting. This focus could be linked to what Zanobini and Usai (2002) describe as the performance orientation of senior mathematics classrooms. We can infer that parent understandings of mathematics learning, most likely connects to their own most recent experiences in performance-orientated mathematics classrooms. Therefore, the parents' and the students' emphases on the importance of being able to work alone and to ask for help when needed can be understood, given the powerful influence Noyes (2004) suggests parents have on the attitudes of their children. At the same time, these factors and the value placed on homework by parents and students reflects notions of improving competencies (Zanobini & Usai, 2002) as well as ensuring what Anderson et al., (2000) describe as academic preparedness. The other factors Anderson and his colleagues drew our attention to in their research (independence, industriousness, and coping mechanisms) were also evident in what the teachers and parents considered important for preparedness for transition. The teachers and parents also placed an emphasis on student awareness of their own mathematical levels and learning needs, risk taking skills, and ability to cope with challenge including an increased workload. However, the parents were a lone voice in considering that a positive attitude to mathematics was essential for successful transition.

The influence of wider policy on the local classroom situation, described by Noyes (2004), is evident in the voices of the students and teachers. Only the teachers outlined a need to teach algorithms as important preparatory steps. We can surmise that this relates to previous policy and classroom practices which contrast with current policy introduced in the New Zealand Numeracy Development Project (MoE, 2008). Likewise, an emphasis placed by students and teachers on the development of a range of numeracy strategies links to current policy. This is a new construct and experience for many parents. Similarly, we can conclude that the prime importance the students placed on aspects of classroom practices and context (for example, working in groups, sharing reasoning, learning from mistakes, and convincing others) was shaped by broader and more recent policy. Moreover, we also need to recognise the discontinuities this poses for students. The culture of mathematics classrooms has changed in recent decades and if we are to ensure fluency in transition then we need to carefully consider how continuity of learning in mathematics is maintained. As Anderson and his colleagues (2000) caution, without paying attention to supporting a successful transition the transition becomes "the beginning of the end rather than a new beginning" (p. 336).

These findings suggest that there should be a shared understanding and recognition of the part that all stakeholders have in the process. We focused on student, teacher, and parent voices yet there are systemic factors that also need to be considered to support successful transitions. The mathematics curriculum needs to be presented and understood so the progressions across the sector are seen as seamless. Differing pedagogical practices need to be respected and understood so that students can be prepared for any change in the learning landscape. Conversations and classroom observations across sectors could strengthen understanding and respect for the changed ‘culture spaces’ to support smooth and positive transitions for all students.

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STUDENT EXPERIENCES OF MAKING AND USING CHEAT SHEETS IN MATHEMATICAL EXAMS

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In many mathematics courses at school and university—including in all South Australian Year 12 mathematics subjects—students are allowed to make a “cheat sheet” to take into their exams. However, despite their widespread use, there is little research on the effect of making and using cheat sheets—whether on the students’ performance, learning or learning experience. As a preliminary stage in researching this issue, students in several first-year statistics courses at the University of Adelaide were surveyed on their experiences of making and using cheat sheets. The results and implications of this survey are discussed in this paper.

Introduction

This paper describes the preliminary stages in an investigation into the effects of using cheat sheets in mathematical exams. A “cheat sheet” in this context is a page of notes that students are allowed to make and take with them into their exam (also known as “crib sheets” or “crib notes”.) It is important to highlight the fact that they are not actually used to *cheat* because they are explicitly allowed by the instructions given to students.

Cheat sheets are common practice in many exams today. In South Australia, cheat sheets are allowed in the exams for every Year 12 SACE mathematics subject; and at the University of Adelaide at first-year level, six mathematics and statistics courses and several science and humanities courses all allow their students to make cheat sheets. These examples alone amount to literally thousands of individual cheat sheets being made every year.

Considering how widespread their use is, there is comparatively little research into the effects that making and using cheat sheets have, whether on the students’ performance in the exam, on the quality of their learning, or on their experience of learning and assessment. Therefore, educators are making decisions about whether to allow cheat sheets mostly without the benefit of existing discussion on the topic; and they give advice to students on how to make and use cheat sheets without really knowing whether the advice is actually helpful. This research is the first stage in beginning to fill these gaps.

Background and aims of the research

This research began when the lecturer for a first-year statistics course commented to us about the students' use of cheat sheets. The lecturer felt that students were not making the best of the opportunity to have a cheat sheet. So, we began to prepare a presentation for the students on how to make and use a cheat sheet effectively.

While preparing this seminar, we quizzed individual students on their experiences of cheat sheets, knowing that most of them had used cheat sheets in their Year 12 exams. The response was always, "I made a cheat sheet, but I didn't use it much in the exam. Still, I think making the cheat sheet helped me to revise, and reduced my exam stress."

As further preparation, we looked for existing resources giving advice to students on cheat sheets, as well as the results of previous research. Both were surprisingly difficult to find. The advice available online for cheat sheets was not directed at helping students learn, and more often was advice on how to actually cheat. Moreover, the previous research did not seem to come to a conclusion about the usefulness of cheat sheets, other than perhaps as a way of reducing exam stress. It also did not consider the issue of what advice might be given to students to maximise the possible benefits of their cheat sheets.

So we finally decided to begin researching cheat sheets ourselves. Based on our experiences with students and the gaps in the research literature, we decided to focus on the following questions for this preliminary research:

How common is the experience expressed to us by students? That is:

- How do students use their own cheat sheets?
- How useful do students find their own cheat sheets?
- Does making the cheat sheet help students revise and how?
- Does the cheat sheet help reduce exam stress and how?

The second question of exactly what advice should be given to students to help them use cheat sheets most effectively is a topic for future research.

Literature review

In the research literature there are several arguments both for and against the use of cheat sheets in exams. The first and most common argument in favour of cheat sheets is that they reduce exam stress. Davis (1993) and Erbe (2007) both advocate the cheat sheet for reducing exam stress, though do not put forward any new research of their own in support. Other authors have confirmed through interviews and questionnaires that having a cheat sheet reduces exam stress for the majority of their students (Trigwell, 1987; Drake, Freed & Hunter, 1998; Theophilides & Koustelini, 2000; Dickson & Miller, 2005). However, Dickson and Bauer (2008) do express the concern that students may study less comprehensively *because* they are less worried about their exam.

Another argument in favour of cheat sheets is that students, with their memory aided by the cheat sheet, will have more time and energy to focus on higher-order thinking skills such as understanding and interpretation (Erbe, 2007). Theophilides and Koustelini (2000) provided support for this argument when the students they surveyed reported study behaviours more consistent with a deep approach to learning when studying for an open-book exam. On the other hand, in a designed classroom experiment, Dickson and Miller (2005) found that cheat sheets had no significant effect on performance in either lower-order or higher-order thinking items. If cheat sheets helped with memory and so allowed for more energy to focus on higher-order skills, one would expect them to affect performance on at least one type of question.

This leads to one of the arguments against cheat sheets, which is that they hinder the students by creating a dependency relationship, rather than helping them learn the course material *before* the exam. Dorsel and Cundiff (1979) and Dickson and Bauer (2008) have supported this argument by showing that students who prepare cheat sheets but are then not allowed to use them do not perform as well in the exam. Further evidence of this dependency is reported by Vessey and Woodbury (1992) who note that students tend to copy things straight from the cheat sheet even if it does not match the question. The hindrance may be worse for students already at risk: In an observational study on open-book exams, Boniface (1985) discovered that those students who continually need to refer to their notes during the exam are the ones who do not perform well.

The final argument presented is simply that performance improves when students use cheat sheets in exams, as reported by Francis (1982), by Skidmore and Aagaard (2004), by Stangl, Banks, House and Reiter (2006) and by Dickson and Bauer (2008). However other authors report no significant improvements (Dorsel & Cundiff, 1979; Trigwell, 1987; Dickson & Miller, 2005). Moreover, it is not at all clear whether it is the cheat sheet, the exam itself, or some other factor that is causing the change. For example, in the study by Skidmore and Aagaard (2004), the exams were on different aspects of the course involved, and it may be that one was more focussed on memory than the other.

Taking the existing literature together, a strong conclusion cannot be made about the usefulness of cheat sheets for performance or for learning. There is a clear need for more research to separate the different variables involved. We also note that the existing research concerns students studying psychology, teaching, English literature, nursing, and research methods. There does not seem to be any research explicitly dealing with cheat sheets for mathematics exams, which may be quite different. Finally, none of the articles reviewed mentioned ways of counteracting the possible negative effects of cheat sheets by giving students appropriate advice.

Research methodology

We felt that it was essential to collect preliminary data in order to inform future research in this area. To this end, we sought to survey students from those first year mathematics courses at the University of Adelaide that allow cheat sheets. Three courses in Semester 2 of 2010 were identified: one financial mathematics course, and two statistics courses offered by different schools within the university. All three courses are compulsory for students studying particular degrees. These courses will be called here FM, StatsA, and StatsB.

All three courses allowed students to bring a single A4 sheet with information on both sides, but each course had its own rules for the format. In StatsA, the cheat sheet had to be handwritten; in FM, students were allowed typed information as long as it was in 11pt font or larger; in StatsB, there were no restrictions on the format of the information.

It is important to mention that we agreed to give a seminar on how to prepare and use cheat sheets to the students in the course StatsA before beginning this research project, and feedback on the usefulness of the advice given was incorporated into the survey.

The questions in the survey contained both multiple choice questions and text response questions and asked students to comment on various experiences of making and using cheat sheets for their exam. Specific details of the actual questions are given in the results section below.

The online survey tool Survey Monkey was used to administer the anonymous survey. Students in the three courses were emailed a link to the survey on the day after each of their exams, with a reminder email was sent a week later. In total 1480 students were sent the link to the survey and 284 responded (a response rate of 19%).

The aim of this preliminary research is exploratory, and so the survey responses were analysed mainly using descriptive statistics. The themes from the text responses were noted, but no attempt was made at this stage to assess statistically the relationship between these themes and the other responses.

Results and discussion

Overall cheat sheet usage

The survey began with questions asking what course the student was enrolled in, and whether they made and then used a cheat sheet. A total of 284 students began the survey, and among these only two said they did not make a cheat sheet. Only nine of those who said they made a cheat sheet said they did not use their cheat sheet. Since the survey was voluntary, we cannot glean from this how many students actually do use cheat sheets.

Twenty-one students only answered these general questions and did not respond to any more of the survey, so these students have not been included in the totals for the remaining analysis.

The usefulness of cheat sheets

The following graph displays the results to the question “How useful did you find your cheat sheet during the exam?”

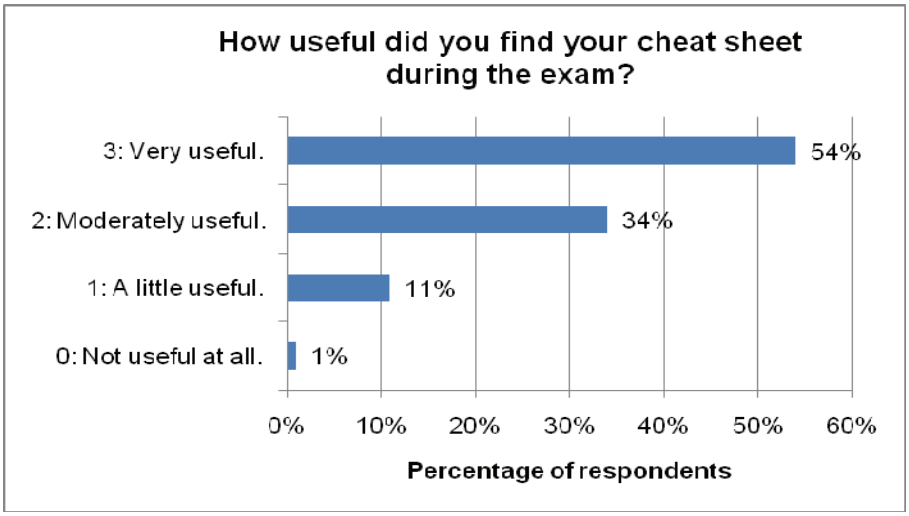


Figure 1. Responses concerning usefulness of cheat sheets.

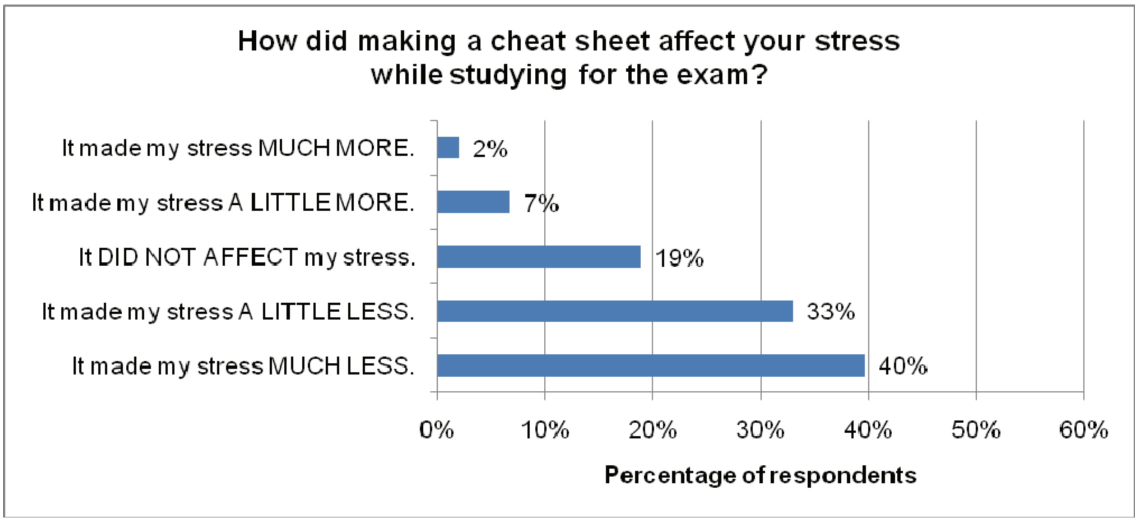
The majority of students found their cheat sheet useful in the exam, which is the opposite of what students have said to us in the past. A possible explanation is that the majority of students surveyed were studying introductory statistics courses, which tend to have a greater need to remember specific formulas and procedures than other mathematics courses.

We also asked students to explain more about why the cheat sheet was useful in order to gain insight into what might cause the increase or reduction of stress. Those who said their cheat sheet was not useful at all felt that they had not prepared properly for the exam. Those who said the cheat sheet was a little useful mainly commented that the majority of what they had put on the cheat sheet was not necessary for their exam. Those who said the cheat sheet was moderately useful mainly said this was because they had formulas on the cheat sheet. Many of these students explained that it was not "very useful" because most of the information they included was irrelevant, because they were able to remember the information anyway without the cheat sheet, and because they left important information off the cheat sheet. Finally, those who said the cheat sheet was very useful said this was because it had formulas, definitions, procedures and examples. Many also commented that it was useful because it helped them to study.

Overall these comments indicate that students have different definitions of what usefulness is, and indeed, what a cheat sheet *should* be useful for. It seems many students believe that they should be referring to the cheat sheet constantly because the exam will require them to regurgitate a lot of information. This does seem to reflect a surface approach to learning as reported by Trigwell (1987), and also an interaction with the perceived usefulness of cheat sheets and the style of exam. Other students see it as merely an aid to the memory of specific details and so are pleased when it does exactly that.

The effect of cheat sheets on stress

The following graphs display the responses from the two questions, "How did having a cheat sheet affect your stress during the exam?" and "How did making a cheat sheet affect your stress while studying for the exam?"



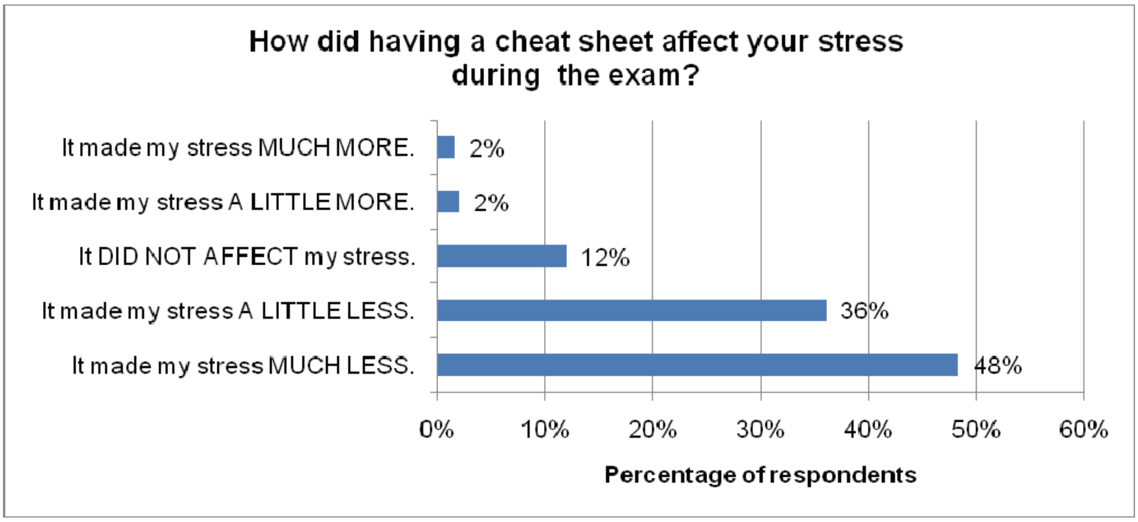


Figure 2. Responses concerning the effect of cheat sheets on exam stress.

According to these two sets of responses, most students do find that their cheat sheet reduces the stress associated with their exam, as described in the literature (Trigwell, 1987; Drake, Freed & Hunter, 1998; Theophilides & Koustelini, 2000; Dickson & Miller, 2005; Dickson & Bauer, 2008). However, it is important not to discount the 9% of students who said that making a cheat sheet actually *increased* their stress while studying—students who might need extra support in order to succeed.

The students were also asked to comment on how the cheat sheet had this effect on their stress. By far the most common reasons given for reductions in stress both before and during the exam were that the cheat sheet increased the students' confidence, or meant they did not have to worry about forgetting things. Almost as many students said the cheat sheet reduced stress before the exam because it encouraged them to study. Some elaborated further saying that the cheat sheet provided them with a way to organise their thoughts and figure out what they really needed to understand, showing that they were using it as an opportunity to take a deeper approach to learning, as reported by Theophilides and Koustelini (2008).

The students who said that the cheat sheet did not reduce their stress most often cited that they had trouble deciding what to put on the cheat sheet and were afraid of leaving something out. They also related how they panicked during the exam when something was *not* on their cheat sheet. These students appeared to believe the cheat sheet should tell them everything they needed to know, rather than merely support their memory of details. Some other students were overwhelmed by having to make a cheat sheet as well as study. This is particularly interesting when compared to the students who used the cheat sheet as tool to *help* them study, and may mean that coaching in study skills is necessary if cheat sheets are to be allowed.

The different uses of cheat sheets

A separate question on the survey asked students to describe how they used their cheat sheet. Almost every respondent said they used their cheat sheet for formulas, for lists of assumptions with hypothesis tests, for exact definitions, or for steps in specific procedures. This indicates that students are using the cheat sheet as an aid to memory of facts in order to help apply the concepts they had learned. A large number of students, however, did comment that they had put fully worked examples on their cheat sheet, or even a whole past exam. They describe reading their exam and looking for similar questions in the examples on their cheat sheet, then copying the working with the new words and numbers. This indicates that these students are merely trying to pass the exam, rather than learn the concepts at all. That is, the students are again using a surface approach to learning. We wonder how many of these students merely copied something even if it did not match, as described by Vessey and Woodbury (1992).

Finally, many students also recounted referring to the cheat sheet for inspiration when they felt confused or stuck. Some commented that they were inspired by things that they put on the cheat sheet for a different purpose, but somehow it formed the connection in their mind. This may support the idea that the creation of the cheat sheet in fact does help students to draw connections between ideas in the course.

Conclusion

Using the results of our survey, we have explored the experience students have of using cheat sheets in mathematical exams. Some students used their cheat sheet as a catalyst for organising and understanding the concepts in their course, while others took a surface approach and used it as a way to *avoid* understanding. Most said it reduced their stress, but some were overwhelmed by it. Finally, most students found their cheat sheet useful, but each appeared to define differently what useful was, depending on their expectations of how cheat sheets could help them.

The respondents are a small subset of students from three particular courses at one university, and as such, the results may not generalise well to other students. However, we feel that one conclusion is clear: different students react differently to the opportunity to make and use a cheat sheet in an exam. Hence, it is unwise to generalise.

Further research is needed into how students go about creating and using cheat sheets, and the effects that these have on their learning and performance. The first stage of this will be to widen the range of mathematical disciplines that we survey in our exploration. After this, future research could include observation of students in actual exams, as well as designed experiments to separate various variables involved. We also feel that future research needs to focus on how those students with negative attitudes and habits can be encouraged to take a more positive approach to the opportunity of using cheat sheets.

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TEACHER KNOWLEDGE ACTIVATED IN THE CONTEXT OF DESIGNING PROBLEMS

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The investigation of teachers' knowledge that informs practice in the mathematics classroom is an important area for research. This issue is addressed in our larger research program which is aimed at characterising the complexity and multi-dimensionality of this knowledge. A report on an earlier phase of this program (Butterfield & Chinnappan, 2010) showed that pre-service teachers tended to activate more common content knowledge than content that is required for teaching. We build on this previous work by examining the kinds of knowledge that a cohort of pre-service teachers activated in the context of designing a learning task.

Introduction

Current reforms and debate about improving the quality of mathematical learning are increasingly concerned with the kind of learning experiences teachers can provide for the learners (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010). The quality of these learning experiences in turn depends on teachers' own knowledge and experiences (Ball, Hill & Bass, 2005). There has been a surge of interest in examining teacher knowledge that drives their actions in the classroom. This study is located within this increasing concern with knowledge that is necessary for the support of deep mathematical understanding.

Context for the study

The performance of teachers has come under increased focus as reflected by accreditation requirements of professional bodies. In order to be accredited by professional bodies such as the NSW Institute of Teachers (NSWIT) and the Queensland College of Teachers (QCT) prospective teachers need to demonstrate that they have achieved a set of minimum knowledge and skills. This development has brought a high degree of urgency among tertiary educators to ensure that their programs and teaching modules are aligned with standards identified by such professional bodies. All these clusters of standards have one thing in common, which is that teachers must develop strong content and pedagogical knowledge. This is the focus of the study.

While the Australian National Curriculum is in various states of implementation a common teaching requirement is the consideration of performance against national standards (ACARA, 2010). This development again has brought the microscope on teaching and teaching knowledge.

Ball, Hill, and Bass (2005) have identified four dimensions of knowledge that are important for teachers to function effectively in a classroom: Common Content Knowledge; Specialised Content Knowledge; Knowledge of Content and Students; and Knowledge of Content and Teaching. These dimensions provide direction for the assessment of teacher knowledge for teaching. The elucidation of this knowledge is somewhat complicated due to the fact that this knowledge is internal. In order to gain insight into this knowledge, it is necessary to externalise the knowledge by providing a range of contexts to elicit this knowledge. It would seem that the richer the context in which the teachers are embedded, the better the quality of teacher knowledge that can be accessed. This logic led us to design a research study in which a cohort of pre-service teachers was asked to develop a complex problem that can be used in Upper Primary classrooms.

Our long term aim is to map the growth of this knowledge during the Graduate Diploma of Education (GDE) program. This study is a follow up of a previous study (Butterfield & Chinnappan, 2010) that was set against the above background concerning teacher knowledge that informs teaching. The results of this study showed that our GDE Pre-Service Teachers (PSTs) tended to access a higher proportion of Common Content Knowledge (CCK) than components of teacher knowledge that are more relevant to their work in the class. Specifically, we found that their knowledge of Specialised Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) were weak. This is not unexpected, as the participants were commencing their studies.

This study is aimed at boosting and assessing the growth of PSTs' knowledge of SCK, KCS and KCT. As described below our lectures and tutorials were modified in order to bring about changes in the above knowledge. This strategy involved guiding the PSTs to construct learning activities that were investigative in nature.

Related literature

Teacher knowledge

Research (Shulman, 1987) on teacher knowledge has spawned a number of studies concerning teacher knowledge and practice (Ma, 1999; Schoenfeld, 2010). In the past decade these studies have attempted to capture the complexity of teacher knowledge under various conditions including that which is played out in the classroom. This body of research has led to a convergence of view that such knowledge is complex and multifaceted. For example, the studies conducted by Ma (1999) showed that teachers need to transform their content knowledge to teach effectively. Concurrent developments in the United States have generated new directions in the way we could conceptualise and study teacher knowledge. Research in the United States has been led by Ball and her associates, which resulted in the development of more refined dimensions of teacher knowledge (Figure 1). The spirit of this research theme has been embraced by others by examining teacher knowledge in a variety of contexts (Mewborn, 2001).

Teaching as problem solving

A major problem for teachers is to design and implement effective learning experiences leading to sound learning outcomes. The problem, defined in this manner, is rather nebulous as there are multiple paths to the solution. If one conceives teaching as a problem solving activity one is open to a range of opportunities for teachers to exhibit and exploit their knowledge. Problem-solving activities involve searching for a solution within a problem space (Newell & Simon, 1972). The nature of problem space and quality of search is a function of the elements in the space. A corollary of this action is that in an open-ended problem such as teaching, the problem space can be expected to be populated by not only more elements but also the search will be supported by the activation of multiple knowledge sources. Thus, it would seem that the kind of knowledge identified by Ball et al. (2005) are better studied in the context of teachers designing problem-solving activities that can be subsequently used to engage learners. In the present study we adopt this approach.

Conceptual framework

Data analysis and interpretations were guided by the following schematic representation of teacher knowledge for teaching mathematics (MKT) (Figure 1) (Hill, Ball & Schilling, 2008, p. 174).

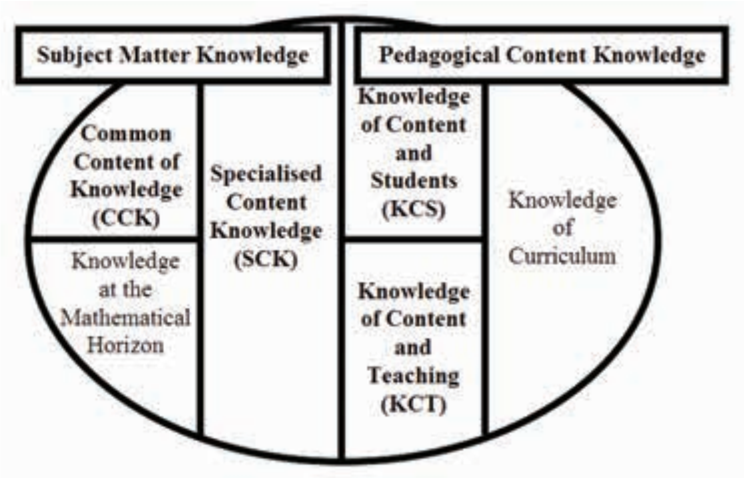


Figure 1. Schematic representation of teacher knowledge for teaching mathematics (MKT).

Four dimensions are defined:

- *Common Content Knowledge (CCK)*: Mathematical knowledge and skill possessed by a well educated adult.
- *Specialised Content Knowledge (SCK)*: Knowledge of how to: use alternatives to solve a problem; articulate mathematical explanations; demonstrate representations.
- *Knowledge of Content and Students (KCS)*: Knowledge that combines knowing about mathematics and knowing about students. Knowledge of how to: anticipate what students are likely to think; relate mathematical ideas to developmentally appropriate language used by children.

- *Knowledge of Content and Teaching (KCT)*: Knowledge that combines knowing about mathematics and knowing about teaching. Knowledge of how to: sequence content for instruction; determine instructional advantages of different representations; pause for clarification and when to ask questions; analyse errors; observe and listen to a child's responses; prompt, pose questions and probe with questions; select appropriate tasks.

Focus questions

The aim of the study was to examine the quality of SCK, KCS and KCT that was activated by a cohort of Pre-service Teachers (PSTs) in the course of designing a problem.

The above aim is reflected in the research questions, seeking

1. evidence of PSTs activating SCK in the context of designing a problem;
2. evidence of PSTs activating KCS in the context of designing a problem;
3. evidence of PSTs activating KCT in the context of designing a problem; and
4. a correlation between the quality of the problem representation and activation of SCK, KCS and KCT.

Methodology

Participants

A cohort of 26 Graduate Diploma of Education students in the final semester of their one-year degree participated in the study. The cohort had completed a numeracy course prior to this mathematics subject, and had also completed professional experience in schools.

Task

Pre-service teachers were required to work in pairs to design a mathematical problem suitable for Upper Primary school children. In designing the task the PSTs were instructed to develop a problem that is isomorphic to the Truss Bridge Problem (Butterfield & Chinnappan, 2010).

Procedures

PSTs were provided with a range of prompts and supports in both the lectures and tutorials before they designed their own problem. The Truss Bridge Problem (TBP) (Butterfield & Chinnappan, 2010) was utilised in a number of tutorials and lectures. This involved discussions about the different problem representations of TBP and how such representations could permit or hinder transfer to other problems by learner. The TBP also highlighted the role and the development of a child's knowledge and skills in Number, Patterns and Algebra, and Space. In addition, we examined the use of appropriate materials and methods (including technology) to solve problems of this type and likely difficulties children could encounter. The TBP, therefore, provided PSTs with a stimulus for hands on activities and reflection on the knowledge components required in subject matter and pedagogy. The PSTs were also given multiple opportunities to explore and solve the TBP. Thus in designing their own new problems we are comfortable in assuming that the PSTs are cognisant of the multiple solution paths and associated representations of the problems.

Representations of TBP and Coding

The TBP (Figure 2) that was developed in the previous study (Butterfield & Chinnappan, 2010) has a certain structure reflecting a hierarchy in the way that it can be represented. The hierarchy is as follows:

1. Concrete – uses concrete materials or physical means to provide a solution
2. Sequential – uses a table to provide a sequential, linear set of solutions
3. Generalisation – describes the pattern that can be used to provide a solution to any given number
4. Transferability – describes how the pattern can be used to solve similar problems

Figure 2. Truss bridge problem.

The hierarchical structure in the Truss Bridge Problem guided us in developing instructions for problems with similar structures. This structure also provided a coding scheme to rate the quality of task developed by the students.

Sources of data

There are two sources of data for the study. The first source involved examining the quality of the problem designed by the students. The coding system is based on the hierarchy of the TBP.

The second source of data involved determining instances of activation of three categories of knowledge (SCK, KCS, KCT). In order to generate this data we analysed PSTs' reflective reports, digital presentations and their responses to questions about the likely difficulties and useful ways to develop children's understanding. The researchers independently coded these instances in order to establish inter-coder agreement.

Results

Participants provided a range of problems that could foster algebraic thinking. The problems designed by student pairs are outlined in Table 1. All problems lend themselves to an analysis of problem representations along the dimensions of TBP.

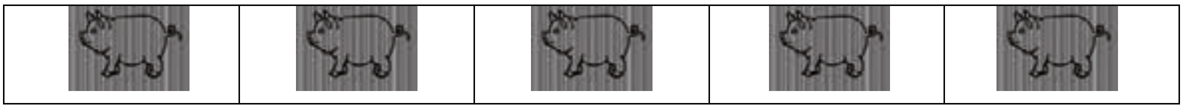
Table 1. Description of problems.

Problem	*Description*
Tricky Trapezium Tables	Number of children seated at a row of trapezium-shaped tables
Multistorey Car Park	Number of beams to construct the front of a multistorey car park
Stair	Number of rail posts for a flight of stairs
Stadium	Number of seats in a stadium
Pig Pen	Number of fence panels in a row of pig pens with shared walls
Jack – In – The - Box	Number of exposed body parts with each wind
Dragon	Number of triangular scales per each body part
Terrace Houses	Number of windows in a row of terrace houses
Fence Posts	Number of fence posts in a rectangular paddock
Path Pavers	Number of pavers in patterned path
Angle Sums	The sum of angles in regular shapes
Hay Stack	Number of cylindrical bales in hay stacks
Mosaic Frame	Number of tiles in a frame with coloured corners

Hierarchy of representations for selected problems

Problem Sample 1

An example of a problem coded 2 for problem representation is the Pig Pen problem (see Figure 3). In this problem PSTs did not identify the potential to generalise the pattern to any number of fence panels.



How many fence panels are needed to construct these pig pens?

Figure 3. Pig pen problem.

The PSTs stated that the children should complete the provided table (see Table 2) and that as teachers they would like their students to communicate, "I saw that the numbers on the bottom line are going up by three". Here the PSTs were able to identify only the sequential patterns.

Table 2. Pig pen problem worksheet sample.

No of pens	1	2	3	4	5
No of panels	4	7	10		

Problem sample 2

An example of a problem coded 4 for problem representation is the Tricky Trapezium Tables (see Figure 4). The problem enables students to generalise and transfer that pattern to a new problem context. The PSTs stated that "generalisations enable students to recognise that similar problems have a common algebraic basis". To support this statement the PSTs wrote that:

> when a child sees the Truss Bridge Problem (see Figure 2) they would say this could be solved by two times the number of triangles plus one, which is the same way to solve the number of people sitting at different shaped tables. For example, the number of people seated around trapezium-shaped table could be determined by counting the number of trapeziums multiplied by three plus two (Number of people = 3*n* +2). This reasoning can be applied to squares.

This type of thinking that resulted in generalisation has been argued to lie at the foundation of algebraic thinking (Bobis, Mulligan, Lowrie, & Taplin, 2004).

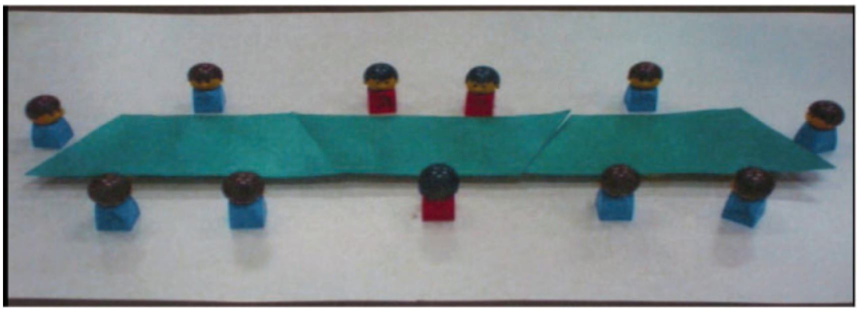


Figure 4. Tricky trapezium tables problem.

In order to generate data that are relevant to research questions 1-3, we analysed the frequency of instances. The mean and standard deviations of this analysis for the four problem representations are given in Table 3.

Table 3. Problem representation and teacher knowledge.

Problem	Representation	SCK	KCS	KCT
1	Mean	12.80	7.80	.60
	Std. Deviation	6.22	4.02	.54
2	Mean	14.50	10.50	1.50
	Std. Deviation	2.12	.70	2.12
3	Mean	18.00	17.50	2.50
	Std. Deviation	11.31	9.19	2.12
4	Mean	31.75	25.50	5.00
	Std. Deviation	2.75	4.04	1.41
Total	Mean	19.69	15.15	2.39
	Std. Deviation	9.95	8.90	2.25

We note the accessing of a higher proportion of SCK followed by KCS and KCT. This pattern is also evident within each representation. There is a significant difference between the number of instances of KCT and the other two categories of knowledge across all four categories of problem representations.

Table 4 shows results of correlation analysis among the four variables. While all three knowledge components are highly positively correlated with Problem

Representations, we note KCS and KCT have higher indices. Thus, there was support for our contention that a qualitatively superior problem representation will involve a higher degree of activation of SCK, KCS and KCT (Research question 4).

Table 4. Correlation analysis.

	SCK	*KCS*	*KCT*
Problem Representation	0.81**	0.88**	0.84**

*** Correlation is significant at the 0.01 level (2-tailed).*

Discussion and implications

The previous study showed that student teachers both individually and as a group tended to activate more CCK component of their subject-matter knowledge of mathematics than SCK. The results were consistent with our expectation that as beginning teachers their content knowledge of mathematics, robust though this might be, would not be translated into forms that were more akin to teaching mathematics to children.

The thrust of this study was to map developments in PSTs' teacher knowledge as a consequence of exposing them to a teaching approach that focused on the design of problems. These teachers had also completed two sessions of their professional experience in the school setting. Thus, our expectation was that the classroom experiences and our guidance in designing problems for deep mathematical learning would assist them to reveal a higher incidence of activation of not only SCK but also understanding of student learning and the demands of teaching via an enhanced body of KCS and KCT.

The results do support our contention that having PSTs design rich learning activities would increase their knowledge and activation of SCK, KCS and KCT. Designing problems that will be used to support children's learning requires a level of sophistication in teachers' conceptualisation of the problem environment as shown by the range of problems in Table 1. The corollary here is that teachers have to understand the mathematics that underpins that activity and insights into how children will grasp the problem. We contend that the complexity of the problems teachers have been asked to design have provided multiple points at which teachers could connect with and activate knowledge relevant to the three categories of knowledge.

While all three knowledge categories were positively correlated with the quality of problem representation, the highest correlation was evidenced with KCS which involved teachers understanding learners. It would seem that problem posing activities could be used to enhance the development of KCS, a point that was alluded to by Chinnappan and Lawson (2005).

Results indicated that (Table 3), a significant number of the participants tended to design problems that from a representational viewpoint were somewhat weak. This group either constructed the physical model of the problem or merely provided a table with numbers indicative of growing dimensions. For example, in Figure 4, student teachers could indicate the growth in number of panels per pen for a small number of pens (1-5). That is, the only pattern they could identify is numbers increasing in threes without being able to extract the *general* pattern that shows the relations between pens and panels. This limitation in the quality of representation, we argue, is the consequence of over reliance on the accessing of procedural knowledge. This outcome is consistent with that reported by Capraro, Capraro, Parker, Kulm, and Raulerson (2005).

A limitation of the present study is that we did not give prominence to KCT as we assume that this is more accessible in real-life teaching contexts. Future studies should focus on this issue. Also, we acknowledge that it is difficult to generate a complete picture of pre-service teachers' pedagogical content knowledge within the confines of one assessment task that was completed for a university subject. Further studies with a greater variety of such tasks might provide more opportunities to examine this knowledge.

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DO INTERESTED STUDENTS LEARN MORE? RESULTS FROM A STATISTICAL LITERACY STUDY IN THE MIDDLE SCHOOL

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An oft-heard comment from adolescents is that “maths is boring”. Such disinterested students are less likely to engage in mathematics related learning tasks than their interested peers and this lack of engagement can result in lower levels of achievement. This study seeks to explore the relationship between interest and achievement in a middle school statistical literacy context. Based on the results of 218 students, a linear regression model indicated that the relationship between interest and achievement is complex and mediated by other factors that include students’ self-competency beliefs. Moreover, the model predicted that interest has a very minor influence on achievement and that factors related to the classroom teacher have much stronger influence.

Introduction

If recommendations by Masters, Rowley, Ainley, and Khoo (2008) are adopted, league tables comparing middle schools in Australia are likely to be based on longitudinal rather than cross-sectional data. Improvements in students’ results in national mathematics tests between Years 7 and 9 will contribute to these comparisons. Schools vying for position on these tables may need to consider their students’ interest, as this period of early adolescence is characterised by declining levels of interest for learning in general (Dotterer, McHale, & Crouter, 2009) and for mathematics in particular (Watt, 2008). Further, interest is known to have an association with achievement in mathematics (Schiefele, Krapp, & Winteler, 1992), some of which is causal, in that interested students are more likely to engage in deeper learning (Schiefele, 1991) and this can contribute to higher outcomes in properly validated attainment tests (Chamorro-Premuzic & Furnham, 2008).

Interest is regarded as an affect with both state and trait properties. Pen and paper measures of interest, as used in this study, assess interest at the trait level where it is defined as “a person’s relatively enduring predisposition to re-engage particular content over time” (Hidi & Renninger, 2006, p. 113). It is believed that students who regularly experience the state of interest during their learning of a subject, such as mathematics, will in time develop a trait-like interest for the subject (Hidi & Renninger, 2006). Although interest is known to be associated with achievement, this association is influenced by a number of other factors. Schiefele et al. (1992), for example, reported that the interest/achievement association was stronger for boys than for girls and argued that the latter are more inclined to conform and thus work hard in subjects that they do not find interesting. Students' self-competency beliefs in mathematics are also known to influence the relationship between their achievement and interest, in that students who believe that they are good in maths are also likely to have an interest for the subject (Trautwein, Ludtke, Köller, Marsh, & Baumert, 2006). As noted, age also has an influence on interest, with declines evident during adolescence.

Although the relationship between students' interest and achievement has been explored in mathematics, it has not been explored in a statistical literacy context. Such literacy is regarded as an ability to interpret and critically evaluate messages that contain statistical elements (Gal, 2003). In Australia, the underlying concepts for this literacy will be introduced in the Statistics and Probability strand of the proposed National Mathematics Syllabus (Australian Curriculum Assessment and Reporting Authority, 2010). This syllabus supports an "across-the-school commitment" (p. 6) to numeracy and students should develop their statistical literacy in other curriculum areas such as the physical and social sciences. For this reason, students' interest and achievement in statistical literacy may differ from their interest and achievement in mathematics. In fact Carmichael (2010) reported that students tended to find the statistical contexts encountered outside mathematics classrooms to be of more interest. Given the distinctive nature of statistical literacy, the aim of the paper is to explore the extent to which middle school students' interest in statistical literacy influences their achievement in statistical literacy.

Method

The study reported in this paper is quantitative in nature. Details regarding the sample of students involved, the data collected, and the analyses undertaken, are reported in the following section.

Background to the study

The student results reported in the study come from the intersection of two larger studies in the middle school statistical literacy context. Purposive cluster sampling was employed in both studies to provide representative samples of the Australian middle school population. Teachers within targeted schools were invited to participate in the studies. These teachers, in turn, nominated classes of students who were then invited to participate.

The first study, described in Carmichael, Callingham, Hay, and Watson (2010a), focussed on the influence of affect in the acquisition of statistical literacy. Measures of students' interest, self-efficacy, and prior mathematics achievement were available from this study, which involved a sample of 791 middle school students from four Australian states. These students were asked to respond on a five-point Likert scale to items comprising the "Statistical Literacy Interest Measure (SLIM)" (Carmichael, Callingham, Hay, & Watson, 2010b) and the "Self-efficacy for Statistical Literacy (SESL) scale" (Carmichael & Hay, 2009). The former contains 16 self-descriptions, such as "I'm interested in getting a job involving statistics" and the latter contains 9 self-descriptions such as "I'm confident that I am able to explain to a friend how probability (or chance) is calculated." Scoring and scaling of student responses to both measures was achieved through the application of the Rasch rating scale model (Andrich, 1978) using the software package Winsteps (Linacre, 2006).

Measures of students' prior achievement in statistics were not available and consequently their prior achievement in mathematics was used instead. Teacher ratings of this achievement were used, with such ratings known to be strongly predictive of actual achievement (Egan & Archer, 1985). More specifically, the teachers rated their students' mathematics achievement on a five-point scale ranging from A, the highest category of achievement, though to E, the lowest category. Such a scale reflects the current reporting scale used in Australia.

The second study, titled "StatSmart" (Callingham & Watson, 2007), focussed on teacher professional development in statistics. It was a large longitudinal study involving more than 50 teachers from 17 schools across three Australian States. A measure of statistical literacy achievement (SLA) was obtained from students of teachers involved in StatSmart on three occasions. Students new to the study did a pre-test, others who had been involved in the study for more than six months took a post-test, and still others who had been involved in the study for longer than 12 months took a longitudinal test. Specific details about the arrangement of these tests are reported in Callingham and Watson (2007), and details on the items and their scoring can be found in Callingham and Watson (2005). Scoring and scaling of 1151 students' responses to the SLA tests was achieved through application of the Rasch partial credit model (Anderson, 1997) using Winsteps.

Student sample

Although 483 students in the interest study attended schools participating in StatSmart, many did not complete the SLA tests and consequently SLA data were available for only 218 of these students. Of these 218 students, 53% were male. The ages of students in the sample ranged from 11 to 17 yrs with a mean of 13.9 yrs. They were in school year levels ranging from Year 6 through to Year 10, with approximately one quarter in Year 7 and one quarter in Year 8. Two thirds of the students attended schools in Tasmania and the remainder attended schools in Victoria. Almost 60% of them attended non-government schools.

Data collected

Interest measures were available for 204 of the students in this sample, as a number of younger students failed to complete the questionnaire. The SLIM scores for these 204 students, measured in logits, ranged from -5.1 to 2.4 with a mean of -0.3. Measures of students' self-efficacy were available for all 218 students and these ranged from -5.0 to 5.1 logits with a mean of 0.0 logits. The SLA measures for the 218 students in the study ranged from -5.5 to 1.7 logits, with a mean of -0.4 logits.

In regards to prior mathematics achievement, data were available for only 215 of the 218 students. Of these students, 64 gained an A rating, 89 a B rating, 47 a C rating, and the remaining 15 students a D or E rating. In order to control for the influence of the classroom on achievement, a relative mathematics grade was also considered. More specifically, students' grades relative to the median grade of their class were determined. Of these 215 students, 52 had ratings below the class median, 105 had ratings equivalent to the median, and the remainder had ratings above the class median.

Analyses undertaken

Simple linear regression was used to assess the influence of interest on achievement and was applied using the statistical software package R (R Development Core Team, 2009). In order to model possible dependence between students in schools and classrooms, a mixed-effects regression model was also applied to the data as described in Faraway (2006). The variables used in the regression models included: students' interest in statistical literacy (*SLIM*), their self-efficacy in statistical literacy (*SE*), their statistical literacy achievement (*SLA*), their mathematics grade (*Mgrade*), their mathematics grade relative to the class median (*RelMgrade*), their age (*Age*), the type of SLA test they did (*Test*), and their gender (*Sex*). Of the 218 students in this study, only 204 had values for all of these variables.

Results

A series of linear models was tested with *SLA* as the response variable. Given the aim of the study was to investigate the influence of interest on achievement, only models with *SLIM* as one of the predictor variables were considered. The modelling process indicated that in the presence of the variable *SE*, *SLIM* ceased to predict *SLA*. In addition to this, the influence of *SLIM* on *SLA* was only significant when *Age* was included in the model. The variable *Sex* was not a significant predictor of *SLA* in any of the models and the variable *Test* was a significant predictor of *SLA* in all models. When the variable *Mgrade* was included in the model, *SLIM* ceased to be a significant predictor of *SLA*. When *Mgrade* was replaced with *RelMgrade*, however, *SLIM* was a significant predictor of *SLA*. After the removal of one student deemed to be an outlier, the resulting model explained 46% of the variance in *SLA*. All predictors in the model, shown as Equation 1, were significant at the 1% level.

Equation 1.

$$SLA = -4.96 + 0.12\,SLIM + 0.29\,Age + 0.36\,Median\ Grade + 0.54\,Above\ Median\ Grade + 0.80\,Post\text{-}test + 0.66\,Longitudinal\text{-}test + \varepsilon \quad [1]$$

Model assumptions appear to have been met. The top plot in Figure 1 shows the sample quantiles against theoretical quantiles and its near linear form supports the normality assumption. The bottom plot in Figure 1 shows the model residuals against the predicted values and its random scatter supports the assumption of homogeneity in the data.

As is seen from the model the relative influence of *SLIM* on *SLA* is minor, with all other predictor variables having a greater influence. Prior achievement in mathematics had a relatively strong influence on *SLA*. With all other factors constant, students with a median grade achievement in mathematics are predicted to score 0.36 logits higher on the SLA tests than those with below median grades, while students with an above median grade in mathematics are predicted to score 0.54 logits higher. The type of test that the students did also had a relatively strong influence on their SLA test result. With all other factors constant, students who completed the post-test are predicted to score 0.80 logits higher on the SLA tests than those who did the pre-test, while students who completed the longitudinal test are predicted to score 0.66 logits higher. It should be noted that the scheduling of these tests was independent of the teaching of relevant statistics units.

The linear model reported above was also tested for teacher and school random effects. In this instance, only the inclusion of random teacher effects in the intercept term contributed significantly to model fit. When this term was included in the model, however, *SLIM* ceased to become a significant predictor of *SLA* and consequently the model is not reported. Nevertheless, the result indicates that in the presence of teacher factors individual interest plays a minor role in students' achievement.

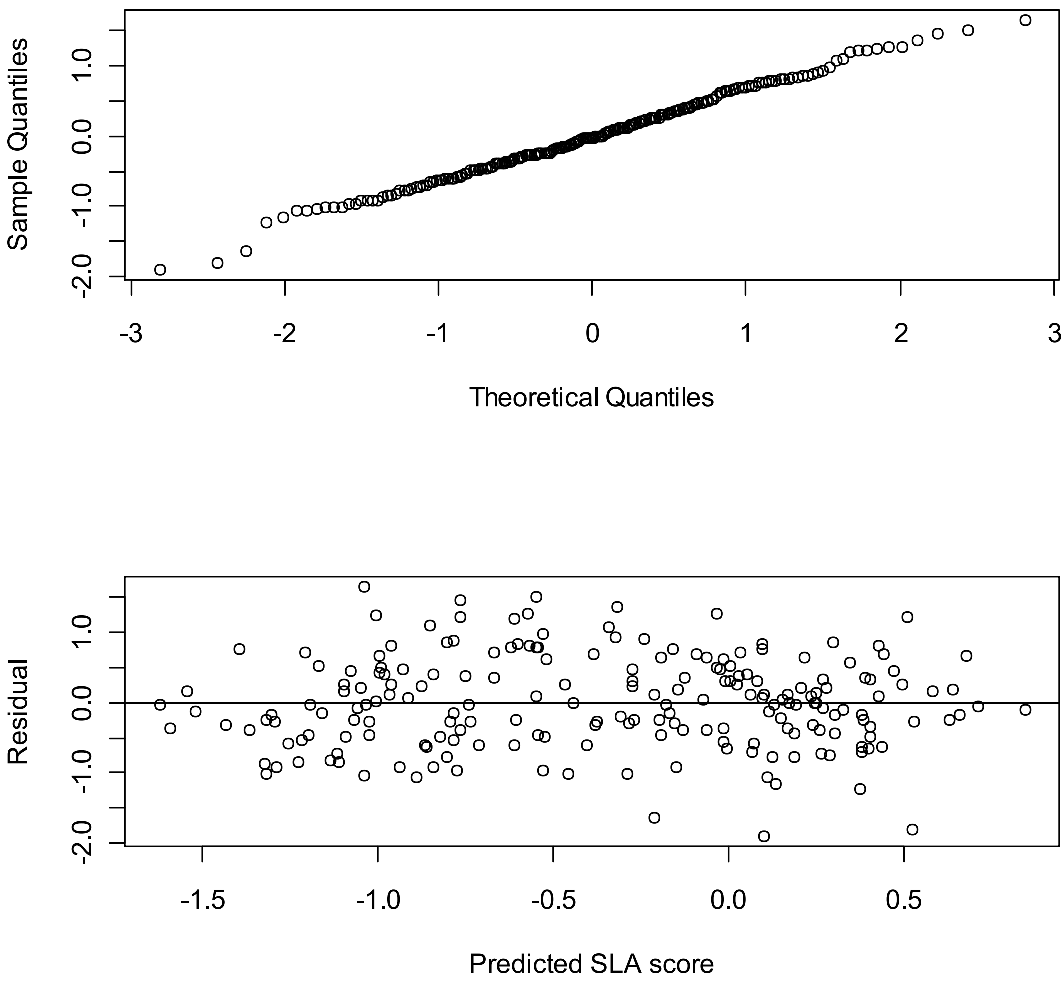

Figure 1. Diagnostic plots for linear model reported in Equation 1.

Discussion

The modelling process confirmed results reported by Trautwein et al. (2006) that student self-competency beliefs, in this instance their self-efficacy, have a strong influence on the interest/achievement relation. Self-efficacious students are likely to be motivated to engage in tasks related to statistical literacy, with such engagement producing higher levels of interest and achievement. The modelling process also revealed that it was only in the presence of relative rather than absolute prior achievement measures that interest significantly predicted achievement. This could be due to the reported association between such relative measures and interest (Trautwein et al., 2006), with high achieving students in classes of even higher achieving students reporting lower levels of interest than low achieving students in classes of even lower achieving students. The lack of a gender effect on statistical literacy achievement was surprising and may indicate changes in educational practices that have minimised earlier reported differences.

A significant result of this study is the relative importance of teacher related factors. When the variance associated with the teacher was modelled, interest ceased to have a significant influence on achievement. Further, the model indicated that even accounting for individual factors such as age, interest and prior achievement; the type of test that students undertook had a significant influence on their achievement. This test was a measure of the length of time that students were taught by teachers associated with the professional-development program and the result provides some evidence for the effect of teacher professional-development on student achievement outcomes.

Conclusion

The results reported in the paper suggest that interest in statistical literacy has a very minor role in predicting students' achievement. It is possible that many of these students focused on achievement outcomes, with interest in the domain an optional extra. Interest, however, appears to be a stronger predictor of re-engagement than of achievement (Wigfield, Tonks, & Eccles, 2004) with high achieving students not participating in senior mathematics courses because of their lack of interest in the subject (McPhan, Morony, Pegg, Cooksey, & Lynch, 2008). Further interest based research in the statistical literacy domain should analyse the influence of interest on participation rates in courses related to statistical literacy rather than in levels of achievement.

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TEACHING SECONDARY MATHEMATICS WITH AN ONLINE LEARNING SYSTEM: THREE TEACHERS' EXPERIENCES

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We studied how three secondary mathematics teachers who had no prior experience teaching with technology used an online mathematics learning system. The teachers received minimal instruction on how to use the system before we observed them over four school terms as they taught with it. We used the Pedagogical Technology Knowledge framework (Thomas & Hong, 2005) to document changes in the teachers' practice. Results show teachers advanced toward using the technology in more sophisticated ways but the improvements were not uniform. We suggest some reasons to explain the variation.

Introduction

Teachers can learn to make more effective classroom use of technology through participation in professional development and training (Bennett & Lockyer, 1999; Bennison & Goos, 2010; Lawless & Pellegrino, 2007). To be effective, professional development should be intensive (Darling-Hammond, 1998), sustained over time (Guskey, 2003), and do more than enhance teachers' technical skills (Watson, 2001). However, creating sustainable, on-going professional development programs which operate effectively across a number of schools can be a challenge (Goos, Dole, & Makar, 2007). As a result, professional development activities are often short-term and sporadic, with an emphasis on learning to operate the technology at the expense of providing guidance on how technology can be used to improve learning and teaching (Fitzallen, 2005).

We report on an investigation into whether the inbuilt structure of an online mathematics learning system can assist teachers who are inexperienced users of ICT develop their ability to teach mathematics with technology in new and different ways that constructively engage students in their learning. Online systems typically incorporate several of the following student learning activities:

- Lesson notes (text material) linked to a particular curriculum;
- Worksheets (activities such as puzzles and games linked to each lesson);
- Lesson Questions (multiple choice and short answer questions, usually graded for different ability levels);
- Timed drill and practice questions (sometimes called a Basic Scorcher);

- Lesson Scorchers (like the Basic Scorcher, but with questions focused on specific lesson topics);
- Walkthroughs (similar to school textbook examples that students complete online while receiving step-by-step explanations and feedback); and
- Widgets (interactive animations linked to a particular topic).

Students can access these features for individual practice and exploration at the computer, or teachers can display them via a data projector for whole class demonstrations and investigations. We based our research on Cambridge HOTmaths (http://www.hotmaths.com.au), an online system incorporating all the activities listed above as well as a student messaging and reporting facility. Our research seeks answers to the following questions:

- What happens in a classroom situation where the online mathematics learning system is implemented with minimal professional development by teachers who have limited experience with technology?
- Do teachers become more confident users of the technology?
- Do teachers shift towards more student-centred teaching approaches?

The research is significant because it shows how teachers who lack technology experience and training begin to use technology in the classroom.

Theoretical framework

Goos, Galbraith, Renshaw and Geiger (2000) conducted a three-year longitudinal study in five secondary mathematics classrooms to investigate the role of graphics calculators in assisting students to conduct mathematical investigations and promote discussion. Goos et al. suggest four roles to describe interactions between the teacher and the graphics calculator: *technology as master*, where teachers' limited knowledge of how to operate the technology means they are subordinate to the technology; *technology as servant*, where technology supports the teacher's preferred pedagogy; *technology as partner*, where familiarity with the class and the technology allow teachers to use the technology more creatively in ways that encourage collaboration in the classroom; and *technology as extension of self*, which features the most powerful, creative and sophisticated uses of technology.

Thomas and Hong (2005) developed the construct of *Pedagogical Technology Knowledge* (PTK) or "knowing how to teach mathematics with the technology" (p. 258). PTK encompasses teachers' recognition of the role of technology in learning and teaching and includes how teachers decide to use technology to assist students learn mathematical concepts and processes. PTK also includes techniques and approaches used to teach mathematics in qualitatively different ways through technology. Teachers who have an advanced PTK will therefore be most likely build on the affordances provided by technology to transform learning and teaching in ways that are not possible without technology (Garofalo, Drier, Harper, Timmerman, & Shockey, 2000).

Method

Data described in this paper are drawn from a larger study of 14 secondary mathematics teachers at 3 schools as they used the online learning system with Years 7 and 8. We deliberately chose schools where the levels of technology use in mathematics lessons prior to the commencement of the study were low.

An experienced teacher from the company that developed the selected learning system presented a half-day professional development session for the teachers at the start of the project. The presenter showed some of the main features of the system and how to navigate around them. She also demonstrated three Widgets, but there was little discussion about how to use them in the classroom.

We studied classroom use of the system in five Year 7 classes. For four weeks at the end of Term 2, two weeks in the middle of Term 3, and four weeks at the start of Term 4. In the following year, we followed three of the Year 7 classes into Year 8 and observed lessons for four weeks at the start of Term 2. Throughout the project, a research assistant [RA] visited each classroom to observe between one and three mathematics lessons per week. We developed a schedule to record teacher and student actions during lessons, and the RA made detailed field notes in an observation log. Following most observed lessons, the RA interviewed the teacher. The teacher interviews were designed to gather data about each teacher's thoughts on how they had used the system rather than mentor them or influence their practice. All interviews were audio-recorded and transcribed for analysis.

At the end of the observation period, the RA submitted her classroom observation notes and teacher interviews for all 14 teachers to the authors. We compiled a profile for each teacher to outline changes in their classroom practices and made an initial comparison between them. We then selected three teachers at Hope Springs who best illustrated the range of variation. We examined their profiles in more detail, re-visiting the original notes and recordings to ensure that our descriptions were accurate and selecting excerpts that best exemplified their behaviour at different times. These profiles allowed us to identify up to three broad stages in the development of teachers' PTK. We also noted some similarities to the technology roles theorised by Goos et al. (2000). Finally, we sought possible explanations for the changes we had documented and formulated some implications for professional development.

Results

Teacher HA

Teacher HA had taught mathematics for over 25 years, including the last seven years at Hope Springs. HA's teaching emphasised student note-taking from the textbook and he was insistent that students kept a neat exercise book. Prior to the study, HA had briefly used the system at home with his own children, but he had not previously used computers for teaching.

HA initially used the system in the computer laboratory where he permitted students to work on anything related to the mathematics topic they were learning. This pattern continued well into the fourth term. Observations of HA demonstrated his lack of awareness about how to support student learning with technology. For example, he did not show students the different levels available in the Lesson Questions or demonstrate how to navigate the system. He continued to use the system exclusively in the computer laboratory, with students working on their own at activities of their own choosing, even when this approach was unproductive.

In an interview near the end of Term 4, HA mentioned using the system in the normal classroom by projecting activities onto the whiteboard so students could work together on a task he chose for them. At the start of Year 8, HA began to teach with the system in the classroom by having students answer Scorcher questions. He displayed the questions one at a time on the whiteboard but sat with his back to the class while operating his laptop. He allowed students to shout out their answers without providing their reasoning. There was much noise as students competed for scores which made it difficult to hear their responses.

HA later commented that the wide range of ability levels in his classes made it difficult for him to keep all students working together when using the system for whole-class discussion, especially because students who could not keep pace with the lesson were prone to become disruptive. He concluded that he would revert to computer laboratory lessons "where they can work on their own at their own pace and those who are off task do not disrupt others or interfere with their learning."

Teacher HB

Teacher HB had a postgraduate research qualification and was the most proficient and confident user of technology. She had been teaching for 12 years, including seven years at Hope Springs. She had explored the system at home and had prepared a lesson with it, although technical problems with the school computers prevented her from giving the lesson as planned.

Observations of her initial use of the system showed a more varied approach than the other teachers in the study. HB used many of the system features in her lessons, although she continued to adopt a very teacher-centred approach. When teaching Year 7 in the computer laboratory, she soon realised that students needed her support to maximise their learning with the system. She spoke in an interview at the end of Term 2 about how students would often go straight to the Lesson Questions without first reading through the accompanying lesson notes and this meant that they could not answer the questions. So HB spent time teaching the concepts before the students were allowed to attempt the related system topic on the computers.

HB was the first to teach with the system away from the computer laboratory. In Term 3, she gave a lesson to Year 7 about surface area using the interactive whiteboard. Her lesson was based on a Widget called "Observing surface areas", which she used in quite a sophisticated manner by combining it with her own explanations to highlight various aspects of the solids she displayed. But she explained everything to the students and her explanations were given too quickly so it was clear that students found them difficult to follow. The RA noted that most students were not paying attention and the activity was not very successful in helping them learn about surface areas.

HB took up a teaching position at another school at the end of the first year of the study and so did not take part in the second phase of the project.

Teacher HC

Teacher HC had first trained as a primary school teacher and had taught for 30 years before moving to Hope Springs and had been at the school for eight years. HC had a good rapport with students and was an excellent classroom manager. She had previously used computers to prepare worksheets and tests for her classes, but had not used technology in the classroom. Based on lesson observations, the RA characterised HC's style as teacher-centred and traditional.

HC's initial use of the system also involved classes in the computer laboratory. But, unlike HA and HB, she prescribed activities for students to complete: Widgets, then the

Walkthrough, the Lesson Questions, and finally the Scorcher. HC said she liked to start with the Widgets because students found them engaging, but it was observed that students often had difficulty identifying what the Widgets were about if they had not been given any prior instruction. Significantly, HC quickly realised the difficulties some students encountered. After just two weeks she started to split the lesson time between the classroom (where she introduced a mathematical concept) and the computer laboratory (where the students used the system to consolidate it).

Her split-lesson method was proving successful and HC saw “some definite advantages in integrating the system with the normal classroom lessons.” But she commented in Term 4 that it was difficult to monitor student behaviour in the computer laboratory and students would visit other websites unless closely supervised. By the following year, HC no longer took her classes to the laboratory and used the system exclusively in the classroom. A Year 8 lesson on fractions demonstrated her more sophisticated approach. She began with an assortment of her own revision questions written on the board. While students attempted these, she set up the data projector. The system was then used to teach fractions, beginning with a Widget on “Representing fractions”. The lesson notes were read aloud by a student but HC interrupted three times to give a further example or explanation on the whiteboard. She also circled or underlined parts of the text to emphasise key ideas. Students then copied HC’s own definitions of the different types of fractions into their exercise books.

Discussion

Our research allowed us insight into how teachers’ PTK develops. We identified three sequential stages to describe the changing roles we observed. We call these roles technology bystander, technology adopter, and technology adaptor. We also conjecture a fourth role of technology innovator. As teachers move from one stage to the next, their technology use becomes increasingly more varied and sophisticated. Our four stages resemble the metaphors of technology as master, servant, partner, and extension of self, proposed by Goos et al. (2000). But whereas Goos’s terms refer to the roles of technology, we focus on teachers’ roles as they use technology to support student learning.

Technology bystanders

All three teachers began teaching with the system in the computer laboratory where students worked individually at their own pace; HA and HB allowed students to work on any aspect of the system they wished, while HC mandated specific tasks for students to complete. Observations in the first weeks of the project showed that teacher-student interactions were minimal, focussing mainly on managing student behaviour or sorting out students’ difficulties in operating the system. There was little actual teaching and the teachers’ PTK did not advance beyond developing an initial familiarisation with the features of the system.

We describe the teachers’ role in these early lessons as *technology bystanders*, because they essentially allowed students to work on their own. We see the initial computer laboratory lessons as an essential first step in developing the teachers’ confidence in their ability to use the system. These lessons allowed teachers to become familiar with the system’s features and to learn how to navigate around them. They also provided opportunities for learning how to deal with relatively prosaic issues such as assisting students who had forgotten their passwords or could not login to the system.

Technology adopters

All three teachers eventually began to deploy the system in conjunction with their usual teaching practices, so that the technology was essentially used to support already established pedagogies. However, the success of the teachers' classroom use of the system was heavily dependent on their general pedagogical skills. HA was the last to begin employing the system away from the computer laboratory and the least successful in doing so. His PTK showed negligible development throughout the project and was characterised by an emphasis on trying to keep students busy regardless of how much they were learning. HB's PTK began to shift from an exclusive focus on helping students operate the system efficiently to a point where she began considering how the system could be used to help student learning. HC supplemented the system activities with her own examples and explanations, reasoning that this would be easier to achieve in the classroom where she could establish a stronger presence at the front of the room.

HB and HC changed their use of the online learning system much more than HA. Not only did they both modify their use of the system in the computer lab, but they also grafted the system onto their preferred classroom teaching method with the intention of making it more effective. Because these two teachers took the system fully into their repertoire of teaching actions, without making significant changes to those actions, we describe them at this stage of their PTK development as *technology adopters*. By contrast, HA essentially remained a technology bystander.

Technology adaptors

HC was the only one of the three teachers who progressed beyond the technology adopter stage. HC's lessons increased in variety and creativity as she became more practised in setting up and using the data projector. Her more sophisticated practice in the second year of the study reflects the early stages of what we call a *technology adaptor* role characterised by a more student-centred approach that teaches *through* rather than *with* technology to promote students' mathematical sense-making and reasoning. HC gradually learned how to integrate the system more successfully into her teaching, as evidenced by the increased number of transitions between the online system and other activities during her lessons. HC's interview comments about how well her students responded to visual images such as Widgets show that she had changed her focus from teaching with the system to consider how the system could assist student learning. Her PTK advanced from an emphasis on the technology to one where using the system to help students learn new mathematical concepts was more prominent in her thinking. The system was becoming an integral part of her teaching rather than an add-on to the lesson.

Technology innovators

A learning system can be used in more creative ways than any we observed in this study. For example, there are many opportunities for setting individual or group work. Students can follow ideas discussed in class to varying levels of complexity, depending on their interests and ability. They can even be set to explore concepts informally before they are discussed in class. We conjecture that some technology adaptors will eventually recognise the affordances of the system and promote a greater focus on problem solving and student-centered learning. We call this role, in which teachers use technology to encourage and support students' mathematical development in novel ways to promote student-generated knowledge, inquiry and reflection, a *technology innovator*.

Implications and conclusions

Our results show that teachers can learn to use an online mathematics learning system to advantage, even after minimal professional development. We conjecture that this may be due to the structure of the system, which appeared to scaffold teachers' technology learning and support their PTK development. The changes we observed in teachers' roles—from bystanders who were subservient to the technology to taking ever greater control over how students used it—were more *evolutionary* than *revolutionary*. This allowed the teachers to gradually build up their confidence in using technology in their lessons.

Our research indicates the kinds of knowledge required by teachers who wish to use online learning systems (and other digital technologies) as tools for learning and teaching. Teachers require a basic familiarity with the structure of the learning system and its various features so they can find activities suitable for different stages in a lesson. Training in these aspects need only be minimal because once teachers become acquainted with the basic operation of the system, the most effective way for them to learn about it is to begin using it themselves.

A far more important role for professional development activities is to assist teachers develop their pedagogical skills in using the system—their PTK. Our results show that online learning systems have the potential to change the dynamics of the teacher-student relationship in the classroom and make learning more student-centred—but only if teachers learn how to use the system in ways that involve students more actively in lessons. But not even a discussion of the potential uses of an online learning system in the mathematics classroom is likely to cause teachers to use it in the most effective way possible. Professional development activities that attempt to demonstrate how technology can transform traditional classroom roles may not be successful unless teachers are already familiar with the particular tool and confident that they can operate it efficiently in the classroom. Instead, training is likely to be most effective after teachers have had time to progress from technology bystanders to technology adopters, since only then will they be in a position to become technology adaptors and innovators.

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MATHEMATICS ANXIETY: SCAFFOLDING A NEW CONSTRUCT MODEL

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Over the years there has been a lack of conceptual clarity and explicit definition of the construct mathematics anxiety. This paper describes a process of building, refining, and validating a construct model of mathematics anxiety using a Rasch Rating Scale.

Background

The purpose for the research project was the growing need to understand in better and clearer ways the construct *mathematics anxiety*. This need was identified from the meta-analysis undertaken by Ma and Kishor (1997) where they noted the confusion caused for many people from the fact that researchers rarely offered explicit definitions of their construct or borrowed instruments from other disciplines, such as psychology without adapting them specifically for use in mathematics education. A further motivation for the project was to understand the manifestation of mathematics anxiety in different situations, at different times, and for different people. Such an understanding is important for people concerned with mathematics education in all its forms.

Early work on mathematics anxiety is associated with Richardson and Suinn (1972) and their development of the *Mathematics Anxiety Rating Scale* (MARS). They also offered a straightforward definition for mathematics anxiety stating that it involved "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (p. 551). Over the years further scales have been devised, for example, the *Revised Mathematical Anxiety Rating Scale* noted by Baloglu and Kocak (2006). The apparent simplicity of the original 1972 definition is questioned as mathematics anxiety has been shown by some researchers to involve multiple dimensions (Kazelskis, 1998) whereas others have confirmed it as having a uni-dimensional structure (Beasley, Long & Natali, 2001). These uncertainties and contradictions on how mathematics anxiety is conceptualised have resulted in no commonly accepted construct model and the need to refine measures.

Experiencing mathematics anxiety

Mathematics anxiety is experienced for many people in a range of ways. Psychological indicators of mathematics anxiety include such things as feelings of tension, fear and apprehension, low self confidence, a negative mind set towards mathematics learning, feeling threatened, failing to reach potential, and a temporary reduction in working memory (Ashcraft & Kirk, 2001; D'Ailly & Bergering, 1992; Jain & Dowson, 2009; Perry, 2004; Richardson & Suinn, 1972; Zohar, 1998.) It is physiologically exhibited by sweaty palms, a feeling of nausea, difficulty with breathing, and for some people heart palpitations (Malinski, Ross, Pannells & McJunkin, 2006; Perry, 2004). Additionally, it interferes with calculating and the solving of mathematical problems in academic, private and social environments (Richardson & Suinn, 1972; Suinn, Taylor & Edwards, 1988). Within a classroom or school environment it is most often associated with children undertaking learning in mathematics, however it is not restricted to children. Some teachers have reported nervousness and lack of confidence when confronted with teaching aspects of mathematics (Malinsky, Ross, Pannells, & McJunkin, 2006). As noted earlier it is a common occurrence in work and everyday life. Often it is seen where people avoid situations and even careers that require the use of even basic mathematical skills (Hopko, 2003).

Many research projects have investigated and tried to understand the causes of mathematical anxiety with the intention of being able to improve attitudinal and cognitive aspects of mathematics learning (Jain & Dowson, 2009; Perry, 2004). The sources of mathematics anxiety are varied, interrelated, and also inconsistent in their effects, for example some studies have noted an influence of gender while others have failed to substantiate this influence (Baloglu & Kocak, 2006). Others, (Furner & Berman, 2003; Jackson & Leffingwell, 1999) have reported that age is also a factor in school student perceptions of mathematical anxiety. One such possible cause of mathematics anxiety, that of teacher behaviour, differed between elementary and high school levels. However, causes of mathematical anxiety may be broadly categorised as attributes of the children, the family, the teacher and instruction, and the nature of mathematics itself.

The construct of mathematics anxiety

While it may be possible and useful to describe the factors influencing mathematics anxiety and also how it is manifested in everyday life, this is insufficient for defining precisely a psychological construct or latent trait. A construct model is normally used to specify the crucial aspects of a latent trait. Wolfe and Smith (2007) noted the following functions of a construct model. It can describe:

- The internal structure of the construct;
- Its relationship to other constructs;
- The incumbent developmental assumptions—levels of proficiency; and
- The incumbent cognitive processing assumptions—cognitive activities and states.

In the following sections these four functions of a construct model will be used as a theoretical framework to review the literature on mathematics anxiety with a view to clarify the construct.

The internal structure of the mathematics anxiety construct

This aspect of construct definition rests on two issues, namely differentiating between mathematics anxiety and test anxiety, and the dimensionality of mathematics anxiety. The data from measures of mathematics anxiety and the measures of test anxiety are often correlated to suggest a similarity between the two anxieties. Others (Kazelskis, et. al. 2000) interpreted mathematics anxiety as a component of test anxiety. However, the extent of correlation between the data is low and often dependent on the measure used. Analysis of data from different anxiety measures provides evidence to suggest that mathematics anxiety is at times multi-dimensional (Baloglu & Zelhart, 2007) while at others it is uni-dimensional (Beasley, Long & Natali, 2001). All the multi-dimensional measures of mathematics anxiety include a 'test anxiety' dimension. There is also little evidence to suggest a single higher order mathematics anxiety construct.

Relationship of mathematics anxiety to other constructs

Measures of mathematics anxiety negatively correlate with measures of mathematical ability, particularly when this is assessed in test situations. D'Ailly and Beregering (1992) noted the small yet significant correlations of mathematics anxiety and mathematics avoidance. Jain and Dowson (2009) used a Motivated Strategies for Learning Questionnaire and noted a direct relationship for self-regulation and self-efficacy with an inverse relationship between self-efficacy and mathematics anxiety. Earlier work by Wigfield and Meece (1988) identified correlations between mathematics anxiety and measures of mathematical ability perceptions, mathematics task demands, mathematics interest, and mathematics performance. These positive and negative associations substantiate theorised similarities and differences between mathematics anxiety and related constructs. These will provide external reference points for the construct model.

Developmental assumptions

It is recognised that latent traits such as cognitive abilities have a developmental aspect. This aspect often has an assumption of a hierarchical development to acknowledge changes and growth over time. The literature on mathematics anxiety is limited in examples of a developmental feature. Prieto and Delgado (2007), however, developed such a model for mathematics anxiety with 'experiencing nausea' indicating a higher level of anxiety than 'my mind goes blank'. Any model that attempts to map the development of anxiety over time must collect data on the same qualities rather than on different qualities.

Cognitive processing assumptions

Cognitive processes and cognitive states govern psychological manifestations of a latent trait, such as mathematics anxiety. While these aspects of the construct are not directly observable, assumptions can be made in the model to note confirmatory evidence that will come forward.

The objective measurement of mathematics anxiety

The following section will explore the idea of applying Modern Measurement Theory in the form of the Rasch model with its potential to resolve some of the difficulty in explaining and operationally defining and building an objective measure of the construct mathematics anxiety.

A construct model of mathematics anxiety

In the following model the horizontal dimension shows three types of anxiety derived from the literature on mathematics anxiety—anxiety when being taught mathematics, anxiety when mathematics knowledge is being assessed, and anxiety when mathematics is required to be applied in situations beyond the classroom. The order of presentation in the model is arbitrary and it is acknowledged that there may be relations between the three. The model suggests eight potential areas of anxiety.

The vertical dimension illustrates levels of anxiety. Extreme anxiety is indicated by somatic (physical and body) factors such as heart palpitations. Cognitive (mental processes) factors indicate high anxiety; such factors here are confusion, and one's mind going blank. Low anxiety factors are generally attitudinal, shown by lack of confidence. It is also suggested that the indicators are cumulative, that is a person showing extreme anxiety will also exhibit those of low anxiety. These lower indicators could be less obvious due to the over-shadowing of the extreme indicator.

Table 1. Situational model of mathematics anxiety.

		Situational types of mathematics anxiety							
		Instruction			Assessment		Application		
Level	**Domains**	Independent work	Group work	Working in a class group	Formal - examinations and tests	Informal - quizzes and worksheets	Other subjects	Home	Work
Extreme anxiety	Somatic indicators								
	Cognitive indicators								
Low anxiety	Attitudinal indicators								

Testing the model

A small study was planned to test the assumptions present in the *Situational model of mathematics anxiety* (Table 1). Specifically, it considered three issues:

1. Can a linear scale of mathematics anxiety be constructed?
2. Are the distributions of scores for different types of mathematics anxiety different?
3. Is the theorised order of the anxiety indicators consistent with the ordering of the anxiety scores?

Research methods

Two forms of questionnaire were designed. One sought information on anxiety when working on mathematics in a class (Form A), and the other on anxiety on completing a test on mathematics (Form B). Both forms comprised the same items in the same order. The items were grouped under three sub-headings of attitudinal (6 items), cognitive (9 items), and somatic (6 items). The larger number of cognitive indicators was included to make the scale more sensitive to students with ‘average’ anxiety.

Table 2. Questionnaire items.

Item label	Item number	Item
Som 1	14	I feel uncomfortable
Som 2	18	I shake or tremble
Som 3	17	I have sweaty palms
Som 4	16	I have difficulty breathing
Som 5	19	My heart beats more quickly
Som 6	21	My mouth becomes dry
Cog 1	7	I am worried about others thinking I am stupid
Cog 2	13	I feel threatened
Cog 3	1	I am aware of previous failures
Cog 4	9	I can’t think clearly
Cog 5	15	I forget things I normally know
Cog 6	3	I am frustrated
Cog 7	2	I am not in control
Cog 8	5	I am confused
Cog 9	20	My mind goes blank
Att 1	8	I am worried about what I am expected to do
Att 2	12	I feel like running away
Att 3	10	I don’t want to be doing this
Att 4	11	I expect to have difficulty doing what is required
Att 5	4	I am not confident I can do what is required
Att 6	6	I am scared about what I have to do

A scoring model describes how qualitatively different observed responses are translated into numerical codes. The scoring model selected for this study used four response categories on a Likert-type scale using 1 for strongly agree to 4 for strongly disagree. Scores were reversed for data entry and missing data were scored 9. Data were then placed into a scaling model so that “… ordinal codes [could be] combined and mapped onto a continuum that represents measurable quantities of the target construct” (Wolfe & Smith, 2007, p.108). The Rasch Rating Scale Model and the computer program RUMM2020 were used for this purpose (Andrich, Sheridan, Lyne & Luo, 2007).

Participants in the study were 50 (27 female and 23 male) children from seven Perth metropolitan primary schools. The children were in Years 5 to 7. They completed both forms of the questionnaire.

Table 3. Sample details.

School	1	2	3	4	5	6	7	Total
Females	10	4	2	3	1	4	3	27
Males	6	2	4	2	3	4	2	23
Total	16	6	6	5	4	8	5	50

Results

Data from the 100 questionnaires were entered into RUMM2020 and a variety of analyses were undertaken. First, the use of the response categories was examined by estimating the thresholds between adjacent response categories. A threshold is a student anxiety score for which there is an equal probability of selecting either of the adjacent categories. Items Att 2 and Cog 5 elicited data with disordered thresholds and these data were removed prior to subsequent analyses.

Next, data fit was examined. When data fit the Rasch model, the observed scores for an item should be similar to the score predicted by the model for persons of similar ability. It was noted that students with low anxiety scores responded more affirmatively than expected while those with high anxiety scores responded less affirmatively than expected. The residual, the difference between the observed score for an item and the score predicted by the model, was high (3.25) for item Att 6.

A Chi Square is estimated to show the interaction between an item and the trait. A low Chi Square probability value is due to poor item trait interaction. Six items were identified with high residual (>2.5) and/or low Chi Square probability values after the Bonferroni adjustment. Data from these items were removed prior to the final analysis.

The data from the remaining 13 items fitted the Rasch Rating Scale model well so it was possible to compare student mathematics anxiety scores and item difficulty. Plotting item difficulty locations and student anxiety scores on the same scale showed students tended to be reluctant in their affirmation of the anxiety indicators. A Principal Components Analysis of residuals after the Rasch measure was extracted provided evidence of multidimensionality.

Data for test anxiety and for classroom anxiety were compared with mean scores and standard deviations were similar, suggesting minimal sample-wide differences between student reports of mathematics anxiety in the two situations. However, when the pairing of scores for individual students was compared, students reporting low test anxiety generally reported comparatively high classroom anxiety. The converse was also apparent. Anxiety experiences in relation to the test and classroom contexts, it is suggested, depend on the individual student.

The difficulty the students experienced in affirming the respective items were estimated as item difficulty logits. From the three categories of indicators it was noted that one group were not more difficult to affirm than the others. It was also noted from analysis of the data on the separate forms that the item location sequences were similar.

Discussion

A model that builds on the original model (Table 1) developed from the literature search and pays attention to findings from the application of Modern Measurement Theory in the pilot study is illustrated in Table 4 (over).

Data from both forms of the instrument (test and classroom) fitted the Rasch model to provide evidence of a trait that is manifest in both situations. They are likely to be manifestations of the same construct and can be defined by the same indicators. In the original construct model there was an assumption of a cumulative relationship between the general indicators, that is some were seen as more difficult than others. This, it is suggested, is not the case and that high anxiety, as well as low anxiety, is indicated by a combination of attitudinal, cognitive, and somatic indicators.

Table 4. Model of mathematics anxiety.

Dominant trait model of mathematics anxiety				
Level of anxiety	Indicators	Attitudinal	Cognitive	Somatic
High anxiety		Scared about what s/he has to do	Worried about others thinking s/he is stupid	Having difficulty breathing
Moderate anxiety		Not wanting to be doing what has to be done	Mind going blank	Heart beats more quickly
Low anxiety		Expecting to have difficulty doing what is required	Being confused	Feeling uncomfortable
Potential applications	*In-class instruction:* Independent work, group work, or whole class *In-class assessment:* Formal exam or tests, informal quizzes *Out-of-class applications:* Other subjects, at home, at work or socially			

The model also acknowledges that mathematics anxiety can arise in any situation in which mathematical skills and knowledge are required. The indicators of mathematical anxiety are common to all situations and the relative ‘severity’ of the indicators is also assumed not to vary across situations. Implicit in this model is the notion that this construct will vary in degree between individuals in different situations.

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INVESTIGATING CHILDREN'S UNDERSTANDING OF THE MEASUREMENT OF MASS

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In this paper, we discuss the use of a framework of "growth points" in early mathematics learning and a related, task-based, one-to-one interview in assessing children's understanding of the measurement of mass. Data are presented from a sample of 1806 children in the first three years of school. An example of a child's responses is given to illustrate the kinds of thinking revealed by interviewing young children about their developing concepts of mass.

Background

The data discussed are from the Early Numeracy Research Project[1] (ENRP), where teachers and university researchers were seeking to find the most effective approaches to the teaching of mathematics in the first three years of school. At the beginning of the project, the research team identified the need for development of a comprehensive and appropriate learning and assessment framework for early mathematics, and a tool for assessing young children's mathematical thinking. The inappropriateness of pen and paper assessment at these grade levels (Clements & Ellerton, 1995) led to the development of a task-based, one-to-one interview schedule. The project team studied available research on the development of young children's mathematics learning in the mathematical domains of Counting, Place value, Addition and subtraction, and Multiplication and division (in Number), Time, Length, and Mass (in Measurement), and Properties of shape and Visualisation and orientation (in Geometry). In this paper, the focus is on the Measurement domain of Mass.

While much has been published about children's concept development in the measurement of Length (e.g., Clements & Sarama, 2009; Lehrer, Jenkins, & Osana, 1998) and Area (e.g., Outhred & Mitchelmore, 1992), little is published about Mass. However, research has provided some insights. For example, in researching the transitive nature of young children's ordinal ability, Brainerd (1974), found that 5 year-olds could arrange three balls of clay according to their mass and could arrange sticks according to their length.

[1] The Early Numeracy Research Project was supported by grants from the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria. We are grateful to our co-researchers in ENRP trial for insights that are reflected in this paper.

Brown, Blondel, Simon, and Black (1995) interviewed 48 Grade 2, 4, 6, and 8 children on four occasions on their understandings of Length and Weight measurement. Their work focused on what they termed underlying general concepts of measurement. Results suggested that “some aspects of competence seemed to progress more smoothly by age than did others” (p. 167) and showed variation in individual performances. The researchers believed that their data supported “a common model of progression in the form of a curriculum and assessment framework” (p. 168). They acknowledged that the results were indicative and tentative and their contention that the data led to a framework was seen as bold and exaggerated (van den Berg, 1995), owing in part to the small sample and idiosyncratic responses by the children.

Spinillo and Batista (2009) conducted a study with 40 children focused on 6 and 8 year olds’ understandings of measurement, and found that children of both ages had an understanding of the relationship between the size of a unit and the number of units needed to measure an object, including for measurement of Mass. They found also that, while Distance and Volume were difficult for children to understand in terms of the relation between units of measure and objects being measured, Mass did not cause such problems. The researchers posited that this outcome was linked to children’s experiences of weighing objects at home from an early age.

The general paucity of research on Mass is reflected in a recent publication on learning and teaching early mathematics (Clements & Sarama, 2009) where, in 325 pages, neither the word “mass” nor the word “weight” appear. Likewise, in the National Council of Teachers of Mathematics yearbook devoted to the learning and teaching of measurement, the place of Mass and Weight was clearly that of “other measurement domains” (Clarke, Cheeseman, McDonough, & Clarke, 2003, p. 75). Reference is sometimes made to Mass when giving an example of a measurement goal (e.g., Clements, Sarama, Spitler, Lange, & Wolfe, 2011) or describing a measurement investigation (e.g., Lehrer, Jaslow & Curtis, 2003), but with no further discussion of specifics related to the concept.

With the limited research on children’s understandings of Mass, the research reported in this paper makes an important contribution to our understanding of this element of measurement. The framework developed for the attribute of Mass followed the same generic form used for each of the measurement domains (see Figure 1).

1. Awareness of the attribute and use of descriptive language

The child shows awareness of the attribute and its descriptive language.

2. Comparing, ordering, and matching with the attribute

The child compares, orders, and matches objects by the attribute.

3. Quantifying accurately, using units and attending to measurement principles

The child uses uniform units appropriately, assigning number and unit to the measure.

4. Choosing and using formal units for estimating and measuring, with accuracy

The child chooses and uses formal units for estimating and measuring, with accuracy.

5. Applying knowledge, skills and concepts

The child can solve a range of problems involving key concepts and skills.

Figure 1. ENRP Generic growth points for measurement.

The purposes of developing the framework for the learning of Mass as it applies to this paper included: to allow the description of the mathematical knowledge and understanding of individuals and groups; to provide a basis for task construction for interviews, and the recording and coding process that would follow; and to allow the identification and description of students' thinking.

Methodology: The interview

Assessment tasks were created to match the framework. The interview was very "hands-on", with considerable use of manipulative materials. Although the full text of the interview involved around 60 tasks in the various mathematical domains listed earlier, no child moved through all of these. The interview was of the form "choose your own adventure", in that given a child's success with the task, the interviewer continued with the next task in the given mathematical domain as far as the child can go with success; but given difficulty with the task, the interviewer abandons that section of the interview. The interview provided information about the growth points achieved by a child in each of the nine domains. It is important to stress that the growth points are "big mathematical concepts and skills", with many possible "interim" growth points between them. As a result, a child may have learned several important ideas or skills *necessary* for moving to the next growth point, but perhaps not of themselves *sufficient* to move there (Clarke et al., 2002; Sullivan et al, 2000).

Of course, decisions on assigning particular growth points to children are based on a *single* interview on a *single* day, and a teacher's knowledge of a child's learning is informed by a wider range of information, including observations during everyday interactions in classrooms (Clarke, 2001).

Interview tasks for Mass measurement

In each case, the instructions to the teacher are given in italics. The equipment needed for the interview questions is listed. The growth point(s) that the interview task addresses has been detailed before each task.

Equipment: tub of at least 20 teddies, 20 gram weight (2 x 20c pieces stuck together with masking tape), a collection of seven objects (a piece of foam, a rock, two plastic containers [short & fat and long & thin], a ball of string, a 1 kg mass or an object which weighs 1 kg [labelled 1 kg], and a tin of tomatoes in a shoe box), a set of balance scales, small film canister filled with water, at least eight ten-gram weights, a set of Salters' Slimmers kitchen scales, 120 g object, 1 kg of brown rice, small scoop.

The first interview task, *What Do You Notice?* was designed to investigate whether a child has an awareness of the attribute of mass, some of the descriptive language associated with weighing objects (growth point 1), and is able to compare masses by hefting and using the balance (growth point 2).

What do you notice?

Please take these things out of the box, and put them on the table.

a) What do you notice about them?

b) Which things are heavy and which things are light?

Push all items aside, except for the two yoghurt containers.
c) Take these two plastic containers *(place one plastic container in each hand for the child to feel).*
Which do you think is heavier?
d) How could you check?
e) Do you know about balances? *(allow some time for the child to become familiar with the balance)*
Use the balance to see which container is heavier.
f) Were you right? How did you know?

The second interview task, *Teddies and Coins* was designed to investigate whether a child could quantify mass accurately, using uniform units appropriately, assigning number and unit to the measure (growth point 3).

Teddies and coins

Place the balance and the tub of teddies in front of the child.
Show the two 20 cent coins wrapped together, and place in the child's hand.
How many teddies weigh the same as this?
(If the child estimates without using the balance, ask "Please use the balance to find out how many teddies weigh the same as this")
What did you find out?

The third interview task, *One Kilogram* was designed to investigate whether a child could use formal units for estimating (growth point 4).

One kilogram

Here is a 1 kilogram weight. I am going to put it in your hand. *(Please do so)*. Here is a tin of tomatoes for your other hand. *(Place the object in the child's other hand.)*
a) Do you think the tin of tomatoes is more than 1 kilogram or less than 1 kilogram weight?
b) Can you check? … What did you find?

The fourth interview task, *Using Standard Units* was designed to investigate whether a child could choose and use formal units for estimating and measuring, with accuracy (growth point 4).

Using standard units

Here is a container. Here are some 10 gram weights. Measure the weight of this container with these 10 g weights.
What did you find? *(To be judged as correct answer including units, the child must say "40 grams" as part of their response. If they say "4" ask "four what?", but even "four 10 gram weights" is not sufficient. We are looking for "40 grams".)*

The final mass interview task, *Using Kitchen Scales* was designed to investigate whether a child could apply their formal knowledge and skills of measurement of mass in context (growth point 5).

Using kitchen scales

Place the kitchen scales and the 120 g object on the table.
Have you seen scales like these before?
a) Please use the scales to weigh this object. What did you find?
If the child gives a number only (without units), ask, e.g., "120 what?"
b) Please use the scales and the scoop to measure out 135 grams of rice.
c) How do you know it is 135 grams?
d) How many more grams of rice would you need to have one kilogram? (865 g)

An example of a child's responses will be used here to illustrate the kinds of thinking revealed by interviewing young children about their developing concepts of mass.

The story of Jack

Jack was interviewed at the beginning of Grade 2. He hefted the plastic containers and could judge which was heavier, and appropriately used the terms heavier and lighter. However, he struggled to think of a way to check his estimate. When given a balance scale he showed interest and, although he said he had never used one, he promptly put a container in each pan and was convinced that his original estimate was correct, that is, that the shorter squat container weighed more. He appeared to interpret the balance tipping to the heavier side correctly. It could be said that Jack had an awareness of the attribute of mass, some of the descriptive language associated with weighing objects (growth point 1), and was able to compare masses by hefting and using the balance (growth point 2). The interviewer continued with the *Teddies and Coins task*. It was soon apparent that Jack was simply adding teddies to one pan of the scales and he did not have the concept of creating equal masses on the balance and using informal units. The Mass interview was concluded there and Jack was considered to have demonstrated growth point 2 in Mass.

Having described the framework and development of the interview protocol, and given an illustrative example of one child's responses, we will now examine the results of an entire cohort of children.

Results

In the domain of Mass, children were interviewed individually by teachers and proceeded through the interview as long as they continued to have success with tasks. Each child's response was recorded on a record sheet for later examination and analysis. Codes were assigned to the responses to reflect the growth point demonstrated by the child on that particular task.

Indicators of growth in Mass

To examine the way the growth points portray the nature of the increasing sophistication of the students' strategies, Table 1 presents a profile of students' achievement over three grade levels.

A random process for choosing students for whom to ask Mass interview questions was provided by the research team and used to provide a "snapshot" of the children's responses to the interview tasks. The data in Table 1 are from a single year of the project, using data from the start and end of the first year of formal schooling (called Prep in Victoria), and the end of Grades 1 and 2.

Table 1. Percentage of students achieving mass growth points over time.

	Prep Mar 2001 (*n* = 533)	Prep Nov 2001 (*n* = 538)	Grade 1 Nov 2001 (*n* = 479)	Grade 2 Nov 2001 (*n* = 256)
Not apparent	17	3	1	0
Awareness of attribute	15	7	2	0
Comparing masses	47	30	17	6
Quantifying masses	21	60	69	50
Using standard units	0	0	10	38
Applying	0	0	1	6

By the end of the Prep year, most students were able to compare masses, and three-fifths were able to use an informal unit to quantify a mass. By the end of Grade 1, virtually all students were able to compare masses, and 69% were able to quantify masses and were ready to move towards using standard units. By the end of Grade 2, over 40% were using standard units successfully, and the rest were ready to move towards that goal. No further growth points seem to be needed to describe growth at this level adequately.

It is noted that with 60% of Prep children being able to quantify masses at the end of the year, it might be expected that a greater number of Grade 1 children would be able to quantify masses or go beyond by this time. We suggest that this might have been due to insufficient experience with use of standard Mass units at the Grade 1 level. Perhaps some children were not being exposed to experiences for which they were ready. This becomes even more apparent when shown in visual form as in Figure 2.

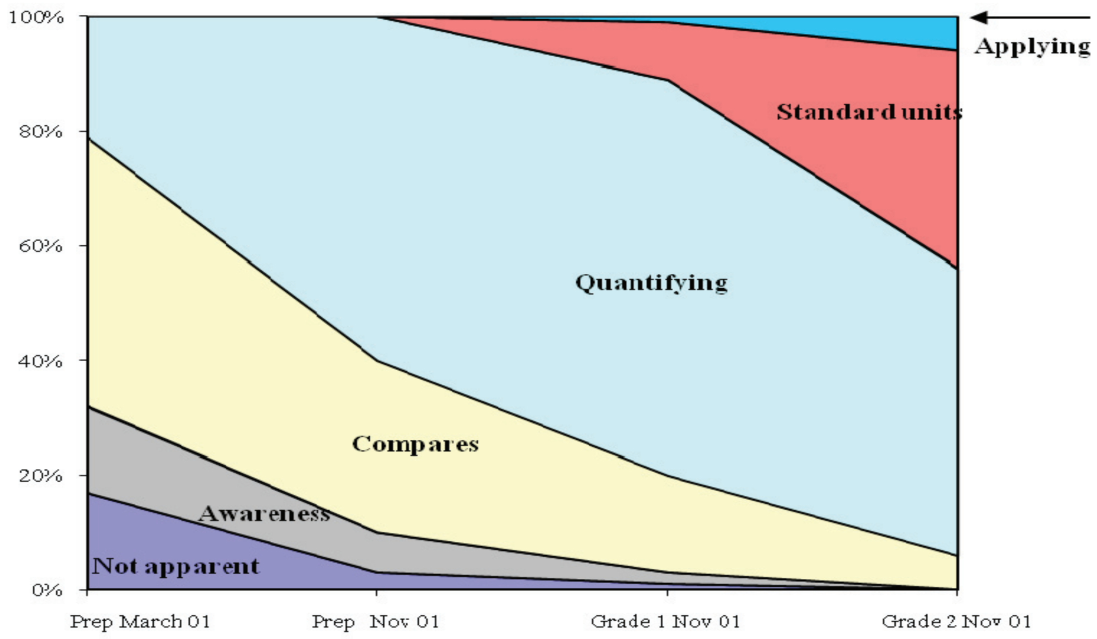

Figure 2. Students (%) achieving Mass growth points over time.

To give a sense of the progress of the students, the percentages of students at each Mass growth point over the four sets of data are shown in Figure 2. To assist with interpreting this representation of the data, it is worth moving the eye in two directions. First, by selecting a year level (Grade 1, Nov 2001), the reader can look vertically from that label, to ascertain the percentage of students achieving each growth point at that time. Second, by moving from the bottom left to the top right, we can see the relative time which students overall spend typically at a particular growth point.

As with Length (Clarke et al., 2002; McDonough & Sullivan, in press), students progress readily through the growth point, Awareness of the attribute. However, two transitions seem to take time: moving from comparing to quantifying; and moving from quantifying to using standard units. It is possible that quantifying Mass is dependent on particular experiences that are beyond the intended curriculum at this stage. The same may well be true for using standard units.

Curriculum expectations

It is interesting to compare the data reported here with the Mass outcomes and indicators in the relevant curriculum (Board of Studies, 2000) of the time. At the end of Prep, the outcomes referred to the attribute of mass, and estimating, measuring and comparing using informal methods. At the end of Grade 2, the outcomes referred to choosing an appropriate attribute, using everyday language, making comparisons, using informal units to estimate, comparing and ordering masses of objects, and measuring by comparing to formal and standard units.

Lately there has been a move towards a national curriculum which is written in broader terms. There are explicit curriculum statements about Mass in the Measurement and Geometry strand of the *Australian Curriculum* (Australian Curriculum Assessment and Reporting Authority, 2010) at Grades 2 to 6 (see Fig. 3).

Grade 2 level	Measure and compare length and capacity using uniform informal and familiar metric units and measure mass using balance scales with familiar metric units (p. 9)
Grade 3 level	Use direct and indirect comparison to order and compare objects by length and develop 'real life' benchmarks for familiar metric units of length, mass, and capacity including centimetre, metre, kilogram and litre (p.10).
Grade 4 level	Use metric units to estimate, measure, and compare the length, mass and capacity of familiar objects reading scales to the nearest graduation (p. 11).
Grade 5 level	Read and interpret scales using whole numbers of metric units for length, capacity, mass, and temperature (p.12).
Grade 6 level	Work fluently with the metric system to convert between metric units of length, capacity and mass, using whole numbers and commonly used decimals (p. 15)

Figure 3. National Curriculum statements concerning measurement of mass.

Clearly there are assumptions about prior learning and mathematical experiences underlying these statements. We hope that the reporting of the "snapshot" of young children's developing thinking about the measurement of Mass will serve to support teachers and mathematics educators as they consider what these prior learning opportunities might comprise.

Identifying targets for teaching Mass

Based on the data reported in this paper, teachers of children in the first year of formal schooling can reasonably aim that nearly all students be able to compare the mass of two objects with use of appropriate language (90%), and begin to move towards quantifying masses by the end of the school year.

Teachers of Grade 1 children could emphasise activities that move the thinking of all students toward the use of informal units to quantify masses, noting that four fifths are either at or moving towards using standard units.

Teachers of Grade 2 children could emphasise activities that stimulate and interest children in using standard units, as 44% were able to use standard units of kilograms and grams successfully. As in other domains, it seems that appropriately chosen activities and experiences can assist students in their development.

In conclusion

In telling the story of Jack, it was noted that his correct use of the balance beam for comparing masses, assuming his statement that he had not used such an instrument previously was correct, may have been learnt during the interview. This finding concurs with that of Brown et al. (1995) who found that

> It was apparent during the interviews themselves that the requirement for pupils to tackle practical problems that they had probably not met before was stimulating learning, since there were many cases where pupils refined their strategies as a result of being asked to explain that they were doing. (p. 165)

These findings point to the value of children having hands on experiences with Mass measurement situations. Indeed, in relation to measurement generally, Cross, Woods and Schweingruber (2009) wrote:

> Even preschoolers can be guided to learn important concepts if provided appropriate measurement experiences. They naturally encounter and discuss quantities (Seo and Ginsburg, 1994). They initially learn to use the words that represent quantity or magnitude of a certain attribute. Then they compare two objects directly and recognize equality or inequality (Boulton-Lewis, Wilss, and Mutch, 1996). At age 4-5, most children can learn to overcome perceptual cues and make progress in reasoning about and measuring quantities. They are ready to learn to measure, connecting number to the quantity, yet the average child in the United States, with limited measurement experience, exhibits limited understanding of measurement until the end of primary grades. (p.197)

From the Mass data reported in this paper, we argue that rich experiences involving measuring Mass are needed, particularly at the Grade 1 level where little progress appears to have been made. The Mass data from the Early Numeracy Research Project also suggest the importance of teachers assessing children’s understandings of Mass measurement and structuring learning opportunities to build on and extend those understandings.

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TEACHERS' STRATEGIES FOR DEMONSTRATING FRACTION AND PERCENTAGE EQUIVALENCE

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Understanding the relationships among fractions, decimals, and percentages is a critical goal of the middle years of schooling. There are many approaches that teachers might take to help students develop this understanding; some capture general principles whereas others only illustrate specific equivalences. In this study teachers were asked to suggest three ways of convincing students that three-eighths is the same as 37.5%. The data reveal a wide range of strategies and show that different approaches may exemplify different features of the fraction-percentage relationship. The explanatory power of the examples is also considered.

Background

The development of thinking about fractional quantities is one of the notorious mountains in school mathematics (e.g., Behr, Harel, Post, & Lesh, 1992; Litwiller & Bright, 2002). Even before worrying about computation with fractional quantities, students need to understand the meaning of these quantities as numbers (Kilpatrick, Swafford, & Findell, p. 235). It is well known that students have difficulty identifying the whole, coordinating the values of the numerator and denominator, and being able to treat the fraction as a single number that can then be related to other quantities. The work of Clarke and Roche (2009) highlighted the difficulties that students have with fraction comparisons, associated with the problems identified above, and pointed out the effectiveness of having knowledge of benchmark fractions such as 1/2 as a point of reference.

Added to these challenges is the fact that fractional quantities can be represented in at least four distinctive ways: as rational numbers in fraction form, as ratios, as decimals, and as percents. Although the last two are closely linked—to the extent that the digits are identical in their representations—it is not always easy for students to appreciate what the % symbol signifies, and that what appears to be a whole number (or, at least, a number greater than 1), is actually a fractional part of 1. What is more, there are many different concrete illustrations of these representations. Students may encounter fractions being illustrated using area models, fraction strips (or fraction walls), number lines, and set models, with each model perhaps highlighting certain aspects of fractions. Sowder (1988) observed that many children are "model poor", having only a circular model to represent fractional quantities. This limited representation restricts access to some of the relationships that are important to establish, not least of which is the connection to decimals and percents. The idea of *epistemic fidelity* is useful here (see, e.g., Stacey, Helme, Archer, & Condon, 2001), since it highlights the need for models and representations to accurately capture the mathematical features of the concept they are trying to represent.

Knowing about and using suitable representations in teaching has long been recognised as a significant component of pedagogical content knowledge (PCK). In his seminal paper, Shulman (1986) wrote:

> [Pedagogical content knowledge includes knowledge of] the most useful form of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. … The teacher must have at hand a veritable armamentarium of alternative forms of representation … . (p. 9)

Thompson and Thompson (1996) also highlighted that teachers need conceptual schemes that incorporate a clear picture of the materials, activities, and explanations that will facilitate the development of mathematical understanding in students. Other researchers (e.g., Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Ma, 1999) have pointed to the significance of teachers being able to make connections among and within topics in order to improve students' learning outcomes.

While brief, the discussion above highlights that fraction teaching has the potential to incorporate a wide range of models or representations, although Sowder (1988) suggests that students use few of them. Through these representations it may be possible to help students develop connected understanding of fractions, decimals, and percents. As Askew et al. (1997) suggest, however, this will depend on the way teachers use the representation or model, and what connections they can make explicit with it. The purpose of the present study, therefore, was to examine what representations teachers draw on when talking about fractions and percentages, and what strategies they have for helping students to develop appropriate connections.

Method

Participants

Four cohorts of teachers—pre-service and practising—provided data for this study. These cohorts are described below, but when referring to all the participants as a single group the term "teachers" will be used, irrespective of whether or not they were employed in that role. Note that the data from the pre-service teachers were initially analysed in Chick (2003), but have been reanalysed for this paper.

Pre-service teachers

DipEd cohort

Participants from the DipEd cohort (N=16) were preparing to become secondary mathematics teachers. They had studied tertiary-level mathematics to at least sub-major level, prior to undertaking a one-year Diploma of Education. The mathematics method unit that they were doing at the time of the study focused on the teaching of secondary level mathematics topics.

BEd cohort

Those in the BEd cohort (N=21) were in the final year of a four-year Bachelor of Education degree preparing to become primary teachers, and prior to this had completed mathematics to at least Year 11. The mathematics units in the BEd program covered elementary mathematics content and pedagogy simultaneously, and at the time of the study the cohort had completed nearly six semesters of such units. The content was mainly concerned with primary school level mathematics, and had included fractions and decimals. The timing of the study, and the fact that participation was voluntary, makes it likely that these pre-service teachers were the more mathematically confident of all the BEd students that year, and so the results for the BEd cohort may be inflated.

Practising teachers

Secondary cohort

Participants from the secondary cohort (N=40) came from three Victorian metropolitan government secondary schools. They comprised all those members of staff in the schools involved in teaching one or more classes of mathematics. As a result the secondary cohort included specialist mathematics teachers, mathematics and science teachers, and some teachers teaching outside their area of formal qualification. Their teaching experience ranged from being in their first year through to more than 20 years in the classroom.

Primary cohort

The practising primary teachers (N=15) were Grade 5 or 6 teachers from a variety of government, Catholic, and independent schools around Victoria. Their paths to their teacher qualification were diverse: some had a four-year education degree, others had a degree in another discipline followed by a one-year education diploma. The range of the number of years of teaching was as diverse as the secondary cohort. These teachers were volunteer participants in a study on PCK in mathematics, and may have a greater degree of mathematics teaching confidence than primary school teachers in general.

The task

All cohorts responded to a written questionnaire addressing a wide range of topics associated with pedagogical content knowledge for mathematics. The questionnaires varied for the different cohorts, but the following item was common to all of them: “Write down three ways of convincing someone that 3/8 is the same as 37.5%”. This item is the focus of the present study. Space was provided on the questionnaire for three written suggestions. The practising teachers were also interviewed about their responses to the questionnaire items, but for the purposes of this paper only the written responses were analysed. There was one exception, a teacher who provided interview data only.

Analysis

As the data were analysed and entered into a spreadsheet, each different method used in a response was given a code to indicate the way in which the relationship between 3/8 and 37.5% was demonstrated (e.g., by doing a division algorithm with 3÷8, by using an area model, by using a dual number line, and so on). Additional annotations were made to record any variations from the standard response types. A total of 32 different methods were observed, although some of these were similar and so the data could, perhaps, have been condensed into fewer categories. In the case where one of the methods suggested by a teacher was essentially equivalent to one of his/her earlier suggestions this was noted as a "repeat" in order to identify distinct strategies.

In addition to having the nature of the method recorded, each method was rated "good", "okay", or "incorrect/inadequate", depending on its adequacy as a "way of showing someone that 3/8 is the same as 37.5%". Although it could be argued that applying the algorithm 3/8 x 100/1 is not a convincing demonstration of equivalence, it is a standard approach to determining the relationship between a fraction and its corresponding percentage value, and so was rated as "good". Indeed, it is the only strategy that is readily applicable in some circumstances, such as with awkward fractions. "Use a calculator" was also rated as "good", on the grounds that, provided the operations chosen are deemed appropriate, the calculator has a sort of computational authority. Other explanations or demonstrations that were clear and had the power to convince were also rated "good". Methods that rated "okay" were those that were partially correct/convincing, incomplete, or difficult to implement. Examples include asserting 3/8 = 0.375 without justification, trying to show 37.5% by dividing a "pie chart" into 100 wedges, or writing "turn 37.5% into a fraction" without discussing how. Finally, the rating of "incorrect/inadequate" was given to responses which were not clear or which were erroneous or lacking key details (such as "guess and estimate").

The data for the present study were initially coded and rated by the second author, and then checked by the first author.

Results and discussion

The results are presented in two parts. First, an overall picture of the number and quality of the suggestions made by the teachers will be provided. This will give an indication of the number and appropriateness of the representations and explanations that teachers had at their disposal for dealing with fractions and percentages. The second subsection reports more specifically on the different kinds of techniques that were proposed, and examines the "explanatory power" of some of the methods.

Number and quality of responses

The number and quality of the responses, for each of the four cohorts, is shown in Table 1, which gives a detailed breakdown of the distribution of "good", "okay" and "incorrect/inadequate" responses. Overall, 82% of the teachers (DipEd 81%, BEd 71%, Secondary 83%, Primary 93%) were able to come up with at least one "good" response, although this may simply have been to apply the fraction-to-decimal computation or to "use a calculator". In contrast, only 38% were able to come up with three distinct "good" or "okay" methods (DipEd 44%, BEd 14%, Secondary 50%, Primary 40%), and fewer than 20% could provide three distinct "good" methods (see line 1 of Table 1). All of the practising teachers were able to suggest at least one "okay" method or better.

Looking at the cohorts together, nearly one-third could not provide what the participant adjudged to be three suitable methods (regardless of whether they were rated by the researchers as suitable or not, or a repeat). The pre-service teachers, in particular, struggled in this area, with over half of them failing to find three methods, and a handful failing to propose any methods. This suggests that experience and professional development *do* provide opportunities for growth in expertise. Having said this, however, it is of concern that a quarter of all the teachers—with this proportion applying to the practising teachers as well—made at least one suggestion that was wrong or seriously inadequate.

Table 1. Percentage of teachers and the number of appropriate methods proposed.

Number of methods provided and their value*	DipEd (N=16)	BEd (N=21)	Secondary (N=40)	Primary (N=15)
Provided 3 or more distinct "good" methods	25%	14%	18%	20%
Provided 2 "good" and 1 "okay" distinct methods	13%	0%	28%	7%
Provided 2 "good" and 0 "okay" distinct methods	13%	14%	20%	20%
Provided 1 "good" and 2 or more "okay" distinct methods	6%	0%	3%	13%
Provided 1 "good" and 1 "okay" distinct methods	13%	33%	10%	33%
Provided 1 "good" and 0 "okay" distinct methods	13%	10%	5%	0%
Provided 0 "good" and at least one "okay" distinct methods	6%	10%	18%	7%
Provided no "good" or "okay" suggestions	13%	19%	0%	0%
Unable to get 3 methods (regardless of correctness or repetition)	44%	57%	18%	27%
Did not provide any suggestions	6%	14%	0%	0%
Provided at least one "incorrect/inadequate" suggestion	13%	38%	25%	27%
Provided 4 "good" or "okay" distinct suggestions	0%	0%	8%	0%
Provided a 4th suggestion (may not have been "okay", and nor may the earlier ones have been)	6%	0%	8%	7%
Number of repeated/equivalent suggestions	6%	5%	13%	20%

Methods for showing the equivalence of 3/8 and 37.5%

In all, the 92 teachers provided 236 methods that they felt were appropriate for showing the equivalence of 3/8 and 37.5%. As mentioned earlier, 32 codes were used to identify the different methods or strategies, but there were some commonalities that allow the methods to be grouped loosely. These categories are described below, and their distributions are indicated in Table 2. In some cases the strategies suggested by the teachers were not described completely with necessary connections made explicit (so that a reader could not be certain that the explanation would be implemented successfully), but if the underlying principle was evident it was grouped into the appropriate category even if it had been rated as "incorrect/inadequate".

Computational approaches

As can be seen in Table 2, the most common approaches were computational (42% of all suggestions), with the prevalent strategy among these to compute 3/8 x 100/1 (suggested in 17% of the responses overall), which the teachers usually did by hand. This approach, like most of those placed in this category, has little explanatory power: the person to be convinced about the relationship has to accept that the computation does, indeed, convert a fraction to a percentage. Other strategies included in this category were using a calculator (proposed in 11% of the suggestions, usually without explaining what operations were necessary), applying a division algorithm to 3÷8 or 300÷8, and converting both 3/8 and 37.5% to decimals. To be placed in this category there had to be a sense of the formulaic application of an algorithm or calculating without deep attention to relationships. This is not meant to devalue the approach, but to highlight what may or may not be conveyed by it. As noted earlier, there are some fraction to percentage conversions that will *only* be possible by such a method, since some of the strategies in the remaining categories below will not work so readily for things like "convert 7/9 to a percentage".

Numerical relationships

Many of the explanations took advantage of the numerical relationships among the quantities, and used these relationships to establish the result. One of the most common of these approaches was to establish a sequence of fractions equivalent to 3/8, from which 37.5% could be obtained (for example, 3/8 = 75/200 = 37.5/100). An alternative was to start from 37.5% and establish the result via 37.5/100 = 375/1000 and then cancel common factors. About 11% of all responses used one or other of these strategies. Still others wrote that "3/8 = something/100" and then used algebra or equivalent fractions to establish the value of "something". The numerical relationships in all of the above methods were, in general, readily established by mental computation. As was the case for the algorithmic/computational approaches, in most cases there were implicit assumptions about the meaning of percentage: the equivalence of 37.5% and 37.5/100 was assumed without explanation. The other family of responses grouped with this category of "Numerical relationships" used benchmark values to establish equivalences, such as working from 50%=1/2 (assumed to be well-known) to establish 25%=1/4 and 12.5%=1/8, and thus 3/8=37.5%. About 5% of all the responses used this benchmark approach.

Diagrammatic representations

About 18% of the responses proposed some diagrammatic or visual representation to establish the relationship. Most of these involved area models but of the 14% of responses that attempted such a representation fewer than half were convincing. Most of the successful ones established eighths in a region (often a circle), and obtained the relationship of 1/8=12.5% from the assumed to be well-known relationship between 50% and 1/2 (see also the discussion about numerical relationships above). One unusual successful example involved a 10x10 grid, in which every eight squares were identified and three of these coloured in. The teacher's description successfully dealt with the four squares remaining at the end. The more problematic examples included (a) attempting to represent 37.5% in a "pie cut into 100" without considering whether this could be done in practice, let alone showing how this is actually the same as 3/8 of the same circle, (b) showing 3/8 of a circle and asserting its equivalence with a square 10x10 grid shaded to show 37.5%, and (b) suggesting "cutting cake" with no further detail. Other diagrammatic approaches used the circumference of a circle rather than the area (not done successfully), or used number lines, with only two teachers proposing an appropriate dual number line. One particularly nice representation used a 1 m measuring tape and folded it into eighths, and then measured the length of 3/8.

Asserted results

In a number of responses (7% overall) the teachers had an appropriate explanation or sequence of computational steps, with the exception of an unexplained jump from 3/8 to 0.375 (or, as was done by some of the teachers, from 1/8 to 0.125). Responses were put in this category if this equality was asserted without explanation. It may be that the teachers envisaged demonstrating the equality on a calculator or by some other means; or it may have been a known fact for them but they may not have realised that they were also assuming that it was known to the recipient of their explanation. Establishing this equality seems to be part of the requirement of the explanation, along with developing the more general connection between a fraction and its associated percentage.

Use of meanings

A small number of responses (3%) made explicit use of the meaning of division in trying to establish the equivalence of 3/8 and 37.5%. For example, two teachers suggested taking 100 objects, sharing them among 8 groups, and seeing how many objects are in 3 groups. The other interesting approach interpreted both the fraction and the percentage as operators and suggested calculating both 3/8 of some number and 0.375 of the same number.

Unclear or tautology

This category was reserved for those strategies in which it was not clear what the teacher intended to do or how, or where the teacher made a tautological assertion. As examples of the former, teachers wrote "Estimation" (secondary), "Compare with a fraction like 4/8 = 1/2" (secondary), "Measuring volume of water" (primary), "Make them divide to two decimal points [sic]" (DipEd), and "Using a protractor" (BEd). As examples of tautological assertions, one teacher wrote "30/80 = 3/8 = 37.5/100" (secondary), with no indication of how these relationships—notably the final one—were established, whereas another wrote

> If you completed a test that was out of 100 and if you received a grade [of] 37.5 which is the same as 37.5%. What if the test was out of 8 instead of 100. 37.5 out of 100 is the same as 3 out of 8. (Primary)

Three of the teachers wrote that the 3/8 = 37.5% relationship holds "because it is" or suggested, "tell them [students] to trust you because you are the teacher".

Table 2. Percentage of methods by type.

Method type	DipEd (N=36*)	BEd (N=43*)	Secondary (N=115*)	Primary (N=42*)
Computational (limited "demonstrative" power)	47%	42%	42%	41%
Uses numerical relationships	33%	12%	22%	21%
Diagrammatic representations (model or illustration)	8%	21%	19%	19%
Asserted a non-obvious result without explanation	6%	5%	9%	5%
Uses meaning of 3/8 or 37.5	0%	5%	3%	5%
Unclear or "it is"	6%	16%	6%	10%

* Here N is the total number of methods proposed by the cohort.

Conclusions

Examining the results across the cohorts, it appears that the pre-service BEd students did not have as many successful strategies at their disposal as their practising and secondary-oriented counterparts. Nevertheless, they suggested appropriate diagrammatic strategies in the same proportion as the practising teachers, and better than the DipEd cohort, perhaps because the latter cohort had reliable personal computational skills at the same time as having had limited opportunities to develop or learn other strategies for assisting students.

Although a wide range of strategies was presented, the prevalence of routine computation was striking. The frequent use of circle models for area reinforces Sowder's 1988 finding of their abundant use, and so perhaps we now know why Sowder's students used circle models almost exclusively: they learned from their teachers.

Finally, some further thought needs to be given to what each of the different methods make transparent and what is obscured. The computational approach works for every possible fraction yet it appears to hide the fundamental relationship between a fraction and its decimal value in a computation that it is possible to conduct—and teach about—almost mindlessly. On the other hand, the use of "nice" relationships and convenient benchmarks to determine the equivalence of 3/8 and 37.5% would only be generalisable to a few special cases. At the same time, however, such methods build facility with mental computation, allow work with benchmarks, and are perhaps better able to indicate the underlying connections between fractions and percentages. Furthermore, these "special" cases may be useful for motivating and justifying the more algorithmic approaches, and for highlighting why they are necessary. The significance of the teaching opportunities that become available because of access to multiple representations cannot be understated.

There are clear implications for teacher preparation and professional learning. Not only is it important that teachers have access to multiple representations that give students alternative models for concepts, but teachers need to realise that some models are better than others for highlighting particular mathematical features. What one model makes obvious may be obscured or difficult to see in another model. Teachers also need to be aware about the generalisability of a representation and of the importance of special cases that can be used to illustrate more general connections that might be difficult to make clear using more arbitrary examples.

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A LESS PARTIAL VISION: THEORETICAL INCLUSIVITY AND CRITICAL SYNTHESIS IN MATHEMATICS CLASSROOM RESEARCH

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This paper addresses three key distinctions central to educational research: (i) the distinction between convergent and critical synthesis; (ii) between data source and data; and (iii) between the compatibility of different theories and the compatibility of the interpretive accounts generated by different theories applied to a common data source. Our capacity to learn from research is precisely our capacity to synthesise the findings of research to inform our actions in particular educational situations. The realisation of such a goal requires a reconceptualisation of research synthesis as partial, purposeful, situated and critical. Mathematics classroom research provides the context for this discussion.

Reconceptualising synthesis

By what processes does the education community integrate research into its communal knowing? That is: How do we learn from research? The process of research synthesis is fundamentally concerned with the challenge of engaging purposefully, strategically and selectively with the body of existing research. Different forms of research synthesis construct communal learning in different ways (c.f. Lather, 1999).

Since a key motivation for educational research is to understand and inform educational practice, strategies are required for the synthesis of the findings of different research studies. Attempts at synthesising research findings have largely focused on a well-defined educational question or issue (for example, the instructional value and optimised use of different types of student group work) disconnected from other aspects of educational settings, and debate has focused on the methodological criteria by which contributory findings might be selected for inclusion in the synthesis process and on the techniques by which the synthesis is conducted. The process itself is most frequently conceived as convergent and either aggregative (meta-analysis) or integrative (best evidence synthesis). The argument being framed in this paper is that it is "critical synthesis" that is most likely to advance our understanding and our advocacy in relation to the mathematics classroom. Suri and Clarke (2009) have already advocated methodologically inclusive research synthesis and elaborated the criteria by which such synthesis might be undertaken. There is a comparable need for theoretically inclusive research synthesis.

We can distinguish quite different approaches to research synthesis; approaches that connect particular methods to distinct purposes:

Meta-analysis has been with us for some time—because of the quite stringent statistical conditions on its use, and its metric orientation, we might characterise it as aggregative synthesis. The product of a meta-analysis is a decision in relation to a stated research question, most frequently involving an explicit comparison.

Best evidence synthesis adopts a much more assimilative or consensus approach—we characterise this as integrative. The product of a best evidence synthesis is likely to be the differentiated portrayal of a situation or an issue.

Critical synthesis would exploit, rather than minimise, differences between the reports; juxtaposing features and findings to reveal elements of context and process that are foregrounded or ignored in each report—such a synthesis is interrogative. The product of such a synthesis would be a critique.

In moving from aggregation to interrogation, we move from the requirement of similarity to the exploitation of difference, from standardisation to differentiation.

Critical synthesis is less familiar and warrants elaboration. Critical synthesis can: highlight the inherently ideological nature of all research (Lather, 1986); "signify reformulated historical narratives, social meanings and problematics, interpellative obligations, analytics and assignments for educational inquiry" (Livingston, 1999, p. 15); illuminate the "ambiguities, tensions and compromises that arise among stakeholders" (Windschitl, 2002, p. 131); explicitly "identify and criticize disjunctions, incongruities and contradictions in people's life experience" (Candy, 1989, p. 7); inform policy by recognizing and fostering overlooked quality published work; reveal "the structures, powers, generative mechanisms and tendencies" within discussions of policy, practice and research in a field (Clegg, 2005, p. 421); and disrupt conventional thinking to construct spaces for new ways of talking about practice (Segall, 2001).

Researching classrooms: Data source and data

Mathematics classrooms offer a rich educational environment, providing recordable instances of language use, a variety of classroom organizational groupings, varied instructional practices (demonstration, lecture, whole class discussion, and collaborative group work), the utilization of a variety of artefacts (both physical and conceptual), the potential for ontological, epistemological, ethical and moral tensions to emerge, and, arguably, a highly diverse range of learning outcomes. It is this richness and complexity that offers the opportunity for the interrogation of current theory and that also poses the greatest methodological challenges.

Multi-camera on-site video technology and post-lesson video stimulated interviews were used in a purposefully inclusive research design to generate a complex data source amenable to parallel analyses from several complementary theoretical perspectives. Data has been generated and analysed from this data source using a variety of theories. While each analysis is demonstrably valuable in itself, in combination, these results demonstrate the inadequacy of any single theory or theoretically-driven analysis to capture the complexity of the mathematics classroom and the corresponding need for inclusive multi-theoretic research designs.

The focus of the analyses was a data source containing video records of classroom and interview interactions, supplemented by digitised student and teacher materials. A three-camera approach was employed (Teacher camera, Student camera, Whole Class camera), including onsite mixing of camera images into split-screen video records used to stimulate participant reconstructive accounts of classroom events in post-lesson student and teacher interviews (Clarke, 2006).

The multiple, synchronised recordings of classroom interaction maximize the sensitivity of the anticipated parallel analyses to a wide range of classroom actions and learning outcomes, and facilitate a form of reciprocal interrogation, where the theories are employed to generate and analyse the data and the comparison of the parallel analyses facilitates the reflexive interrogation of the theories.

The point has been made in various earlier publications, that if early researchers had access to the same tools for data collection and analysis as are available today, the general view of classroom interactions would be quite different (Clarke, Mitchell & Bowman, 2009). The most striking of these differences concerns the role of students in classrooms. Single-camera and single-microphone approaches, with a focus on the teacher, embody a view of the passive, silent student at odds with contemporary learning theory and classroom experience. Research done with technologically more sophisticated approaches has described a quite different classroom, where different students are active in different ways, exercising purposeful agency in the classroom, and contributing significantly to their own learning.

We have an obligation as researchers to accept responsibility for the constructed nature of our data—and to document the process of data generation, identifying the points at which decisions were made regarding inclusion and exclusion. This is not always easy—particularly when the acts of exclusion are made for us by the technology, the method, or a theoretical frame that attends to some aspects of the setting and ignores others.

Theory is embedded in research from the inception of any project. The choice of setting, of content domain, of participants, of targeted data types and the means by which these will be generated are all a consequence of the researchers' theoretical position (explicit or implicit). Consider video: Every decision to zoom in for a closer shot or to pull back for a wide angle view represents a purposeful act by the researchers to selectively construct a data source optimally amenable to the types of analyses anticipated and maximally aligned with the particular research questions of interest to the researchers. As the discourse of the classroom acts to position participants in ways that afford and constrain certain practices, so the discourse of educational research acts to position participants in ways that afford and constrain certain interpretations.

Parallel analyses within the Learner's Perspective Study

In the Learner's Perspective Study, the aim is to employ a variety of theoretical and analytical approaches to explore data generated from a common data source. This approach was intended to realize two very specific aims:

- Understand the setting/s: To maximize the sensitivity of the combined analyses to a wide range of classroom actions and learning outcomes, and
- Understand the theory/ies: Through the combination of theoretical perspectives, to identify what is attended to by each and what is excluded, and to consider the extent to which the interpretive accounts generated by use of the various theories are complementary, mutually informing, or, perhaps, incompatible.

Compare the following four parallel analyses undertaken as part of the Learner's Perspective Study:

Variation theory (Häggström & Emanuelsson, 2010)

This study is guided by an interest in teachers' opportunities to make distinctions in relation to what students understand in the mathematics classroom. The analysis is made of "question episodes" in grade eight mathematics classrooms from four countries in the LPS video dataset. A question episode is a section of interaction that typically starts with the teacher asking a question and ends when the topic of discussion is changed. *Variation theory* and the concept of *Responsiveness* are combined to capture two different aspects of classroom instruction—the *pattern of variation* and the *pattern of interaction*—which are analyzed and compared in relation to teachers' opportunities to learn about the students' conceptions. The findings suggest that both the character of the tasks used, as well as the ways in which teachers ask questions, and perhaps even more importantly how follow-up questions are phrased and aligned to student responses, influence how the students' knowledge becomes visible in interaction. Our analysis suggests that cultural differences between mathematics education in "the East" and "the West" are complex and far from clear-cut.

Cognitive reductionism in interactional analysis (Ohtani, 2010)

Recent research has a common and persuasive vision of classrooms as a site for discursive practice. This study investigates how Japanese linguistic conventions are performed in classrooms in ways that may privilege certain participation structures in classroom practice. Japanese value implicit communication, requiring speaker and listener to supply the context without explicit utterances and cues. In Japanese discourse, agency or action are often hidden and left ambiguous. This tendency is typically found in leaving sentences unfinished. Such culturally specific linguistic traits are different from English. In English, when introducing a definition, the teacher might employ a do-verb: "We define". In Japanese classrooms, the teacher often introduces a definition in the intransitive sense as if it is beyond one's concern. Such differences in the location of agency, embedded in language use, constitute a different participation structure in classroom practice. Analytical tools of ethnomethodological conversational turn allocation in the classroom may need to be reexamined in light of Japanese culturally-grounded linguistic traits. The reflexive relationship between discourse structure and participation structure is evident from analysis of the data generated using the LPS research design.

Conversation analysis: Epistemic stance (Sahlström & Melander, 2010)

This analysis contributes to the growing body of work within Conversation Analysis (CA) on learning, knowing and remembering, by investigating the ways in which participants display their epistemic stance, i.e. their ways and claims of knowing, by investigating how the epistemic stances change within and across situations and by investigating whether there are differences between classrooms in different cultures. A comparative analysis was conducted of 15 lessons and post-lesson interviews in eighth-grade mathematics classrooms in Sweden, USA, and Australia. The results show an abundance of epistemic stance markers (such as "think", "know", "say") in both teacher and student talk. In more precise analyses, epistemic stance changes were studied in relation to the same content, both within and across situations. The results show that there are substantial possibilities in this approach. We can study changes in "participation" in a systematic, practical and concrete way, pursuing the analysis of learning in interaction on the basis of the evidence participants themselves offer as evidence for their ways of knowing. The further development of this approach requires expansion of current notions of epistemic stance beyond the verbal and word-oriented work done so far.

Discursive practice and learning (Clarke & Xu, 2010)

Mathematics learning can be conceptualised in terms of participation in forms of social practice, where discourses form key components of that practice. Language plays a central role in mediating and constituting this participation, which is performed as classroom discourse. Adopting this perspective promotes mathematical discourse from the role of a mere instructional means to that of the object of learning. Traditionally regarded as only auxiliary to thinking, active mathematical communication is nevertheless believed to enhance mathematical learning. It is a useful exercise, however, to conceptualize mathematics as a special form of communication. Indeed, from this perspective, the term "learning mathematics" becomes tantamount to developing mathematical discourse. The classroom settings (and practices) analysed in this study (in Shanghai, Seoul, Hong Kong, Tokyo, Singapore, Berlin, San Diego and Melbourne) suggest that discursive practices are culturally-situated to a profound extent and that differences in these practices can be associated with distinctive learning outcomes.

Each of the above researchers applied their own analytical perspective to select elements within the data source, thereby generating a distinct data set for each of the intended analyses. Each data set, so constructed, was then analysed using the same theoretical framework that guided the construction of the data set. In this respect, each analysis resembles any mono-theoretic research design in that the constructs privileged by the chosen theory were matched to data types and a research design constructed that employed methods suitable to the generation of the targeted data. Each independent analysis is vulnerable to the same accusation of circularity or pre-determination that can be levelled at any mono-theoretic research design. Once available, however, the results of the parallel analyses can serve several purposes:

- By addressing different facets of the setting/s and providing a richer, more complex, more multi-perspectival portrayal of actors and actions, situations and settings;
- By offering differently-predicated explanations for documented phenomena and differently-situated answers to common research questions;
- By increasing the authority of claims (and instructional advocacy), where findings in relation to the same question or the same phenomenon were coincident;
- By qualifying the nature of claims, where findings of the parallel analyses in relation to the same question or phenomenon were inconsistent or contradictory; and
- By providing a critical perspective on the capacity of each particular theory to accommodate and/or explain particular phenomena, in comparison with other theories employed to conduct analyses related to the same events in the same setting.

The derivation of all findings from the same data source and the application of all analytical approaches (and therefore all findings) to the same setting/s (the mathematics classroom) greatly strengthens the project's capacity to realise these five purposes.

In relation to the first of the two stated goals of multi-theoretic research design (above), all four analyses relate to the one data source generated through implementation in the classrooms of many different countries of a common research design, but they address different aspects of that data source. The question to be addressed is "How might these analyses of the same data source and pertaining to the same educational setting (the mathematics classroom) be synthesized?"

The pragmatism of theoretical inclusivity and the challenge of synthesis

Specifically, for the purposes of this paper, I am interested in establishing under what conditions we can synthesise across different parallel analyses of data generated from the same data source if those analyses are grounded in different theories.

In 2001, the book *Perspectives on Practice and Meaning in Mathematics and Science Classrooms* (Clarke, 2001) reported ten parallel analyses, undertaken from different theoretical perspectives, of a common body of classroom data drawn from eight mathematics and science lessons. Since that time, an emergent pragmatism within the education research community has seen a growing acceptance of multi-theoretic inclusive designs, at least in principle (Cobb, 2007). The interdependence of theory and research findings has been explored in several recent studies. For example, the work by Even and Schwarz (2003) compared the two theoretical perspectives: Cognitivist theory and Activity theory in their investigation of a mathematics lesson. Their study demonstrated that the two approaches suggested different interpretations of the situation and different origins for the learning difficulties identified. While cognitivist theory pointed to student difficulties in integrating information from different representations, the analysis using Activity theory suggested that the teacher and the students participated in the same lesson but in different activities guided by different motives, goals, beliefs, and norms. Even and Schwarz concluded their paper with a note of caution regarding any attempt to harmonise and integrate different theoretical approaches towards the development of a new radical theory.

Each theory brings with it a vocabulary that privileges certain constructs and downplays or ignores others. It is not the compatibility of the two theories that should be considered, but of the interpretive accounts generated by their application to a common representational record of the same classroom events. So, the relevant question is "Under what conditions are the interpretive accounts compatible?" This contingent compatibility focuses our attention on the use of theories as interpretive tools.

Some principles in the use of theory in classroom research

- No one type of analysis can provide the definitive account of classroom practice.
- Each theory privileges certain aspects (and outcomes) of the mathematics classroom and discards other aspects (and outcomes).
- Each mono-theoretical account contributes a particular and potentially valuable perspective.

- It is the critical synthesis of these accounts that should optimally inform our understanding of the classroom.
- Inclusive designs optimise the validity of such critical syntheses.
- No single synthesis can provide the definitive account of classroom practice.

It is not necessary to aspire to a single all-inclusive theory. If theories are seen as tools with descriptive and explanatory capacities, then the essential critierion for the application of a particular theory in research is “Does it address the question or phenomenon of interest in a way that accommodates the salient features of the setting, including living and non-living participant elements and the mechanisms by which they contribute to the setting, and the embracing cultural, historical or political contexts, as these are realised performatively as affordances or constraints of the setting?” Which is no more than to say, “Does the theory resonate with our values and research interests?” Assuming the answer to both questions is “Yes,” the insights made tangible by such a theory will still be partial with respect to the setting. And this is inevitable.

Theoretically inclusive critical research synthesis attempts to reduce this partiality, by the juxtaposition of theoretically-situated descriptions and explanations. Through this juxtaposition, theoretically inclusive research synthesis highlights discrepancy and agreement, voice and silence, inclusion and omission, and helps us to set the bounds on our aspirations in using theory.

In conducting such critical syntheses, our goal is not completeness but relevance. Our critique is not of the compatibility of the theories but of the accounts that they produce of events or relationships in that particular research setting. It may be that the ontologies or epistemologies underlying the various analyses are fundamentally different, and where this is the case, the warrants for any findings will be differently drawn. This may lead to the useful juxtaposition of both the finding and the warrant. It is conceivable that two sets of findings cannot simultaneously be true, because the warrants to which they appeal are grounded in incompatible ontologies. We then find ourselves in the fortunate position of being able to interrogate not the setting but the theories, since the findings derive from and relate to similar settings drawn from the same data source.

Summative remarks

The deliberate and strategic use of parallel analyses informed by complementary but distinct theoretical frameworks offers a form of safeguard against the possibility that commitment to a single analytical framework might render our research insensitive to potentially salient considerations and significantly reduce its explanatory potential.

It is essential that we broaden our conception of research synthesis, both in relation to its methods and its goals. As we become more methodologically inclusive in our primary research, so we must extend this inclusivity to our attempts to learn from our research and the research of others. This process of knowledge construction from multiple research studies can rightly be called “synthesis.” But a commitment to inclusivity is not a commitment to consensus or to convergence. Inclusivity also requires the recognition and even the celebration of difference. In the context of research synthesis, this brings with it the acceptance of a bounded pragmatism in our attempts to learn from educational research.

To re-state the point made earlier: It is not the compatibility of any two theories that is being considered, but of the interpretive accounts generated by their application to a common representational record of the same classroom events. So, the relevant question is "Under what conditions are the interpretive accounts compatible?" This compatibility is contingent on particular conditions governing the application of the theories in a particular setting for a particular purpose, and focuses our attention on the identification of these *contingencies for compatibility.* Theoretically inclusive research designs and critical synthesis may help us to understand both setting and theory, and thereby facilitate the reflexive refinement of both.

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MASTERING BASIC FACTS? I DON’T NEED TO LEARN THEM BECAUSE I CAN WORK THEM OUT!

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Knowing basic facts is critical for expediency in computational mathematics. By the time students reach the age of eleven some teachers are finding that groups of students are still counting with their fingers or resorting to the use of calculators, tricks with fingers, charts, or asking someone the answers to times tables. The question to be answered is why, after all the years at school, that students cannot remember 55 simple facts? An intermediate school in New Zealand has been investigating ways to motivate the students‘ learning of basic facts. This paper explores the improvement of student achievement through an action research plan.

Introduction

Carr and Kemmis‘ action research model (1986) is well known in education communities and there is a plethora of papers to confirm its use. Often individual teachers will be involved with an outside researcher in a project that encompasses action research but less frequently they are involved in their own school collaborative action research process. Carr and Kemmis define action research as:

> … simply a form of self-reflective enquiry undertaken by participants in social situations in order to improve the rationality and justice of their own practices, their understanding of these practices, and the situations in which the practices are carried out (p. 162).

It is essentially a spiral model involving four steps: planning, acting, observing, and reflecting. As a process to improve teachers‘ practice, it has many benefits but most importantly it happens in the classroom. This paper focuses on the work of teachers of a school in Dunedin, New Zealand and follows their journey as they ascertain a problem common to all, plan, implement, make observations, and reflect on their outcomes. Within this context it must be clear that this is not the result of an outside researcher‘s work. The story is the school‘s one although they are referred to as ‗they‘, ‗the staff‘, or ‗the school‘.

Background

Educational settings for students aged 5 to 13 in urban New Zealand are generally primary and intermediate schools. The intermediate school concerned in this paper has 15 teachers and 485 students aged from 11–13 years who vary in academic, behavioural, and social backgrounds. Previously, classes were streamed for mathematics according to students‘ ability. However, the school has undertaken significant changes in their mathematics programme due to the New Zealand *Numeracy Development Project* (NDP). A mathematics extension programme caters for a small number of students who are working at Level 5 in the New Zealand Curriculum but, in the main, students are no longer streamed and individual differences and needs are catered for within the students‘ own classes. Since the completion of the numeracy project they have continued to sustain and enhance the changes they have made.

Today visitors can walk through classes in the school and see an environment for mathematics learning. Of note is the mathematics discourse: teachers‘ questioning that extends students‘ thinking and shared ideas about problems solved; modelling books that record students‘ responses; and the use of equipment by students to demonstrate understanding. These are some of the aspects that less than a decade ago would not have been seen in an intermediate school. Teachers within the school are well supported by their principal and deputy principal with school-wide data discussed, targets set, and further development in understanding of the New Zealand Curriculum and the *New Zealand Framework* (Ministry of Education of New Zealand, 2007).

A major part of the professional development in which teachers participated was to focus on how students develop strategies. As numeracy facilitators focused their energies on improving the teaching of strategies it seemed less importance was placed on the teaching of number knowledge and basic facts (found in NDP, Book 1). The school responded to this in 2008 by revising its mathematics programme and explicitly stating that the interdependence of number knowledge and the teaching of strategies should go hand in hand. Strategies create new knowledge through use and knowledge provides the foundation for strategies.

Since then Johnston, Thomas, and Ward (2010) have provided evidence that points to the importance of strategy development but acquiesce that the importance of knowledge should not be underestimated because strategies require knowledge as a pre-requisite for their effective use. This position sat comfortably with the school as staff had explicitly stated they needed to ensure students‘ knowledge as a prerequisite to introducing new strategy teaching. For example, it did not make sense to offer problems that involved partitioning fractions if the students had little or no knowledge of place value or fractions.

Research has shown that the transition from primary to the intermediate school can show a dip in achievement (Young-Loveridge, 2007). It could be argued that it is perfectly understandable when about 485 pre-adolescent students are feeling, some for the first time, anxious about their learning. Teachers in Intermediate Schools have to work extremely hard to create a positive learning environment for pupils who will be with them for only two years. In an endeavour to create a more harmonious and cohesive teaching unit within their school, the staff turned to action research with the intention of informing and changing aspects of their practice for the improvement of children‘s achievements.

Previous to 2009 some of the staff had been involved independently in an action research model. However, it was felt that raising achievement of all students demanded a coherent, collaborative process from the whole school staff. Whilst many problematic issues were raised, written up on a board for a couple of weeks and discussed at length, it was decided it would be best to start with a simple question. Observations by teachers in classrooms identified a common problem in lack of multiplication facts when solving multiplication or division problems. Looking for an issue that was common to all was easy to identify: basic facts.

What are basic facts?

In the previous Mathematics in New Zealand Curriculum published in 1992, basic facts had been defined as addition and subtraction facts to 10 and times tables to 10 x 10. However, with the advent of NDP in New Zealand between 2000 and 2009, basic facts have been redefined to include other useful facts such as 4 x 25 = 100 and compatible numbers such as 52 + 48 = 100. Further information about basic facts can be found in *The Numeracy Development Book 1* under the *Knowledge Framework* (Ministry of Education of New Zealand, 2007).

Refining the problem by reflecting where we are now

School wide data, compared with the global stages in the New Zealand Framework indicated the majority of the students at the school ranged from Stages 5–7 (an additive stage to a multiplicative stage) with a few students experiencing learning difficulties at Stage 4 (counting 1 by 1 from a set held). In the knowledge framework, students at stage 5 are expected to know their 2s, 5s, and 10s multiplication facts and at Stage 6 to know all multiplication basic facts up to the 10 times tables.

When discussed further the problem was not how to teach the basic facts but how to motivate the students to be able to recall them instantly. The issue was that students had mastered the strategies of how to work out the answers to problems involving single digits but had not mastered the knowledge of instant recall. For example, to work out 6 × 7 they would use a known fact such as 5 × 7 = 35 and then add on another 7 to reach 42.

Van de Walle (2004) advocates the mastery of basic facts as ―the development of fluency with ideas that have already been learned" (p. 156). Many of the 485 students had efficient strategies but were not facile. In fact when some students were asked why they didn‘t learn their tables the response was frequently ―I don‘t need to ‘cause I can work them out".

> Fluency with basic facts allows for ease of computation, especially mental computations, and, therefore aids in the ability to reason numerically in every number related area. Although calculators and tedious counting are available for students who do not have command of the facts reliance on these methods for simple number combinations is a serious handicap to mathematical growth. (van de Walle, 2004, p. 156)

The ‗traditional‘ way of rote learning multiplication basic facts has long been a popular and somewhat successful process, and there has been resurgence in the popularity of teaching the basic facts in this manner in recent times. Steel and Funnell (2001) have suggested, in a study of students between the ages of 7–12, that they will memorise the multiplication facts quicker using rote learning, but they did place a caveat by saying that it was important that they do understand what they are and how they function.

Van de Walle (2004) also supports the idea of drill when students have learned through a good basic facts programme: ―teach for understanding, consolidate through practice and apply through investigations" (p. 157). The idea is to focus on drill when automaticity is a desired outcome. With the students already showing very good understanding about multiplication and how to derive facts from what they knew, the staff felt without this immediate recall of their times tables their students would not reach their full potential in their multiplicative strategies. That was the defining moment: the decision to concentrate on the fluency of recalling multiplication tables.

The research question for this study is ―How can the staff, through a whole school approach and home school partnerships, raise an awareness of the importance of learning multiplication facts and increase the multiplication tables knowledge of the students?"

Process

Planning

The process up to this stage took three weeks but once the problem was identified planning came relatively quickly. There were some aspects that staff felt they needed to concur with before they could move forward. If they did not all hold common aspirations, there was a chance their expectations would not be met. Everyone agreed:

- that knowing basic facts is critical to success in solving multiplication and division problems in mathematics, science and technology;
- to improving student achievement in basic facts across the whole school;
- to raising the fluency in recall of basic facts, with regular checks, and quality teaching;
- to fostering closer home school partnerships; and
- to be committed to working towards the expected outcomes: (a) that students will become fluent with their knowledge of their times tables, and (b) the whole school average will be in the mid-80th percentile.

With shared beliefs and the question identified the staff deliberated on how best to plan for optimum success. To begin with, they prepared themselves by becoming familiar with literature that was written around the teaching of multiplication tables. Most readings dealt with how to teach for understanding but very few dealt with influencing children to want to learn the facts. Van de Walle‘s (2004) work on mastering basic facts resonated with their philosophy. It was easy to read and couch this in practical terms.

The next matter was how to gather data so that the school-wide trends could be identified from the subsequent analysis. School-wide testing as an issue was debated fiercely, but because teachers saw a common good they felt some testing was appropriate. From their readings, staff had realised that constant drill and testing could be detrimental to some children‘s self-efficacy. Whatever they did they had to be sensitive to students‘ needs. Sessions were to be as enjoyable as possible with challenges and successes for all. Practice and drill had to be meaningful for each student. However, baseline data needed to be accumulated particularly as Year 7 students came from several schools with various assessment profiles.

The following step was to plan what was going to happen in their classrooms. How were they going to go about it? It was decided that they would aim for a period of four weeks and then reflect on what had happened after analysing school wide data. Each class was committed to as much time of the mathematics hour as they needed, five days a week.

The school resources were checked and teachers undertook to search for new websites, games and ideas that may help motivate the students. The task was not just about finding some engaging websites; it was ultimately about selling the idea that ‗it is worthwhile having instant recall of multiplication tables'. Until the students could see that it was plausible and rewarding for them, they would not buy into it. One straightforward idea was to put the students into pairs with one asking a set of ten timed easy multiplication tables (2s, 5s, or 10s) and then the other using a calculator to work out the same questions, also timed. Students quickly saw how much faster it was by recall than using the calculator.

It was realised that students spend a large amount of time with their families and that the school needed supportive links with each family in order to affect the maximum outcomes. Bull, Brooking, and Campbell (2008) found that ―parental involvement makes a significant difference to educational achievement" (p. 1). In their best evidence synthesis Anthony and Walshaw (2007) also confirmed what many educators and researchers believe: if parents are involved in their students' education there will be positive outcomes. Information sheets with suggested activities went home to parents. Individual check sheets were developed to foster a home/school partnership, including the student's progress and ‗tough' facts that needed to be learnt. Surveys on whether or not their parents had helped them at home, time spent on their tasks, as well as the students' feedback on the month, were to be given to all children. Incentives were to be held at a school-wide level in terms of a school swim, and often at class levels in terms of free time or shared lunch.

Finally there was just one more thing to do: decide on a name—the hardest part.

Acting

Mega Maths Month, as it was named, started with a baseline test during Mathematics period. Every student had three minutes to complete the questions, thus ensuring knowledge rather than strategisation of solutions. The baseline test was set for a Friday at 8.55 a.m., to make certain all students did the test. From their results, students identified five of the tables they got wrong or struggled to remember and recorded those on their check sheet. They were the facts they practised during the next week. The class results were sent to the Deputy Principal to enter and find the baseline average for the school. They repeated these steps for the following three weeks. During the four weeks, teachers spent time each day with various activities to encourage the students to become more proficient with the multiplication basic facts they each had according to their ability. At home, the students were expected to devote more time to remembering their multiplication facts, and parents were encouraged to support them.

Observation

The whole school results after one week quickly jumped from 68% to 87% in 2009, and from 72–86% in 2010. A pleasing outcome was that the year 8 children (in 2010) held their knowledge from 2009. Results showed that in week 3 of 2009 students showed an average improvement of only 1%, with five classes actually going backwards between one and four percent. In 2010 student results increased by 4% between weeks 1 and 2. In both years the students collectively moved more than 15 percentile points but were never able to get into the 90th percentile.

That may have been due to the inability of students to learn the hardest facts of the multiplication tables. LeFevre and Liu (1997) report correlations of error rates with product size. While problems with products greater than 40 comprised 17% of the problems in their study, they accounted for 45% of the errors. Salvo (2006) found similar results on a pre-test that she administered. ―Nine of the 10 most missed problems had products greater than 40 and both factors greater than 5. The nine problems, in order from the most missed, were 8 × 7, 8 × 6, 7 × 9, 6 × 9, 6 × 7, 7 × 7, 9 × 8, 8 × 8, and 9 × 9" (p. 583). They comprised 25% of her test items, accounted for only 12% of the correct responses but 40% of the errors and omissions. They were also the problems students at their school consistently identified as their ―hard ones".

According to the teachers, incentives appeared to be motivating. However, the students surveyed held different opinions. They felt incentives made little difference to their motivation. Some had even forgotten that there were incentives in place.

Once results from students were compared to their surveys it was found that when parents helped their children they improved the most. It was nearly a 50/50 split of parent support from the whole sample but it was very clear that home support was greatest for students who improved by 30 or more percentage points.

The classrooms which had the greatest success often discussed class and individual targets. One class found the use of a spreadsheet to show the classroom average at a point in time and what it would be if students set and reached an improved score they thought they could attain. That demonstrated to them clearly that if everyone made small improvements it would work towards achieving their overall goal of an improved class average.

Reflection

It is hard to identify the one key thing that brings success to a student. The school's process indicated that the best results come from a combination of ensuring students' understanding of what multiplication tables are, practice, and family support.

No teacher interviewed for this action research project felt they had found a defining tool. Flash cards, basic rote learning, computer games, and testing each other were all common practice tools. Students themselves identified personal flash cards and games as valuable tools for learning their multiplication tables.

The teachers felt they started out a little disjointed but along the way valued the individual input and team commitment to the process, which gave them a sense of ownership. It has made them look at the activities they have used and question ‗Why that one?' They have differentiated some of the tasks they have used for students of differing ability. They have collaborated with colleagues, shared their successes, and talked about improvement. Most importantly they have been the drivers of their own critical reflection.

Questions that have arisen from the process are:

- Is there a need to spend more than one month a year on *Mega Maths Month*?
- What aspects can we make an integral part of our practice?
- Should gender/ethnicity results be looked at more closely?
- How can relationships with the parents who are ‗invisible' be improved?

Success in 2011 could come through looking at one of three options:

1. A differentiated programme where some children continue with *Mega Maths Month* and the others who know all their tables become the mentors/partners for the children who are experiencing more difficulty than usual;

2. A ‗Maths Matters' programme which focuses on the multiplication strategies being taught, drill (especially on the identified harder multiplication facts), and either an intrinsic (family/peer expectations to do well) or an extrinsic (reward) incentive for automaticity of multiplication facts, which is still under debate; and
3. Students identifying their own style of learning and planning their own programme to help them learn their multiplication tables.

Conclusion

The school has made considerable changes to their teaching and learning of mathematics through discussion, debate, and professional development. Since 2009 the staff has worked hard to find aspects that, as a school, they could focus on in a collaborative way. What was significant was just how much difference they could make when teachers, parents and students all worked together. It was their first process as a whole staff and it is that story that makes it worth telling.

It was not a huge undertaking but it was manageable and therefore did not seem too onerous a task. It is hoped that by sharing their journey other schools will be tempted to find a problem that they can tackle through the simple action research process. The new knowledge, improvement in practice, communication, and the relationship building with teachers, children, and parents that transpires through the process cannot help but make it a worthwhile endeavour.

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SUPPORTING YOUNG CHILDREN'S MATHEMATICS LEARNING AS THEY TRANSITION TO SCHOOL

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It is now acknowledged that children start school with a wealth of mathematical knowledge and experiences (e.g. Aubrey, 1993; Perry & Dockett, 2004; Young-Loveridge, 1989), and that recognition of this rich resource by the new entrant teacher may facilitate the smooth transition of the child into school (Perry & Dockett, 2004). Positive transitions directly impact on children. This paper investigates how the mathematics content, understanding and practices of the new entrant classroom align with the learning children experience within early childhood settings. In particular it reports on the supportive practices provided by two schools for young children"s mathematical learning as they begin school. Results from the study show tenuous links in mathematical practices between these sectors.

Background

As a direct result of recent research interest in areas of early mathematical learning there has been a surge of interest in the development of mathematics in early childhood. Researchers now recognise the „mathematical power" young children possess on entry to formal schooling (Clements & Sarama, 2007; Perry and Dockett, 2005). Furthermore, understanding that the child"s competence in mathematics at the end of the first year of schooling is a strong predictor of later success in mathematics has contributed to a focus on early mathematics. We questioned how schools support young children"s mathematical development and how that support connects with the support provided within the early childhood settings?

Transition from early childhood to school can pose difficulties for new entrant (NE) children (Eyers & Young-Loveridge, 2005; Perry & Dockett, 2004) and has a long-term impact on school achievement (Timperley, McNaughton, Howie, & Robinson, 2003). Kagan and Neuman (1998) suggest there are high costs when there is a lack of continuity between sectors; this results in lower success rate at school, difficulties in making friends and vulnerability to adjustment problems. It has been argued that for successful transition the differences and discontinuities between the sectors need to be addressed, as "starting school is not a simple process" (Margetts, 2007, p. 106).

Transition to school calls for the development of "higher mental functions" (Broström, 2007, p. 61) if a successful move from a play focus to a more formal school learning system is to be achieved. Furthermore, it is suggested that the differences between the requirements of early childhood and school settings may invite problems related to adjustment (Kienig, 2002). Broström (2002) has noted that these requirements are a consequence of different social and academic goals between the school and those of the pre-school setting. Tensions arise as a result of change from a learning environment based on socio-cultural and co-constructivist ideas of learning (Bronfenbrenner, 1979) to more structured activities and formal instruction (Pratt, 1985), and in which there are very different expectations by teachers within early childhood education (ECE) and the primary school sector (Timperley et al., 2003).

Arguably, barriers to smooth transitions vary depending on the individual contexts, and in particular on relationships that have developed among ECE services, schools and parent/caregivers. Successful transition to the school setting has been described as an ecological transition between two "microsystems" (Bronfenbrenner, 1979). A comprehensive framework for understanding the complexity of child development has been provided by Bronfenbrenner (Margetts, 2007) and adapted as a "Levels of Learning" framework by the New Zealand early childhood curriculum *Te Whāriki* (Ministry of Education, 1996. p. 19). Here the learner and his or her engagement within their immediate environment (or microsystem) are situated as the first level of learning (Peters, 2003). The second level (or mesosystem) extends to the relationships between the immediate learning environments. In the context of early childhood this relates to the home and family, the early childhood setting and the people within these contexts. Level three (exosystem) encompasses the influence of the adult‟s environment on their capacity to care and educate. Wider social beliefs about the value of early childcare and education form the final level (macrosystem). *Te Whāriki* is mainly concerned with these first two levels whilst acknowledging the influences of the other two. In Margetts‟ (2007) view it is this combination of the child‟s personal characteristics, their experiences, and the interconnections between home, prior to school settings and school that ultimately determines how the child adjusts to school.

At the early childhood level teaching involves "reciprocal and responsive interaction with others", building on the "child‟s current needs, strengths, and interests by allowing children choices and by encouraging them to take responsibility for their learning" (Ministry of Education, 1996, p. 20). The child is viewed as a competent learner and communicator and „dispositions to learning‟ is included as an important outcome. "Dispositions are a very different kind of learning from skills and knowledge. They can be thought of as habits of the mind, tendencies to respond to situations in certain ways" (Katz, 1988, p. 30). The child‟s dispositions towards learning are reflected in the nature of assessment undertaken in early childhood settings. Narratives of incidences of a child‟s/children‟s learning are often in the form of a „learning story‟ (Carr, 2001); they focus on dispositions such as curiosity, trust, perseverance, confidence and responsibility rather than specific content areas and achievement objectives.

A strong influence on mathematics teaching and learning in New Zealand schools is the Numeracy Development Project (Ministry of Education, 2001). A key focus of the project is on developing teacher‟s pedagogical knowledge and mathematics content knowledge, and improving the performance of all students. The Number Framework (Ministry of Education, 2001) provides a framework for the development of number knowledge and mental strategies. Professional development for teachers promote effective mathematics pedagogy together with the provision of teaching booklets, activities and resources, and on-going professional support.

The latest curriculum reform for schools, *The New Zealand Curriculum* (Ministry of Education, 2007) acknowledges and celebrates the development of dispositions in the form of „key competencies‟ (p. 12) that "young people need for growing, working, and participating in their communities and society" (p. 38). An underlying theme within this curriculum is a stronger cohesion between the two sectors through a focus on key competencies (Young-Loveridge & Peters, 2005). Although in its infancy, the implementation of this document heralds within a „formal curriculum‟ a focus on children‟s competencies in developing capabilities for living and life-long learning. Competencies are viewed as "not separate or stand alone" and are "the key to learning in every learning area" (Ministry of Education, 2007, p. 12). The alignment of dispositions and key competencies may also develop a continuity of the learning environments across the sectors (Carr, 2006). The ways in which the practices of the new entrant classrooms align with the practices of the ECE services is the focus of this paper.

Methodology

The research was a two year study which investigated the existing transition practices, in a small town in New Zealand, between four early childhood education (ECE) services and two primary schools with regard to mathematics learning and teaching. The research was centred on one key question: What ECE and new entrant practices facilitate positive transitions in mathematics between early childhood settings and primary schools? A case study approach allowed the researchers to focus on interactions between specific instances or situations and to study in depth the transition practices in mathematics within focussed time frames. Evidence was systematically collected enabling the relationships between variables to be studied over time. Our data collection method involved observations in both sectors, teacher interviews, documentation including a range of artefacts, teacher planning, policies relevant to teaching programmes and transition, copies of newsletters, copies of assessments, and photographs of children involved in mathematical experiences. All documentation was analysed and categorised by major themes related to transition and teacher practice using the theoretical framework of Bronfenbrenner‟s (1979) analogy of the child‟s learning environment as „interconnected systems‟. In the study the five key themes analysed within this framework were: structural provisions for mathematics, the assessments that are made with regard to children‟s mathematical understanding, how information is conveyed between sectors, process and provisions for transition, and parental perceptions and expectations.

Findings from Phase 1 explored practice in four ECE services and findings (see Davies & Walker, 2008) provided a baseline of practices for comparison with the school sector. We were interested in investigating how "this new stage in children‟s learning builds upon and makes connections with early childhood learning and experiences" (Ministry of Education, 2007, p. 41) This paper reports on the second phase where transition practices were investigated to determine the extent to which "schools can design their curriculum so that students find the transitions positive and have a clear sense of continuity and direction" (Ministry of Education, 2007, p. 41).

The second year of the research was undertaken in two primary schools (Nikau and Punga) to which many of the children from the four ECE involved in Phase 1 had transitioned. Both are large primary schools. School Nikau is a decile[1] 4 school with 480 pupils from new entrant (NE, or reception class) to Year 10, across 20 teaching classrooms. School Punga is a decile 3 school with 440 pupils enrolled, consisting of NE to Year 8, across 16 teaching classrooms. In New Zealand children can generally start school on the first school day after their fifth birthday, which results in a continual arrival of children in the NE classroom. School Nikau had two NE classes continually filling whereas School Punga had one NE class already full (25 children) from the beginning of year, and a second NE class filling. All four teachers and classes were involved in the research project.

This paper focuses on two key themes of the research: structural provisions for mathematics, and the assessments that are made with regard to children‟s mathematical understanding in school settings (for full report see Davies, 2009). The examples are chosen to illustrate the range of transition practices and are representative of findings within this case study. Results of this study are relevant to these project sites and may not be able to be generalised.

Results and discussion

Structural provisions

One key theme of the study was the differences and discontinuities in the structural provisions (i.e. the approach to teaching and learning, and use of resources) between the early childhood settings and the schools. The approach to learning in ECE is holistic in nature based on Bronfenbrenner‟s (1979) idea of the child engaging with the learning environment. Children are immersed in rich learning experiences across a range of subject curriculum areas with a strong focus on the child‟s interest often embedded in play. The approach to learning in a school setting may be viewed as a change in focus from personal, social and emotional development of the ECE to the formal beginning of specific subjects and content prescribed in the form of „achievement objectives‟ from the national curriculum (Stephenson & Parsons, 2007). In Bronfenbrenner‟s framework the move is towards the second level of learning. The children were being affected by what happens outside their own „microsystem‟.

Lessons contrasted greatly from the children‟s socio-cultural experiences promoted in the early childhood settings. While teachers expressed a belief in the importance of learning through play, they did not reflect this in practice (Sherley, Clarke, & Higgins, 2008). Authentic social contexts for learning which the new entrant previously experienced were not provided through whole class learning and through the activities provided in the resource materials (Belcher, 2006). There was a strong belief in both schools that games or activities from the Numeracy Project replicated the children‟s earlier experience of learning through play.

> I suppose that helps them transition. I suppose we just expect them to start participating in the games (Nikau Teacher, 1).

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[1] The decile rating of a school is based on a Government assessment of the school in terms of the nature of the school community, particularly regarding the predominant socio-economic make up of that community, with 10 being the highest.

> I think there is an expectation of when they come [pause] well how they behave when they are at school and numeracy time is a set time … So we cater to those children by doing games (Nikau Teacher, 2).

It has been demonstrated that children in classes where teachers have used more developmentally appropriate practices exhibit less stressed behaviours (Margetts, 2007). Stephenson and Parsons (2007) emphasise that play should continue to have an important part in developing children as learners in the first few years of schooling.

> Yes they are allowed to have free choice not so much in maths time because I do prefer them to use more appropriate activities that tie in with what they have been learning (Punga Teacher, 1).

Although it has been suggested that school teachers should be responsive and reflective to the diversity of backgrounds in the early weeks of schooling (Margetts, 2007), little evidence was found of this. Concerns have been raised that children become impassive and disempowered with more formal approaches to teaching which may lead to anxiety and low self-esteem (Stephenson & Parsons, 2007). It was commonplace, in all four classrooms visited, for children to be placed in ability groups from their first day at school. Formal whole class teaching followed by group rotations using a range of teacher selected independent activities was widespread. Children in the non-contact group, although having some control over their learning through their choice of resource, had little opportunity to interact with the teacher. Similar to the findings of Belcher (2006) the teacher was unable to scaffold or respond interactively to children‟s initiations because they were predominately engaged in instruction or classroom management.

> We have ability groups. We have two rotations. One rotation they see me and two they do an independent activity. That goes for four days a week and on the fifth day we have a maths circuit (Nikau Teacher, 2).

> They [non contact groups] will either be activities to reinforce previous learning or to help with current learning or a sheet [photo copied work sheet]. More formal type activity for counting. Something where they have got to record (Punga Teacher, 1).

However, one classroom teacher provided practical experiences and opportunities for structured play with opportunities for children to experience confidence and success and to maintain their perception of themselves as effective learners.

> Because I try to make an easy transition from pre-school to school. So you are not from day one sitting down and doing this, this, this. You‟ve got to have free time and activities where the children can unwind and relax. Because they can‟t stay full on all day (Punga Teacher, 2).

> It is a bit of both really. That is where I have developmental type activities - so they have a little bit of structure on the mat. Then they have freedom of other activities at the same time they are learning that rotation process (Punga Teacher, 2).

Mathematics learning in primary classrooms was teacher initiated with predetermined learning intentions. The four teachers had similar fixed ideas as to the particular needs of new entrant children and planned and directed the children‟s learning according to their predetermined intentions. As in a study by Sherley, Clark, and Higgins (2008) teachers were in control of the learning environment providing activities to „plug the gaps‟.

> Well, we see were the gaps [in children‟s knowledge] are taken out of the numeracy project book and we just follow that (Nikau Teacher, 1).
>
> I just stick them in a bottom group for a start to see what they can do and normally you can recognise straight away if they can recognise numbers or count (Nikau Teacher, 2).
>
> I guess you are really quite restricted but you have your planning and guidelines for numeracy project so usually that really controls most of what you do (Punga Teacher, 1).

Our findings confirmed earlier views that the professional development project does not allow teachers to develop a comprehensive understanding of the pedagogy appropriate for transitioning children. Belcher (2006) suggested that the children‟s experiences in numeracy were largely influenced by the teacher‟s belief and understanding of the numeracy project. This may be attributed to a lack of confidence and knowledge of teachers on how to teach numeracy through play (Stephenson & Parsons, 2007).

Assessment

The second theme of the research in this report was the assessment made with regard to children‟s mathematical understanding in school settings. Narrative assessments were the most common form of documentation in all the ECE. These tended to document, in written and photographic form, the dispositions exhibited by the child rather than acknowledging the development of content knowledge. Very different assessment practices from those at ECE were undertaken at the school. Similar to a study by Sherley, Clark, and Higgins (2008) the teachers did not attend to the knowledge and skills the children already had on entry to school. All teachers had limited understanding of mathematics teaching and learning in ECE.

> A huge jump for children who didn‟t know anything when they started. I think early childhood provides for all the opportunities it is just that if the children don‟t choose to take those up then when those children then come to school with nothing and then you already have a gap (Nikau Teacher, 1).
>
> I think there is a lot of maths going on in all the different areas but very much depends upon the teacher being there at the moment to facilitate it. … I think they might do a lot of rote counting, that sort of thing. But when they come to us I see always a gap in number recognition and sometimes 1 to 1 counting (Nikau Teacher, 2).

New Entrant teachers indicated their use of either the „I can …‟ checklists (Ministry of Education, 2005) or the „Numeracy Project Assessment‟ tool [NumPA] (Ministry of Education, 2006) to assess children early in their schooling.

> We do observation assessment for the first six weeks and then in the sixth week we do the NumPA Form A … and after that we carry on with a tick chart, one from the numeracy project stage that they are at (Nikau Teacher, 1).

These checklists provide a guide for teachers with their planning. However they provide little attention to the situated nature of learning experienced by children prior to school. Concerns have been made regarding the use of such tools with its focus on narrowly defined goals and checklists (Peters, 2004) at a NE level. The resulting categories and classifications fail to recognise the richness of children‟s mathematics learning resulting from holistic experiences prior to starting school. The new entrant teachers referred to filling the gaps in children‟s knowledge and tended to overlook the competencies earlier documented within the ECE narratives.

> We get the same thing playing games and you get an idea of stage and what group children would fit into. I guess the NumPA just confirms … and also it finds the gaps that maybe you don"t always find in games (Nikau Teacher, 2).

> So they come out at 0 [Stage 0 of Number framework] so they don"t know any of the things (Nikau Teacher, 1).

Conclusion

The richness of mathematical learning experiences that children bring with them to school has been well researched (Aubrey, 1993; Perry & Dockett, 2004; Young-Loveridge, 1989). Perry and Dockett (2005) analysed the many mathematical experiences children have in prior-to-school settings demonstrating "immense knowledge … including mathematics" (p. 36) and the mathematical power of young children"s skills in mathematising, making connections and argumentation. There was limited recognition of this mathematical power among NE teachers, and little attempt to nurture it by providing learning experiences that made connections to their existing mathematical understanding by the primary school teachers.

Involvement in the numeracy project dominated the teaching of mathematics in the new entrant classes. Children experienced structured numeracy lessons involving whole class mat-time followed by ability group rotations. The use of numeracy project activities and games varied between classes for non-teacher contact groups. Structured mathematics games were believed to replicate the learning approach of the ECE.

Narrative assessments in ECE were very holistic in nature focussing on dispositions to learning. On the other hand the Numeracy Project assessment tool and "I can …" checklists were the main methods of school assessment and these failed to assess the richness of children"s previous mathematics learning. There was a failure to recognise the rich holistic experiences of children"s mathematics learning prior to starting school. The new entrant teachers referred to the filling the gaps in children"s knowledge and tended to overlook the competencies earlier documented within the ECE narratives.

It was evident that connections between the mathematical practices and experiences within early childhood setting and the new entrant classroom were tenuous. Little flexibility was shown in the extent to which the new entrant teachers were prepared to adapt teaching approaches for transitioning children. The NE teacher directed learning rather than being responsive to children"s previous ECE experiences. Activities were structured with a specific learning focus. However, this limited the opportunity for children to engage in exploration and play. Assessment practices were narrow in focus and did not connect with the „mathematical power"" demonstrated by the children in ECE settings.

Further effort is needed in order that "this new stage [the transition from ECE to school] in children"s learning [that] builds upon and makes connections with early childhood learning and experiences" (Ministry of Education, 2007, p. 41) becomes a reality. Findings from this study indicate that a reform of transition practices is needed to ensure that "schools can design their curriculum so that students find the transitions positive and have a clear sense of continuity and direction" (Ministry of Education, 2007, p. 41). Only when that occurs will children"s mathematical experiences be optimised as they transition from early childhood to school.

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LOCATING THE LEARNER: INDIGENOUS LANGUAGE AND MATHEMATICS EDUCATION

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Indigenous language speaking students in remote Northern Territory schools are expected to learn mathematics in English and are assessed in English. Most teachers in these schools have little knowledge of the mathematical concepts with which their students start school. This paper reports on the initial findings of a project which is investigating spatial concepts in Iwaidja, an Indigenous language spoken in the NT. Examples of spatial frame of reference preferences in Iwaidja and related languages are compared with those taken for granted by English speakers. Implications for mathematics teaching are explored in the context of an Australian Curriculum.

Introduction

In a country such as Australian with a mobile population and a nation-wide assessment program, a national curriculum makes sense. For smaller jurisdictions, such as the Northern Territory, it will enable more access to teaching resources with explicit links to the curriculum, something that is difficult to generate for a small population. However, one danger of a national approach is that the specific needs of special groups may be overlooked. The focus of this paper is on particular needs of Indigenous Language Speaking (ILS) students in remote areas of the Northern Territory. These students make up a substantial proportion of the students in the Northern Territory and are widely represented as underachieving in numeracy.

Indigenous education strategies and policies now focus on “Closing the gap” between Indigenous and non-Indigenous numeracy outcomes (Ministerial Council on Education Employment Training and Youth Affairs [MEETYA], 2010). This study is part of an attempt to bridge another gap: teachers’ understanding of the distance (Berry, 1985) between their mathematical language in English and in the Indigenous languages of their students.

First this paper will contextualise the study in terms of expectations and requirements for teaching mathematics in English that apply in a remote community such as Minjilang, the site of the study. Then it will describe spatial frame of reference from a cross-linguistic perspective and why it is relevant to the Early Years mathematics curriculum. It will describe some of the findings from the investigation of spatial frame of reference in Iwaidja, one of the languages spoken at Minjilang. It will then analyse the terminology and sequencing in the location area of the Northern Territory

Curriculum Framework and the Australian Curriculum from the perspective of spatial frame of reference.

Teaching and learning in English in a remote school

New teachers arriving at remote schools in the Northern Territory are faced with many challenges. Many of them have no formal English as a second language (ESL)[1] training and have never heard a living Indigenous Australian language spoken. Entering a classroom of ILS students, they are entering an environment of vastly different cultural expectations and traditions than their own.

At the same time, pressures on teachers in these communities have never been greater, educationally speaking. While some of the trappings of "remoteness" have decreased with improved infrastructure and electronic communication, teachers in remote Indigenous schools such as this are under increasing expectation to assist their students to achieve benchmark levels in the National Assessment Program: Literacy and Numeracy (NAPLAN). NAPLAN results of Indigenous and non-Indigenous students are frequently compared both in the media and in official reports (e.g., MEETYA, 2008). All components of NAPLAN, including numeracy assessment, are conducted in English.

Numeracy, "the capacity, confidence and disposition to use mathematics" (National Curriculum Board, 2009, p. 5), could arguably be achieved in any language, subject to the development of a mathematics register (Roberts, 1998). However there is a powerful perception that it needs to be achieved in English (Commonwealth of Australia, 2000). To this end, the Northern Territory Government's contentious *Compulsory teaching in English for the first four hours of each school day policy* (2009), was explicitly directed towards Indigenous students, banning bilingual education.

The research is being conducted at Minjilang community, on Croker Island in North West Arnhem Land. It arose out of firsthand teaching experience in the school. Traditionally a multilingual region, the main languages spoken in the community are Iwaidja, Mawng, and Kunwinjku, as well as several dialects of English (Standard Australian and Aboriginal). The language of the school is Standard Australian English, although local Indigenous assistant teachers speak to the students in local languages. The main mathematics program followed is *Count Me in Too*. There are ESL support materials for the teaching of literacy, but a lack of targeted curriculum support for teaching mathematics to ILS students from an ESL perspective.

Spatial language

The goal of the project is to investigate some aspects of mathematical language in one of the languages of the community and to make links between that and the mathematics curriculum in the Early Years. The spatial area was chosen as a focus for several reasons. Spatial thinking is a perceived strength amongst Indigenous students (Harris, 1991). Also, spatial language and thinking underpins many numerical and logical

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[1] Although most of the students in these remote communities could be more properly classified as English as an Additional Dialect or Language learners (EAD/L), I use ESL here as it is the more widely used term for a range of teaching strategies.

processes. Finally, there is a body of cross-cultural cognitive linguistic research into spatial language that allows comparison with other languages.

It was not feasible for this project to investigate all the languages spoken at Minjilang, so Iwaidja was chosen for a number of reasons, some of which were political and logistical rather than purely educational. Iwaidja is considered by the inhabitants to be the language of Croker Island, whereas Mawng and Kunwinjku have their homes elsewhere. Iwaidja is not the most frequently spoken language in the school although there are some similarities between the spatial language of Iwaidja, Mawng and Kunwinjku, as we will see.

In order to understand the role of language in mathematical learning when the language of instruction is different from the student's preferred language, Berry (1985) describes two types of difficulties. The first, most obvious, type has to do with level of fluency in the language of instruction. The second type of problem can be more subtle, and arises when there is a mismatch between the student's cognitive structure and that taken for granted by the teacher.

An example of the first type of problem can be drawn from the 2010 NAPLAN test for Years 3 and 5. One question showed a diagram of a bedroom and asked, "What is between the bed and the toy box?" Understanding the concept of 'between', which may exist in the students' home languages—in Iwaidja it is *balarra*—is different from knowing this word in English. Thus "this item is as much a test of English as it is of mapping skills" (The Australian Council of TESOL Associations [ACTA], the Applied Linguistics Association of Australia [ALAA] and the Australian Linguistic Society [ALS], 2010, p. 19). It is this type of difficulty that the *First Four Hours in English policy* was intended to address.

Frame of reference

The main focus of this project is on the second type of problem, the cognitive mismatch between the teacher and student. Spatial thinking has often been assumed to be based on a natural, innate perception of the world (e.g., Piaget & Inhelder, 1956). But the cross-linguistic research of the Cognitive Anthropology Research Group at the Max Planck Institute for Psycholinguistics [CARG] revealed unexpected differences in the ways that people talk and think about space and location. In particular this involved what is termed "spatial frame of reference"—the manner of talking about where one thing is located in relation to another in a horizontal plane. A typology was developed that described three main frames of reference: intrinsic, absolute, and relative (Pederson, Danziger, Wilkins, Levinson, Kita & Senft, 1998). Some languages, such as English, have all three frames of reference. One can variously say "the man is in front of the car"—intrinsic, using the front of the car as a reference, "the man is to the north of the car"—absolute, using a fixed system that is larger and external to the described scenario, and "the man is to the left of the car"—relative, using our own body as the reference. But although English has all these frames of reference, there are patterns of use linked to context. In small-scale space, the speakers of European languages such as English prefer the relative over the absolute and over the intrinsic (Barton, 2009; Levinson, 2003). The absolute is generally only used in large-scale spatial description, such as reading maps.

The pattern of acquisition of spatial language for English speakers reflects these preferences. Children learn first the intrinsic frame of reference such as ‘in front’ and ‘behind’, then left and right, with north, south, east and west regarded as somewhat specialised and not part of everyday speech. Mathematics curricula also reflect this. This will be discussed in more detail below, but generally early years mathematics curricula have a strong focus on the acquisition of left and right well in advance of the cardinal points.

It has long been known that in many Indigenous languages of Australia the terms for left and right can be used only about a person’s body and not projected onto a scene or non-human object (Harris, 1991). It has also been known that some languages such as Warlpiri not only use cardinal directions frequently in small-scale space but that this use is compulsory in spatial description (Laughren, 1978). Some of the implications of this for mathematics teaching in schools have been previously recognised (Harris, 1991). What the CARG researchers did was move from these observations to a general typology of spatial language. They also demonstrated links between preferred frame of reference and spatial memory (Pederson et al., 1998).

This study contends that children who use different frames of reference to those preferred in English might benefit from a different sequence of mathematics teaching that more closely reflects these preferences.

Man and tree game

The ongoing project combines a cognitive linguistic approach to investigating spatial frame of reference in Iwaidja with teacher interviews, ethnographic observation, and an action research approach to improving mathematics teaching in the early years classroom.

To elicit verbal frame of reference, the “Man and Tree” game (CARG, 2003) was used, a barrier task for two participants involving photo matching. The photos show a toy man and tree that differ in spatial location and orientation. The aim of the game is for one person to choose a card and describe it and the other person to find the identical card. Gesture is not permitted. Since the cards show exactly the same objects, spatial description is necessary to distinguish them. The “Anne Senghas” set of 16 cards was used, in which the man could be in one of four orientations to the tree and one of four standing positions in relation to the tree. The cards are named R*xy*, where *x* refers to the facing direction of the man and *y* to where he stands with relation to the tree. The game was conducted with four pairs of speakers.

Findings

Iwaidja

The data revealed use of all three frames of reference, with variation between speakers.

Absolute

There was extensive use of absolute terms, with common terms including *abalkbang manyij* ‘east (sunrise)’ and *wurrying manyij* ‘west (sunset)’.

Warrkbi wakaldakan abalkbang manyij. [1]
“The man is on the east side.” (dvR_100512 25:29 AB) R43

Ruka warrkbi ari yawukan wurrying manyij. [2]
"This man is standing looking over west." (dvR_100513 03:34 RN) R34

Some local landmarks are also used as absolute terms, such as *mayinmul* 'headland' which refers to the headland at the north of Croker Island, and which is used as a term for north.

Relative

Unusually for Indigenous Australian languages, speakers of Iwaidja do sometimes use the terms 'left' and 'right' beyond the scope of their own bodies.

Baraka arlirr ari maruj. [3]
"The tree is standing on the left." (dvR_100512 00:26 AB) R12

Warrkbi rayan nurlinurli or maruj? [4]
"Is the man looking right or left?" (dvR_100512 43:05 DG) R11

However, this was not a popular strategy and did not often lead to the correct card being found. It was remedied with absolute or intrinsic information being added. Body parts were also used to describe the orientation of the man with respect to the speakers.

Riki arrumbukung rtamburryak. [5]
"This one, he gave us his chest (He's facing us)." (dvR_101115_2 13:30 CM) R12

Intrinsic

Body part descriptions were also used to describe the orientation of the man with respect to the tree, such as *rukung kirrwarda* 'he gave it his back'. More frequent was the use of the terms *wurdaka* 'in front' and *warrwak* 'behind, after'.

Kabanayan baraka warrkbi ari wurdaka lda arlirr warrwak? [6]
"Can you see the one where the man is standing in front and the tree is behind?" (dvR_100522 17:16 JW) R24

These terms are of particular interest and a more detailed analysis will appear in Edmonds-Wathen (2011). As with their English equivalents, these terms can have both intrinsic and relative applications. They can describe a situation where the man is in front of the tree with respect to the viewer or they can describe a situation where the man is in front of the tree by virtue of having his back to it. One of the interesting features of Iwaidja is that these terms are frequently used when from the speaker's perspective the man is to the left or right. In example (6) above, card R24 shows the man on the left side of the card with his back to the tree, which is on the right.

Other languages of Minjilang

Mawng

Mawng is another language from the Iwaidjan family. It shares some vocabulary and grammatical structures with Iwaidja, with possibly up to 70 percent coming from a shared origin (Teo, 2007). Speakers of Mawng also use a mixture of strategies in small scale spatial descriptions. Common absolute terms include *kinymalkpa muwarn* 'east (sunrise)' and *kinyuryi muwarn* 'west (sunset)' as well as landmark terms such as *matanti* 'mainland' and *wungijalk* 'deep ocean'. *Matanti* is used for south, and *wungijalk* for north. *Inyjaku* 'left' and *wurulwurul* 'right' are also used (Ruth Singer, personal communication, 18 November, 2010).

Kunwinjku

Kunwinjku is widely spoken in the community and appears to be a language gaining strength and speakers. It is one of a chain closely related, mutually intelligible dialects known variously as Bininj Kun-Wok, Mayali, or Kunwinjku. It is only distantly related to the Iwaidjan languages. In Kunwinjku, the cardinal directions are also used frequently. For example, a story in the Manyallaluk Mayali dialect about hunting freshwater crocodile describes hunters hidden in the water in a waterhole and other people hitting the water to stir up the crocodiles. When they see a crocodile, the people call out to the hunters:

> "Gumeke! Walem!" gareh "gakbi!" o "goyek! Ngale gareh garri!" [7]
> " 'Over there! To the south!' or maybe 'North!' or 'East! Maybe to the west!' " (Evans p. 676)

In a similar context English speakers would be more likely to call out the relative directions "To your left!" or "Behind you!"

Child language

The next stage of the project is to further investigate children's use of frame of reference. Three pilot versions of the "Man and Tree" game have been conducted with adult-child pairs, with the children aged from seven to nine years old. There was a strong emphasis in each of these trials on which way the man was looking. One parent frequently used the absolute terms *abalkbang manyij* 'east (sunrise)' and *wurrying manyij* 'west (sunset)'. Another favoured *wurdaka* 'in front' and *warrwak* 'behind, after'. Some of the parents also used *maruj* 'left' and/or *nurlinurli* 'right'. There is not enough data yet to draw conclusions.

Cognitive effects—Animals in a Row

A non-verbal task was conducted with some of the speakers. The "Animals in a Row" task was developed by CARG to demonstrate the effect on cognition—specifically on memory—of spatial frame of reference preference (Pederson et al., 1998). It was designed to demonstrate differences between absolute and relative thinking. Participants were shown a row of three animals all facing in one direction, either to the participant's left or right. They were instructed to remember the animals. They were rotated 180 degrees and taken to another table with identical animals lying on it and instructed to "make it the same". If the stimulus showed the animals facing relative left/south, a relative response would be to lay them out facing left (which would now be facing north due to the rotation. An absolute response would be to lay them out facing south (which would now be facing right).

For the Iwaidja speakers, however, the task appeared to demonstrate a preference for intrinsic thinking. Most of the participants placed the animals facing all the same way in four or five of the five tests. That is, each time, they placed them facing south/relative left, regardless of whether the stimulus has been placed north/relative left or south/relative right. One speaker placed the animals in a row facing away from his body, at a right angle to how he had viewed the stimulus. This was definitely an intrinsic response.

Location in the early years curriculum

Northern Territory Curriculum Framework

The Northern Territory Curriculum Framework (NTCF) is an outcomes based document. These outcomes are brief. For example, the Key Growth Point 2 (school entry level) outcome for location is “describe the position of nominated everyday objects in familiar locations” (Northern Territory Department of Education and Training, 2009, p. 3).

The 2009 NTCF introduced a section of key vocabulary. This begins at Key Growth Point 2 with topographic and intrinsic concepts such as ‘in’, ‘on top’ ‘beneath’ and ‘behind’, ‘in front’. The relative words ‘left’ and ‘right’ are introduced at Band 1 and compass points ‘north’, ‘south’, ‘east’, and ‘west’ at Band 2. This sequencing correlates to how English speakers are taught, acquire and use the language of location. It does not correlate to how speakers of many Australian Indigenous languages may acquire and use spatial language.

Another drawback of this type of curriculum is its size. Location is a small area of the mathematics curriculum but in this document at Key Growth Point 2 alone there are 12 separate indicators. These were not intended to be a checklist, but in practice teachers often feel that they should all be attempted and achieved.

The Australian Curriculum

The Australian Curriculum is far more concise. The location outcome at Foundation level is “Describe position and movement” (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010, p. 15). There are only two elaborations. Far fewer examples and specific terms are given than in the NTCF. ‘Left’, ‘right’, and the compass points are not specified at any stage. This is potentially more inclusive of ILS students. One of the elaborations, though, uses the phrase “everyday language of location and direction” (ACARA, 2010, p. 15). It is precisely at the Foundation level that the teacher needs to know more about the everyday language of their students if it differs from that of the teacher (Edmonds-Wathen, 2010).

Interestingly though, ‘clockwise’ and ‘anticlockwise’ are specifically identified as important in the Year 1 elaborations. Understanding these requires an understanding of ‘left’ and ‘right’. It would be possible to “give and follow directions to familiar locations” (ACARA, 2010, p. 17) using absolute phrases such as “turn towards the sun” rather than “turn clockwise”. The argument is not that ILS students should not learn ‘clockwise’ and ‘anticlockwise’, rather that it may be better to focus on achieving the early years outcomes in the frames of reference they are more familiar with, especially while they are learning basic English, and move onto those terms later.

Northern Territory Diagnostic Net

There is also the danger that the curriculum elaborations will be used in a prescriptive rather than illustrative manner. The *Northern Territory Diagnostic Net* is an attempt to define minimum standards for each year level that students must achieve to progress at school. For Year 1 minimum standards, it specifies that all students *must* “know the meaning of ‘anticlockwise’ and ‘clockwise’” (NTDET, 2010, p. 47). This has been lifted straight out of the Australian Curriculum.

Conclusion

In the area of location, the Australian Curriculum as it stands may be more suitable than the Northern Territory Curriculum Framework for Indigenous Language Speaking students who have different frame of reference preferences. By specifying less of how teachers are to achieve outcomes, it may enable more scope for teachers to target their teaching programme to the specific needs of their students. Nevertheless, the new curriculum still makes assumptions about the sequencing of spatial learning that has been drawn primarily from the language acquisition and concept development of children from European language backgrounds. Children who are learning Indigenous languages such as Iwaidja, Mawng, and Kunwinjku are acquiring spatial language and concepts with different foci. The study of spatial frame of reference and its uses is a field that aptly demonstrates some of these differences. Further investigation is required into the actual acquisition of spatial frames of reference by the children of Minjilang community.

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DATA MODELLING IN THE BEGINNING SCHOOL YEARS

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This paper argues for a renewed focus on statistical reasoning in the beginning school years, with opportunities for children to engage in data modelling. Some of the core components of data modelling are addressed. A selection of results from the first data modelling activity implemented during the second year (2010; second grade) of a current longitudinal study are reported. Data modelling involves investigations of meaningful phenomena, deciding what is worthy of attention (identifying complex attributes), and then progressing to organising, structuring, visualising, and representing data. Reported here are children's abilities to identify diverse and complex attributes, sort and classify data in different ways, and create and interpret models to represent their data.

Introduction

The need to understand and apply statistical reasoning is paramount across all walks of life. One only has to peruse the daily newspapers to see the variety of graphs, tables, diagrams, and other data representations that need to be interpreted. With unprecedented access to a vast array of numerical information, we can engage increasingly in democratic discourse and public decision making—that is, provided we have an appropriate understanding of statistics and statistical reasoning.

Young children are immersed in our data-driven society, with early access to computer technology and daily exposure to the mass media. With the rate of data proliferation has come increased calls for advancing children's statistical reasoning abilities, commencing with the earliest years of schooling (e.g., Franklin & Garfield, 2006; Langrall, Mooney, Nisbet, & Jones, 2008; Lehrer & Schauble, 2005; National Council of Teachers of Mathematics [NCTM], 2006; Shaughnessy, 2010).

We need to rethink the nature of young children's statistical experiences and consider how we can best develop the important mathematical and scientific ideas and processes that underlie statistical reasoning (Franklin & Garfield, 2006; Langrall et al., 2008; Leavy, 2007; Watson, 2006). There has been limited research, however, on developing young children's statistical reasoning. One approach in the beginning school years is through data modelling (English, 2010; Lehrer & Romberg, 1996; Lehrer & Schauble, 2007; Lehrer & Schauble, 2000).

In this paper, I first argue for the need to review young children's statistical experiences, with a focus on data modelling. I then address some findings from a data modelling activity implemented in second-grade classrooms during the second year of a three-year longitudinal study. In reporting some findings, I consider children's:

1. Recognition of diverse and complex attributes;
2. Identification of ways to sort and classify their data;
3. Models created in representing their data and their interpretations of their models.

Modelling

Data modelling is a developmental process, beginning with young children's inquiries and investigations of meaningful phenomena, progressing to identifying various attributes of the phenomena, and then moving towards organising, structuring, visualising, and representing data (Lehrer & Lesh, 2003). As one of the major thematic "big ideas" in mathematics and science (Lehrer & Schauble, 2000, 2005), data modelling should be a fundamental component of early childhood curricula. Limited research exists, however, on such modelling and how it can be fostered in the early school years. Indeed, the majority of the research on mathematical modelling has been confined to the secondary and tertiary levels, with the assumption that primary school children are not able to develop their own models and sense-making systems for dealing with complex situations (Greer, Verschaffel, & Mukhopadhyay, 2007).

Generating and selecting attributes

Early experiences with data modelling involve selecting attributes and classifying items according to these attributes (Lehrer & Schauble, 2000). As Lehrer and Schauble (2007) noted, it is not a simple matter to identify key attributes for addressing a question of interest—the selection of attributes necessitates "seeing things in a particular way, as a collection of qualities, rather than intact objects" (p. 154). Moreover, children have to decide what is worthy of attention (Hanner, James, & Rohlfing, 2002). Some aspects need to be selected and others ignored, the latter of which could be salient perceptually or in some other way. Frequently, however, young children are not given experiences in which they need to consider attributes in this way.

Classification activities presented in the early school years usually involve items with clearly defined and discernable features, such as green square shapes, blue triangular shapes etc. (Hanner et al., 2002). It is thus rather easy for children to classify items of this nature. In contrast, problems involving the consideration of more complex and varied attributes, which could define more than one classification group, present a greater challenge to young children.

Structuring and displaying data

Models are typically conveyed as systems of representation, where structuring and displaying data are fundamental—"Structure is constructed, not inherent" (Lehrer & Schauble, 2007, p. 157). However, as Lehrer and Schauble indicated, children frequently have difficulties in imposing structure consistently and often overlook important information that needs to be included in their displays or alternatively, they include redundant information. Providing opportunities for young children to structure and display data in ways they choose, and to analyse and assess their representations is important in addressing these early difficulties.

Constructing and displaying their data models involves children in creating their own forms of inscription. By the first grade, children have already developed a wide range of inscriptions, including common drawings, letters, numerical symbols, and other referents. As children create and use their own inscriptions they also develop an “emerging meta-knowledge about inscriptions” (Lehrer & Lesh, 2003), which diSessa, Hammer, Sherin, and Kolpakowski (1991) termed, metarepresentational knowledge. These developing inscriptional capacities provide a basis for children’s mathematical activity. Indeed, inscriptions are mediators of mathematical learning and reasoning; they not only communicate children’s mathematical thinking but they also shape it (Lehrer & Lesh, 2003; Olson, 1994). Developing a repertoire of inscriptions, appreciating and assessing their qualities and use, and using their inscriptions to explain or persuade others, are essential for data modelling. Yet children are often taught traditional representational systems as isolated topics at a specified point in the curriculum, without really understanding when and why these systems are used.

Role of context

The nature of task design is a key feature of data modelling activities. Stillman, Brown, and Galbraith’s (2008) notion of “modelling as vehicle” (p. 143) is applicable here. Such modelling involves choosing contexts in which stimuli for the desired mathematics learning are embedded. Genuine problem situations are used as vehicles for students to construct significant mathematical ideas and processes rather than simply apply previously taught procedures. Furthermore, the mathematics that students engage with in solving such modelling problems usually differs from what they are taught traditionally in the curriculum for their grade level (English, 2003a; 2008; Lesh & Zawojewski, 2007).

When solving data modelling problems children need to appreciate that data are numbers in context (Langrall, Nisbet, Mooney, & Jansem, 2011; Moore, 1990), while at the same time abstract the data from the context (Konold & Higgins, 2003). Research has shown that both the data presentation and the context of a task itself have a bearing on the ways students approach problem solution—presentation and context can create both obstacles and supports in developing students’ statistical reasoning, emphasising the need to consider carefully task design (e.g., Cooper & Dunne, 2000).

Methodology

The participants were from an inner-city Australian school. In the first year of the study, three classes of first-grade children (2009, mean age of 6 years 8 months) and their teachers participated. The classes continued into the second year of the study, the focus of this paper (2010, mean age of 7 years 10 months, n=68).

A teaching experiment involving multilevel collaboration (English, 2003b; Lesh & Kelly, 2000) was adopted here. This approach focuses on the developing knowledge of participants at different levels of learning (student, teacher, researcher) and is concerned with the design and implementation of experiences that maximise learning at each level. The teachers’ involvement in the research was vital; hence regular professional development meetings were conducted. This paper addresses aspects of the student level.

Activity: Baxter Brown's Shop

The initial activity implemented in the second year of the longitudinal study continued the story context (purposely created) from the first year of activities. The context involved the adventures of Baxter Brown (a "westipoo"—West Highlander X toy poodle). The children requested more stories about Baxter Brown in the second year of the study; hence the Baxter Brown's Shop was created. The Baxter Brown stories, presented as picture books, were read to the children in a whole class setting.

The Baxter Brown's Shop story told of the mischievous supermarket expeditions Baxter took with his owners, Mr and Mrs Brown. The dog created various forms of mayhem as he raced down the supermarket aisles. Following the story, the children were shown a simple table of data indicating the different types of mayhem he had created. As a whole class, the children were to determine whether Baxter Brown was becoming more mischievous as his week in the supermarket progressed. In the second component of the activity, the focus here, it was explained that Baxter Brown was subsequently banned from the supermarket and thus ended up creating his very own shop in his bedroom. The children were given an A3 chart comprising illustrations of 16 supermarket items that displayed diverse attributes (the items were a carton of milk, a frozen pizza, apples, coco pops, pasta, a tin of sliced pineapple, fresh carrots, a packet of cheese, a packet of bread rolls, a packet of biscuits, a container of apple juice, a carton of eggs, a tin of dog food, a packet of fish, packaged chicken, and a packet of Cheezels). Working in groups, the children responded to a number of written questions, including: (a) What are some things you notice about the shopping items? Make a list of these. (b) There are lots of things you have noticed. To help Baxter Brown here, what are some ways in which you might sort and classify your data? (c) Which way do you like best? (d) To make it easier for Baxter Brown, how might you represent your data? What are some different ways? (e) Which way do you like best? Why? (f) Now represent your data on your sheet of paper, and (g) What are some things that your representation tells Baxter Brown? On completion of the activity, the groups presented class reports. Children's responses to questions (a), (b), (f), and (g) are reported here.

Data collection and analysis

In each of the second-grade classrooms, two focus groups (of mixed achievement levels and chosen by the teachers) were videotaped and audiotaped. There were 17 groups of children, five in one class and six in each of the remaining classes. The range of data collected was analysed using iterative refinement cycles for analysis of children's learning (Lesh & Lehrer, 2000), together with constant comparative strategies (Strauss & Corbin, 1990), where data were coded and examined for patterns and trends. For questions (a), (b), and (g), some groups gave more responses than others. For question (f), one group created two models while all other groups created just one model. The analysis of data addressed the total number of responses for each question across all groups.

Selection of findings

Children's identification of attributes

The children's responses to question (a), asking for things that they noticed about the given items, were analysed iteratively with two main categories of responses identified, namely, attributes that were primarily qualitative in nature and those that comprised a quantitative element. Qualitative attributes were favoured over quantitative ones. Of the 41 group responses to this question, 28 were of the former type and 13 of the latter. Examples of the qualitative attributes included "dinner foods, brekky foods, dessert," "foods to cook, foods not to cook," "healthy/non-healthy food," and "cans, bottle, plastic, bags, boxes." Quantitative attributes included, "Only one dog food," "all items are under $10 apart from the chicken," "there are only two tins," and "there are more foods than drinks."

Children's identification of ways to sort and classify data

Half of the 24 responses to question (b) ("What are some of the ways you might sort and classify your data?") referred to complementary categories such as "drinks and food," healthy and not healthy," "cereals and not cereals," and "fruit and non-fruit." Two groups offered three or more categories, such as "Dinner, snack, lunch, and breakfast." A further two groups suggested putting like items together such as "drinks together, cans together, things that are in boxes together." One other group recommended sorting by cost ("highest price to lowest,") and another suggested sorting by shape and size ("big food, small food;" "thickest and thinnest").

Children's model creations

Three main models were evident in the children's recorded representations of their sorting and classifying of data. These were models that comprised (a) lists of items in labelled columns, (b) sets of items enclosed in a curve, and (c) items grouped in two divisions (horizontal or vertical) on the A3 sheet provided.

Items in columns

Five of the 17 groups developed models that displayed two to five labelled columns, with the names of the items recorded one under the other in the respective columns. One of these models used the shop context to define the categories, namely, "fridge" and "cupboard," while another model listed healthy and unhealthy items with prices attached. The model that listed items in three columns displayed interesting categories, namely, "dry," "wet," and "dry and wet." In their class report, one group member explained, "I said how about things that are dry and things that are wet so we decided to put that down, and then I thought about pasta and I said if pasta, if you just had pasta by itself it would be dry but if you cooked it, then it would be wet, so then it would be both, so it depends on if you cooked it or if you kept it raw." Another interesting model that comprised five columns displayed items listed under the categories of shapes, namely, "rectangular prism," "sphere," "cone," "cylinder," and "pyramid." The children explained that there were no items that were a pyramid shape and so that column was left blank.

Items enclosed in a curve

Four groups created a model with item names enclosed within a curve. One of these models comprised a large oval, divided into four, with the divisions labelled “healthy,” “not healthy,” “both,” and “dog food.”

Horizontal/vertical division

The most common models were those that comprised either a vertical or horizontal division of the A3 sheet of paper (either with or without actual dividing lines) and displayed item names or illustrations or both. Nine groups developed models of this nature, with one of the groups creating two models, the other being labelled items within a curve. Interesting classifications were evident in these models, such as “combination [of foods]” and “things that taste good by itself.” The influence of task context was also visible in these models, such as in the last group who drew two tables (one at the top of the paper and the other at the bottom) with illustrations of items lined up across the tables and prices attached. Another group used a shop context of “fridge” and “cupboard.” Their model displayed a drawing of a fridge (labelled “fridge” on the top left-hand corner) with illustrations of cold items stacked on shelves and a cupboard (labelled “cupboard” on the top right-hand corner) with non-cold items illustrated on shelves. Another group incorporated a food pyramid within their model.

Children’s interpretation of their models

Due mainly to lack of time, not all groups provided responses for question (g), where the children were asked what their representation tells Baxter Brown. Of the 18 responses, 13 focused primarily on a contextual interpretation (with a common focus on healthy eating) and a literal reading of their model, that is, there was limited interpretation of the data (cf. Curcio, 2010, first level of graph comprehension). Examples of such responses included, “There’s a healthy aisle and an unhealthy aisle,” “there’s dry food and wet food and both,” and “It tells him what to put in the fridge and what to put in the cupboard; if he doesn’t put it in the right place he might get sick.” Only five responses referred to any relational observations (cf. Curcio’s second level of graph comprehension [“reading between the data”]). Such responses included: “There are more healthy things than unhealthy things, “there is more food than drinks and less drinks than food,” and “Our grid tells Baxter Brown what aisles the shape foods are in. It tells him there are no pyramid shape items like the camera stands-tripods. He would have to go to another shop.”

Discussion and concluding points

This paper has addressed the need for a renewed focus on statistical reasoning in the beginning school years, with a focus on data modelling. In such activities, children interpret and investigate meaningful phenomena involving the identification of diverse and complex attributes, in contrast to the standard attributes they usually experience in early mathematics curricula. Building their data models engages children in organising, structuring, visualising, and representing their data in ways that they, themselves, choose.

Evident in the children’s responses was the influence of task context, which appeared to present both support and obstacles in the children’s reasoning. For example, the children's familiarity with the task context appeared to enable them to identify a diverse range of attributes, some quite unexpected, such as a consideration of food combinations, items for different meal times, and items that were identified as "dry," "wet,' and "dry and wet." On the other hand, the children were very aware of healthy and non-healthy foods (from their health lessons) and this could have overshadowed a possible broader range of attributes being identified. The shop context also influenced some of the groups' model creations where fridges, cupboards, and tables were used to represent data. The impact of task context was also evident in the children's interpretations of their models, where there was a focus on shop aisles and also food storage.

The nature of the task items appeared to have a further impact on the children's identification of attributes and ways to sort and classify the data. Qualitative rather than quantitative features were considered. Nevertheless, the children did identify a wide range of qualitative features despite not making many numerical comparisons. The children's models were not as varied as those created in the first year of the study, (e.g., English, 2010), where the story context focused on Baxter Brown cleaning up his very messy room. Children were given various sets of multiple cut-out items to work with and generated a range of representational models including graphical formats. Perhaps the use of multiple manipulative items might have broadened the children's responses in the present activity. Nevertheless, the children did display an awareness of representational structure in developing their models, with their use of inscriptions enabling clear interpretation and communication of their models. The role of task context in young children's mathematical learning has been highlighted over the years (e.g., Watson, 2006) but this study suggests that further consideration is needed.

Young children need rich opportunities to develop their statistical reasoning abilities; data modelling activities provide such opportunities. However, despite the increased calls for renewed attention to statistical learning in the early school years, research examining young children's statistical developments remains in its infancy.

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MATHEMATICS PRESERVICE TEACHERS LEARNING ABOUT ENGLISH LANGUAGE LEARNERS THROUGH TASK-BASED INTERVIEWS

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This paper explores what two cohorts of middle school mathematics pre-service teachers (PSTs) report they learned after interviewing two middle school English Language Learners (ELLs) using four measurement tasks. The written responses of the PSTs to the question about what they learned from the experience were coded and classified into three overarching themes—accommodations, conceptions about ELLs, and the role of language in mathematics. Implications of these themes in future teacher preparation are also discussed.

Introduction

The school population of English Language Learners (ELLs) has seen rapid growth in the US. Between 1979 and 2008, the number of 5–17 year olds who speak a language other than English at home increased from 9% to 21% in the US (NCES, 2010). ELLs require teachers who are familiar with the unique challenges that these students face as they try to learn new content in a language they are still learning. With the increasing number of ELLs in the US, there is a great need for better trained teachers as more mainstream teachers work with ELLs in their classrooms. Of the 41% of teachers working with ELLs in their classroom, only 13% are adequately prepared (National Center for Education Statistics [NCES], 2002). The underperformance of ELLs in the National Assessment for Educational Progress (NAEP) (Martiniello, 2008), known as the ―nations report card", is indicative of the critical need for all teachers to be trained to work with ELLs.

In this article I report on initial results from an exploratory project that I conducted, for two semesters, in mathematics content courses for middle school pre-service teachers (PSTs). The goal of the project was to foster an understanding, among the PSTs, of the challenges faced by ELLs as they learn mathematics in English. The PSTs engaged in a semester long project in which they interviewed two ELL students on four measurement problems and wrote a structured report on each occasion. In this paper, I will focus on one of the guiding research questions from this project: What do PSTs report about learning from mathematics task-based interviews with ELLs?

Most of the PSTs in my classes were monolingual English speakers and reported having minimal to no interaction with ELLs. I conjectured that engaging these PSTs in task-based interviews would provide the needed platform for them to rethink some of the preconceptions that they may have about ELLs.

Literature review

The educational needs of ELL students are complex as they are learning content in a language in which they are still developing proficiency (Lucas, Villegas, & Freedson-Gonzalez, 2008). Cummins (2000) differentiated between the informal conversational language used in everyday interactions and the more formal academic language that is usually used in content areas like mathematics. Conversational language is acquired through interactions with others in different settings, requires limited vocabulary and meaning making is assisted through the use of contextual cues. Academic language, on the other hand, refers to the special language of a content area that is usually acquired in school, employs special vocabulary and discourse features and is devoid of context. Conversational language fluency is acquired in one to two years whereas five to seven years are needed for development of academic language.

Part of learning the academic language requires that students become proficient in the ―mathematics register" (Halliday, 1978) which refers to the special vocabulary and linguistic structures that are used to express and discuss mathematics. Spanos, Rhodes, Dale, and Crandall (1988) further elaborated the syntactic, semantic, and pragmatic features of the mathematics register that could pose a challenge to all students, but especially ELLs. Syntactic features included comparisons of size (X is greater than Y) or conditional relationships (if X, then Y). Other syntactic features of the mathematics register include the use of passive voice (e.g. When 15 is added to a number, the result is 21. What is the number?), which is known to be challenging for ELL students (Abedi & Lord, 2001). ELLs need to master the semantic features of the mathematics register which include the use of special words like *coefficient* and *denominator* or the use of everyday terms like *square* and *rational*, which have a special meaning in a mathematical context. The last category of pragmatics referred to the use of language in particular contexts where students would need to know culturally specific meanings to understand and solve a problem (e.g. Campbell, Adams, & Davis (2007) discuss a problem where the student would need to understand the word Laundromat).

Future teachers have to understand the challenges that ELLs face learning the academic language if they are to assist these students in learning mathematics (Lucas et al., 2008). However, in most cases, PSTs are usually white and monolingual, and they may not have experienced the same linguistic challenges that ELLs face in English only classrooms; hence it is hard for them to empathize with these students. In the absence of formal experiences with ELLs, the PSTs' views about ELL students are most likely shaped by media or the public leading to the creation of stereotypes (Walker, Shafer, & Liams, 2004). For example, PSTs may assume that an ELL student possesses a disability if they are proficient in conversation but struggle with the mathematics content. Currently, there is an over representation of Latinos, who constitute 85% of ELLs, in special education (Gandara & Contreras, 2009). Providing PSTs with the proper experiences during their preparation is important so that they have opportunities to re-examine prior conceptions and acquire views that are more aligned with research (Sleeter, 1995).

In multicultural education, experiences with diverse populations like living in communities that are culturally different to one's own or tutoring such students have shown promise in challenging PSTs existing beliefs about such diverse students (Sleeter, 2001). In particular, Griego-Jones (2002) found that PSTs who had tutored or worked with ELLs had beliefs that aligned most closely to those in the research literature about second language acquisition. Youngs and Youngs (2001) also found that increased exposure of PSTs to ELLs promoted positive attitudes towards these students. Mathematical field-experiences, like tutoring students, have also shown promise in changing PSTs initial beliefs about mathematics teaching (Ambrose, 2004; Vacc & Bright, 1999). A note of caution about field-experiences is that they can reinforce preconceived notions that PSTs have about diverse students (Grant, 1991; Grant, Hiebert, & Wearne, 1998). So it is imperative that the experiences are followed up with guided reflection from the instructor for them to impact the PSTs (Cabello & Burstein, 1995; Mewborn, 1999). Teacher reflection is also tied to what they notice and attend to in the experience (Mason, 2002). In interviews with ELLs it is possible that the PSTs attend to mathematical aspects of the student's thinking and not pay attention to the influence that language may have on the mathematical thinking. Mason (2002) outlines techniques that can be used to foster noticing, the key being recording incidents of interest and probing them. Mason refers to the descriptions of the incidents as providing ―Accounts-of” which ―describe as objectively as possible by minimising emotive terms, evaluation, judgements and explanations” (p. 40). By holding off judgement, there is potential to get a fresh look at the incident of interest. On the other hand, ―Accounts-for” ―introduces explanation, theorising and perhaps judgement and evaluation” (p. 40). By providing ―Accounts-of” and ―Accounts-for”, there is potential for a person to notice things that they might not otherwise pay attention to. Overall, field-experiences, like task-based interviews, in conjunction with structured reflection, guided by noticing, has the potential to foster an awareness of the unique challenges ELLs face as they learn mathematics content in English.

Methods

The study was conducted in my data analysis and probability course in spring 2010 and in a geometry course that I taught in fall 2010. As part of the courses, I engaged the 17 PSTs (spring 2010) and 32 PSTs (fall 2010) in two task-based interviews with fifth- and sixth-grade ELL students at a local school. The PSTs worked in pairs, with one interviewing the student and the other taking notes. They switched roles for the second interview. The interviews in fall 2010 were also videotaped. Four measurement tasks from the NAEP were selected by me based on my own experience interviewing ELL students with them.

In the second week of class the tasks were presented to the PSTs who solved them on their own. Two video clips from a prior study were used to illustrate the questioning process and highlight aspects of the language that could challenge ELL students. After watching the clips, the PSTs brainstormed in their groups and developed an interview script for each of the four problems. The PSTs were instructed to first allow the ELL student time to solve the problem independently and then engage them in a discussion about their solution. If they could not start the problem or sought help, the PSTs were asked to intervene appropriately. The PSTs were advised to allow the ELL student to do most of the thinking for the task without explicit hints.

After the interviews were conducted, the PSTs were required to write a detailed report about the process. In spring 2010, the pair of PSTs that interviewed the ELL student submitted a joint report (there was one group of three PSTs). However, in fall 2010, each PST submitted an individual report. Guiding questions, based on Mason's (2002) notions of ―Accounts-of" and ―Accounts-for", were provided to assist the PSTs notice features that went beyond the superficial. The questions encouraged the PSTs to describe the interview in detail and then analyse specific aspects like the student's challenges, resources they used to solve the problem, their oral communication, writing, and their mathematical thinking.

The PSTs turned in the reports which were graded and in most instances I provided feedback and asked them to produce a revised version. Later in the semester, the above process was repeated with another ELL student. The data for this article consist of the PSTs' responses to the question that asked them to list at least three things they learned from the task-based interview experience.

For data analysis, all the PSTs' responses to the question about what they had learned were compiled into four documents (two from each semester) using NVivo 9, a qualitative analysis software. I printed and read all the documents multiple times and did *content analysis* to first outline broad themes that captured what the PSTs were saying about their learning (Patton, 2002). I kept refining these themes to arrive at a final list of codes. I used this list to code the four documents in NVivo which allowed me to observe the number of instances a certain code occurred and I dropped the less frequent ones (less than five) from my analysis. I looked at the final set of codes and determined subsets of them that could be related under an overarching theme.

Results

Based on the above analysis, the PSTs' reports about what they learned from the task-based interviews could be captured in three overarching themes—accommodations, conceptions about ELLs, and the role of language in mathematics. I discuss each of these below.

Accommodations

After interacting with the ELLs, most of the PSTs observed that they would have to make an extra effort to ensure that the ELLs understood the content. Some of the accommodations discussed included: spending one-on-one time with the students, relating the concepts to the ELL students' experiences, breaking down the question, slower speech, assistance with the mathematical vocabulary, and accepting different ways of demonstrating understanding that went beyond the traditional written format. For example, one PST mentioned that ELLs will need extra support and slower paced instruction as they try to negotiate the language and the content.

> In order for an ELL student to excel in math, they need additional support and explanation. Also, a teacher may need to slow instruction for students who are English language learners. These students need more time to process information than those whose native language is English. (Keith, Interview 2, Fall)

Another PST reported the need to incorporate aspects of language like reading and writing in the mathematics class to support the development of ELL students in the class.

> I also learned that it is very important to make sure that reading, writing and math are all covered in class. It is important for a student to be able to solve the math, write out the answers and explain themselves and be able to read and interpret what is being asked. All three of these things should be focused on when teaching lessons on different topics. Also, many opportunities should be given to each student with all three of these elements so they can improve or master each. (Sandra, Interview 1, Fall)

Most of the PSTs see these accommodations as necessary if the ELLs are to develop an understanding of the content and keep pace with the other students in the class. One group of PSTs observed that it was easy for the ELL students to remain quiet or for their conversational fluency to mask understanding of the content. They recommended a proactive role for the teacher when working with ELLs.

> For our classrooms, we now see that we will need to take extra steps such as one on one time to make sure that all students clearly understand, particularly ELL students. We learned that they can easily coast through the class just by being quiet or proficient in the English language. It would be the greatest failure as a teacher to have a student come through the classroom and go through the entire school year being years behind in mathematics. (Clair, Janet, and Karen, Interview 1, Spring)

A majority of PSTs espoused the benefits of using visuals and concrete materials to assist ELLs with understanding and also as a means for students to explain their solutions. These recommendations could be traced back to the materials that were part of the Area Comparison problem and the String problem. The Area Comparison problem provided cut-outs and involved comparison of the areas of a right-triangle and square where the side of the square was equal to the height of the right-triangle and the base was twice the height. Two cut-outs each of the square and the right-triangle were provided for the students. The String problem asked the students to describe how they would instruct another student to divide a length of string into four equal pieces. For this problem, the PSTs provided a piece of string to the student after they worked on the problem independently. Based on their experience with the String problem, a PST reports,

> I also learned that manipulatives can be helpful to an ELL student when learning math. He was able to solve the third question easily after he had the string [String problem] in his hand… the manipulative helped him understand the problem. I think that manipulatives are good for any student, but it may be even more helpful for a student who does not quite understand what all the words in the question mean. (Betty, Interview 2, Fall)

The PST observes that manipulatives could play an important role for all students, but especially ELLs. Another PST reports on the usefulness of the cut-outs [Area Comparison problem] and string to assist the student with their explanation.

> I can't stress enough how helpful the string and the cutouts were…. Not only did they help her solve it, but they were a big factor in her communicating how she did it. The same would be said with the string problem. Where her writing was a little confusing, she was able to demonstrate using the string very clearly…. I think the availability of concrete materials to aid in understanding and communicating are vital for these students and should be used extensively in the classroom. (Tess, Interview 2, Fall)

Other accommodations that the PSTs reported included making the directions of the problem clear. The PSTs related the confusion that the ELL students were having solving a problem to the wording of the question. One PST pointed to the lack of clarity—"It is very important when teaching not only ELL students, but all students in mathematics, that the questions are worded very clearly and simply to ensure they are not confused by what is being asked of them" (Mills, Interview 2, Fall). Another PST went further to recommend that teachers should pay attention to the language of the questions they plan to ask and think of ways the question could be misunderstood by ELLs, and modify them accordingly.

Conceptions about ELLs

The interactions with the ELLs helped the PSTs rethink some of the conceptions that they may have had about ELLs. One conception involved associating ELLs with having a disability in mathematics.

> We also learned that just because a student is classified as ELL does not mean that they will have problems in school. A lot of people tend to see ELL as a "disability" while in many cases it can be a sign of a high intellectual ability. Not many people are bilingual and these students can read, write, speak, and comprehend in two languages. (Karen and Janet, Interview 2, Spring)

Another PST reports,

> If anything we learned that you cannot be stereotypical about a student and think that they are instantly going to have a problem with mathematics because they are learning English as a second language. (Elisa, Interview 1, Fall)

A majority of the PSTs understood that the language ability of the ELLs could interfere with their mathematical performance. One PST reports,

> First I learned that ELL students sometimes just need a little guidance to get to the solution. Often, they understand how to solve the problem once they comprehend what the problem is asking. This may be a huge indicator that these students are struggling with the language and wording rather than the math. (Clair, Interview 1, Fall)

In some cases, the PSTs observed that the ELLs tended to interpret the problem differently and this could lead them on a different solution path. For example, the String problem stated "Brett needs to cut a piece of string …" which some ELL students thought referred to a part of the string rather than the whole string. The PSTs observed that the ELLs could understand the requisite mathematics but struggled with deciphering the question due to their still evolving language skills.

Another conception that some PSTs seemed to have was that ELLs would have difficulty speaking in English.

> Overall, I learned a lot more about ELL students after this interview. I learned that not all students are the same, including ELL students. Teachers should not assume that if the student is ELL that they will not be able to understand English. This student was classified as an ELL student, but was able to read and understand all the questions. (Betty, Interview 2, Fall)

Given the limited exposure that these PSTs had to ELLs it is understandable that they might assume that ELLs were a homogenous group who did not speak English. In other cases, there were some PSTs who assumed that conversational fluency displayed automatically implied academic fluency and considered the ELLs to be no different than non-ELLs.

> First of all, I don't know if you could really call these kids ELL students because it seems like they already know the language fluently. So to me, these kids seem just like all other American school children. So for ELL kids that are so American and not even that ELL, so to speak, I can't say I have learned anything about teaching children who are learning English. (Richard, Interview 2, Fall)

Role of language in mathematics

The experiences with the ELL students helped the PSTs see the possible role that language played in the mathematical performance of these students. One PST notes,

> I learned that ELL students' difficulty with language does affect their math…. The student I worked with had difficulty understanding the language of the question which made it almost impossible for her to answer the question correctly. But, once the student understood the question she was able to mathematically think correctly and figure out the answer to the question. (Betty, Interview 1, Fall)

The PSTs also understood that their role as future teachers of ELL students should extend to assisting the students with the language in addition to the mathematics content.

> I also learned that language is an issue when working with ELL students and as teachers we need to be able to bridge language and math so that the student is able to learn and comprehend. We as teachers also need to make sure the language we use is effective and understood by the student. (Linda, Interview 2, Fall)

In contrast, there were some PSTs who believed that mathematics was a universal language and their knowledge of English should not pose a major barrier to learning mathematical concepts in English.

> …regardless of how you say the words, Math is a very structured subject. By this I mean that two plus two is four no matter what language or dialect you speak, how long you've been in the US or where you come from. Two pesos plus two pesos is still four pesos just like two quarters plus two quarters is four quarters. You don't have to worry so much about the subjectivity of the translation. (Mark, Interview 2, Fall)

This PST assumes that language plays a minimal role in the mathematics education of ELLs. Here the PST discounts the language used by the teacher and the students in the classroom interactions and how that impacts student learning.

Discussion

The PSTs responses to the question about their learning from the experience can be classified into three overarching themes—accommodations, conceptions about ELLs and role of language in mathematics. The PSTs in this study observe the importance of incorporating language goals like vocabulary, reading and writing into mathematics lessons. They also understand that as future teachers they will have to take the extra step to teach ELLs. Their notions of accommodations agree with the literature on the best practices of teaching ELLs, which calls for instruction to be scaffolded for such students with an emphasis on both the language and the content (Gibbons, 2002).The PSTs see the role of concrete materials as making the abstract math ideas more accessible to the ELLs, which is also an idea elaborated by Cummins (2000). The role of the cut-outs as part of the Area Comparison problem helped in this respect. The PSTs emphasis on clarity in language is also discussed by Abedi and Lord (2001) who found that modifying the language in test questions allowed ELL students to improve their performance. Interactions with the ELLs helped the PSTs rethink some of the past conceptions they had about these students, such as associating ELLs with a disability or not being able to speak English. Gandara and Contreras (2009) pointed out that such conceptions about Latino students (85% of the ELL population) have led to lower expectations for these students and as a result they are over represented in special education classes. Finally, most of the PSTs saw that language played a role in the mathematics learning of ELLs, but there were some who still viewed mathematics as universal. The latter group was more likely to associate ELL's mathematical difficulties to the ability of the student.

Overall, task-based interviews are beneficial for PSTs learning about ELLs. However, they are not enough, and PSTs need exposure to the research about ELLs to complement this experience. Further, getting the PSTs to reflect on the interview is crucial to the learning process. In this regard, Mason's (2002) techniques of ―Accounts-of" and ―Accounts-for" are key in building a capacity for PSTs to notice aspects of language and how this impacts the mathematical performance of ELLs.

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PROMOTING AN UNDERSTANDING OF MATHEMATICAL STRUCTURE IN STUDENTS WITH HIGH FUNCTIONING AUTISM

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In a study of Year 2 students, performance on a novel open-ended Make 10 task was one of two strongest predictors of students diagnosed at risk of mathematical learning difficulties (MD) on the Queensland Year 2 Diagnostic Net (Finnane, 2007). Students who performed poorly on this task produced few combinations, gave counting sequences or figurative responses featuring physical embellishments of the numeral 10, compared to a range of flexible responses by normally achieving students. This paper demonstrates the application of the Make 10 task to facilitating the conceptual and skill development of a Year 4 student with high functioning autism who was facing significant mathematics anxiety and pervasive mathematical leaning difficulties.

Introduction

In a special issue on mathematics in the *Journal of Learning Disabililities,* authors of a research review on early identification and interventions for students with mathematical learning difficulties described the field as ―in its infancy” (Gersten, Jordan, & Flojo, 2005). Gersten et al. stressed the importance of identifying the best predictors of early difficulties in mathematics as a guide to designing effective interventions for struggling students. Mazzocco (2005) further drew attention to the need to fully understand the nature of the mathematics learning difficulties of students with other significant cognitive and processing difficulties.

One group of students whose mathematics learning needs may prove challenging to teachers are students with high functioning autism. While a proportion of students with high-functioning autism (HFA) may have outstanding mathematical abilities, research suggests that up to half of students with HFA may face significant difficulties in learning mathematics (Chiang & Lin, 2007; Mayes & Calhoun, 2006; Reitzel & Szatmari, 2003). Chiang & Lin (2007) raised the need for assessments which can adequately measure the strengths and weaknesses of students with HFA. The cognitive profiles of students with HFA suggest that they may have difficulty in detecting patterns and distinguishing relevant details, and may find it difficult to conceptualise numbers as abstract concepts of comparative quantities. In addition, students with HFA are prone to anxiety (Attwood, 2007), which may further disrupt their ability to make mathematical connections. On the other hand, students with high functioning autism may be expected to show strengths in sustained attention and ability to master facts (Sansoti, Powell-Smith, & Cowan, 2010).

There is little available research to guide effective interventions for students with HFA and mathematics difficulties. Research on assisting academically low achieving students in mathematics focuses on the importance of developing number sense and rich mathematical concepts (Dole, 2003; Gersten & Chard, 1999; Gersten et al., 2005; Woodward, 2006). There is a danger that promoting rote learning by students with high functioning autism might inhibit the development of a meaningful understanding of mathematical concepts.

Australian researchers have identified important developmental frameworks and constructs which are helpful in establishing priorities for intervention for students with mathematics learning difficulties. Wright, Martland, and Stafford (2000) highlighted the critical role of mastery of forwards and backwards counting and fluent numerical identification skills in developing essential concepts of numbers as composite units and efficient strategies for solving basic additions and subtractions.

Mulligan, Mitchelmore, English, and Robertson (2010) have further demonstrated that progress in students' mathematical understanding depends on an understanding of underlying mathematical structure. Using the construct of Awareness of Mathematical Pattern and Structure (AMPS), Mulligan, Mitchelmore, and Prescott (2005) have shown that low achieving students have more difficulty in perceiving and representing visual patterns and mathematical structure and, most importantly, that these problems may be associated with weaknesses in multiple counting, partitioning, equal grouping and equal units of measure.

The present paper aims to contribute to the continuing growth of the field of early identification and interventions for mathematics learning difficulties by:

1. Presenting the results of a research project on early predictors of mathematical learning difficulties.
2. Applying the research findings to an intervention to support the mathematical development of a Year 4 student with high functioning autism, anxiety problems, and pervasive mathematics difficulties.

Method

The paper is presented in two parts: Part 1 identifies early predictors of mathematical learning difficulties, while Part 2 reports a case study of an intervention with a Year 4 student.

Part 1. Early predictors of mathematical learning difficulties

As part of a large study exploring early indicators of mathematics learning difficulties, a comprehensive set of mathematical, memory and processing tasks was administered to 68 students (mean age 7.1 years) in three Year 2 classes in metropolitan Brisbane, Queensland (Finnane, 2007). The mathematical tasks included forwards and backwards counting, numeral identification and strategy use for solving basic additions and subtractions, as assessed on the Learning Framework in Number (Wright et al., 2000). On the *Make 10* task that is the focus of this paper, students were asked: ―How many different ways can you make the number 10?" The task was administered after the author had modelled making a variety of number combinations for the numbers 7 and 9 using smaller numbers.

A subset of students (n = 17) was identified at risk of mathematical learning difficulties, determined by an independently administered state-based assessment process—the Queensland Year 2 Diagnostic Net (Education Queensland, 2007). The author has previously presented the results of t-test comparisons which showed significant differences in verbal memory capacity between students identified in the Net and normally achieving students (Finnane, 2008). Regression analyses were used to determine which of the measures were the best predictors of students who would be assessed at risk of mathematical learning difficulties on the Year 2 Net, with a view both to identification and intervention.

Results

Make 10 was one of two strongest predictors of the 17 students who were caught in the Queensland Year 2 Diagnostic Net as at risk of mathematical learning difficulties (Finnane, 2007). The other best predictor was the stage of early arithmetic learning (SEAL) as assessed on the Learning Framework in Number (Wright et al., 2000). Student responses on *Make 10* depicted different levels of number understanding, which were consistent with their stage of arithmetic learning. Net students tended to provide a number sequence 1, 2, 3, …, 10 or to focus on figural features of numerals only. Figure 1 shows the response of a student caught in the Year 2 Net, who depicted 10 as the initial two numerals of 4-digit numbers, with differing physical embellishments. Similarly, at a figural level, this student said you could ―make 9” by painting a 9, or you could ―make 7” by two people lying down, one horizontally to form the top ¯ and the other diagonally to form /. This student was assessed at SEAL 2 (Figurative stage) on the Learning Framework in Number, where he was using his fingers and a count-all strategy to solve basic additions with sums less than ten, but did not know how to solve combinations (e.g. 9 + 6) with sums greater than ten.

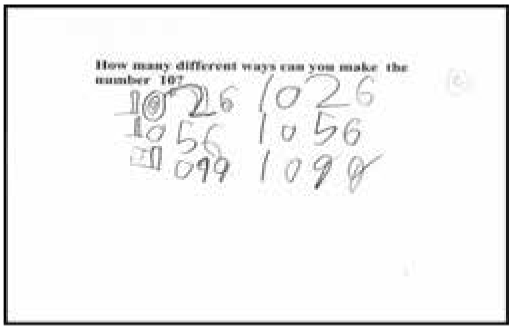

Figure 1. Make 10 *Figural level response of a Year 2 Net student.*

Normally achieving students showed a range of responses involving partitioning skills (see Figure 2) or a flexible use of operations (Figure 3), indicating a more advanced concept of numbers as composite units (Fuson, 1988). The responses of these students were consistent with their further progression to more fluent counting, numeral identification and advanced strategy use on the Learning Framework in Number.

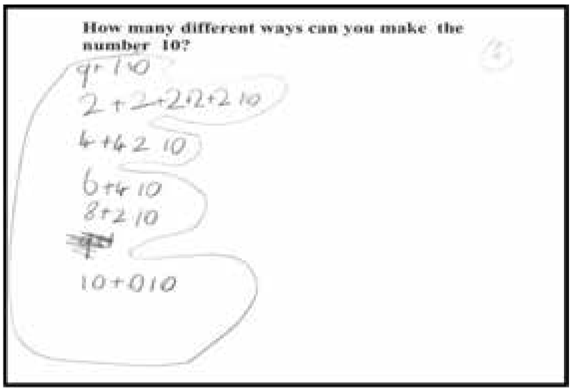

Figure 2. Make 10 *response of a normally achieving Year 2 student showing partitioning skills.*

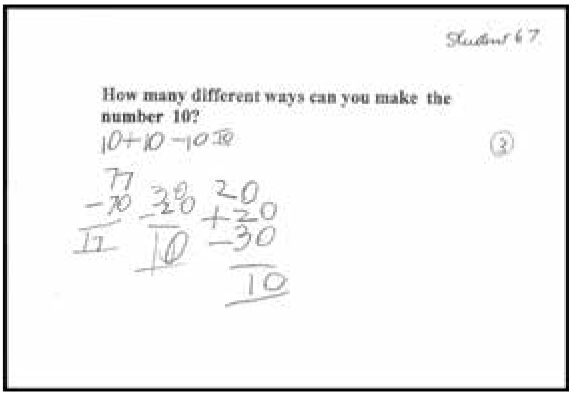

Figure 3. Make 10 *response of a normally achieving Year 2 student showing flexible use of operations.*

Part 2. Case study intervention

Assessment.

A Year 4 student aged 9.8 years with diagnoses of high-functioning autism, anxiety and learning difficulties was referred to the author by his Paediatrician for a mathematics assessment. The student will be referred to as Will, a pseudonym. Will was showing signs of significant stress during mathematics lessons within the classroom. His mother reported that he showed particular distress in relation to problem solving, did not seem to understand the concepts needed, and would cry on the way home from school in anticipation of his mathematics homework. The student was assessed using the mathematics subtests of the Wechsler Individual Achievement Test - Second Edition (WIAT-II), the Learning Framework in Number (Wright et al., 2000), and the *Make 10* task (Finnane, 2007).

Will performed in the Well Below Average range for his age on both the Mathematics Reasoning (1st percentile) and Numerical Operations (8th percentile) subtests of the WIAT-II. In Year 4, he was still unable to solve 2-digit additions and subtractions with regrouping and expressed a very high level of anxiety in relation to written number questions. Will showed persisting confusions between teen and -ty numbers (e.g., 13/30) in both oral sequence counting on the Learning Framework in Number and in written algorithms. Will's responses showed that he had developed only an initial concept of 10, where he focussed on the individual items that make up ten rather than ten as a unit. He was unable to match a quantity meaning to 2-digit algorithms, or to interpret and solve word problems.

Intervention

An intervention was designed for Will to address his mathematics anxiety, and to build his number sense, place value understanding, addition and subtraction concepts and problem solving skills. Will attended sessions on a weekly or fortnightly basis, with activities to complete between sessions. Will's parents were very supportive, providing assistance as needed. During the course of the intervention, Will's class teacher reduced the level of his set word problems from Grade 4 to Grade 2 level. This paper reports only one aspect of the follow-up intervention with Will, involving the *Make 10* task described above, and related number tasks (e.g. *Make 20*, *Make 100, Make 120*). *Make 10* was chosen as an integral part of the intervention, as it was observed in the assessment that Will enjoyed this task and he showed an initial level of familiarity with the Ten facts as components of 10.

The *Make 10* task was used on a repeated basis during the intervention to encourage Will to explore number composition and base-10 structure. After providing Ten Facts and multiplicative solutions to *Make 10* as he had done before, Will was excited when he thought of a subtractive solution 11–1 = 10. Figure 4 shows how Will maintained his attention on the pattern of subtracting the Ones from the teen numbers in order up to 19-9, and then expended considerable effort to subtract Tens up to 100.

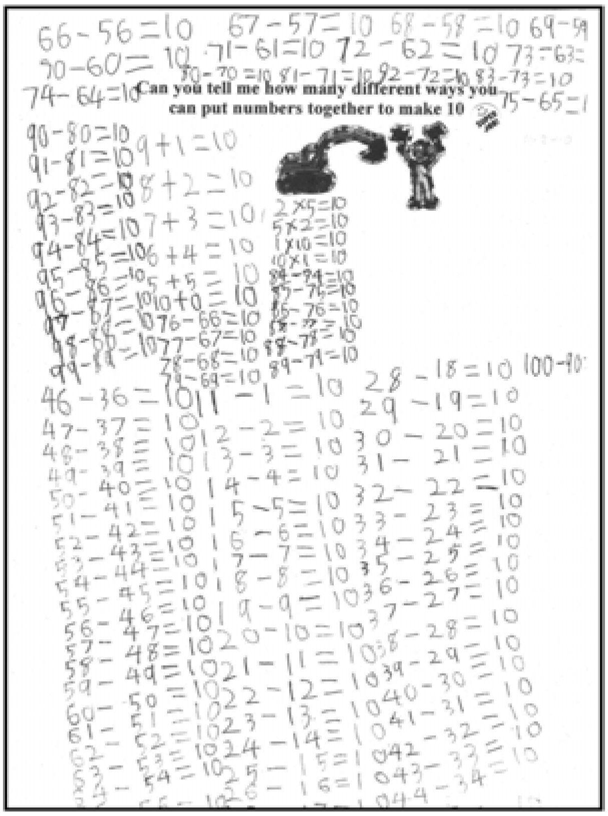

Figure 4. Will's response on the Make 10 task when he first discovered a subtractive solution.

This was an important session for Will. By using order to produce his responses in sequence, he gained a better understanding of the tens/ones composition of 2-digit numbers. Will was later able to generalise this ordering strategy for constructing ten to enable him to construct 3-digit numbers.

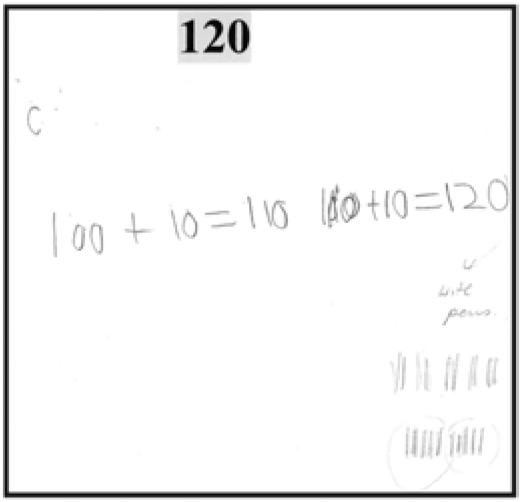

Figure 5. Will's initial response when asked to Make 120.

Figure 5 shows that, initially, Will found it very difficult when he was asked to *Make 120*. He became stuck and needed prompting to explore the composition of the missing quantity, after adding one 10 to 100 to make 110. Will understood this was not a sufficient response, but he could not work out what was missing. When I suggested he could use pens to find the missing amount, Will explored the component parts of 10 by dividing the pens into 5 groups of two, and then 2 groups of five. This partitioning enabled him to realise 10 was the missing part or addend of 120 he needed (Figure 5).

In the following session one week later, Will used order to produce several combinations to *Make 120*, after first taking 1 away from 120 to give the parts of 119 and 1 (Figure 6). He had started to see the inverse relationship between addition and subtraction more clearly. By sustaining his attention on the whole (120), Will was able to shift his attention to producing the different component parts, and he enjoyed depicting the decrementing and incrementing relationships. To finish, Will reproduced the combination 110 + 10 that he had initially struggled with in the previous session.

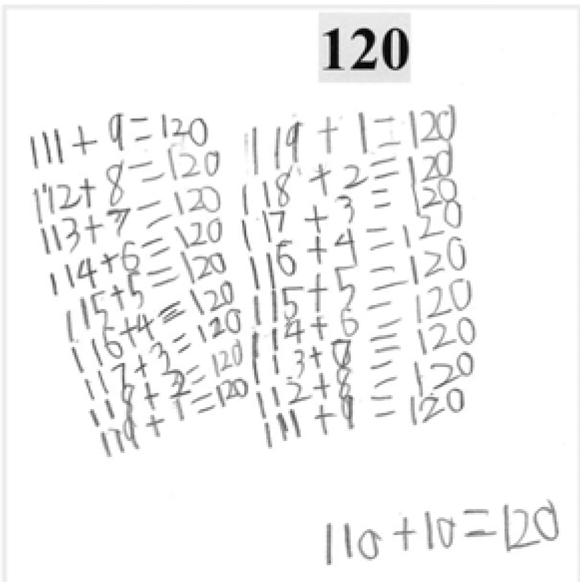

Figure 6. Will's second response when asked to Make 120.

An example from the next session (see Figure 7) illustrates Will's increasing flexibility in deconstructing and constructing numbers. Given a 2-digit number *Make 20*, he is no longer reliant on order to increase and decrease his responses in a unitary manner.

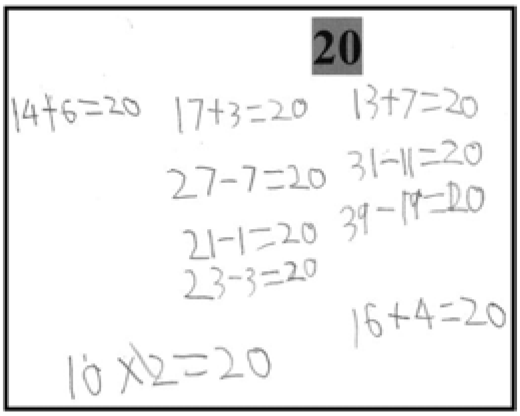

Figure 7. Will's flexible solutions when asked to Make 20.

Will's part-part-whole number concept development is further illustrated in Figure 8. Here he was able to use mental computation successfully to decompose numbers in multiple steps to perform a variety of subtractions. On the first item, Will explained that he broke up 800 into 700 and 100 and subtracted 80 from 100 to give 20, then added back the 700. By this stage, Will had developed the conceptual understanding of numbers as composite units which enabled him to carry out this mental computation confidently, and also to manage regrouping in subtraction up to 4-digits. By mid-year 5, after 20 intervention sessions, Will was in a mainstream mathematics class 90% of the time and achieving at Year 5 level. His teacher reported that "his whole mathematics has changed".

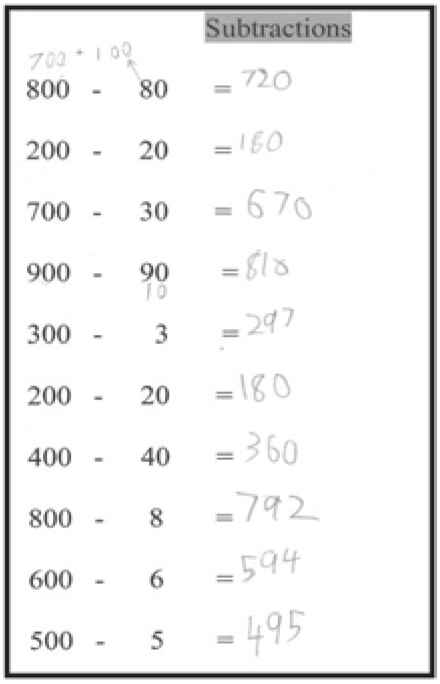

Figure 8. Will's responses to a subtraction task where he used partitioning and mental computation.

Discussion

The research described in this paper confirms the validity of counting stage on the Learning Framework in Number (Wright et al., 2000) as a predictor of performance on the Queensland Year 2 Net, and describes another discriminating task *Make 10* (Finnane, 2007) which has significance for interventions with students facing significant mathematics learning difficulties.

The paper illustrates how an open-ended assessment task (Make 10) assisted a Year 4 student with high functioning autism to explore mathematical structure and part-part-whole relationships in a way he had previously been unable to do. The opportunity to make the number 10 in as many ways possible on multiple occasions allowed the student to gradually discover and apply his understanding of addition and subtraction as inverse relationships. This understanding enabled him to develop a schema he could apply successfully to additive and subtractive problem solving (Xin & Jitendra, 2006). The intervention also had a marked impact on reducing the student's anxiety and in significantly reducing familial stress associated with mathematics homework. It is argued that the open-ended nature of the task together with a specific limited instruction was empowering for the student in accessing his existing knowledge and allowing him to make new connections to this knowledge.

While students with high-functioning autism might be able to learn facts by rote, particular attention should be paid to their level of conceptual understanding of number. The *Make 10* task can provide a useful intervention tool for facilitating students' development from a unitary concept of number to a flexible understanding of part-part-whole number structure. Future research can determine whether the open-ended nature of *Make 10* provides a useful tool for lowering anxiety in relation to a range of written mathematics topics in highly anxious students.

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GRAPH CREATION AND INTERPRETATION: PUTTING SKILLS AND CONTEXT TOGETHER

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Although the creation of graphs to display data has been part of the school curriculum for some time, the call in the new Australian Curriculum for "numeracy across the curriculum" provides both the opportunity and challenge to link the skills of graph creating with the understanding of context in order to produce meaningful interpretation of the messages held in graphs. This paper reports on classroom experiences of and follow-up interviews with 12 grade 5/6 students who were introduced to the software *TinkerPlots* to assist in graph creation. The focus is on their success at graph creation and interpretation in contexts that provide potential links to other subjects in the school curriculum. Implications for the curriculum and teaching are drawn from the students' experiences.

Introduction

As statistics continues to occupy a place in the Australian Curriculum—Mathematics (ACARA, 2010), the justification for this place is often built upon its application "in context" (Mooney, 2010). In fact, many years ago Rao (1975) claimed, "Statistics ceases to have a meaning if it is not related to any practical problem" (p. 152). These statements suggest that the cross-curriculum opportunities of statistics should make it a major component of numeracy across the curriculum, as set out for the Australian Curriculum by the National Curriculum Board (NCB, 2009). The question for those developing the pedagogy for statistics is, "How do teachers link the development of skills with the provision of motivating context?" In turn, the question for the assessment of student learning is "What mix of skills and understanding of context is expected of students when they are drawing informal statistical inferences?" These questions may be further complicated by the stipulation by the NCB (2009) that technology be employed across the curriculum as appropriate. The Australian Curriculum–Mathematics (ACARA, 2010) carefully avoids mention of specific technology, and in particular of statistical software packages. Acknowledging the importance of software to enhance the teaching and learning of statistics, this study employed the software for middle school, *TinkerPlots* (Konold & Miller, 2005), to explore the issues raised in these questions. Twelve students' experiences with *TinkerPlots* and data sets are explored in two settings: four teacher-led classroom sessions and a one-to-one assessment interview with researchers.

Underlying model

The model used for the analysis of the data in the study was developed by Watson and Fitzallen (2010) in a research report commissioned by the NSW Department of Education and Training. The model posits three cycles of development of graph understanding for the purpose of drawing informal inferences while employing graphical representations of data sets. The model is based broadly on the work of Biggs and Collis (1982, 1991) in cognitive psychology. Although acknowledging other factors that influence learning, this model is based on the combining of basic cognitive content elements to construct the understanding required to be successful in each of the three cycles. The elements involved for each cycle are derived from the research in the field and are combined in various ways to produce the understanding necessary to create meaningful graphs.

The first cycle consists of building up the Concept of Graph from the elements of Attribute, Data, Variation, and Scale. Because of the several types of complexity associated with the data that are represented in graphs, there are two parallel second cycles in the overall model. One cycle considers multiple attributes and the other, large data sets. For the purpose of this study, only the cycle related to multiple attributes is employed in the data analysis. The Concept of Graph for Multiple Attributes is based on the following elements: Concept of Graph (from the first cycle), Types of Attributes (e.g., categorical and numerical), Two-dimensional Scaling, and the Relationship of Two Attributes to a Single Case. The aim of this study is to explore the creation of graphs representing two or three attributes. The third cycle, Graph Interpretation is based on the elements: Concept of Graph (from cycle 1 or cycle 2), Concept of Variation, Concept of Average, Context, and Critical Questioning Attitude.

In this study variation and average were not explicitly taught in the classroom but the introduction of hat plots, where the crown covers the middle 50% of the data and the brims extend to the extremities, was intended to develop intuitions about these concepts. Context played an important role in motivating the students in both the classroom and interview settings and students were encouraged to make hypotheses and be critical of what was represented in the graphs.

Research questions

With reference to the questions asked at the beginning of this paper and the underlying model, the data for the 12 students were analysed to answer the following research questions.

1. Within a grade 5/6 classroom teaching setting, what skills for Graph Creation for Multiple Attributes leading to the use of contextual understanding for Graph Interpretation are developed?
2. Within a one-to-one computer-based interview setting employing software, what skills for Graph Creation for Multiple Attributes leading to the use of contextual understanding for Graph Interpretation are transferred and applied?

Methodology

The subjects in the study were 12 students from a grade 5/6 class at a rural primary school (K–6) in Tasmania, Australia. The 12 were members of a class of 26 who participated in four lessons with the first author using *TinkerPlots* to analyse the class' resting and active heart rates. Throughout the experience, the students were introduced to bins, stacked dot plots, reference lines, hats, scaling, and scatterplots. Students were asked to save the graphs they produced and the text boxes in which they wrote their interpretations and conclusions from their graphs.

A month later the students were videotaped as they were interviewed by the second author or another researcher in a study to explore the effectiveness of paper-and-pencil versus on-line assessment of students' statistical understanding (Watson & Donne, 2009); these students were among the 24 on-line subjects using *TinkerPlots* as the software in that study. Transcripts of the videotapes of the students' interviews and *TinkerPlots* files were used as the data for this study. Earlier analyses of related data from these students are found in Fitzallen and Watson (2010).

Student data

In the classroom, students had access to their own heart rate data both before and after a jump-rope activity. They could hence identify the data cards and icons with their fellow students or themselves. The other attribute in the classroom was gender. For the interview the data set consisted of data for 16 students, named (but unknown), along with their age, eye colour, weight, favourite activity, and number of fast food meals eaten per week (data found in Chick & Watson, 2001).

Analysis

All students in this study satisfied the criteria for the first cycle of the model of Watson and Fitzallen (2010), understanding the Concept of Graph, a basic element of the second cycle, creating a Graph for Multiple Attributes. The other elements of the second cycle all contribute to the forms of graphs that were created by the students. The types of attributes, categorical or numerical, determined the types of plots that could be created in *TinkerPlots*, either with "bins" or "scaled" axes. Bins are used for categorical data (such as gender) or for numerical data that are grouped together (such as age ranges of 7–13 years and 14–20 years). A scaled axis can only be used with numerical data.

In considering two attributes, it was hence possible depending on the type, to create six different plots in *TinkerPlots*. These are shown in Figure 1 on the following page. Students' skills in graph creation and interpretation are displayed as they create these different plots with the software and discuss what they mean.

For the third cycle of Graph Interpretation the most significant element beyond using the second cycle of creating Graphs for Multiple Attributes for this study is the element of Context. In relation to the tasks provided to the students during the classroom sessions and interviews, there were three aspects of student introduction of context. In the first case either the individual icons on the plot, or the data cards themselves, were used to provide context for the individual people represented in the data set. In the second case, the graph was used to provide a contextual description about trends observable in the graph (not individuals). In the third case, students went beyond the information in the graph to create stories about the people in the data set, to speculate about why relationships were seen, or to give advice about how to mitigate a trend seen in the graph.

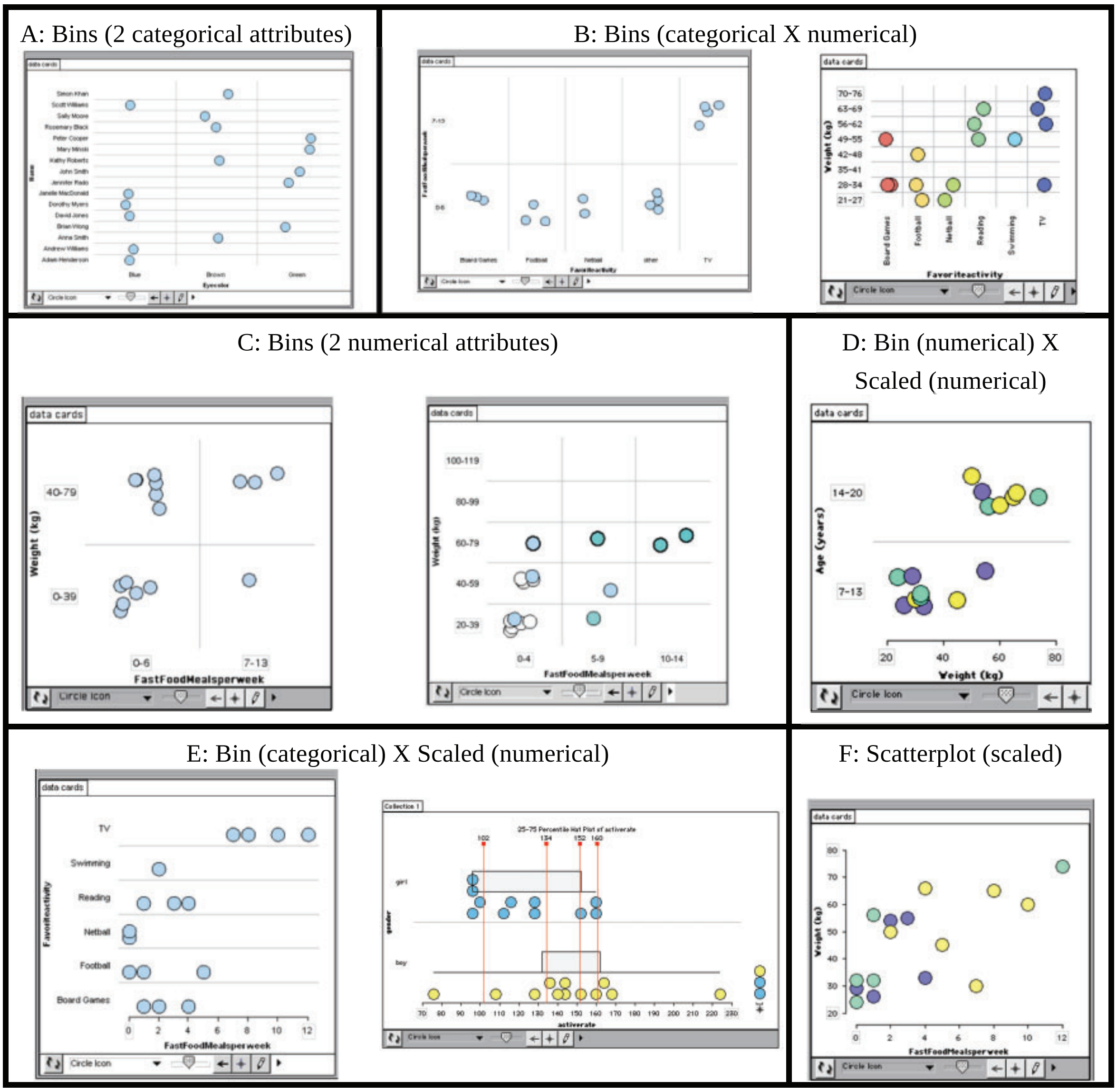

Figure 1. Six different types of graphs for two attributes in TinkerPlots.

The two groupings for skills of Graph Creation and use of context for Graph Interpretation were used to categorise the responses of students in the two situations: first the work created by the students as part of their four classroom lessons and second as part of their follow-up interviews.

Results

In the classroom (Research Question 1)

The students created four of the types shown in Figure 1 (B, C, E, and F) as well as using bins or stacked dot plots for single attributes. It was not possible to create bins for two categorical attributes (A) because only gender was available and because of their concurrent experience with scaled stacked dot plots, it would not be expected that students would combine bins with scaled axes for the numerical attributes (D). The text boxes saved in the files provided the material to assess the extent to which context was used in the interpretation of the graphs created.

Although there is no doubt that to some extent the files represented interaction with the teacher and perhaps other students, they present a picture of the students' beginning appreciation and understanding of the data handling task and its purpose relating to heart rates under different conditions. Even graphs of the same type had different appearances in different students' files because of the flexibility of plot creation in *TinkerPlots*.

Except for one student who forgot to save his file, the other 11 students created between 3 and 6 graphs, for a total of 48. Fifteen graphs were based on a single attribute, with two displaying categorical attributes (name or gender), three displaying the numerical attributes in bins (hence showing ranges), and 10 stacked dot plots of one or other of the heart rate data sets (with scaled axis). Thirty-three of the graphs produced showed the relationship between two attributes. Four graphs used bins for gender and one of the heart rate sets (B), 2 used bins for numerical data on both axes (C), 16 used bins for gender and scaled the heart rate data (E), and 11 were variations of scatterplots (F).

The students' interpretations of the graphs recorded in the text boxes in *TinkerPlots* were placed in three categories. From their graphs, three students reported individual values for interesting participants; e.g., the highest or lowest rate, or the rates for their teachers. Ten of the students wrote more generally about what the graph portrayed about the data set, e.g., there is a larger range for the boys, and 50% (or most) of the people are between 80 and 110. These comments were based specifically on reading trends or clusters in the graph. Going further than the graphs, using their personal knowledge, six students added other interpretations to the data. Some created explanatory text, such as the heart pumps more blood for the active rate, and "less average" for the girls so girls are more healthy. In contrast to this declaratory text, some students speculated on causes, e.g., the difference might be due to boys being into video games or Mrs. M. might be skinnier than everyone. Some of the students made more than one type of comment for a graph.

From classroom to interview (Research Question 2)

In the interviews, the six attributes (three categorical and three numerical) provided the opportunity for students to choose up to six types of plots. No suggestions of graph types were made by the interviewers, so all were "spontaneous" from the students. The 12 students each produced between 2 and 5 plots for a total of 42 graphs. Only one student plotted two categorical attributes in bins whereas 10 used bins to consider one attribute of each type (e.g., favourite activity and weight) (a total of 16). Six students produced 8 plots with bins using two numerical attributes. Seven students also created 8 plots with a categorical attribute and a scaled numerical attribute. Four students produced a mix of bins and scaled plots for two numerical attributes (a total of 5) but only one went on to produce a scatterplot. Two other students produced scatterplots having elsewhere created a scaled numerical and categorical plot (a total of 3).

In the interview, 10 students provided context from the individual data values, such as the highest and lowest values for an attribute. One student clicked on icons in a scatterplot to discuss values for a third attribute found on the data card. Nine of the twelve students made more general comments about the data set specifically from the relationships portrayed in the graphs they produced. These included trends such as more sport means less weight, younger are more active, and those who weigh more eat more fast food. Six students went on to add more to their explanations from their life experience. Some created stories to match what they observed, such as those who watch TV see junk food, and those who watch TV are lazy. Others speculated on reasons for the trends, rather than making declarative statements, such as maybe they are lazing around and watching TV and not getting up and doing anything. Two students gave advice to students in the data set, e.g., they need to go out more and they should cut down on fast foods and eat vitamins.

The performance of the students in the interviews where they encountered twice as many attributes as in the classroom indicated that they had internalised many of the graphing skills presented there. They showed flexibility by creating plot types that they had not used in the classroom. At times students were content to use a graph with bins rather than scaled axes for numerical attributes (type C in Figure 1) to interpret the data because the particular trend could be seen easily there. Although perhaps more experienced graph creators might always use scaled axes for numerical attributes, these students showed that such conventions are not always needed.

The students' use of their contextual knowledge was more varied in the interviews. They made more comments on individual cases than in the text boxes created in the classroom. This was most likely because (i) there would have been initial oral discussion of classroom data among students as they were entered; and (ii) the data in the interview were all new and hence it was interesting to check out the characteristics of the people represented. The greater number of attributes in the interview data may have contributed to more extensive discussions and the students may have also found it easier to articulate their thinking verbally than in text boxes.

Implications for the curriculum and teaching

This study shows that students in grade 5/6 are able to pick up graphing skills and application of contextual understanding in meaningful ways. Of particular interest is the ability to deal with two attributes in relationships, with bins or scatterplots, when scatterplots are not specifically mentioned in the Australian Curriculum – Mathematics until Year 10 (ACARA, 2010, p. 46). This seems to be an oversight. The first mention of considering two attributes, by creating side-by-side column graphs in Year 6, appears to be mentioned later and be more basic than is necessary. From the results of this study, it is seen that students can deal with two attributes and create and interpret relatively sophisticated representations by this time. The mention of considering two attributes in the curriculum is often done with no suggestion of the types of graphs that should be employed (e.g., in Year 7, where "investigating issues" has elaborations about two attributes but with no specific graph types suitable for the level).

The fact that the curriculum mentions creating graphs with (and without) digital technology from Year 3 is appropriate and encouraging. After the initial classroom introduction, the students in this study required virtually no assistance in working with *TinkerPlots* to create graphs.

The first specific mention of context in relation to Statistics and Probability in the Australian Curriculum – Mathematics is in Year 5, with reference to interpreting different graphs (ACARA, 2010, p. 29).

Context is implicit, however, in all grades in the elaborations, often from Year 4 in relation to examples presented from the media. The evidence from this study supports this recommendation—but teachers need to be aware that discussion of context needs to be developed from two standpoints: that of interpreting contextual information specifically available within the graph itself (e.g., trends observed) and that of the extra real-world understanding brought to the data set by the students' life experiences. Both aspects of context are essential for students to become critical statistical thinkers (Mooney, 2010). The link to other areas of the curriculum is crucial to provide the background for students to be able to go beyond the information provided in a graph and perhaps ask further questions.

The students in this study are typical of students from working class and rural schools in Australia. They hence challenge the writers of the Australian Curriculum—Mathematics to acknowledge that with creative software, it is possible to progress rapidly to meaningful representations of two attributes and make sense of these in real-world contexts.

Particularly at the primary school level, the opportunities to introduce cross-curricular contexts that link to the statistics component of the Australian Curriculum—Mathematics abound, especially if constructivist software is available for students to use. After developing the basic concept of graph, software can assist in the rapid consideration of various representations of data sets, in order for students to choose the one that best suits the story they wish to tell. By the time students reach secondary school, they should have a repertoire of graph types that can be used in various subject areas, and the ability to use them. It is then up to teachers across the curriculum to collaborate so that all teachers, not just teachers of mathematics, are aware of the possibilities that can be further developed with their students.

The Australian Curriculum—Mathematics mentions the use of media extracts in several places to motivate critical thinking in statistics. These opportunities not only may be based on unusual or inappropriate use of statistics but also may lead to in-depth cross-curriculum collaboration involving current real-world contexts. This will happen most effectively if all teachers are aware of the possibilities, including the application of "digital technologies."

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TWO AVATARS OF TEACHERS' CONTENT KNOWLEDGE OF MATHEMATICS

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In this research we explore pre-service teacher knowledge for teaching mathematics by focusing on the development of the conceptual and procedural knowledge of a cohort of pre-service teachers. In the first phase of this study, we found that a previous cohort of pre-service teachers utilised procedural rather than conceptual knowledge when completing fraction operations. We aimed to address this imbalance by targeting the development of conceptual knowledge through modelling. This paper reports the results of this approach with a subsequent cohort of pre-service teachers, where our expectation of greater conceptual knowledge was achieved and procedural knowledge was maintained.

Introduction

The role of teacher knowledge has been acknowledged as vital in teachers doing their jobs. This issue has been a central concern of the mathematics teaching community both in Australia and elsewhere. We take up this issue in the present study.

Teacher knowledge for teaching mathematics

Shulman's (1986) seminal work on teacher knowledge identified a range of different types of knowledge necessary for teachers to teach effectively. While he acknowledged the essential role of pedagogical knowledge, he highlighted the importance of content knowledge, which he categorised into subject matter knowledge, pedagogical content knowledge and curricular knowledge.

Teachers' mathematical content knowledge affects the quality and nature of their teaching (Schoenfeld, 2000) and has been found to positively predict student achievement (Hill, Rowan, & Ball, 2005). There is little disagreement that teachers need to acquire and understand mathematics in order to teach it effectively.

In acknowledging the multidimensional character of teacher content knowledge, Ball and associates (Ball, Hill, & Bass, 2005; Hill et al., 2005) refined and developed four dimensions of this knowledge: Common Content Knowledge, Specialised Content Knowledge (SCK), Knowledge of Content and Students, and Knowledge of Content and Teaching.

SCK refers to the particular way teachers of mathematics have to understand their content. This involves, among other things, a 'repackaging' of their formal mathematical knowledge. The current study is aimed at better understanding the SCK of prospective teachers. Specifically, we focus on two subsidiary components of SCK, namely, procedural and conceptual knowledge within the domain of fractions. We refer to these strands of knowledge as constituting two *avatars* (Sanskrit word for manifestation) of teacher knowledge in the sense that each of the components are incarnates or embodiments of one key knowledge form, namely, SCK which is the focus on the present study.

Procedural versus conceptual knowledge

Broadly speaking, procedural knowledge involves understanding the rules and routines of mathematics while conceptual knowledge involves an understanding of mathematical relationships. The relationship between procedural and conceptual knowledge, and the dependency of one on the other, continues to be a legitimate concern for mathematics teachers and researchers alike. Schneider and Stern (2010), in examining potential interconnections between the two, suggest that teaching and learning research needs to examine their parallel developments. Within the context of primary mathematics, and in particular fractions, Mack (2001) suggests that children's use of strategies for representing and solving fraction problems are based on both these knowledge strands.

The relationships amongst and the relative role of these two main dimensions of knowledge that is relevant to decoding and solving fractions problems needs further clarification if we are to better inform teachers and knowledge underlying teaching. The debate on this issue appears to proceed along three lines. One view is that children learn conceptual knowledge of fractions before procedural knowledge (Groth & Bergner, 2006). A second view is that children learn procedural knowledge before conceptual knowledge (Baroody, Feil & Johnson, 2007). Finally, it would seem that children's conceptual knowledge and procedural knowledge grow in tandem with one building on the other (Schneider & Stern, 2010). While this debate is continuing, recent research by Hallett, Nunes and Bryant (2010) suggest that a) some children rely on procedural knowledge to inform conceptual knowledge and b) those who rely on conceptual knowledge of fractions tend to have an advantage over those who rely on procedural knowledge. Taken together, these findings suggest that teachers need to have a sound understanding of both these knowledge categories that involve fractions. That is, despite a growing call from some quarters to underplay the role of procedural knowledge in favour of conceptual knowledge (Rittle-Johnson, Siegler, & Alibali, 2001), teachers need to develop a repertoire of both these streams of knowledge as these are legitimate and necessary parts of the corpus of knowledge used by learners that teachers need to know. In this sense conceptual and procedural knowledge are important components of teachers' SCK, and the investigation of this knowledge is a major aim of this study.

Theoretical and Conceptual framework

The aim and analyses of data in the present study are guided by two broad theoretical constructs. In the first instance, we draw on Ball et al.'s (2005) dimensions of teacher knowledge that inform mathematics teaching. Secondly, we examine the interplay between conceptual and procedural knowledge within the *Representational-Reasoning* (RR) model of mathematical understanding provided by Barmby, Harries, Higgins and Suggate (2009). According to this model, the quality of mathematical understanding can be captured by a) the type of representations that learners construct; and b) the robustness of reasoning that is used in establishing or justifying relations among the representations. We see the RR model as somewhat unbiased in the interpretation of the relative roles of conceptual and procedural knowledge, as both components can be foregrounded in the representations and reasoning.

Issues and aim

The review of literature on teacher knowledge and teachers' performance in relation to children's numeracy levels has highlighted the need to research and monitor the developing knowledge of mathematics teachers who are in practice and those who are in training. Initially we investigated this issue by analysing the procedural and conceptual knowledge of fraction operations of a cohort of pre-service teachers (Forrester & Chinnappan, 2010). The results of this analysis demonstrated clearly the dominance of procedural knowledge over conceptual knowledge in this group, with almost four times the number of pre-service teachers activating procedural knowledge in comparison to those that demonstrated conceptual knowledge in their solution attempts. About one fifth of responses evidenced neither procedural nor conceptual knowledge.

While both knowledge categories are important, the dominance of one over the other would seem to be unhealthy for classroom practice, as teachers will have to support the development of both procedural and conceptual knowledge in their students across all strands of primary mathematics including fractions. This line of reasoning motivated us to modify our teaching strategies with the view to enhancing the conceptual component of our pre-service teachers' knowledge of fractions.

The aim of this study was, therefore, to ascertain the impact of a model based teaching (MBT) approach on the development of procedural and conceptual knowledge in the domain of fractions. This guided us in the development of the following research questions:

1. Does a model-based teaching approach have an impact on the development of pre-service primary teachers' procedural knowledge of fractions?
2. Does a model-based teaching approach have an impact on the development of pre-service primary teachers' conceptual knowledge of fractions?

Methodology

Participants

Two hundred and twenty-four students (37 males and 187 females) participated in the present study. They were enrolled in a first year compulsory subject, which is generally completed in the second semester of a four-year Bachelor of Primary Education degree. Prior to entry into the program, the participants had a range of mathematical backgrounds.

Procedure

Model-based teaching

Subsequent to the analysis of the 2009 cohort of pre-service teachers' conceptual and procedural knowledge of fraction operations discussed earlier (Forrester & Chinnappan, 2010) changes were made to the delivery of the subject in 2010. Utilising Barmby et al.'s (2009) notion that robust mathematical understanding is demonstrated when learners can construct and utilise multiple representations of mathematical ideas and can justify the relationships among representations, we focused on enabling our students to develop models of fraction operations and appropriate explanations of these models (MBT approach).

Tasks

The following tasks were two parts of one question in a fifteen-question examination. They were selected from a pool of thirty-five questions given to students in their subject outlines at the beginning of the semester. These particular tasks were chosen to examine students' mathematics content knowledge in terms of their conceptual and procedural knowledge of fractions and fraction algorithms. While the fractions were different from those given in the subject outline, the format of the questions was identical and students had been able to engage with similar questions throughout the session to consolidate their procedural and conceptual understandings.

Task 1: Division problem involving a mixed number and fractions with different denominators

$$1\tfrac{1}{2} \div \tfrac{1}{4}$$

Two algorithms could be used to complete this task. Firstly students could: change the mixed number into an improper fraction; invert the divisor; multiply the numerators and denominators; check if the answer can be simplified. Alternately students could: change the mixed number into an improper fraction; identify a common denominator of the dividend and divisor; change the dividend and divisor to equivalent fractions; divide the numerators and denominators; check if the answer can be simplified.

One conceptual understanding of this task involves the notion that $1\tfrac{1}{2} \div \tfrac{1}{4}$ involves finding how many $\tfrac{1}{4}$s are in$1\tfrac{1}{2}$. Partitioning $1\tfrac{1}{2}$ into quarters and counting the number of quarters will achieve an answer of 6.

Task 2: Addition problem involving a mixed number and fractions with different denominators

$$1\tfrac{5}{6} + \tfrac{2}{3}$$

Again, two algorithms could be used to complete this task, both involve most or all of these procedures: changing the mixed number to an improper fraction; identifying a common denominator of the addends; changing the addends to equivalent fractions; performing the addition; checking if the answer can be simplified. A conceptual knowledge of this task involves these elements: when the addends are modelled visually the *wholes* to which they relate are the same size; equivalent fractions e.g., $1\tfrac{5}{6}$ is the same size as $\tfrac{11}{6}$, $\tfrac{2}{3}$ is the same size as $\tfrac{4}{6}$, $\tfrac{15}{6}$ is the same size as $2\tfrac{3}{6}$ which is the same size as $2\tfrac{1}{2}$; addition involves joining two or more quantities together.

In undertaking the tasks, students were asked to complete the calculations and provide models and explanations of their models. They needed to use an appropriate algorithm for carrying out the required operation with fractions. The successful use of an appropriate algorithm would indicate that students have a procedural understanding and concomitant use of procedural knowledge. Conceptual understanding of these tasks involves demonstrating the nature of fractions (equal parts of a whole object or group) including the meaning of the common fraction symbol—as opposed to the misconception common among children that the numerator and denominator are simply two whole numbers (NSW Department of Education and Training, 2003). Additionally, a conceptual understanding of the tasks involves grasping what happens when dividing and adding fractions, including the relationship between the fractions involved.

Coding scheme

Students' responses to each of the two problems were analysed in terms of their demonstration of conceptual and procedural knowledge, and coded using a ten code scale (see Table 1). This coding scheme is a refinement of the one used to analyse the data collected and analysed previously (Forrester & Chinnappan, 2010) which was developed using the theoretical framework of Barmby et al. (2009) and Goldin's (2008) analysis of problem representations. We wanted to modify our codes to allow for greater differentiation of responses, in terms of conceptual and procedural knowledge. This scale also includes a code for a category of responses that did not occur in the previous data (Code 8), where a correct solution was achieved through a conceptual model and no algorithm was utilised.

Table 1 - Coding Scale.

Code	Algorithm	Model/Explanation
0	None provided	None provided
	Explanation: Where there is a response it is just an answer with no algorithm or model/explanation.	
1	Inappropriate	None or incorrect conceptual representations
	Explanation: An algorithm has been used but it is not appropriate for the problem. If a model/explanation has been provided it is incorrect conceptually.	
2	Inappropriate with correct elements	None or incorrect conceptual representations
	Explanation: While the algorithm used was inappropriate to the problem, important fraction processes were used e.g., making equivalent fractions, changing mixed numbers to improper fractions. If a model/explanation was provided it was incorrect conceptually.	
3	Appropriate but errors made	None or incorrect conceptual representations
	Explanation: The algorithm used was appropriate for the problem but an error occurred in its use. If a model/explanation was provided it was incorrect conceptually.	
4	Appropriate, used correctly	None or incorrect conceptual representations
	Explanation: The algorithm used was appropriate for the problem and achieved a correct answer. If a model/explanation was provided it was incorrect conceptually.	
5	Appropriate but errors made	Some level of conceptual representation
	Explanation: The algorithm used was appropriate for the problem but an error occurred in its use. The model/explanation utilises some level of conceptual representations.	
6	Appropriate, used correctly	Thorough procedural representation No conceptual representation
	Explanation: The algorithm used was appropriate for the problem and used to achieve a correct answer. The model/explanation was a detailed representation of the algorithm but did not demonstrate the concepts involved in fraction operations.	
7	Appropriate, used correctly	Some level of conceptual representation
	Explanation: The algorithm used was appropriate for the problem and used to achieve a correct answer. The model/explanation utilises some level of conceptual representation.	
8	None provided	Strong conceptual representations.
	Explanation: No algorithm was used. A correct answer was achieved using a conceptual model.	
9	Appropriate, used correctly	Strong conceptual representation
	Explanation: The algorithm used was appropriate for the problem and used to achieve a correct answer. The model/explanation was conceptually correct.	

Inter-rater reliability analysis

In order to determine the reliability of the coding scheme, we assessed the extent to which two coders agreed when they independently categorised students' responses. The two researchers coded twenty-five students' responses independently. The inter-coder reliability analysis, using the Kappa statistic, was performed to determine consistency. The inter-coder reliability was found to be Kappa = 0.77 ($p < 0.001$), 95% CI (0.504, 0.848), indicating substantive agreement (Landis & Koch, 1977) in the way the students' responses were coded by each researcher. Potential areas of disagreement were analysed which helped us to improve the distance between the codes, thereby reducing areas of ambiguity.

Data and analysis

Quantitative data analyses were conducted with the aid of SPSS version 18. Our analyses focused on the above ten categories of problem representation; the scale of our data was nominal.

In this paper we report the results of our analysis of student examination responses following a semester of lectures and tutorials that focused on developing conceptual and procedural knowledge through modelling and explanation (2010 cohort). We compare these results with those reported previously (2009 cohort) (Forrester & Chinnappan, 2010).

The data were analysed in terms of the two research questions:

1. *Does a model-based teaching approach have an impact on the development of pre-service primary teachers' procedural knowledge of fractions?*

The proportion of pre-service teachers who were able to find correct solutions to the fraction operation tasks using algorithms was not considerably different over the two years. Of the 2010 cohort, 74.6% (÷) and 76.4% (+) of participants were able to demonstrate the competent use of appropriate algorithms to achieve correct solutions in the division and addition tasks respectively (See Figures 1 and 2 - Codes of 4, 6, 7, 9). Within the 2009 cohort, 79.6% (x) and 72.6% (−) of participants were able to demonstrate the competent use of appropriate algorithms to achieve correct solutions in the multiplication and subtraction tasks respectively.

The proportion of pre-service teachers unable to achieve a correct answer using procedural or conceptual knowledge decreased slightly over the two years: Of the 2009 cohort, 20.4% (x) and 27.4% (−) of participants did not achieve correct solutions in the multiplication and subtraction tasks respectively. Within the 2010 cohort, 17.9% (÷) and 22.9% (+) of participants did not achieve correct solutions within the division and addition tasks respectively (See Figures 1 and 2 - Codes 0, 1, 2, 3, 5).

The impact of model-based teaching on pre-service teachers' procedural knowledge is somewhat unclear because the data collected in 2009 and 2010 are not markedly different.

2. *Does a model-based teaching approach have an impact on the development of pre-service primary teachers' conceptual knowledge of fractions?*

The majority of pre-service teachers in the 2010 cohort exhibited strong conceptual understanding, with 65.6% (÷) and 55.8% (+) (See Figures 1 and 2 - Codes of 8 and 9) being able to successfully model and explain the mathematical concepts involved in division and addition operations. A further 1.8% (÷) and 4.9% (+) were able to demonstrate some level of conceptual understanding (See Figures 1 and 2 - Code of 7).

Interestingly, 7.6% of participants (17 students) were able to achieve a correct answer to the division task (÷) utilising a conceptual model without utilising an algorithm. Two participants (0.9%) achieved a correct solution to the addition task (+) utilising a conceptual model without using an algorithm (See Figures 1 and 2 - Code of 8). Given that students were required to provide a calculation, any omission in providing evidence of procedural knowledge can be interpreted as not having this knowledge.

In comparing these results with those of the previous cohort (2009), it seems that a model-based approach to teaching has contributed to the substantial differences in our pre-service teachers' demonstration of conceptual understanding of fraction operations over the period of this research. There are difficulties in making direct comparisons between these sets of data as we examined multiplication (x) and subtraction (−) in 2009 and division (÷) and addition (+) in 2010. However, many of the concepts and procedures in finding a solution for these tasks are the same. In 2009, 11.8% of participants could demonstrate conceptual knowledge in multiplication (x) and 18.8% in subtraction (−). In the present study, 65.6% of participants evidenced conceptual knowledge in division (÷) (58% demonstrating both conceptual and procedural knowledge) while 55.8% could demonstrate conceptual knowledge in addition (+) (54.9% demonstrating both conceptual and procedural knowledge). We regard this as supporting our expectation of the positive impact of the model-based approach.

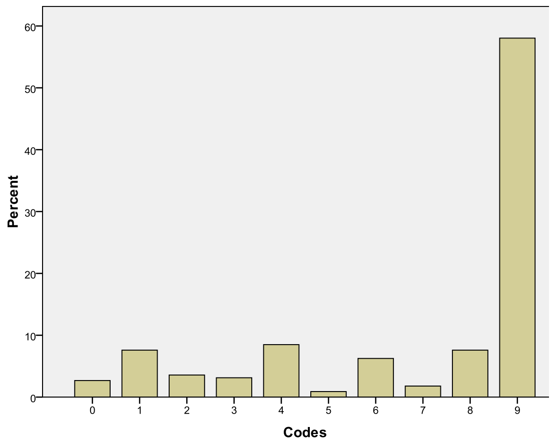

Figure 1: Coding frequency for Division.

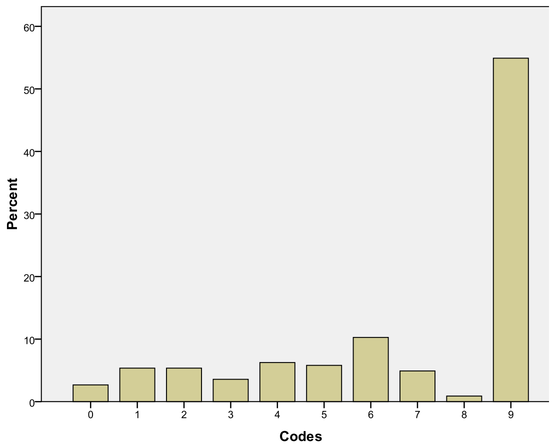

Figure 2: Coding frequency for Addition.

Discussion and implications

The study was grounded on the assumption that teachers' Specialised Content Knowledge of Mathematics (Ball et al., 2005) needed to have both procedural and conceptual characteristics in the domain of fractions. While conceptual knowledge may subsume procedural knowledge and indeed contribute to a better understanding of related procedural knowledge, it is important to capture and support both strands of knowledge for future teachers of mathematics.

The research questions were concerned with the impact a teaching approach that was based on modelling would have on the development of pre-service teachers' procedural and conceptual knowledge of fractions. The results here suggest that, while there was no tangible effect on procedural knowledge, our teaching had a positive effect on pre-service teachers' conceptual knowledge. The design of the MBT was guided, in the first instance, by an analysis of the state of pre-service teachers' knowledge within a narrow domain of context-free fraction problems. This analysis, it would seem, is critical for the design of MBT for fractions or similar approaches for other areas of primary mathematics. MBT was also framed around the notions of representations and reasons (Barmby et al., 2009) which aided us in visualising the role of conceptual and procedural knowledge in comprehending and making progress with the fraction problems.

The role of Barmby et al.'s (2009) framework in the development of MBT constitutes an important outcome of this research. We found the framework to be useful in drawing the distinction between procedural and conceptual knowledge, and how these two strands of knowledge interact and constrain the construction of representations.

The MBT approach was based on the assumption that pre-service teachers who had developed robust conceptual knowledge could be expected to exhibit strong procedural knowledge. This appears to be the case with most of our participants. However, there were a number of pre-service teachers who demonstrated conceptual knowledge but failed to activate the corresponding procedural knowledge. This raises a question about the character of conceptual knowledge in subsuming and supporting procedural knowledge. This issue needs further analysis and the subject of future investigations.

The results showed that pre-service teachers have developed strong conceptual understanding of division problems. However, the robustness of this understanding needs to be the subject of further research including the analysis of prospective teachers' representations and solutions of division problems that are contextualised. The representation of division problems, both from a conceptual and procedural point of view, could inform us about pre-service teachers' ability to discriminate measurement versus partitive interpretations which have been shown to be a problematic area for teachers and students (Flores, 2002; Siebert, 2002).

Is conceptual knowledge better than procedural knowledge for practice? We suggest that there has to be a balance and that teachers' SCK ought to exhibit both these characteristics. Equally, teachers need to be facile in articulating their relationships.

Our previous study (Forrester & Chinnappan, 2010) provided the impetus for this project. In that study we examined procedural and conceptual knowledge in the context of subtraction and multiplication problems. The current study, however, involved the investigation of addition and division problems. This could be seen as a limitation. We contend that in both situations, there is a structural similarity (inverse relationships) among these operations, both algorithmically and conceptually.

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FORMATIVE ASSESSMENT TOOLS FOR INQUIRY MATHEMATICS

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Inquiry in mathematics is well suited to address authentic, ill-structured problems that are encountered in everyday life. However, available formative assessment tools are typically not designed for an inquiry approach. An exploratory study using Design Research aimed to understand and improve assessment practices of mathematical inquiry. Data collected from one classroom provided detailed examples of these assessment practices in action. Results from the initial stage and future directions of the project will be presented.

Reform efforts have for several decades worked to improve the curriculum, pedagogy and assessment of mathematics in schools. An inquiry-based approach shows particular promise for improving student learning in mathematics. Duckworth (2006) contends that the most wonderful ideas and understandings of children are revealed when adequate time to explore is provided and learning activities are designed to allow conflict and reconsideration of ideas. There is a professional need to capture those learning moments and to use the evidence to enhance student understandings and extend their new and emerging ideas. Little is known about how assessable information from these activities can be better identified, stored and used to inform future teaching and learning experiences. In order for inquiry to be used more widely, there is a need to recognise ways to assess students that value multiple types of understandings, rather than focusing and reporting only on narrow content. One potential resource for addressing this problem is a framework from The Programme for International Student Assessment (PISA) (OECD, 2009). PISA identifies a reflective cluster of competencies to assess mathematical literacies that value processes used by students to solve open-ended problems. These hold promise as a design framework for making assessable information from inquiry available to educators.

This paper will discuss results of a pilot PhD study from one teacher‟s experiences in trialling and implementing innovative assessment tools. The case study builds an initial foundation for supporting teachers in adopting new assessment tools. The study addresses an important gap in the field as little is known about the kinds of formative assessment tools that can support mathematical inquiry. Until the field is able to capture and record students‟ mathematical learning of inquiry processes, there is little hope of inquiry becoming a normative practice in school mathematics.

LITERATURE

Inquiry

Makar (2007) discusses the shift in paradigm in the teaching and learning of mathematics away from a primary emphasis on skills, facts, and procedures in isolation. An increased stress is on integrating these within the development of children‟s mathematical conceptions, and proficiency at applying mathematical tools to new situations, in particular, open-ended, complex and everyday problems. Her work describes cycles of statistical and mathematical inquiry as investigations that immerse learners in open-ended problems where phases of investigating and reporting are repeated to refine understandings through improved knowledge tools (e.g., statistical concepts, technology support). Ill-structured problems in mathematics inquiry, like real-life problems, can generate discussion to identify characteristics of a phenomenon and how to capture its possible qualities (Makar & Fielding-Wells, in press). Learning contexts that provide such rich problems, such as in an inquiry classroom, have the potential to provide rich assessment opportunities that reveal students‟ level of mathematically literacy.

Assessment

An understanding of how assessment is used in a primary classroom will help readers to better understand that many assessment methods do not match teaching and learning experiences in an inquiry classroom. The Queensland Curriculum, Assessment and Reporting Framework (QCAR) (Queensland Studies Authority, 2009) includes assessment development as a process to support the planning of teaching and learning experiences and offers guidance on how and when to provide feedback. With such an importance placed on assessment, and with so many opportunities available in a math inquiry classroom, effective assessment tools should be fore-grounded and consideration given in how to capture these learning moments.

Assessment can inform in two ways, summatively and formatively. When evidence is used to adapt the teaching to meet student needs it becomes formative assessment. Wiliam (2007) highlights how formative assessment can support learning and even refers to this type of assessment as assessment *for* learning. Teachers must assess their students while learning is in progress in order to adapt instruction so that it is successful in helping students achieve learning goals (Black & Wiliam, 1998; Furtak & Ruiz-Primo, 2008). Black and Wiliam (1998) point out that there is convincing evidence that formative assessment can raise standards of achievement, this being an important educational priority.

Challenges arise in a primary mathematics inquiry classroom when using formative assessment. Furtak and Ruiz-Primo (2008) analysed formative assessment prompts for their effectiveness in eliciting valuable assessment information. They categorised timeliness in how student responses were collected, and teacher-responses shared, to complete a successful feedback loop. Formal or informal prompts can be recorded through student writing, eliciting students‟ conceptions and offering an opportunity for students who are less sure of answers to share their ideas (Furtak & Ruiz-Primo, 2008). Yet in a classroom context, analysis of these reflections can result in a delayed teacher response. The PISA assessment framework may assist in faster analysis and turnaround time to ensure a successful feedback loop.

PISA Framework

The PISA (OECD, 2009) mathematics framework is provided to describe and illustrate the PISA mathematics assessment. In PISA, mathematisation is used to describe the process students use to transform complex real-life problems into ones which can be solved with mathematics. In order to engage successfully in mathematisation, students need to possess a number of mathematical competencies. Eight of these have been identified and can be possessed at different levels of mastery.

The mathematical competencies are further organised according to three clusters reflecting conceptual categories of broadly increasing cognitive demand and complexity, summarised below in Figure 1. Categorising competencies into these three clusters offers a description of the cognitive activities students undertake when completing the mathematical problems. The reproduction cluster highlights those basic mathematical processes, knowledge and skills of common problem representations, commonly required on standardised assessments and classroom tests. Students can build on these skills and apply them to situations that are not routine as part of the connections cluster. Assessment items that require integrating, connecting, and an extension of practised material measure the connections cluster. In the reflection cluster, students reflect on the processes required and may have to plan solution strategies for unfamiliar problem settings.

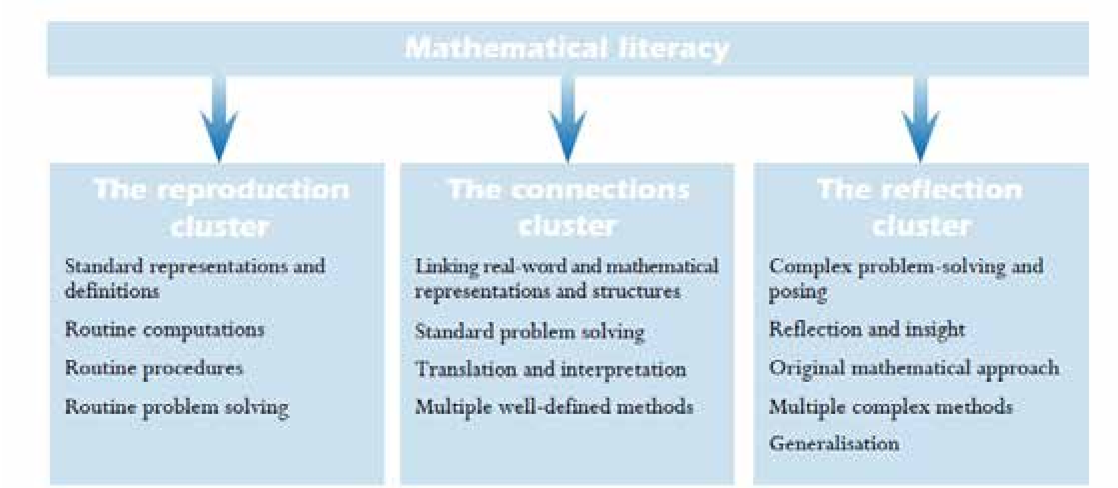

Figure 1. PISA Competency clusters to assess mathematical literacy (OECD, 2009, p. 115).

METHOD

Participants and context

This pilot study uses a case study method (Merriam, 1998) to examine the learning environment, student artefacts, and observations from a primary, inquiry mathematics classroom. The pilot study involved a laptop class from a middle-class suburban primary school with 28 Year 6 students (age 10-11) taught by the author. The students were confident in using their own laptops while working in mathematics inquiries. The study aimed to improve, record, and reflect on the assessment practices during mathematical inquiry. In the data presented here, the students were engaged in a mathematical inquiry entitled *How much is 1m³?* (Figure 2) as part of the normal activities and assessment practices within an inquiry-based classroom. In this inquiry, students had to demonstrate their understanding of a cubic metre and its relationship to 1000 litres and to one tonne. The ambiguity of „How much?" lends itself to the open-endedness of the question as students could choose multiple pathways to present this.

How much is 1m³?
You need to show:
- The cubic metre you make,
- How you made the cubic metre,
- And how you worked out that it was a cubic metre.

Figure 2: Inquiry question.

Data collection

Students were given many opportunities to reflect on their learning throughout the inquiry. Whichever stage of a mathematical inquiry students are working in, reflection can offer valuable insight. In this inquiry, students were encouraged to use a *Quickwrite* (Dodge, 2009) strategy to record their reflections in a timed two-minute written reflection that can take place at any time within the learning experience. These included thoughts, questions, and ideas about the topic, as well as reflections on what had been achieved in the lesson or a prediction of what the lesson might look like. The students individually recorded these Quickwrite-inspired ideas and conceptions using their laptops over time as an electronic learning journal.

The nature of the electronic learning journal made the information easy for students to access and manipulate. Students could delete, backspace, and insert thoughts and ideas without „rubbing" any work out. They could also use menu options such as Synonyms, Look up (Online dictionary), or Images (internet or clipart) to better understand relevant vocabulary or to improve their own explanations. Enthusiasm meant that students would eagerly format and edit their work with a purpose to reflect on their mathematical understandings. As the inquiry progressed, additions were made to their original thoughts and were dated to help show the audience progression and development of ideas. Different choices in fonts, text sizes, and colours also made the progression clearer. Students were adept at saving their work in a number of places to ensure access in any physical location.

As well as working individually on the computer, students worked in small groups of no more than three to solve the problem, *How much is 1m³*? Many of the ongoing ideas formulated in groups were recorded on A3 sheets of poster paper. These were a second set of important artefacts of group collaboration with a focus on communicating this information later to others.

Finally, a research journal was used by the teacher to record discussions with students and observations of their work. Teacher reflections were also created to record the context and circumstances at iterations within the inquiry. For the teacher as researcher, reflection on these experiences could provide evidence of categories that may require further investigation. This reflective tool assisted the teacher to inform and improve upon teaching and learning.

Analysis

The electronic learning journal became important documentary evidence of students‟ learning yet how the teacher was to best interpret and use this information to inform teaching and learning was not initially clear. The PISA assessment framework was trialled as a way to analyse and assist in making teacher judgements of student work in the journals. This would highlight the mathematical language, modelling and problem-solving skills used, as organised in terms of competency clusters. Processes or thinking skills particular to this inquiry were Thinking and Reasoning, Problem posing and Solving, Communication and Modelling. Although examples of other competencies could be found, they are not a focus for this paper. Examples of the processes or competencies were identified in the students‟ reflections and rated according to each competency cluster: Reproduction, Connections, or Reflection with a 1, 2, or 3 respectively (Table 1). In the PISA assessment framework, there is considerable overlap in the processes or competencies students will use to work mathematically, as is common when working through mathematical inquiries in the classroom. Identifying which cluster students are working in can provide evidence of higher order thinking skills used, highlight gaps or topics that may need further explanation, and can inform teaching practice.

Peer analysis of the electronic learning journals occurred in the classroom also as students defended each stage or juncture in the inquiry. An interactive whiteboard allowed groups to display their reflections, calculations and ideas and to edit this work while on display. As ideas were shared, other students posed questions that critiqued the work. Also, other students analysed their own efforts and the effectiveness or appropriateness of their ideas.

Table 1: Rubric comparing competencies and clusters.

			Cluster		
			1 Reproduction	2 Connections	3 Reflection
Process/Description	1	Thinking and Reasoning			
	2	Argumentation			
	3	Communication			
	4	Modelling			
	5	Problem posing and solving			
	6	Representation			
	7	Using symbolic, formal and technical language and operations			
	8	Use of aids and tools			

Evidence of students working collaboratively can be found in their journals, but also in the written work completed together. The A3 poster papers recorded the inquiry process throughout all iterations and were used to also communicate findings and generalisations with other groups. Using a rubric (see Table 1) comparing competencies with clusters, again using the PISA framework, analysis could capture evidence of mathematical processes used and when learning moments moved between clusters.

Teacher reflections of learning moments provided further explanation of the findings in the electronic journals and poster papers. Discussions between teachers and students, and students with other students, were analysed using the PISA framework to find further evidence of competencies that had been developed through mathematical inquiry, and opportunities for competencies to be further developed if gaps were identified.

Results

Electronic journals

When the class began their mathematics inquiry unit on the cubic metre, it was clear some students were not confident in their understanding of volume and the relationship to mass. Students had previously completed a one-week planned unit of work exploring this concept and had experienced a range of activities that included manipulating and viewing materials and practise of routine operations in particular contexts. The kinds of answers or conceptions evident in the inquiry often did not match those in the previous unit of work, where both aimed to develop the same mathematical understanding of a cubic metre. The cognitive mathematical competency (PISA) of *thinking and reasoning* was identified in the reflections in the electronic learning journals where students demonstrated an ability to pose mathematical questions and have knowledge of the kinds of corresponding answers mathematics can offer. When asked to pose a question where the answer was $1m^3$ (a pre-assessment task), one student wanted to explore a problem about filling an area with grass and a flower bed. This conception of volume did not match the *thinking and reasoning* already hoped to have been developed in the previous unit of work. *Thinking and reasoning* and *problem-solving* skills used by the student were only based in the Reproduction Cluster, where the student was working with contexts familiar to them. They were still developing their understanding of practised routine procedures regarding area and were not yet making links to less familiar, real-world contexts. Nor were they solving problems using independent problem-solving approaches. Later in the inquiry, the same student changed their mind to explore how much cement would be needed to fill and build the slab under a shed; their understanding of volume was beginning to develop.

Identification of the thinking processes students use can be of interest when students who perform reasonably well in pencil and paper tests (in the previous unit of work) display low thinking and reasoning ability, generally in the Reproduction cluster, when applying the understanding to an inquiry context. For example, reflections in one student"s electronic learning journal indicated an additive understanding of a cubic metre as opposed to a multiplicative understanding ($1m \times 1m \times 1m = 1m^3$). A problem was posed by the teacher (see below) at the beginning of a lesson to orientate the students to the next phase of the mathematical inquiry. This student, "added all of the measurements together to check if it was over, under, or equal to 3m". This student had earlier produced results that indicated a sound understanding of volume.

> A child"s wading pool measures 1.3m wide, 1.5m long and 75cm deep. What is the volume of the pool? How much water is needed to fill it up?

A3 poster paper

Aiming to present findings to the class, students used A3 sheets of poster paper to record mathematical conceptions and ideas collaboratively. Evidence of *thinking and reasoning*, *modelling,* and *problem posing and solving* as competencies were identified as being used in this process. The rubric (Table 1) was used to rate evidence of learning in each competency to gain an initial label of one through to eight. A rating of one to three was a second score earned which indicated in which cluster the students were working. Working in the Reproduction cluster received a score of one, whereas, evidence of working in the Reflection cluster received a score of three. Scores therefore indicated which process was evident, and the level of thinking e.g. a score of (1, 2) would indicate evidence of the first competency, *thinking and reasoning*, and that the student was working in the Connections cluster.

As students began to work on the inquiry topic, evidence of problem posing and solving was rated in the Reproductions cluster as students reproduced standard, closed problems, which could typically be solved in the one way. For example, one group had recorded on their poster the following question: "I had $6m^3$ and I had another $5m^3$ and subtracted them and got a result of $1m^3$." This lower-order thinking was not challenging the students to think beyond the problems already practised in class. To push students beyond these questions, the teacher was then able to discuss with the class what types of questions display a mathematical understanding of the concept. A basic equation like $1m \times 1m \times 1m = 1m^3$ would demonstrate some basic understanding of volume, but the equation was not highly creative nor did it demonstrate a good mathematical understanding of the concept. A new criterion was jointly constructed to guide students to think about how their responses might be more creative or reflect a good mathematical understanding of the concept. Solutions began to move away from the reproduction-style problems that practised standard problem solving. Assessing which cluster students are working in can highlight areas for improvement and can help students to use more rigorous mathematical processes.

Researcher field journal

Identifying where students are working can help teachers know where to go next. For students who do not like to write, posters did not reflect much thinking. It was useful to then look at the individual student"s reflections in their electronic learning journal, where an enthusiasm for computers meant that more effort was made in recording their ideas. Many assessment opportunities are missed when students are working in groups and discussion is not recorded. In this inquiry, one group of students who generally did not like to record their reflections in writing became a focus for the teacher to record anecdotal evidence. This information was recorded in the teacher journal. Anecdotal observations of student work have long been an assessment method valued by teachers with notes being recorded in different ways and being stored in various locations for later use at report card time. Using the PISA assessment framework, the teacher in this instance was able to reflect on the teacher journal to identify clusters students were working in and areas to improve teaching and learning. One reflection noted how students moved between clusters when *modelling* their ideas. Initially, the group had sketched a box with the dimensions 0.5m x 0.5m x 0.5m with an assumption that this was half a cubic metre. Once the students were encouraged to use the cubic metre model in the class to show this they were able to visualise that about eight of the boxes would be able to fit into one cubic metre! Students were interpreting back and forth between models and their results, and this communication was evidence of students now working in the Connections cluster. Knowing where these students were working using the PISA assessment framework provided the teacher with direction and focus.

Discussion

The student-centred nature of a mathematical inquiry unit of work can mean that the direction learning travels is not always as expected. Mathematical inquiry is complex and open-ended (Makar, 2007) and intended content areas are not always realised. This makes it a difficult task for teachers to foreground assessment. Often new and exciting mathematical topics need to be introduced to assist in answering students‟ questions. Using the PISA assessment framework to analyse student work made it possible for the teacher to quickly check the cognitive level the students were working in and provided some direction in how to push their reasoning beyond those experiences. A faster turn-around of feedback on rich, written reflections ensures a more successful feedback loop (Furtak & Ruiz-Primo, 2008). It offers an alternative to earlier pencil-and-paper-style assessment with tasks that give a clearer indication of the level of a student‟s mathematical reasoning. In this inquiry, there were three levels of evidence to show how students were working: individually through the use of electronic learning journals, collaboratively by analysing poster sheets student groups worked on, and with a focus on differentiation as the teacher reflected and added anecdotal notes to their journal.

In analysing these three areas, the framework provided useful feedback and information to both students and teachers. In the electronic learning journals, the feedback was for the individual student as it identified which clusters the student was working in or developing. It allowed opportunities for teachers to formatively assess their own teaching practice and to offer feedback to students that was meaningful and could guide them further in their inquiry. For small groups, analysis highlighted which competencies students were working in. This information could be used formatively to provide feedback to the whole class, encouraging groups to use higher order thinking skills. Identifying the competencies and clusters in the teacher journal further informed teaching and learning pedagogy and could assist the teacher to make more informed judgements of the level of students‟ mathematical reasoning.

Although this case study was not a typical classroom, the results provide insight into ways to capture and analyse assessment opportunities in a primary classroom using mathematical inquiry. Using this information can inform teachers and students how to move away from lower-order thinking processes of posing and solving familiar and practised problems (Reproduction cluster). Teachers and students can aim to apply problem-solving processes, knowledge and skills to situations that are not routine (Connections cluster) or with an element of reflectiveness (Reflection cluster) with problems that contain many elements and may be more unfamiliar.

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MODELS OF MODELLING: IS THERE A FIRST AMONG EQUALS?

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Given the variety existing within mathematical modelling enterprises, it is not surprising that different perspectives have found their way into educational practice and research. A variety of genres, (or variations within genres), has emerged within education communities, who use the term 'mathematical modelling' with different emphases, and in some cases with different meanings. This presentation will review the origins and purposes of several articulations of mathematical modelling. Tensions will be identified, and some inconsistencies and misplaced inferences illustrated. Different approaches will be linked to underlying purposes that are not always made explicit, and some specific issues will be highlighted.

Introduction

Mathematical Modelling, its practice, research, and curricular implications continue to engage members of the mathematical and mathematics education communities. In Australasia recent foci are found in the *MERGA Review of Research (2004—2007),* in the recent special issue of the *Mathematics Education Research Journal* (Stillman, Brown, & Galbraith, 2010), and through the ongoing published work of individuals.

Practitioners and researchers inhabit different sections of the respective communities, as well as the interface between the two. Hence it is not unexpected that different perspectives share similar terminology when talking and writing within the field—resulting in a variety of genres, and variations within genres among those who use the term 'mathematical modelling'. Confusion is generated when individuals lay a particular meaning over writings and other scholarly products that have been constructed within a different genre, while more fundamentally, value judgments concerning the purposes and features of application and modelling initiatives stand to be distorted by generalisations made on the basis of limited experience or understanding, or indeed selective referencing.

Structure and purpose

This paper first reviews the characteristics of several articulations of mathematical modelling and applications as found within the mathematics education community. Its lens focuses on mathematical modelling as it interacts with curricular purposes within mathematics education, rather than analysing particular variations viewed from within the modelling field—as in Kaiser & Sriraman (2006). Some criticisms of mathematical modelling will be illustrated, tensions identified, and inconsistencies and misplaced inferences illustrated. Different approaches will be linked to underlying purposes that are not always made explicit, and some specific issues highlighted. Finally reference is made to a stated aim of the proposed Australian Curriculum—Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010): that "mathematics aims to ensure that students are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens".

It will be argued that at most two contemporary approaches that use the term 'mathematical modelling' can hope to contribute decisively to such an aim.

Models of modelling

For present purposes the focus is on terms and meanings associated with 'mathematical modelling' that are recognised within the international community of practitioners and researchers in the field (Blum et al., 2007). Six approaches to the use of mathematics with connections to the real world are considered below.

1. Using real problem situations as a preliminary basis for abstraction

Two studies that used practical contexts to motivate and develop the linear relationship ($y = mx + c$) at respectively years 9 and 8 levels, are reviewed in (Bardini & Stacey, 2006; Bardini, Pierce, & Stacey, 2004). Symbolic, numerical, and graphical representations of the relationship were introduced by considering the cost of hiring trades people, where the given conditions included a flat 'call charge', together with labour charges on a per-hour basis. With the year 8 above average ability students, graphical calculators were introduced to facilitate the learning. Axial intercepts, slopes, points of intersection, and intervals required interpretation in context, across a variety of problem settings. The students learned to write algebraic rules in conventional formats, were comfortable selecting symbols that made sense in terms of the problem settings, and showed understanding of the function property of expressing one variable quantity in terms of another. Problematic was the time factor—five weeks seems a very heavy investment for the achieved outcomes. Since the approach had to cater for a pre and post testing format, perceived clashes between research requirements, and authenticity of problem solving were resolved at the expense of the latter. For example students made decisions about contextualised problems on their own, where in reality a decision about which plumber to hire would usually be a collaborative (e.g. family) decision reached after some discussion of competing quotes. This is all about the team nature of aspects of a modelling process, whose goal is to obtain and justify the solution to a problem. Some useful outcomes were achieved in both studies—the time commitment was problematical, and expedient rather than authentic modelling practices were imposed at times.

2. Emergent modelling

Emergent modelling (Gravemeijer, 2007; Doorman & Gravemeijer, 2009) is an instructional design heuristic, developed as a component of a domain-specific instruction theory generated within the Reality in Mathematics Education framework in the Netherlands. ‘Emergent’ refers both to the nature of the process by which models emerge from students’ experience, and to the process by which these models support the emergence of formal mathematical ways of knowing - that are no longer dependent on the support of the original models. That is, there is emphasis on a search for models that can be developed into entities of their own, and subsequently into models for mathematical reasoning. Gravemeijer (2007) summarises the process as one of “abstraction-as-construction” in which mathematical knowledge is grounded in earlier experiences that are meaningful and applicable. In that they are familiarised with a mathematical take on everyday life situations in the process, students are incidentally prepared for more serious application and modelling adventures in the future - indeed Gravemeijer has referred to emergent modelling as a precursor to mathematical modelling. Emergent modelling can also be viewed as a more organised and theorised approach than that described in the previous section, which typified approaches aimed at using contextualised mathematics to motivate and attain proficiency with the form of a basic mathematical relationship.

3. Modelling as curve fitting

This approach has become increasingly significant with the availability of regression menus in software and graphical calculators. A model generated by this means can become a purely technical artefact whose parameters vary with the particular data set, and which can be generated in complete ignorance of the principles underlying the real situation—indeed undertaken without knowledge of where a table of data comes from. It raises a profound theoretical issue—the relative authority of data as such, versus the theoretical structure underpinning its generation In one example curves were fitted to population data by using successively the full suite of regression choices available on a graphical calculator—with no apparent realisation that data generated by births deaths and migration should have an underlying exponential pattern. Curve fitting remains an important activity within the modelling enterprise, but when used mindlessly it creates a dangerous aberration of the modelling concept. Riede (2003) demonstrates good modelling practice when relating weightlifting records to weights of athletes. An inverted parabola was postulated to model the data, on the grounds that weight lifted at first increases with body weight, but ultimately (beyond the super heavyweight class) begins to decrease as body weight impairs the ability to lift. The subsequent fit was excellent.

4. Word problems

Vershaffel (e.g. Greer & Vershaffel, 2007; Vershaffel &Van Dooren, 2010), has been writing and researching insightfully, for many years on the subject of student approaches to word problems. Studies in a variety of countries have consistently demonstrated the propensity of students to ignore contextual factors, and apply (often incorrect) actions based on perceptions of what school mathematics is about—such as being divorced from reality. His work with colleagues has included a focus on the suspension of sense making by students while working on word problems, so that aberrations are produced that the same students would never contemplate in their real lives outside the classroom. Various intervention studies to identify problems and stimulate improvement have been designed and implemented, with varied outcomes (see Vershaffel & Van Dooren (2010) for a summary of some of these). Attention is drawn to the impact of classroom culture in seeking change, for not only are different types of problems needed, but improvement "would also imply a classroom culture radically different from that which typically exists in many mathematics classes." The difficulty of producing change may well be compounded by the types of intervention materials proposed—more realistic word problems in text books do not address the cultural issues that learning from text books in this area themselves reinforce. If the medium remains the same a different message is difficult to promote.

5. Modelling as a vehicle for teaching other mathematical material

When used as a *vehicle* (Julie, 2002), modelling contexts are chosen so that the mathematics of interest is embedded in the associated examples. The principal driving force in defining the boundary of the activity is curriculum content, and genuineness of applications is made subservient when necessary (not always), to achieve this perceived need. This particular emphasis is most clearly expressed by Zbiek & Conner (2006).

> … we recognise that extensive student engagement in classroom modeling activities is essential in mathematics instruction only if modeling provides our students with significant opportunities to develop deeper and stronger understanding of curricular mathematics. (pp. 89–90)

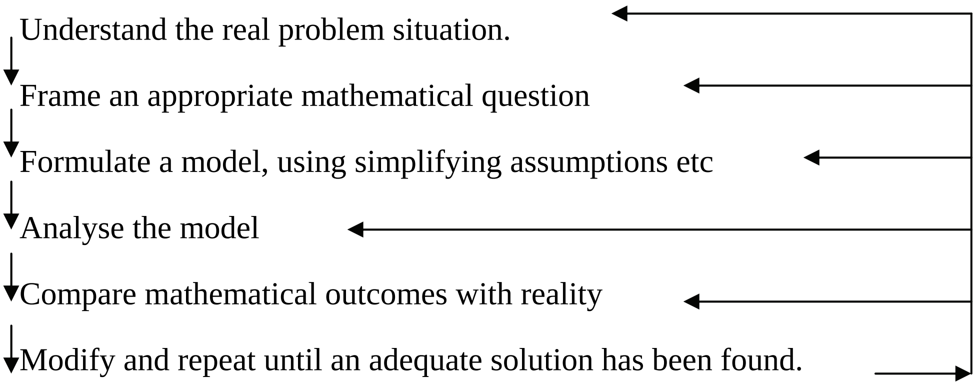
These authors describe the implementation of a problem involving the siting of a hospital to service three large (actual) cities in north western USA, so the data were real. The subjects (student teachers) were left to address the problem in their own way, and later interviewed to probe their use of assumptions, strategies, parameters, interpretations, and justifications. All these aspects are involved in real world modelling, except here it was the evocation of these separate mathematical and problem solving entities as such that was of central interest. In fact the authors' did use a modelling process that was included in the paper but not shared with the students.

English (2010) also working within the *vehicle* mode, used environmental material as a stimulus for engendering data based modelling involving classification and display of attributes, with first grade children. We are reminded that, as has been pointed out many times, the capacity to learn from modelling examples is not a function of age or the amount of mathematics that is known—although the types of modelling activities that are suitable clearly are impacted by experience and knowledge.

While the approaches described within 1 and 2 (above) also use real contexts, this genre is much more thorough in using a modelling process to generate information of value, even if (as with Zbiek & Conner, 2006) the students are not made aware of this. Lesh and associates (e.g. Lesh & Doerr, 2003), use carefully constructed Model Eliciting Activities (MEAs) to elicit mathematical concepts for consolidation and enhancement.

6. Modelling as real world problem solving

This perspective differs in some important respects from those discussed so far, firstly because its origins lie substantially with those who have used mathematics to model problems in professional fields outside education, and in their personal lives. Some, such as Pollak (telecommunications), Burkhardt (physics), and other early ICTMA progenitors, have taken their experience and insights specifically across into modelling initiatives in education, while others (e.g., Pedley, 2005) without specific intention to do so, continue to provide external reference criteria for those working within the educational field. An essential goal is for students to develop and apply modelling skills to obtain mathematically productive outcomes for problems in their world with genuine real-world connections. Modelling in this vein has two concurrent purposes—to solve a particular problem at hand, but over time to develop modelling skills, that empower individuals to apply to problems in their world. That is, to become productive users of their curricular mathematical knowledge. Characteristic of this approach is a cyclical modelling process—containing elements such as the following drawn from the Presidential address to the *Institute of Mathematics and its Applications*, (Pedley, 2005). Note that Pedley was not addressing an audience of educators.



The arrows on the right indicate that iterative back tracking may occur between any phases of the modelling cycle when a need is identified. This diagrammatic translation of Pedley's message is a compact version of the modelling chart that appears in various representations in many sources (e.g., Galbraith & Stillman, 2006). Such diagrams describe the modelling process, but also act as a scaffolding aid for individuals or groups as they develop modelling skills through successive applications. The labels do not represent vacuous generic properties, but attributes that find specific and different instantiations, depending on the context in which the modelling takes place. In this genre the solution to a problem must take seriously the context outside the mathematics classroom within which it is introduced, and its evaluation involves returning to that context. It cannot live entirely in a classroom.

It is argued that two substantive strands can be identified that contribute to this approach to modelling. One of these involves using MEAs (Lesh & Doerr, 2003) as 'modelling development tutors' when used as orchestrated activities within a carefully planned sequence. When used thus, the identification of relevant mathematics with which to model becomes a central feature, an aspect that is reduced when MEAs are used in close proximity to a cognate topic to elicit or consolidate particular concepts—where their purpose is more often in the vehicle mode. Note that this amounts to creating a strength out of the use of MEAs as 'stand alone activities', perceived as a weakness by Yoon et.al., 2010 when used in isolated and uncritical ways.

The other strand typifies the emphasis of the ICTMA group (e.g. Blum & Leiss, 2007), in which a modelling chart is used as a scaffolding aid with dual purpose. One purpose is to assure that real world problem solving in education contains the procedures, and checks and balances that professional mathematical modellers endorse and apply. The second purpose is to provide a means for individuals to build and test modelling expertise, including use of curricular mathematics, through engagement with a selection of appropriately chosen problems, including at times their own. The emphasis is on learning how to identify problems, and to formulate related mathematical questions that can be addressed with existing mathematical knowledge—developing this ability is one of the most significant challenges new modellers face.

Critique of modelling

This section selects and reflects on some criticisms made regarding mathematical modelling. Jablonka & Gellert (2007) argue that there is no straightforward way to move from a real problem context to a mathematical model, because it is virtually impossible to quantify non-mathematical characteristics, and relate them mathematically in *one* step. There is confusion here between a procedure (step) and a phase in the modelling process—the latter may contain several steps and will vary in complexity with the sophistication of the problem. They further argue that there can be no *validation* because a result is not put back into a 'real' *real situation*. We will return to this criticism in the concluding section.

Arleback (2009), when introducing modelling to upper secondary students, could find no evidence of the cyclic process widely described in modelling research: "the discrepancy with what actually happens is palpable". This was a strange observation, as shortly before he had identified sub-processes that characterised the students' work: reading; making a model (structuring and mathematising); calculating; validating; and writing. All these are essential elements of the modelling process, and validating cannot occur without reviewing a solution in terms of the original problem statement (reading). This alone completes a cycle, even without further cycles introduced through the checking and reviewing that inevitably takes a solver back through earlier phases in producing a defensible solution. This comment is enigmatic, as the paper in general is carefully constructed, and the description of the research is excellent.

Sfard (2008) claims that, the minute an 'out of school' problem is treated in school it is no longer an 'out of school problem', and hence the search for authentic real world problems is necessarily in vain. There are several examples in the literature where individual students have, on their own initiative, used mathematical modelling techniques learned in school, to address situations in their personal lives outside school—as authentic as one could wish. Again we will return to this point in the final section.

In a similar vein Barbosa (2006), argued that "since students and professional modellers share different conditions and interests, the practices conducted by them are different." While there are differences of course, what both groups need for success are modelling competencies that can be applied effectively and sensitively, including the ability to work productively both as individuals and as team members. The following questions are relevant for both groups. Is it important to be able to: Define a problem from a real-world setting? Formulate and defend an appropriate mathematical model to address it? Solve the mathematics involved in the model? Interpret the mathematical results in terms of their real world meanings and implications? Evaluate and report the outcomes of the model both for mathematical validity, and in terms of their relevance to the original question? Revisit and challenge material produced within any part of the modelling process in the interests of improved outcomes? Can any of these 'stages' be omitted from a seriously constructed modelling endeavour? Is the ordering of the stages arbitrary? If as we contend the answer to every question except the last two is “yes”, and to those two is “no”, we have a process that characterises essential modelling activity that is as relevant to school learners as it is to those doing modelling professionally or for personal reasons.

In a very recent paper Jablonka & Gellert (2011) begin with the assertion that “Modelling approaches are propagated to enhance the quality of the outcomes of mathematics education by providing students with generic competencies and thereby creating a flexible work force”. This is a sweeping and mistaken generalisation, as motivations are various, and include centrally that of student empowerment, as in: “… for students to spend years learning mathematics without any sense of how to apply it in the world around them, is inappropriate” (Stillman, Brown, & Galbraith, 2010). The paper is a mixture of observations, assertions, and arguments. It raises some important issues concerning equity, but a drawback is the dependency on selections chosen seemingly to support the ideology of the critique, rather than a representative spectrum from the field. For example, the authors allege that modelling conceptions do not see associated competencies as ‘culture bound and value driven.” Yet an introduction in Blum et al., (2007) points out that “the best route for a new freeway”, implies that “best” must be interpreted, and this implies not only considerations such as “most direct”, or “cheapest”, but also “least disruptive to communities”. Again the authors assert “…contextuality of all knowledge is (mis)interpreted in a way that leads to the contention that mathematical concepts can be meaningfully learned only within a ‘real life’ context”. Compare this with:

> … neither the content nor vehicle approach argues in some abstract sense that all mathematical curricular content must be justified in terms of relevance - mathematical modelling has a role to play in meeting certain important goals, but other significant mathematical skills and purposes are important as well. (Stillman et.al., 2008, p. 145)

And reasoning that argues against the use of contextualised problems on the grounds that they may be initially more familiar to some students than others should also argue against teaching any new mathematics, because some students will be better prepared to benefit than others. What this paper and others provide, is the cautionary tale that there are many versions of modelling out there, that cover the full range of good, bad, and indifferent implementations. But it is imperative that the theory of mathematical modelling, its purposes and possibilities, are kept conceptually separate from poor implementations, and abuses. There is no question that the latter exist, but they must not be used to undermine arguments for what is possible when the best is undertaken.

Concluding reflection

So, to return to the question posed in the title! It is not reasonable to expect a single definitive answer because not all the ‘models’ considered have the same priorities. The use of contexts to introduce new mathematical relationships like $y = mx + c$ need to be analysed in terms additional to those raised earlier in this paper. Wrestling with symbolic representations such as m and c, at the same time as embodiments meant to motivate their abstraction creates issues of cognitive load (Chinnappan, 2010) that need to be specifically considered. Emergent Modelling as a package is well constructed by its practitioners, who emphasise and explain what it does and does not set out to do. It is important to realise the curricular implications of its 'total package' aspect. Curve fitting will have an increased presence as technology enhances the capacity to use messy real data. It remains a significant *part* of many modelling enterprises, but a challenge is to eliminate any separation between the search for a mathematical relationship, and the nature of the data involved. Word problems will continue their presence, and contain aspects of mathematics congruent with some components of the modelling process. Generally their simplicity and association with text book mathematics, limits their capacity to add decisively to modelling capability.

The last two 'models of modelling' share some common ground, and although different in purpose, are not antagonistic. While 'modelling as a vehicle' has the prime purpose of eliciting and consolidating *new* mathematical concepts, such entities then enlarge the field of problems that can be addressed. While 'modelling as real world problem solving' has the prime purpose of helping students to access and use their *existing* store of mathematical knowledge to address problems, the mathematics evoked is often used in novel ways, and as such contributes to enhanced conceptual understanding. Returning to the aim of the new Australian curriculum, emphasis is on the ability to use mathematics creatively in "personal and work lives and as active citizens". This requires the ability to formulate mathematical problems out of contextualised settings, and to go through a systematic process of solving, testing, and evaluating. It is not an ability that is acquired by osmosis or transfer on the basis of 'seeing'—it requires direct structured experience. Formulation of a mathematical problem from a messy real context is arguably the most difficult aspect of learning to use mathematics, and only the two approaches illustrated in this last of the 'models of modelling' contain formulation as a major component. It is not surprising that they both resonate with those who apply mathematics outside education.

Finally some comments are needed in response to issues in the previous section raised by Barbosa, Jablonka & Gellert, and Sfard. What each is doing is privileging their conception of what school mathematics is about, and what mathematics teaching and classrooms are allowed to be—then requiring that modelling fit the stereotype and be subject to associated practices. By contrast, what modelling properly conducted can do, is to challenge some of those norms, assumptions, and stereotypes—mathematical, situational, and pedagogical. In that modelling as real world problem solving involves intersections between the values and methods of more than one community of practice, it challenges the boundaries of the existing education industry.

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MEASURING ACADEMIC NUMERACY: BEYOND COMPETENCE TESTING

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Academic numeracy consists of three critical elements: competence, confidence, and critical awareness of students' own mathematical knowledge and the mathematics used in students' future professions. This definition is used to frame pre-test assessment in a first year nursing program. Competence and confidence were measured using a paper and pencil test. Critical awareness was measured via students' reflections on their own performance, their relationship to mathematics, and their understanding of how mathematics relates to nursing. Results show issues related to professional numeracy practices including relatively low understanding of the connection between mathematics and nursing.

Introduction

The word "numeracy" was first coined in the 1959 in the UK from the Crowther report (1959) and redefined in the Cockcroft Report (1982) to reflect literacy. More recently it has been hijacked by the school lexicon, largely in Australia, to the extent it is often seen as a replacement for "mathematics", particularly in primary schools. The term is also used in adult education, where it has taken on a context of basic skills in the workforce or everyday life. Academic numeracy, however, has attributes of both school numeracy and particular professional numeracies. Students need to situate mathematics learned in school to their imagined future context. This shift is hopefully aided by the courses students enrol in at university where they are exposed to contextualised mathematics. Students also need to situate the mathematics in the particular academic context. While the mathematics strongly reflects the professional context, it is not the same. The academic context has particular extra attributes, such as reading and critiquing journal articles, or investigating particular topics in depths students may never experience in their professional careers. These contexts may also need mathematically based skills. Academic numeracy is not the same as professional or school numeracy.

The term "academic numeracy", modified from Yatsukawa and Johnston (1994), was used by Galligan and Taylor (2005) to clarify the skills necessary for success in the university context and defined as:

> … a critical awareness which allows the student to situate, interpret, critique, use and perhaps even create mathematics in context, in this case the academic context. It is more than being able to manipulate numbers or being able to succeed at mathematics. (p. 87)

However, academic numeracy needs to highlight two further attributes: confidence and competence (Coben, 2000). Thus, I propose academic numeracy has three elements:

- *mathematical* competence in the particular context of the profession and the academic reflection of the profession at the time;
- *critical* awareness of the mathematics in the context and in students' own mathematical knowledge and involves both cognitive and metacognitive skills; and
- *confidence*, highlighting its deeply affective nature.

While mathematical competence testing is common, both at university and school, few incorporate a broader definition as proposed here. These three elements are needed to assess students' numeracy in a particular professional path—whether it is in nursing, engineering, business, or education. This paper will outline an approach taken to test nursing students' academic numeracy at the beginning of their degree. This research is part of a larger study that investigated the development of nursing students' academic numeracy over their first semester of study.

Method

In 2008, 206 first year nursing students at USQ undertook a first semester course to build nursing attributes of mathematics and computing. Students attended one two-hour tutorial per week. One of these tutorials was taken by the researcher. Three instruments were used to assess academic numeracy in the first two weeks of the course. The first was an online discussion forum where students wrote about their mathematics experiences; second was a mathematics competence and confidence test, and the third was students' reflections on the results of the questions in the test.

Online discussion forum: In the very first tutorial, students discussed their past experience with mathematics and were encouraged to reflect critically and honestly. The informal discussion in class and the time given to actually writing, prompted well thought out responses. In this first tutorial, a discussion was also held on the advantages and disadvantages of different scales in nursing, using a pain intensity scale (Figure 1a). Students were then asked to rate their relationship with mathematics on a five-point scale (Figure 1b).

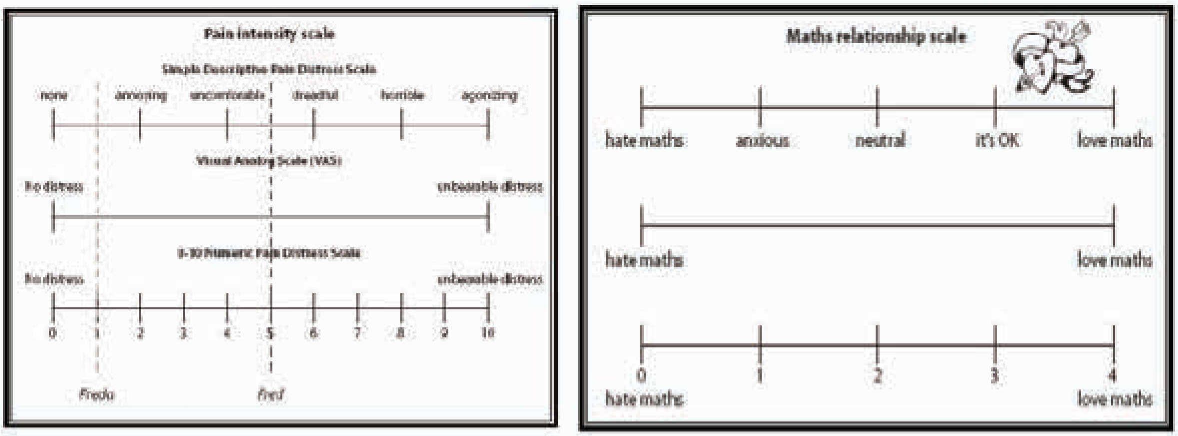

Figure 1. (a) Pain intensity scale (left) and (b) mathematics relationship scale (right).

They were also directed to articles on the relationship between mathematics and nursing. Using these two exercises as a basis they were asked to reply to the questions, "Describe your previous experiences with mathematics in a couple of sentences" and "How do you think mathematics…will be important for you as a nursing student, and later as a professional?" in an online forum, as part of a larger suite of questions about introductory academic studies.

Mathematics competence and confidence pre-test consisted of 32 (Table 1) items that was equivalent to tests undertaken previously by first year nursing students at USQ. Students were given a paper-based version of the test in tutorial classes. They then started the test in class and could finish it in their own time. Once they completed the test on paper they submitted their answers online via a Computer Managed Assessment system (CMA). The test also included a 5-point Likert style section on confidence levels.

Student reflections: The pre-test was then used as a stimulus for student reflection. When the test had been completed online, students were sent their results, question by question, in a table via automatic reply email. Students then copied and pasted this result into a word document and they then added two columns: a reflection on each question, and a strategic response about what to do next. Examples of the type of reflections and strategies were provided online, in the study book, and in class. Students then emailed these results and reflections to their tutor. While this was an assessment piece, marks were allocated on completion of the test and reflective comments and did not depend on how correctly they answered the questions.

Results

Replies on the discussion forum were classified into one of five categories. The categories: Hate, Dislike, Neutral, Like, Love were used as they reflected the sentiments in Figure 1b. The results are shown in Figure 2.

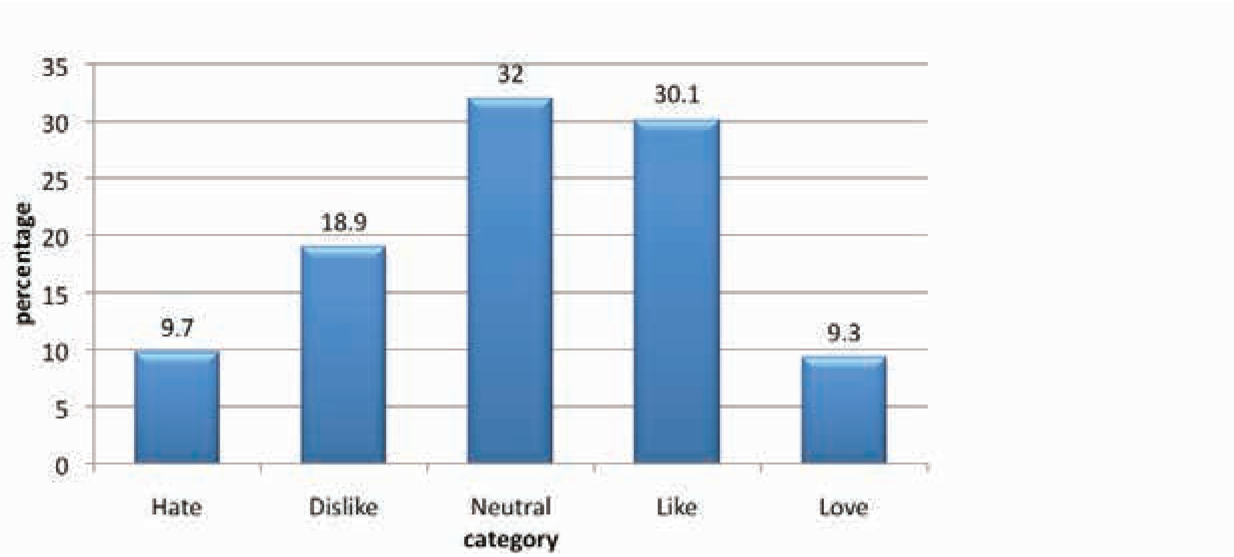

Figure 2. Relationship of nursing students to mathematics (n = 206).

A sample of students comments coded from “Hate” to “Love” is provided:

- I ended up changing from Maths B to A but not before developing a loathing for it!! (Hate)
- Maths I do not like it. For me it’s like great mystery (Dislike)
- Sometimes numbers just don’t make sense to me…I do blame the teachers and their inability to find why maths perplexed me so much, but in high school I do blame myself … now I find maths much better, I am no longer afraid of numbers and can grasp the theory and at least try to put it in practice—even if the answer is wrong. (Neutral)
- Maths has always been an enjoyable experience for me. (Like)
- I learned mathematics at school and really love mathematics … (Love)

Students also described other relations with their mathematics. For example, 14 students explicitly commented on teachers’ influence on their mathematics, mainly negative, but there were some positive comments, for example:

> I dreaded mathematics at school. It was my worst subject but that was due to never having a constant teacher. It wasn’t till year 11 and 12 when I had the one teacher throughout that helped me a lot that I began to enjoy it a little.
> I dreaded to do maths because I found it boring and the teachers were not always that helpful.

Pre-test results were generated from the CMA system and included both an overall percentage correct for each question and an individual student-by-student response. The pre-test results were analysed in four ways: competence, comparison with previous semesters, relationship between confidence and competence, and error.

Overall 192 students completed a pre-test. The pre-test had 32 questions. Table 1 shows the questions, the percentage correct, and mean confidence levels per question. There were seven questions where fewer than 52% of the students were correct: question 6 on estimation (23% correct); question 10 on average (31% correct); question 20 on conversion from hours to minutes (50% correct); question 22 on conversion from grams to milligrams (51% correct); question 25 on substitution into a formula (44% correct); question 26 on solving an equation with the unknown on the denominator (48% correct) and question 27 on reading a scale on a syringe (48% correct). The median mark of students was of 25 out of 32 and the middle 50% of students were in a range from 21 to 28. While the test showed the majority of students doing relatively well, and were comparable to previous years, nursing students are expected to be fully competent in various aspects of nursing that require numeracy and in particular drug calculations. During their nursing degree they will undertake a specific medical calculations course, and have multiple instances for testing their numeracy skills. Students ranked their confidence with each of the 32 questions (Table 1, next page) from 1 = no confidence, to 5 = very confident. Students particularly lacked confidence with three questions.

Question 26 (3.39 out of 5). The poor result is not surprising as students could not use an intuitive approach (e.g., doubling or dividing by a whole number) and in general research suggests students show an inability to scale by non integer (Steinthorsdottir & Sriraman, 2009). If using an algebraic approach, the unknown on the denominator is a more difficult question than one where the unknown is in the numerator (related to poor manipulation skills, Poon & Leung, 2009).

Table 1. Results of pre-test and confidence levels (1 = none; 5 = complete) S1 2008 (n = 192).

Question	% correct	Confidence	
		mean	Std dev
1. Write … in numerals: *Twenty thousand two hundred and six*	83	4.842	0.456
2. 102 - 36 =	97	4.770	0.512
3. 1 048 + 21 376 =	96	4.751	0.552
4. 23 × 145 =	92	4.626	0.699
5. 168 ÷ 12	96	4.487	0.819
6. Estimate 512 × 174	23	4.011	0.976
7. Round 495 to the nearest 10	79	4.399	0.831
8. 7 + 2 × 3 =	76	4.569	0.679
9. 3/4 = 15/?	88	4.160	1.078
10. Find the average (mean)…: 21.3, 22, 24.7, 20.4, and 19	31	4.005	1.134
11. 15.8 x 0.2	83	4.295	0.871
12. Express 3/4 as a decimal	94	4.513	0.905
13. Express 80/480 as a fraction in simplest form	78	4.166	1.145
14. 7.42 ÷ 100	84	4.293	0.950
15. Find 30% of 25	86	4.080	1.109
16. Express 0.5 as a fraction in simplest form	91	4.452	1.004
17. Calculate: 2 mL -1.34 mL	78	4.353	0.924
18. Calculate: √81	95	4.419	1.104
19. Express 7 hours 20 minutes in minutes	88	4.516	0.817
20. Express 1.2 hours in minutes	50	4.235	0.980
21. 360 mL = ?L	73	4.208	0.955
22. 1.23 g = ?mg	51	3.984	1.085
23. The chart … When was his temperature the highest?	88	4.652	0.606
24 What was his temperature the last time it was taken?	66	4.436	0.671
25. If $b = \frac{w}{h^2}$ find b if w = 2, and h = 4. Answer as a fraction.	44	3.810	1.229
26. $\frac{10}{4} = \frac{8}{x}$: x=	48	3.387	1.335
27. …... How much fluid is in the syringe? 	48	4.396	0.734
28. Energy is measured in Kilojoules (kJ). Margarine contains 32.2 kJ/gram. How much energy is in 500g tub of margarine?	75	3.935	1.118
29. Unit in qn 27?	86	4.060	1.159
30 A clock gains 15 secs in a day. How long does it take to gain 2 mins?	85	4.307	0.946
31 A Paediatric patient weighing 25 kg is ordered Augmentin 10mg/kg. If Augmentin is supplied as a syrup containing 125mg/mL, how much syrup is to be measured out?	57	3.720	1.227
32 Unit in question 31?	73	3.914	1.181

Question 31, an in-context proportion word problem, had an average confidence level of 3.72. It too had a relatively low pass rate (57%), reflecting school students' difficulty in

this problem solving area (Stacey & MacGregor, 1993). Nursing students have expressed anxiety in undertaking word problems (Galligan & Pigozzo, 2002). This, combined with the unfamiliarity of the terms, as the test was undertaken at the beginning of the first semester of their degree, would account for the low confidence level.

Question 25 (3.81). While only 44% of students were correct with this question, 70% had either 1/8 or 0.125 as an answer. This suggests that the algebraic symbols or the squaring on the denominator may have created the uncertainty.

Figure 3 shows the relationship between confidence levels and pre-test results. Overall, there was a positive correlation with about 38% of the variation of the pre-test results being accounted for by confidence level. While all the questions below the regression line could be identified as 'overconfident', question 6 (on estimation), question 10 (on average) and perhaps question 27 (on reading a syringe), in particular appear to be overconfident (circled points in Figure 3).

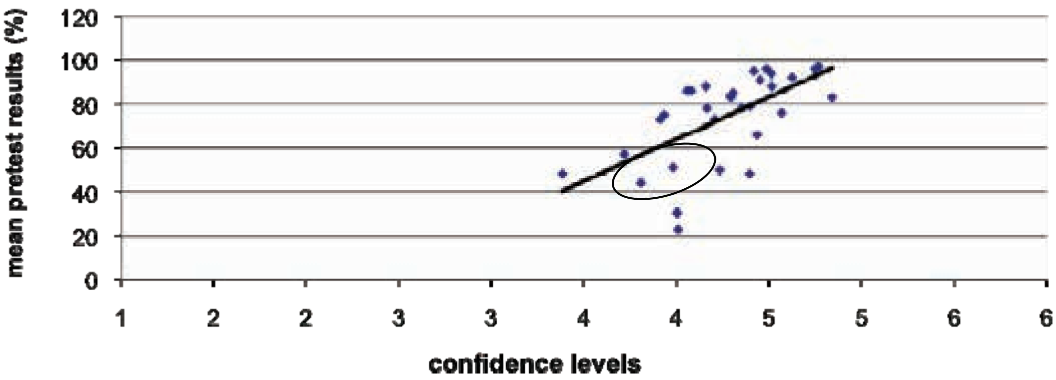

Figure 3. Scatterplot of confidence levels and pre-test results.

An aim in the nursing numeracy course is to improve confidence and competence in numeracy ensuring students are neither confident and wrong, nor underconfident and right. The confident and wrong category is of particular concern, as they may not know they are wrong and may be less likely to check their calculations. There were 183 students that had at least one question confident (4 or 5) and wrong. On average, these students got 5.01 question incorrect (sd 2.94), and 17 students had more than 10 questions wrong in this category. Questions that were 'overconfident' in at least 50 cases are shown in Table 2. These seven questions match the data points below the line of best fit in Figure 3. The other two questions below the line were question 1 on numerals (83% correct) and question 8 on order convention (76% correct).

Table 2. Questions where there was significant overconfidence.

Question	Details	Number confident & wrong (4 or 5 in the Likert scale)
6	estimation	111
10	average	86

20	hrs to minutes	68
22	grams to milligrams	54
23	reading graph—what temperature	58
25	$b = \frac{w}{h^2}$	50
27	reading the syringe	82

The test results and comments from the researcher's class were collated into one spreadsheet that detailed students' responses and reflections for each question. Sixteen of the students submitted comments and results. These were analysed for themes, question by question. One preliminary scan and two complete scans of the document produced seven themes. The themes were also coded positively or negatively and whether the student was correct or not, thus creating four alternatives for each question. From the 512 possible comments (16 students × 32 questions), 697 identifiable points were extracted. Themes and some samples can be seen in Table 4.

Table 3. Themes from student reflections.

Theme	Example
Students' ability or understanding (30%)	I never understood fractions (negative/ wrong); I found this easy however I got it wrong (positive/wrong).
Confidence (20.6%)	I have never been that confident with multiplication. Especially when it is with large numbers (negative/right); I feel confident with this type of question (positive/right)
Complacency/checking (12%)	Care needs to be taken to ensure answer received make sense (positive/right); That again is just terrible adding up and not taking the time to check. Check and re-check my answers (positive/wrong)
Calculator (13.3%)	Fairly confident No calculator (did not use/wrong)
Knowledge/Remembering/ Thinking/Method (19.4%)	Once I remembered what the symbol that was it was easy.. try to unlock my suppressed maths from my brain (Positive/right);. Don't know what the sign over the 81 meant (negative/wrong).
Affect (happiness/ enjoyment/relieved); Importance/life experience (3.9%)	I didn't read the syringe properly … dangerous I could of killed someone!! (positive/wrong). Percentages don't seem to agree with me, I didn't think I would get it right (negative/wrong)
Silly error (1.2%)	I didn't read the question properly, silly mistake

From the data of errors made in one class and an analysis of errors of the whole cohort, a profile of error was starting to emerge with some points to note:

- Students sometimes had a variety of solutions that, in the context of nursing would probably be acceptable, so care needs to be taken to ensure the mathematics is marked appropriately or the question is worded realistically.
- Students need to be aware of reading questions correctly. This can often be vital in nursing where prescriptions and directions from doctors are exact. While entering units in an answer when the question asked for no units may appear trivial, it does reflect a hidden issue of reading instructions in general.
- There appears to be an underlying issue of understanding of some mathematics concepts mainly decimals and fractions and perhaps an awareness of estimation.

While students appear to be aware of their lack of understanding of fractions, there may be a limited awareness of the issue of decimals. This is particularly true when students are using their calculator.

- The pre-test appeared to focus on mathematical skills with few of the questions having an explicit nursing context. However, there were very few reflective comments from students suggesting a problem with their mathematics related to a clinical setting. This is despite explicit articles students were directed to read in the first two weeks of semester.

Conclusion

The instruments used in this study aimed to identify students' numeracy when they first started the course. Numeracy was framed in terms of competence, confidence, and a critical awareness of the mathematics in context and in students' own mathematical knowledge.

Confidence and critical awareness included an understanding of students feeling towards mathematics. Only about one quarter of students disliked or hated mathematics. Many students described changes in feelings towards mathematics, sometimes from liking in primary to disliking in high school; others were the other way around. When directed to reading specific articles on mathematics in nursing, most students could see the relationship between mathematics and nursing in general but were unable to articulate that in specific mathematical skills.

In analysing students' results from a test designed to investigate nursing students mathematics knowledge needed in nursing, students did fairly well. The percentage of students correct ranged from 23% correct for a question on estimation, to 97% for a question on subtraction.

Students' confidence in their answers varied from 3.38 (out of 5) for solving an algebraic problem, to 4.84 for writing a number in numerals. However, a successful nurse needs to be confident and competent with this level of mathematics, so something close to 100% would be the aim for these students in competence and 5 for confidence. In this first test there was a significant issue with both under-confidence (right but not confident) and over-confidence (wrong and confident) with their answers.

Since 2008 the implementation of tests have been refined to further engage students in being more critically aware of their own mathematical skills and the mathematics needed for their degree and to make the test easier to administer using a variation on Self-Test (Taylor, 1998). It is also planned to extend this approach to the final year of students' degrees where they can again appraise their competence, confidence, and awareness of the mathematics needed in the career in which they are about to embark. Variations of this approach have also been used in other degree programs, in Economics, Engineering and Education where the focus is not just on competence at a particular moment in time, but with an aim to assess their academic numeracy.

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TEACHER PROFESSIONAL LEARNING IN NUMERACY: TRAJECTORIES THROUGH A MODEL FOR NUMERACY IN THE 21ST CENTURY

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This paper reports on a year long, state wide research project that aimed to assist primary and secondary teachers to improve their teaching and learning practices through engagement with a new model of numeracy. Data collection included sequence maps of participants' development as teachers of numeracy as elements of the model became more prominent in their thinking and planning. Semi-structured interviews were also used to clarify and expand upon teachers' perceptions of their own development. Findings include a propensity to begin with the dispositions element of the model but responses showed that in most cases all elements were eventually addressed by each teacher.

Introduction

The importance of developing numeracy capabilities, in addition to acquiring purely mathematical competence, has been acknowledged in national reports (Human Capital Working Group, Council of Australian Governments, 2008), pending national curriculum documents (Australian Curriculum Assessment and Reporting Authority, 2009) and through the inclusion of contextualised mathematics problems in the assessment frameworks of international testing regimes (e.g., OEDD/PISA, 2003). Numeracy is increasingly seen as fundamental to developing students' capacities to use mathematics to function as informed and reflective citizens, to contribute to society through paid work and in other aspects of community life (Steen, 2001).

While there is a substantial body of literature devoted to the nature of and importance of numeracy education and to effective approaches to professional development in mathematics teaching (Loucks-Horsley, Love, Stiles, Mundry & Hewson, 2003), far less is known about how teachers learn about, appropriate and then create effective mathematics teaching practices. This paper reports on a year long research and development project that investigated approaches to assisting teachers to plan and implement numeracy strategies across the curriculum in the middle years of schooling (Years 6-9). The aim of this paper is to examine teachers' perceptions of their own professional learning in relation to a rich model of numeracy and to map how these perceptions changed through the duration of the project.

Theoretical framework

Numeracy, which is also known as quantitative or mathematical literacy in some international contexts, has been recognised internationally through the OECD's Program for International Student Assessment (PISA). According to PISA's definition, mathematical literacy is:

> an individual's capacity to identify and understand the role mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2004, p. 15)

Within Australia, increasing importance is also being placed on the need for individuals to have the capacity to use mathematics in the beyond school world. This way of using mathematics is captured in the following definition, which has gained general acceptance in Australia, "To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life" (Australian Association of Mathematics Teachers, 1997, p. 15).

While these definitions capture the broad thrust of the concept of numeracy, they lack the detail necessary for teachers to implement numeracy based approaches in practice. More recently, however, Goos (2007) has proposed a model of numeracy which encompasses four essential elements which are enacted within a perception of mathematics as knowledge-in-action. The model incorporates attention to real-life contexts, the deployment of mathematical knowledge, the use of physical and digital tools, and consideration of students' dispositions towards the use of mathematics. These elements are embedded in a critical orientation to the use of mathematical skills and concepts which emphasises the evaluative and judgemental aspects of numeracy practice, for example, the capacity to evaluate quantitative, spatial or probabilistic information used to support claims made in the media or other contexts. The elements of the model and the critical orientation within which these elements interact are described in Table 1.

Table 1: Descriptions of the elements and critical orientation of the numeracy model.

mathematical knowledge	Mathematical concepts and skills; problem solving strategies; estimation capacities.
contexts	Capacity to use mathematical knowledge in a range of contexts, both within schools and beyond school settings
dispositions	Confidence and willingness to use mathematical approaches to engage with life-related tasks; preparedness to make flexible and adaptive use of mathematical knowledge.
tools	Use of material (models, measuring instruments), representational (symbol systems, graphs, maps, diagrams, drawings, tables, ready reckoners) and digital (computers, software, calculators, internet) tools to mediate and shape thinking
critical orientation	Use of mathematical information to: make decisions and judgements; add support to arguments; challenge an argument or position.

The elements of the model are represented as the net of a tetrahedron surrounded by and bound together by a critical orientation (Figure 1, below).

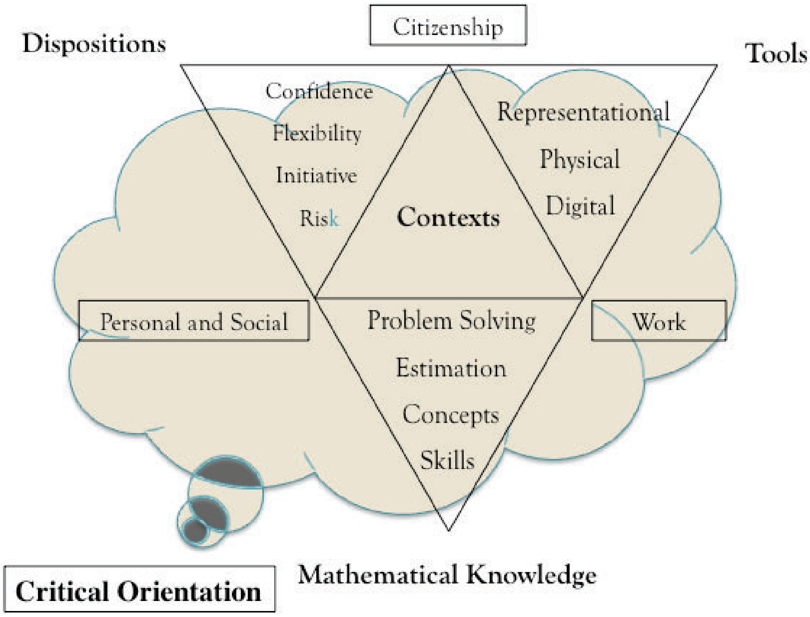

Figure 1: A model for numeracy in the 21st century (Goos, 2007).

This model offers a broad interpretation of the role of mathematics in bridging the gap between school mathematics and the wider world and has been used as a framework to audit mathematics curriculum designs (Goos, Geiger & Dole, 2010) and for analysis of teachers attempts to design for the teaching of numeracy across the curriculum (Goos, Dole & Geiger, 2010). The numeracy model was also used to promote teacher professional learning and, in particular, to assist teachers to reflect upon their own practice.

Teacher professional learning in numeracy

Ball and Bass (2003) and Ma (1999) have both identified the importance of personal and professional identities to reform in mathematics teaching. In supporting these identities and associated teacher self-efficacy issues, Millett, Brown and Askew (2004) emphasise the vital elements of time, talk, expertise and motivation. These elements were deemed as essential in promoting teachers' sense of agency when attempting to make fundamental changes to their teaching practice.

In a synthesis of literature related to effective teaching in numeracy, Muir (2008) identified the following practices as being central: making connections; challenging all pupils; teaching for conceptual understanding; facilitating purposeful discussion; maintaining a focus on mathematics; and possessing and instilling positive attitudes towards mathematics. While it is helpful to identify such practices, Muir (2008) does not attempt to describe how teachers decide to change their current practice or how to support them in the process of change. In a study of teachers' numeracy pedagogical practices in Tasmanian schools, Beswick, Swabey, and Andrews (2008) found that most teachers focused on the creation of supportive classroom environments but there was a disconnect between the aims of the mathematics curriculum and teachers' actions in relation to numeracy specific pedagogical approaches.

These studies highlight the need for ongoing research into understanding how teachers come to identify and then appropriate new pedagogical practices specific to numeracy, especially those practices which are different from those practices specific to the teaching of mathematical skills.

Research design

Participating teachers self-identified as volunteers in response to a request for expressions of interest in a cross-curricular, middle school (Years 6 to 9) numeracy project which was distributed to every government school within a single Australian state. Participants were selected in order to provide coverage across metropolitan, provincial and remote schools and to capture a mix of primary (K – Year 7), secondary (Years 8 – 12) and area schools in rural areas (Years 1 – 12). In addition, because the focus was on the teaching of numeracy across the curriculum, efforts were made to include teachers who had specialist mathematics knowledge and those who did not. This meant that participants included generalist primary teachers who taught across the curriculum and also secondary teachers with specialised subject knowledge (e.g., mathematics, science, English) Participating schools nominated two teachers in order that these teachers could collaborate on, and support each other with, their contributions to the project.

The project was conducted between January and November 2009 and included both teacher professional learning and research components. As action research is an appropriate methodology for supporting educational reform through collaborative partnerships between participating teachers and university researchers (Somekh & Zeichner, 2009), in this case, the embedding of numeracy throughout the school curriculum, this approach was adopted for this study. A series of project meetings and school visits were conducted to support teachers through two action research cycles of plan, act, observe, reflect in order to replan and continue through the next cycle.

Teacher professional learning activities included whole project teacher meetings (March, August and November) which were led by the project researchers. In these meetings, elements of the numeracy model were explored and examples of classroom activities which embodied these elements were demonstrated. After the initial meeting, whole project meetings were also used by teachers to showcase work in progress and to seek feedback on ideas they were preparing for implementation from other project teachers as well as the researchers. Between whole project meetings, members of the research team visited each participating school and provided feedback and advice on the introduction of numeracy based approaches to teaching in their specific school contexts. This included, for example, feedback on an observed lesson using the numeracy model as a guide or providing support in assisting teachers to understand aspects of the numeracy model they were struggling to comprehend.

The research component of the project was based on data gathered during whole project meetings and school visits. During whole project meetings teachers were asked to: outline their initial conceptions of numeracy; complete a survey on teachers' confidence with numeracy teaching; and to map their personal progress in numeracy by using the numeracy model as a lens. Researchers' visits to schools involved: recording field notes for lesson observation; pre- and post-lesson teacher interviews and post-lesson interviews with students; collection of student work samples. An outline for both teacher professional learning activities and research data collection appears in Table 2.

Table 2: Outline of professional learning and research activities.

Event	Activity
1st whole project meeting	Professional Learning: Researcher led orientation to numeracy model, exemplar activities. Research: Teachers' initial conceptions of numeracy; numeracy confidence survey.
School visit 1	Professional Learning: Researcher feedback on teaching programs and lesson observations. Research: Field notes of lesson observations; student work samples; audio recorded pre- and post-lesson teacher interviews; audio recorded post-lesson student interviews.
2nd whole project meeting	Professional Learning: Evaluating implementation of the initial numeracy unit that the teachers had taught; setting goals and planning for the second action research cycle.
School visit 2	Professional Learning: Researcher feedback on teaching programs and lesson observations. Research: Field notes of lesson observations; student work samples; audio recorded pre- and post-lesson teacher interviews; audio recorded post-lesson student interviews.
3rd whole project meeting	Professional Learning: Showcase of four different professional learning trajectories. Research: Learning trajectory mapping; repeat of conception of numeracy activity and numeracy confidence survey.

As this paper is concerned with teachers' perceptions of their own professional learning, the data examined here are drawn from the learning trajectory mapping activity conducted in the final whole project meeting and the final post-lesson teacher interview conducted during the second school visit.

Teacher trajectories through the numeracy model

The professional learning trajectory activity required teachers to indentify the element of the numeracy model which represented their initial focus at the beginning of the project and also to indicate those elements that assumed greater importance to them as the project progressed. Teachers were provided with a copy of the numeracy model and asked to annotate the model in a way that indicated their professional learning journey over the duration of the project. For example, Karen annotated her copy of the numeracy model in the following fashion (Figure 2).

This annotated copy indicates that Karen began with a focus on supporting students' development of mathematical skills as she believed that, once acquired, these skills would be "naturally" adapted for use in out-of-school contexts. She comes to realise that it is the use of the skill in real-world contexts she wants to promote and not just the skill itself, which leads her to take account of the importance of the other elements of the model as time progresses.

Of the 20 teachers involved in the project, 18 completed the mapping task in the way we requested. Figure 3 shows these teachers' starting points and the direction in which they indicated they had developed as the project progressed. Of the 18 valid responses, 8 teachers indicated that they had entered the project with a concern for students' dispositions. Their annotations suggested that they were uneasy with students' negative feelings towards mathematics and wanted to devise numeracy learning experiences that would have a positive impact. Seven teachers indicated that their starting point had been students' mathematical knowledge and skills, and their annotations suggested that they believed that if students had appropriate mathematical knowledge and skills, they would be successful in applying these as required in context. Only 3 teachers indicated that they started the project with an emphasis on contexts, stating that this approach allowed students to apply their mathematical knowledge in meaningful situations. None of the teachers said they came to the project with a primary interest in tools or a critical orientation.

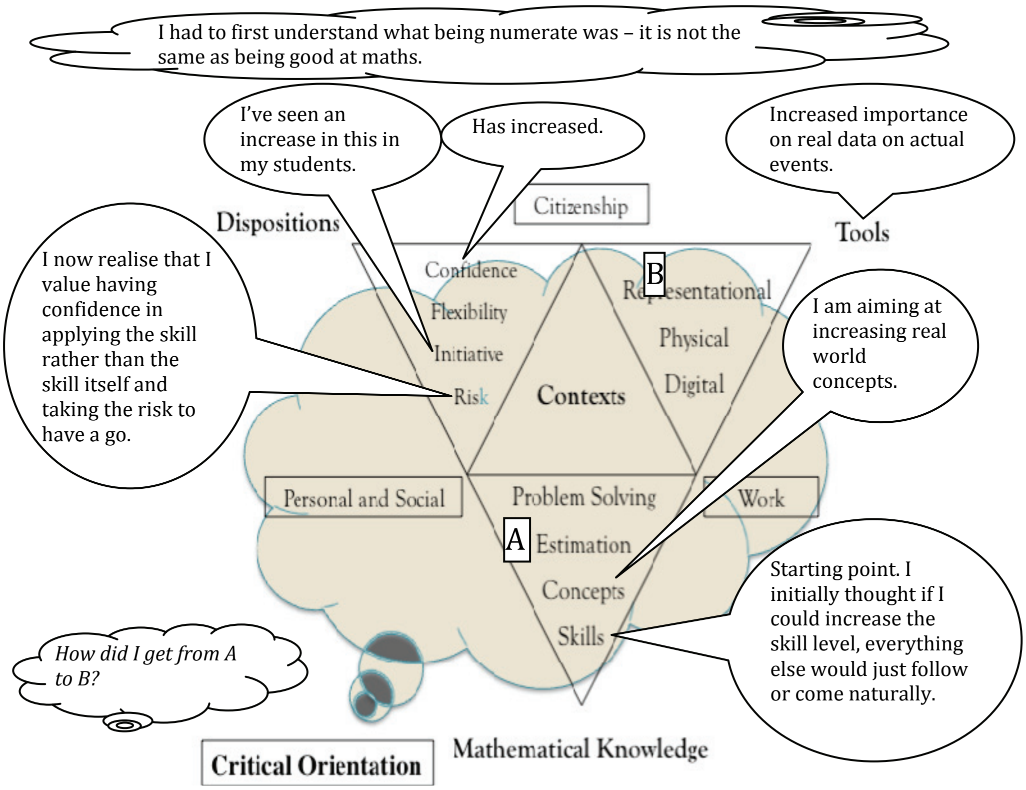

Figure 2: Karen's trajectory through the numeracy model

Although varied, teachers' trajectories through the model showed some patterns of similarity (see Figure 3). Knowledge to dispositions (K – D) and dispositions to knowledge (D – K) were common patterns, possibly indicating teachers' beliefs about the connection between success in using mathematical knowledge and a positive disposition.

Dispositions (D)	Knowledge (K)	Context (C)
D – C	K – D (2 teachers)	C – K – CO
D – C – T	K – D/C	C – K – D – T
D – C – K (2 teachers)	K – D – T	C – All
D – K/T/C	K – T – D (2 teachers)	
D – K/T – C	K – C – D	
D – K/T – C/CO		
D – K/T/C – CO		

Figure 3: Starting points and trajectories in engaging with the numeracy model.

For the latter pathway, tools were linked often with knowledge. Only four teachers indicated that they considered the critical orientation aspect of the numeracy model, and this was their end point. One teacher, indicated by C – All in Figure 3, put the starting point as contexts, but then annotated the model comprehensively to show how integrated and equally important all these elements were.

Although the teachers identified different starting points and trajectories through the numeracy model, at least half of the valid responses to the mapping task indicated they had attended to four of the model's five components during the life of the project: 16 teachers annotated knowledge, 16 dispositions, 13 contexts, and 9 tools.

It was interesting to observe that teachers' most common starting point in engaging with the model was a concern for student dispositions. It appears that teachers may have initially paid most attention to components of the model representing student characteristics of concern to them, such as dispositions and mathematical knowledge, and then explored the use of contexts, tools, and, less commonly, a critical orientation as a means of enriching their numeracy teaching.

Vignettes

Figure 3 indicates that teachers took different directions in developing their approaches to the teaching of numeracy. These directions varied in relation to both their starting points and also the order in which they developed an appreciation for the other elements of the numeracy model. Three different teacher learning trajectory cases are now presented to illustrate the types of reasoning teachers used to make decisions about the directions they chose for their own development.

Catherine

During her final interview, the researchers asked Catherine (a middle school teacher with an English specialisation) to reflect on her changing understanding of numeracy in terms of the model presented early in the project. She explained that her desire to improve students' dispositions marked her entry point to the model, and she attempted to do this by exploring the numeracy demands of different curriculum and real world contexts. This necessitated a change in teaching practice towards a less directive and more inquiry-oriented approach, a "letting go" process that Catherine found difficult but more effective for enriching students' mathematical knowledge and promoting a critical orientation to evaluating information and answers. Once she began to give students more responsibility for their learning, she became more willing to experiment with unfamiliar tools, such as spreadsheets, for problem solving. While her entry point into enhancing her students' numeracy was through attempting to improve students' dispositions, Catherine, over the duration of the project, addressed all aspects of the numeracy model and through this process changed her approach to teaching in a fundamental way.

Maggie

When Maggie (a secondary mathematics teacher) was asked what were the key factors in developing her new understanding of teaching numeracy, she said she began with a desire to improve her teaching by increasing her focus on embedding student learning in engaging contexts. She believed this was a vital precondition before she could convince students of the need to acquire relevant mathematical knowledge. Through the course of the project Maggie noticed her increased focus on developing activities that provided a critical orientation towards the use of mathematics. Once this element was introduced, she realized the role dispositions played in encouraging students to try approaches to solving a problem for themselves rather than expecting Maggie, as the teacher, to simply provide solutions. Finally, Maggie had increased her use of tools, particularly digital tools, through the project as she could see there were advantages in using these tools in exploring and analysing authentic contexts she could bring into her classroom. Although she had taken a different pathway from other teachers in the project, Maggie addressed all aspects of the numeracy model, resulting in a deeper understanding of what it means to be numerate and in a more targeted approach to developing numeracy capacities in her students.

Sarah

Sarah (a generalist primary teacher) came to this project as an experienced and successful classroom teacher who incorporated literacy development in her teaching at every opportunity. She knew that numeracy should also be promoted across all learning areas, but believed that numeracy stemmed from proficiency in mathematics knowledge and skills – and this had been the predominant emphasis of her mathematics program. Indeed, at the beginning of the project she said that she looked at the elements of the numeracy model and saw them as a blur, in that she knew they were all important but felt that the model had little clarity to guide her planning for numeracy. Her journey started by using a context to extract mathematical knowledge, with the result being an artificial imposition of mathematics in unnatural and irrelevant contexts (e.g., what pattern makes up the floor of the War Memorial when the focus of the unit was on history, heroism and the horror of war). Through critical self-reflection, Sarah saw how the learning area provided the context, not the topic, and through the learning activities the numeracy elements of mathematical knowledge and tools could be developed. The context also enabled students to develop a critical orientation as they explored particular topics in depth. (e.g., How many young men died serving the war? What percent of the population was this?) By developing mathematics knowledge through such meaningful contexts, Sarah noted the growth in her students' positive dispositions towards mathematics and confidence in their desire and ability to apply mathematics as required. Sarah claims she now sees the importance of all elements of the numeracy model that we presented at the start of the project, and is confident in developing units of work that integrate this vision of numeracy into her natural teaching style.

Conclusion

Even though teachers' involvement in this project began with an introduction to a single, specific model of numeracy, their own development as teachers of numeracy varied considerably in relation to elements of the model they initially chose to emphasise and in the order they chose to adopt other elements. It would appear that, as their own understanding of the nature of numeracy developed, so did their appreciation for other elements of the model. This, in turn, led teachers to incorporate other elements of the model within the duration of the project.

Teachers also noted the inter-related nature of the elements and often reported how beginning with one element led to the incorporation of another in their teaching. This was the case with Sarah, who noted how the use of an authentic context led naturally to the incorporation of a more critical approach to the use of mathematical skills as there was a need to use quantitative methods in order to resolve important questions which arose through her unit of work.

This study has provided insight into possible approaches to assisting teachers to find their own directions in developing effective numeracy pedagogies, but further research is required into how to best support teachers in finding directions and trajectories most suited to their own circumstances.

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INSIGHTS FROM ABORIGINAL TEACHING ASSISTANTS ABOUT THE IMPACT OF THE BRIDGING THE NUMERACY GAP PROJECT IN A KIMBERLEY CATHOLIC SCHOOL

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As part of the Bridging the Numeracy Gap Project, four Catholic schools in the Kimberley appointed Key Aboriginal Teaching Assistants in Numeracy who, along with a classroom teacher from the school, participated in a 6-day professional learning program aimed at developing their mathematics teaching and leadership. At the end of 2010, audio-taped conversations took place to gain insight about the impact of the Project on learning and teaching mathematics at the school. Analysis of these data demonstrated that Aboriginal Teaching Assistants had clear views about the positive impact of project and of how to improve Aboriginal students" opportunities to learn mathematics at school.

Introduction

Aboriginal people across Australia advocate strongly for children, and speak passionately about the important role that education plays in breaking the cycle of poverty experienced by many Aboriginal families. Sadly, Aboriginal and Torres Strait Islander students continue to have lower scores on national mathematics achievement tests than non-Indigenous Australians, and lower rates of secondary school completion.

One recent Federal Government initiative that aimed to reduce this education gap has been the *Literacy and Numeracy Pilots* (Department of Education, Employment and Workplace Relations [DEEWR], 2010). This paper reports on one of these pilot studies *Bridging the Numeracy Gap in Low SES and Aboriginal Communities* (Gervasoni et al., 2010) that involved 42 school communities across Victoria and Western Australia, including four schools in the Kimberley.

The focus of this paper is an analysis of the views of three Aboriginal Teaching Assistants (ATAs) from a participating Catholic School in the Kimberley. Their perspectives are examined to provide insight about the challenges facing our country as we learn to bridge the numeracy gap.

Bridging the numeracy gap

Both at national and state level there is concern about the size of the „gap" between the results of non-Aboriginal and Aboriginal students on the national benchmark tests in numeracy (Australian Association of Mathematics Teachers [AAMT], 2009; DEEWR, 2009; Perso, 2002). Many researchers argue that that this is due to the difference between the mathematics of Aboriginal people and Western mathematics, and the ways in which Aboriginal children learn (e.g., Perso, 2002). Jorgenson (2010) explored an inclusive pedagogy model in Kimberley schools, based on successful practices in the United States, but noted that the pedagogies they aimed to develop had little effect or were inappropriate in this context. She concluded that the changes needed to Indigenous education are profound and urgent, but that such changes must be considered in light of the needs and cultures of the people with whom we, as researchers and educators, work. This finding suggests that drawing upon the views and expertise of Aboriginal people is critical. Indeed, Howard, Cooke, Lowe, and Perry (2011) argue that enhanced educational quality and equity for Aboriginal students can only occur through purposeful curriculum change, quality teaching, increased student participation, and the engagement of the Indigenous community. They also highlight that programs aiming to improve education outcomes for Aboriginal people need to consider the social, cultural and community contexts of the Indigenous learners and their families, as well as the mathematical characteristics of the material to be learned (Howard et al., 2011).

Constructs for evaluating programs involving Aboriginal people

Matthews, Howard, and Perry (2003) identify seven constructs that they argue are important for evaluating programs involving Aboriginal people: Social Justice; Empowerment; Engagement; Reconciliation; Self-determination; Connectedness; and Relevance. These constructs are used to analyse the transcript excerpts examined in this paper so need to be well understood. For this purpose, they are outlined below.

> **Social justice** is about treating all people with dignity and respect. It is about a community recognising and acknowledging injustices and the development of appropriate actions and processes to address these injustices for individuals or groups so that there is a degree of equality in the overall outcomes. It is about a freedom of choice. It is about living with your own rights and beliefs and not those imposed from others. It is about your right to be who you are.
>
> **Empowerment** is gaining the necessary knowledge to impact upon change that is essential for effective educational outcomes. It is about Aboriginal people making decisions and sharing their knowledge and skills with others. Being empowered is about making a difference.
>
> **Engagement** is being able to interact purposefully with the discourse around mathematics learning. It is about being excited about what you are doing. It is about being treated as a capable learner. It is about respect and positive interactions.
>
> **Reconciliation** is about walking in someone else"s shoes. It is about taking the time to listen and to care. It is about working together. It is about sharing and understanding the diversity of culture. It is about appreciating people and their values, language and learning styles. It is about recognising and appreciating difference.
>
> **Self-determination** is political. Aboriginal people are a minority people in their own country. To achieve self-determination, there need to be Aboriginal people in control and making decisions. It cannot happen when there is always a non-Aboriginal person with the power to say „yes" or „no" as to what can happen. Individually it can be achieved - you can determine for yourself if you have access to health, education and support.
>
> **Connectedness** is a sense of belonging. A feeling of being accepted, knowing that you have as much right to be in a place as any other person. The need for Aboriginal students

to know that people [teachers] like you, relate to you for who you are. It is about the need to implement the talk. It is about honesty, integrity, being a critical friend in what you bring to any given situation as an important person within the Australian society.
Relevance is about bringing the Aboriginal students‟ environments into the mathematics classroom. It is about providing Aboriginal students with the necessary mathematical skills to enable them to look beyond their horizons. It is about Aboriginal country, Aboriginal nations. It is about where an Aboriginal student lives and using that country in mathematics curriculum, teaching and learning. It is tokenistic to think of relevance being only the application of Aboriginal motifs to classroom materials. The relevance is in how, why, and who make the motifs and how materials are used. (Matthew et al., 2003, pp. 23–24)

Insights about learning mathematics

As part of the *Bridging the Numeracy Gap Project* evaluation, two members of the research team met with three ATAs at School K to discuss their views about the impact of the project and advise about how to assist students to learn mathematics effectively. The conversation was digitally recorded and transcribed, then analysed in terms of the seven constructs identified by Matthews et al. (2003): *Social justice*; *Empowerment*; *Engagement*; *Reconciliation*; *Self-determination*; *Connectedness*; and *Relevance*.

First, the transcript was broken into 44 naturally occurring segments according to the topics, ideas, and issues discussed during the conversation, and then each segment was tagged according to the construct with which it was most strongly associated. The number of segments associated with each construct is shown in Table 1.

Construct	Number of Segments
Social Justice	2
Empowerment	4
Connectedness	5
Engagement	12
Reconciliation	10
Self-determination	6
Relevance	5

Table 1. *The number of transcript segments associated with each construct.*

The most common associations were with the engagement, reconciliation, and self-determination constructs. The Aboriginal Teaching Assistants provide many insights about the challenges faced by children when they are learning mathematics. The following section presents illustrative examples of these insights in relation to the seven constructs.

Social justice

Social justice is about treating everyone with respect and dignity. It is about acknowledging and addressing injustices so that there is a degree of equality in the overall outcomes. It is about living with your own beliefs and not those imposed from others. However, this is not always the experience of Aboriginal students, parents, and teachers. One ATA raised the issue of how important it is that teachers have high expectations of Aboriginal students:

> Perceptions, and that (Aboriginal children) have potential. They can excel at anything. Because a lot of people, when you have an Aboriginal child that can read really well, is really good at maths, it"s like, "Oh!" They"re surprised by it. Why should they be surprised? It"s like, "Yeah we can do it."

This issue of low expectations of Aboriginal students is a social justice issue that needs to be addressed by school communities and school systems.

Empowerment

As part of the Bridging the Numeracy Gap Project, each Catholic School in the Kimberley nominated an ATA and a classroom teacher to participate in a six-day course focused on learning and teaching mathematics. One of the ATAs explained that this experience enabled her to gain important knowledge from other Aboriginal people.

> It"s been really good actually „cause there"s a lot of things that I didn"t sort of know were happening. „Cause I"ve been talking to other ATAs who are involved in this, like Robert who"s from St Clare"s and he was saying that they were … taking on the more *hands on* approach and he was saying how it works there, and he had a lot of stories and so did Lisa from (up north) … and it was good in that way, just that networking, and we could all talk about it … Some of the data stuff though, Alice sat down and had to actually explain to me, but otherwise, other than that, it was really good. I enjoyed it.

She also gained knowledge through observing the classroom teacher assess students using the *Mathematics Assessment Interviews* (Gervasoni et al., 2010), and was impressed that this assessment used practical tasks rather than written questions.

> I"ve seen *Linda* do a couple (of Mathematics Assessment Interviews). Like she"s invited me into actually watch her test … It"s been good „cause she"s got everything set up and they"re just doing each question with the different equipment that"s needed. And again, testing hands on, that"s something that, you know. When you look at testing, "Oh it"s just a written piece of paper, it"s a written test" but testing hands on, you know, it"s great!

Lucy recognised the importance of her role for making a difference in the community.

> Sometimes I think about being a teacher but then I also think maybe I do a lot of good as an ATA as well.

ATAs are encouraged to pursue further studies in education, but many believe that the role they play in connecting with the community would be difficult if they were teachers and worry that the community would view them differently.

Engagement

Children"s engagement in mathematics was an issue of considerable interest and concern for the ATAs. They had a clear view that Aboriginal students learn well through visualisation and hands-on activities, such as they experienced with Linda.

> Oh it"s awesome. The kids love when Miss Linda comes over to Year 2 because there"s that hands on, that visual, the fun atmosphere about learning about Maths, not just the blackboard and the paper you know. I"ve noticed that when … Aunty Linda has come to our classroom, the kids love learning Maths without even realising it"s Maths time. They"re just straight into it. It"s good.

However the ATAs explain that, as the children move through the school, their experiences of learning mathematics change; they become less engaged.

> Getting them to participate, that‟s something we do really well at in (the) Early Childhood Years, and that shows in the attendance rate and yet …(in) middle primary and in senior school and the high school, slowly it drops away because they‟re not having fun anymore. You know they‟re not learning the way that the... How can we teach one way *hands on*, and then all of a sudden they hit Year 2 and Year 3 and then straight away (the teachers) are banging in the worksheets there, and the text books there, and writing from the board, reading, and all that sort of stuff.
>
> And it goes back to the point, if you can‟t read the question, then you can‟t answer the story, so why are you giving them a Maths book … if they can‟t understand it, then they‟ll just sit around. Whereas if you‟ve got the hands on stuff that we do with Linda, and then that‟s good and everybody gets involved.

The views of the ATAs highlight that creating mathematics classrooms where students are engaged and enjoy learning is critical.

Reconciliation

Reconciliation is about listening and caring, working together, and appreciating people and their values, language and learning styles. This theme of reconciliation was apparent in the ATAs‟ conversation around children, parents, teachers, and themselves.

They highlighted how important it is for education that connections are made between students and teachers.

> It‟s making the connection with the kids and then knowing who they are, and what they can do, not kind of labelling them under the levels and abilities.

A number of the families live in a community some 30 minutes from town, and seldom visit the school. However, the ATAs were critical in helping the teachers to appreciate this community, and parents to appreciate the school.

> We went out to *Willow Creek* again at the end of last term and we actually took out all the (children‟s) reports and portfolios out there and we actually sat down and went through the reports with the parents, because, I mean reporting is very, it‟s really hard for a parent, especially one that isn‟t as educated, and even, like the teachers at the school, we don‟t understand all the dots and the shapes (in the reports). You know, how can we expect a parent to? So we sat down with (the parents) and we actually went through it saying, "He‟s good at this" or "He needs work on this" and this sort of thing…. And they enjoyed it, and we actually got invited back again.

The ATAs felt highly valued and appreciated by the principal, and made several comments about this.

> We all feel valued and we know that we‟re valued and even … Mike [principal] will come to us and ask us questions. We‟ve never had that sort of a principal before. And it‟s that feeling valued and knowing that your opinion counts.

The ATAs also highlighted how important it is for classroom teachers to listen and learn about the community from all those about them.

> And I think too, with new graduates … „cause I‟ve had a string of graduates sort of straight out of university, it‟s just listening to who you have in your community. Who you have in your school as well, ‟cause ATAs are a good source of information, as are you know people that have been there for a while, as well and the Indigenous parents.
>
> Our role isn‟t just confined to the classroom … we‟re a member of the community, but we‟re also, we have a lot of other input, and … value to the rest of the school. But a lot of

teachers think, oh you‟re just there to assist them, that‟s it. But it‟s not. … There‟s a lot more to our role than that.

Self-determination

Self-determination is about politics and about voicing your opinion and making decisions about what can happen, without someone else having the power to say „yes‟ or „no.‟ The ATAs have many insights and opinions related to education, and exercise their agency.

> We‟re very passionate about what we do and we put 100% into our positions. If we see that there‟s something not quite right, then we‟re quite willing, and you know, very open to voice our opinion.

They also voiced the opinion that things need to change if the education gap is to be bridged, and argue that a *hands on* approach is needed in mathematics.

> (Our) whole thinking needs to be changed, because obviously what we‟re doing now isn‟t helping. It‟s not working. That‟s why the gap is there with the Indigenous and non-Indigenous students with their education. And I think we need to look at it as a country, "Okay, so this isn‟t working". We need to bring in something that will work, and that would be with this, *hands on* (approach).

One ATA highlighted the importance of Aboriginal parents being involved in decision making about their child‟s education.

> When my son was in Year 1 … it took me a long time to convince the teacher that he could read really well. I knew what he could do at home and then I could see what he was bringing home and I said, "No he can read." and she had him as a level 1 … and I‟m not into levels or anything, but I think, "He‟s going into Year 2 as a level 1 and he hasn‟t been assessed or anything" and I actually got him assessed with his reading test and he was a level 18 ... This teacher had this assumption that he couldn‟t read, you know. It took me as a parent to say, " No. I know he can be better at that." That‟s just keeping him down there when he could be ...

This excerpt also highlights the importance of the teacher listening to parents and assessing students at the beginning of each year to determine their current knowledge so that appropriate instruction can be designed to meet students‟ learning needs.

Connectedness

Connectedness is about belonging and being accepted. The ATAs explained that connections and relationships were very important, particularly for the parents. It was also clear that the ATAs played a critical role in building connections and relationships between the school and family members.

> Just to know each other and get an understanding. Like get (parents) to understand where we come from at the school, and what we do, and then how they feel at home, like you know, if they‟re feeling left out of the loop; then it‟s kind of like for us to explain it to them. Like that connection … If … they feel they don‟t want to speak to the teacher about it, then there‟s always us there, and you even actually get the connection between non-Indigenous parents coming up to us as well … I think you feel that connection as soon as you start talking, as soon as you know everybody in your community, then it‟s a good, like, fostering that relationship.

The ATAs also developed and co-ordinated a Backpack Program for families that included mathematics content. This was another way in which they built connections between the school and home.

> I think also the backpacks are good for parents to take an interest in what their children are doing at school. It‟s like, “Oh so this is what you‟re doing, it‟s great you know”. And just being a part of that and they‟re doing it at home. They don‟t have to come into the school to see what their child is doing, you know what they‟re learning and that. Those backpacks are at home, it‟s in their environment where they‟re most comfortable probably, so that‟s good as well.

However, the ATAs also cautioned that schools are unfamiliar environments for some students, and they were very concerned about this.

> This is the school, it‟s a foreign environment and if they‟re not going to be happy with their learning, well then, we‟ll try and do something else and take them out and do something different …

Researcher: Schools shouldn‟t be foreign environments should they?

Interviewee: No. They shouldn‟t be. … I think for Indigenous kids. A lot of parents keep their kids at home for a very long time. They don‟t look at pre-primary as being compulsory and Year 1 as being compulsory. It‟s … up to the school to try and get those kids in. But we have kids that might turn up on one day, and that day they might not like it, so they don‟t come for the rest of the term. So … to them it‟s like, “What am I doing here?” and “I don‟t want to be here.” So it is a foreign environment, and I know it shouldn‟t be, but that‟s usually it to them.

Surely Aboriginal children should not be experiencing schooling as a foreign environment in their own country.

Relevance

Relevance is about bringing Aboriginal students‟ environments into the mathematics classroom and enabling students to use mathematics knowledge and skills to look beyond their horizons. The ATAs described a program in the western desert that made school relevant for students.

> The perfect example would be Fitzroy Crossing. The kids that we were talking about before, *Melbourne College*, they took those kids out of the Fitzroy School and took them out to Leopold. Now those kids went to school every day but they weren‟t just in the classroom, they were doing a lot of practical things, they were learning at the same time. Now (some of) those kids are going to *Melbourne College* but they actually enjoyed it because it was the practical things that made it enjoyable for them.

The ATAs were also aware of the mathematics students used outside of school. However, they noted that the students didn‟t always connect this knowledge to what they saw in textbooks and worksheets.

Researcher: Do the kids do much Maths outside of school? There‟s a lot of the card games that go on and then, what else?

Interviewee: Well they know about money. They understand money, like they know what two dollars is and a dollar and 50 cents and they know if they want to buy an ice-cream, they have enough money to buy an ice-ream. So they understand that, I suppose they‟re doing that all the time.
I think the difference is whether they understand the concept of money, oh yeah here‟s money, here‟s two dollars, I can go and buy this lolly, like

> the price tag of two dollars. But as soon as you come into the classroom, it‟s just that different atmosphere of learning and how do you put the two and two together, but they‟re still the same thing as when you‟re shopping.

This last comment highlights how importantly the ATAs view students‟ development of mathematical concepts, as opposed to their simply being taught procedures for calculating that they do not understand.

Conclusion

The perspectives examined in this paper highlight the critical role played by Aboriginal Teaching Assistants in helping school communities in the Kimberley provide high quality learning environments for students and their families. Although they are sometimes viewed as just helping out the itinerant teachers in the school, Aboriginal Teaching Assistants are often the only permanent members of the school staff, and play an essential role in building community connectedness and relationships between teachers and families.

When associating the transcript excerpts with the seven constructs used to examine their perspectives, it was found that the Aboriginal Teaching Assistants were particularly concerned about: (1) student engagement in mathematics learning; (2), the importance of the school community appreciating people‟s values and learning styles (reconciliation); and (3) involving Aboriginal people in decision-making about their children‟s education (self-determination). Connectedness (belonging), and relevance were also highly represented in their discussion.

Overall, the views expressed by these Aboriginal Teaching Assistants lead to a vision of education in which: Aboriginal students and families feel part of the education system and are highly involved in decision-making; students are engaged in a relevant and engaging curriculum that they enjoy, and that enables them to learn successfully through visualising, modelling and practical experiences, with minimal use of worksheets and textbooks; teachers believe in the potential of Aboriginal students, have strong relationships with students and their families, high expectations for students as learners, and are able to meet students‟ learning needs through culturally appropriate instruction; and school communities draw upon the expertise of Aboriginal Teaching Assistants, invest in their professional learning, and acknowledge their critical role in building community connectedness and advocacy for Aboriginal students and their families.

Learning from each other and working together to bring about this vision for education is what Reconciliation is all about.

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INSIGHTS ABOUT CHILDREN'S UNDERSTANDING OF 2-DIGIT AND 3-DIGIT NUMBERS

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Five interpretive place value tasks were added to the *Early Numeracy Interview* (ENI) to gain further insight about students' construction of conceptual knowledge associated with 2-digit and 3-digit numbers. The researchers hypothesised that even though some students were successful at reading, writing and ordering numbers, interpreting multi-digit numbers for problem solving remained a struggle for them. Analyses of students' responses showed that the new tasks distinguished students who previously were assessed as understanding 2-digit or 3-digit numbers, but who could not identify 50 or 150 on a number line or state the total of collections reduced or increased by ten. The new tasks assist teachers to identify students who need further instruction to fully understand 2-digit and 3-digit numbers.

Introduction

Most children learn to read and write 2-digit and 3-digit numbers fairly easily, but interpreting the cardinal value of these numbers is the greater challenge. Research during the *Early Numeracy Research Project (ENRP)* in Australia (Clarke et al., 2002) found that being able to read, write, order and interpret 2-digit numbers was a difficult growth point for young children to reach. In a later study involving over 7000 Victorian primary students, Gervasoni, Turkenburg, & Hadden (2007) also highlighted the number of students in Grades 2–4 who were yet to fully understand 2-digit numbers. If we are to improve young children's whole number learning it is important to understand the challenges children face in coming to understand multi-digit numbers. This is the issue explored in this paper that reports on the refinement of the *ENRP Early Numeracy Interview (ENI)* and framework of Growth Points (Clarke et al., 2002) as part of the *Bridging the Numeracy Gap Project* (Gervasoni et al., 2010). The research team aimed to refine and extend the *ENI* and associated Growth Points, originally designed for use in the first three years of schooling, to address issues such as the Place Value dilemma, and so that they were more appropriate for assessing students across all primary school years. The aspect of the research reported here is the refinement of the assessment tasks for Place Value Growth Point 2 (GP2) — reading, writing, ordering and interpreting 2-digit numbers, and Place Value Growth Point 3 (GP3) — reading, writing, ordering and interpreting 3-digit numbers.

Early Numeracy Interview and Growth Points

The *Early Numeracy Interview* (ENI*)* developed as part of the *Early Numeracy Research Project* (Clarke, Sullivan, & McDonough, 2002), is a clinical interview with an associated research-based framework of Growth Points that describe key stages in the learning of nine mathematics domains. Teachers reported that the *ENI* provided insights about students that might otherwise remain hidden (Clarke, 2001). The data discussed in this paper were drawn from the ENI and Growth Point Framework, so both need to be understood.

The principles underlying the construction of the Growth Points were to: describe the development of mathematical knowledge and understanding in the first three years of school in a form and language that was useful for teachers; reflect the findings of relevant international and local research in mathematics (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Wright, Martland, & Stafford, 2000); reflect, where possible, the structure of mathematics; allow the mathematical knowledge of individuals and groups to be described; and enable a consideration of students who may be mathematically vulnerable. The processes for validating the Growth Points, the interview items and the comparative achievement of students are described in full in Clarke et al. (2002). The following are the growth points for the domain of Place Value.

1. Reading, writing, interpreting and ordering single-digit numbers.
2. Reading, writing, interpreting and ordering two-digit numbers.
3. Reading, writing, interpreting and ordering three-digit numbers.
4. Reading, writing, interpreting and ordering numbers beyond 1000.
5. Extending and applying Place Value knowledge.

Each growth point represents substantial expansion in knowledge along paths to mathematical understanding (Clarke, 2001). The whole number tasks in the interview take between 15-25 minutes for each student and are administered by the classroom teacher. There are about 40 tasks in total, and given success with a task, the teacher continues with the next tasks in a domain (e.g., Place Value) for as long as the child is successful. Children's responses are recorded on a detailed record sheet.

The challenge of understanding multi-digit numbers

Many studies have provided insight about the challenges involved in understanding and using multi-digit numbers. One important finding is that children who have not constructed grouping and Place Value concepts often have difficulty working with multi-digit numbers (Baroody, 2004). Another finding is that being able to interpret numerals to order them from smallest to largest is another difficulty for some children. Griffin, Case, and Siegler (1994) observed that this involves integrating the ability to generate number tags for collections, and make numerical judgments of quantity based on the construction of a mental number line (Griffin & Case, 1997; Griffin et al., 1994).

Grouping and place value concepts

Studies have found that successful problem solving with 2-digit numbers depends on children's ability to construct a concept of ten that is both a collection of ones and a single unit of ten that can be counted, decomposed, traded and exchanged for units of different value (e.g., Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989; Steffe, Cobb & von Glasersfeld, 1988). Cobb and Wheatley (1988) found that some children develop a concept of ten that is a single unit that cannot be decomposed, and proposed that this type of concept is constructed when children learn by rote to recognise the number of tens and ones in a numeral, but do not recognise that the face value of a numeral represents the cardinal value of a group.

Fuson et al., (1997) identified five different correct conceptions of 2-digit numbers and one incorrect conception that children use, several of which may be available to a given child at a particular moment and used in different situations. These six conceptions provide researchers with a detailed model to analyse children's use of 2-digit numbers and were considered by researchers when developing the *ENRP* Place Value framework of growth points and the associated *ENI*. However, for the *ENRP*, researchers opted for a less complex model than the Fuson et al. model that they hoped would be more user-friendly for teachers. Ten years on, in refining the *ENRP* assessment interview and framework of growth points as part of the research reported in this paper, it seems important to consider whether the Fuson et al. model better explains the difficulties that some children experience in coming to understand 2-digit, and consequently 3-digit numbers. The six conceptions of 2-digit numbers are explained in detail in Fuson et al. (1997). They are the: Unitary Multi-Digit Conception; Decade and Ones Multi-digit Conception (noticing word parts); Sequence of tens and ones conception (noticing the advantage of counting by tens associated with partitioning in tens); Separate Tens and Ones conception (noticing the number of tens and the number of ones); Integrated sequence-separate tens conception (noticing that the number of tens is linked to the number name); and the Incorrect Single-Digits Conception (viewing each digit as representing ones).

Fuson et al. (1997) contend that for full understanding of number words and their written symbols, children need to construct all five of the correct multi-digit conceptions, with the Integrated Sequence-Separate Tens Conception being the most sophisticated understanding. This requires considerable experience and time. Thus, we believe that the refinement of the *ENI* needs to ensure that teachers can identify students who can use the Integrated Sequence-Separate Tens Conception of 2-digit numbers. To this end we included three new tasks that require students to demonstrate this understanding when increasing or decreasing a given quantity by ten.

Constructing a mental number line

Another important characteristic of number learning is the forming of a mental number line. Griffin, Case and Siegler (1994) proposed that success in early arithmetic depends on the formation of a mental number line in association with understanding the generative rule that relates adjacent cardinal values (i.e., each adjacent number in the number line is one more or one less than its neighbour); and understanding the consequence of the previous idea: that each successive number represents a set which contains more objects, and thus has a greater value along any particular dimension.

One way to help children develop a mental number line for use in problem solving is to engage them in activities involving an empty number line. This is a strategy widely used in the Netherlands and aims to link early mathematics activities to children's own informal counting and structuring strategies. "The choice of the empty number line as a linear model of number representation up to 100 (instead of grouping models like arithmetic blocks) reflects the priority given to mental counting strategies as informal knowledge base" (Beishuizen & Anghileri, 1998, p. 525). This emphasis in the research literature on the importance of the mental number line and empty number line as a means of interpreting numbers is not reflected in the *ENI* until Place Value Growth Point 5 (GP5). When refining the *ENI* we included two new number line tasks earlier in the interview to determine whether students who reach Growth Point 2 (GP2) and Growth Point 3 (GP3) are able to interpret numbers on a 2-digit and 3-digit number line.

Refining assessment tasks for 2-digit and 3-digit numbers

This paper examines students' place value knowledge and the effect of the five new tasks designed to identify students who were assessed at GP2 or GP3, but who may not interpret successfully the quantitative value of 2-digit and 3-digit numbers. These tasks were added to the ENI as part of a refinement process. The data examined are drawn from 2011 assessment interviews with approximately 2000 Grade 1 to Grade 4 students (5-9 years old) from 42 low SES school communities in Victoria and Western Australia who are part of the *Bridging the Numeracy Gap Project* (Gervasoni et al., 2010). This is a Federal Government funded project aiming to bridge the numeracy gap for low SES and Aboriginal and Torres Strait Islander students, and is collaboration between the 42 school communities, Catholic Education Offices in the regions of Ballarat, Sandhurst, Sale, and Western Australia, and Australian Catholic University. The new tasks are shown in bold in Figure 1 (GP2 tasks) and Figure 2 (GP3 tasks).

Pop-Sticks Bundling Tasks – Interpreting 2-Digit Numbers
Ask the child to unpack the icy pole sticks.
Here are some icy pole sticks in bundles of ten. *(Offer the chance to check a bundle if it seems appropriate).*
Here are some more loose ones. (*Show white card for 36.)*
a) Get me this many (icy pole) sticks. *(If the child starts to count all in ones, interrupt and ask them if they can do it a quicker way with the bundles).*
Tell me how you worked that out.
b) Please put one bundle back. How many sticks are there now? How do you know that?

2-Digit Number Line – Interpreting 2-digit Numbers
***Show the child the mauve 2-digit number line card.* Look at this number line. Please tell me the largest number (100). *Point to the little mark*. What number would go here? *(50 – acceptable number range is 45-55).* Please explain.**

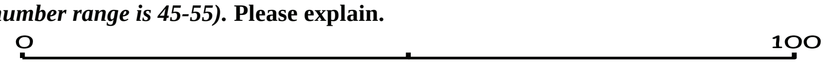

Figure 1. New Growth Point 2 tasks (in bold). Students' place value knowledge.

Part b of the *Pop-Sticks Bundling Task* (2-digit), and the *Ten More* and *Ten Less* questions (3-digit) were designed to distinguish those students who use the Integrated Sequence-Separate Tens Conception strategy when interpreting multi-digit numerals. Inclusion of the 2-digit and 3-digit number line tasks reflects the emphasis in the research literature of the importance of students developing a mental number line to interpret quantities when problem solving.

3-Digit Number Line – Interpreting 3-Digit Numbers
(Show the child the white 3-digit number line card.) **Look at this number line. Please tell me the largest number** ***(200.)*** *Point to the little mark.* **What number would go here?** *(150 – acceptable number range is 130-170).* **Please explain.**

Ten More – Interpreting 3-digit numbers
Show the child the white 592 card. Pause for a couple of seconds for the child to look at the number.
Tell me the number that is ten more than this number.

Ten Less – Interpreting 3-digit numbers
Show the child the white 408 card. Pause for a couple of seconds for the child to look at the number.
Tell me the number that is 10 less than this number.

3-digit Chart Task
Show the child the white 3-digit chart card. This is a number chart. Look at the way the numbers go on this number chart. *Point to the shaded square.* Tell me which number goes in this square *(540).* Please explain.

Figure 2. New Growth Point 3 tasks (in bold). Students' place value knowledge.

A key issue for the research reported in this paper was to determine students' Place Value Growth Points, and whether the new GP2 and GP3 tasks identified students who were not successfully interpreting the quantitative value of 2-digit and 3-digit numbers. Figure 3 shows the distributions of ENI Place Value Growth Points at the beginning of the 2011 school year for 1920 Grade 1–4 students. Each student was assessed by their classroom teacher, and the growth points were calculated independently by trained coders to increase the validity and reliability of the data.

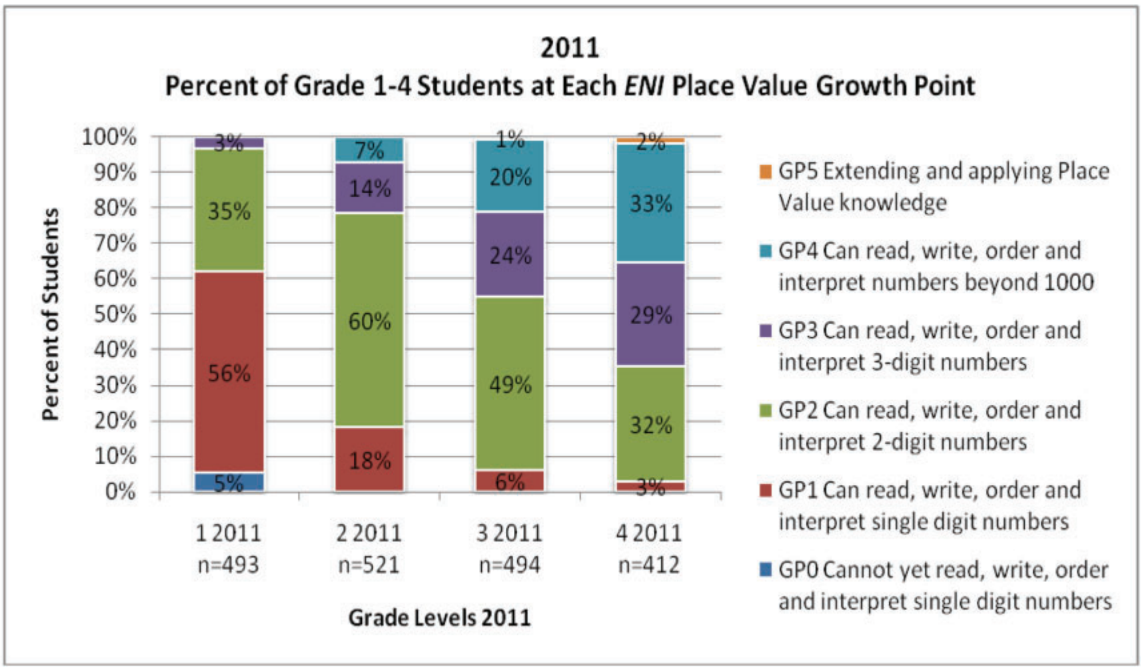

Figure 3. Place Value growth point distribution for Grade 1– 4 students.

An issue highlighted in Figure 3 is the spread of growth points at each level. This has been noted elsewhere (e.g., Gervasoni & Sullivan, 2007; Bobis et al., 2005) and confirms the complexity of the teaching process and the importance of teachers identifying each student's current knowledge and knowing ways to customise learning to meet each student's needs.

The *ENI* data indicate that more than half the Grade 1 students are at GP1, so the initial focus for Place Value instruction for most students is GP2—reading, writing, ordering, and interpreting 2-digit numbers. By the beginning of Grade 2, most students reach GP2. However, by Grade 3, half the students remain on GP2. Examination of the assessment tasks for GP3 and GP4 indicate that students cannot reach these growth points unless they interpret the quantitative value of numbers. We also noted that with the ENI tasks, students could reach GP2 and GP3 successfully using only procedural knowledge to read, write, and order numbers, collect 36 pop-sticks, and identify a 3-digit number on a number chart. The original tasks did not actually require conceptual knowledge to interpret quantities, although conceptual knowledge was assumed.

Analysis of new assessment tasks

Next we examined the data to assess the ability of the new GP2 and GP3 tasks to identify any students who were not interpreting the quantitative value of numbers in the tasks. As the majority of students in Grades 2, 3, and 4 had reached GP2 at least, students in these grades who were assessed at GP2 and GP3 respectively were selected for further examination, and their responses to the two new tasks analysed.

The first new 2-digit task required students to identify the value of a quantity that was reduced by ten (Pop-stick Bundling task). Only students who were judged to be using Fuson et al.'s (1997) Integrated Sequence-Separate Tens Conception strategy were deemed to be successful. This provided confidence that students were able to use all five correct conceptions of 2-digit numbers. The second task required students to interpret a 2-digit number line by identifying the number that was half way between 0 and 100, where a number between 45 and 55 was deemed to be successful.

The data presented in Figure 4 demonstrate that these tasks did identify students who were assessed at GP2, but who did not successfully interpret 2-digit numbers in the *'One Less' Bundle* and *Number Line* tasks.

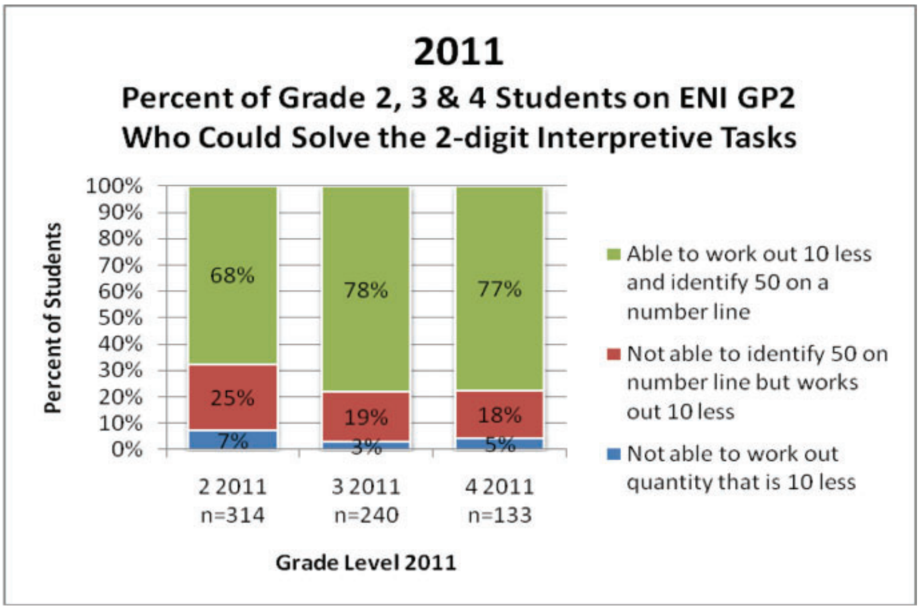

Figure 4. Percent of Gr 2, 3, & 4 students on ENI GP2 who could solve the new 2-digit tasks.

About one third of the Grade 2 students and one quarter of the Grade 3 and Grade 4 students on GP2 were not able to solve both new tasks. This highlights that interpreting 2-digit quantities is an issue for a significant number of students. The number line task was the more difficult of the new tasks. The most common incorrect response was 10, with students counting by ones along the number line until they reached the half-way mark. Of the remaining students who were successful, analysis of their responses to the 3-digit assessment tasks showed that none of these students were successful with the 3-digit interpretive tasks, although many could read, write and order 3-digit numbers. This inability to interpret quantities was the reason why students did not progress to GP3.

Data presented in Figure 5 show that the 3-digit tasks also identified considerable numbers of students who were assessed at GP3, but who could not successfully interpret 3-digit numbers in the 3-digit number line and 10 more/10 less tasks. Only a quarter of the Grade 3 students and 20% of the Grade 2 and Grade 4 students on GP3 were able to solve all 3-digit interpretive tasks. Further analysis showed that the 3-digit *Number Line* task and the *10 Less than 408* tasks were the most difficult of the new tasks. For those students who got three out of the four 3-digit interpretive tasks correct, half were unable to complete the *Number Line* task, and just under half were unsuccessful with the *10 Less* task. Of those students who got only two 3-digit interpretive tasks correct 88% were unsuccessful with the *Number Line* task and 75% were unsuccessful with the *10 Less task*. All these students could read, write and order 3-digit numbers, and all but 5% of these students could successfully complete the original 3-Digit Number Chart task.

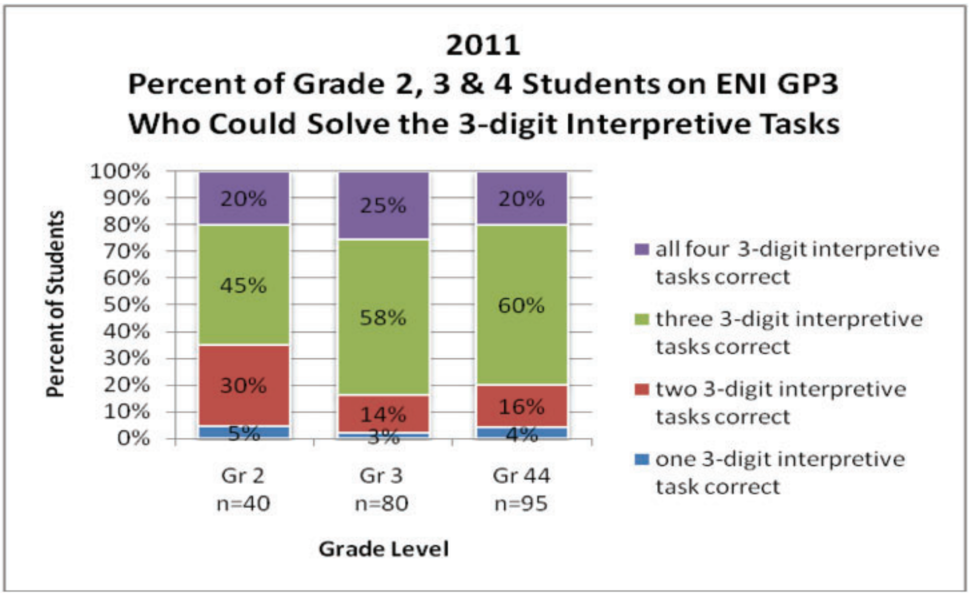

Figure 5. Percent of Gr 2, 3, & 4 students on ENI GP3 who could solve the new 3-digit tasks.

Conclusion

Analysis of 687 Grade 2–4 students' *ENI* responses to the new 2-digit interpretive tasks, and 215 Grade 2–4 student's responses to the new 3-digit interpretive tasks showed that these tasks distinguished students who were assessed as understanding 2-digit and 3-digit numbers respectively, but who in fact could not reliably identify numerals on a number line or state the total of a collection reduced or increased by ten. These additional tasks assist teachers to identify students who need further experience with multi-digit numbers to construct full conceptual understanding, and highlight the importance of teachers focusing instruction on interpreting quantities and developing a mental number line, and not simply reading, writing and ordering numerals. Most children learn to read and write 2-digit and 3-digit numbers easily, but interpreting the cardinal value of these numbers is the greater challenge. Interpretation of quantity and relative quantity are essential for conceptual understanding and problem solving with multi-digit numbers. Perhaps the fact that the *ENI* has not included tasks that identify students who do not fully interpret 2-digit and 3-digit quantities has given teachers an inflated impression of some Place Value GP2 and GP3 students' understanding. We argue that a significant number of these students need further instruction focused on their development of 2-digit and 3-digit number conceptions, including an understanding of quantity, relative quantity and the development of a mental number line.

An implication of these findings is that learning trajectories associated with Place Value and the development of whole number concepts need to adequately account for students' interpretations of quantities. We believe that the *ENRP* Place Value growth points and the associated assessment interview (*ENI*) needs to be modified accordingly, and recommend that the new tasks that were piloted are now included in the *ENI*. Such a refinement will give teachers more certainty about students' current number knowledge and assist them to design more precise instruction.

Acknowledgements

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TEACHING LINEAR ALGEBRA: ONE LECTURER'S ENGAGEMENT WITH STUDENTS

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Linear algebra is a difficult introduction to advanced mathematical thinking for many students. In this paper we consider the teaching approach of an experienced lecturer as he attempts to engage his students with the key ideas embedded in a second course in linear algebra. We describe his approach in lectures and tutorials using visualisation and an emphasis on language to encourage conceptual thinking. We use Tall's framework of three worlds of mathematical thinking to reflect on the value of these activities. An analysis of students' attitudes to the course and their assessment results help to answer questions about the value of such an approach, suggesting ways forward in teaching linear algebra.

Introduction

Research examining the teaching of mathematics at university is a growing but relatively new field and, compared with school-based research, outputs are still relatively modest (Selden & Selden, 2001). Further, the research that has been conducted has rarely examined the daily teaching practice of mathematicians (Speer, Smith, & Horvath, 2010). Three possible reasons for this research lack are enunciated by Speer et al. (*ibid*) as: lecturing is a teaching practice rather than a common instructional activity in which teaching takes place; the professional culture of mathematicians tends to obscure differences in teaching; and strong content knowledge, well-structured for students is considered sufficient for good teaching. Among studies that have been conducted, Rowland (2009) documents the way a university teacher's beliefs about mathematics led her to implement changes to her style of teaching, avoiding a paradigm of exposition and note-taking. Instead she introduced an interactive environment in which class session exercises, testing of conjectures and sense-making were commonplace. Establishing such a community of inquiry in any classroom requires all involved to believe that all participants are learners (Jaworski, 2003). Another study, focussing on university linear algebra teaching (Jaworski, Treffert-Thomas & Bartsch, 2009), examined teaching from a community of practice perspective. This research highlighted that how to deal with the common difficulty of a didactic tension between an abstract/conceptual approach and one that emphasises computational facility is not well understood at university. We agree with the authors that: "Awareness of didactical challenge and a didactic tension can illuminate practice more broadly." (*ibid*, p. 256) and that doing so through a community of inquiry is likely to be a productive way forward. The research described in this paper investigated, through a community of inquiry, how linear algebra may be taught to promote both procedural and conceptual understanding and thinking. Linear algebra demands a more formal approach than calculus, making it difficult for undergraduates to understand the subject (Dorier & Sierpinska, 2001) and research suggests that many students have a minimal understanding of concepts, manipulating matrices instead to pass examinations.

Theoretical framework

Tall's (2004, 2008, 2010) developing theory of three worlds of mathematical thinking seems highly relevant for analysing linear algebra students' thinking processes. It introduces a framework for development of mathematical thinking based on three mental worlds of mathematics: *conceptual embodiment*; *operational symbolism*; and *axiomatic formalism* (Tall, 2010). The embodied world is enactive and visual. It contains embodied objects; it is where we think about the physical world, using " ... not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery" (Tall, 2004, p. 30). The symbolic world is the world of *procepts*, where actions, processes and their corresponding objects are realized and symbolized, and the formal world comprises defined objects (Tall, Thomas, Davis, Gray & Simpson, 2000), presented in terms of their properties, with new properties deduced from objects by formal proof. All three worlds are available to, and used by, individuals as they engage with mathematical thinking. In particular, the three worlds of mathematical thinking combine so that "three interrelated sequences of development blend together to build a full range of thinking" (Tall, 2008, p. 3). A pedagogical implication is that the framework is not proscriptive, and "(a)lthough embodiment starts earlier than operational symbolism, and formalism occurs much later still, when all three possibilities are available at university level, the framework says nothing about the sequence in which teaching should occur" (Tall, 2010, p. 22). For example, Tall claims that many students learning mathematical analysis are happy to think and operate entirely in the formal world, whereas others prefer a more natural approach and think in terms of thought experiments and concept imagery. Thus no single approach is privileged over another, instead decisions should be based on the objective of each course "and not to inflict formal subtleties on students who are better served by a meaningful blend of embodiment and symbolism" (Tall, 2010, p. 25).

Method

The research reported here employed a mixed-methods approach, partly an action research project in which the first-named author (referred to as 'the lecturer' or 'John' in what follows) worked with the other authors as he tried to determine the effectiveness of certain aspects of his teaching of introductory linear algebra, forming a community of inquiry to discuss the teaching openly, and partly a case study of the students. The project involves cycles of planning the relevant teaching episodes, implementing them, and then reflecting on and evaluating the results. Data was collected in 2010 from a second year linear algebra course at the University of Canterbury taught by the lecturer. About 170 students took the course, almost all of them majoring in science or engineering. The lecturer was interviewed (in a discussion mode) twice by the other two researchers, in connection with each of the stages of the project. Thus some of the discussion focussed on his overall goals for the course, and in particular to determine how these goals related to Tall's framework of three worlds (embodied, symbolic and formal) of mathematical thinking, or to the relationship between language and understanding. In addition, other questions dealt with the day-to-day implementation of these goals during lectures and tutorials (the lecturer kept a diary of what happened after each class). Finally, some questions dealt with how the course was measuring up against the intended goals. There were also regular Skype discussions after the lectures had finished as part of the community's discussion of these issues. The interviews and the Skype discussions were audio-recorded and later transcribed for analysis. Data for the case study providing a student perspective comes from several sources. A good number of students (48 out of 170) allowed us to examine their responses to test questions, some of which had been designed to elicit information about their acquisition of the language of linear algebra, or about their relationship to the kinds of thinking described by Tall's framework. About 100 students filled in a survey about the lecturer's teaching and three-quarters of these gave responses to some open-ended questions. Finally, a small number (nine) of students volunteered for individual semi-structured interviews two weeks after the completion of the course. They were asked about definitions (Can you give me the definition for any of these terms in the first question? Were you confident with the definitions during the course?), geometry (Which of these terms in part A can you describe geometrically? Would geometry help you to understand it better?) and general questions (How did you find linear algebra in general? How did you learn the concepts?). Some of the points that came up in discussion with the lecturer were: I wondered if you'd like to tell us how you see the role of the tutorial; I think you like to get the definition motivated by what you're doing in solving equations. Can you tell us how that works?; What's your view of the use of technology in general in this course; How confident were the students in speaking the linear algebra language?; What was your thinking behind setting the exam questions?

Results

In this paper we principally describe the outcomes in terms of the students, and their reactions to the style of the course and evidence of their consequent learning of linear algebra in relation to the lecturer's expectations. The course was founded on the value of language, visualisation, technology (Matlab) and writing and problem solving in tutorials to give students the tools to think about mathematics for themselves. This was all part of what John called trying to put across the "big picture". Both the lectures and the tutorials had to fit in to this overarching aim, and an example of a section of a tutorial, to show the general philosophy behind them, is given in Figure 1.

> The tutorial exercises look at span and linear independence for typical vectors in 2-space and 3-space, and also look at the geometric meaning of span and linear independence.
>
> 1. (a) i. Let $\mathbf{u_1}$, $\mathbf{u_2}$ be two vectors in 2-space. Does $\mathbf{u_2}$ usually belong to the span of $\mathbf{u_1}$? *Hint:* Use Matlab's rand command to construct random pairs of vectors. Use rref if you need to solve any systems of linear equations. ii. Does $\mathbf{u_2}$ always belong to the span of $\mathbf{u_1}$? Give an example of each possibility. iii. Interpret your results geometrically ... [Formatting changed]
>
> **To be handed in:** Write a short report (at most one side of A4 paper) describing your results.
> Your report should consist entirely of English sentences, with no symbols or equations.

Figure 1. An example of a section from one of the tutorials.

The overall aims of the tutorials were expressed to the students by the lecturer as: Learn the technical terms used in linear algebra; Get a feel for what usually happens in linear algebra, but be aware of exceptions; and Be able to describe what happens in linear algebra using ordinary English. As John said in his first interview:

> The main reason I'm in the job is I like helping people. I'm curious about how people think, why they do the things that they're doing. I'd like to show them other ways of thinking, but I'd really prefer that they went out into the world thinking for themselves, and if I could give them some tools that will do that, that will be really nice.

The student interviews were revealing about the lecturer and student perspectives on each of the areas of language use in tutorials, definitions, and visualisation, and these are considered below.

An emphasis on language

For this year's version of the course, John decided to put greater emphasis on gaining the "big picture" through getting the students to use and understand the language of linear algebra. Some of his motivation for this surfaced while he was trying to use a theoretical framework to analyse an incident during lectures where he had been telling the class about how the tutorials were going to work "[My] actual goal [here] is getting students to use written language to describe mathematical ideas, events, etc. because I think this will help them to learn or understand the new ideas." In order to promote this aspect of writing about mathematical ideas, John put some thought into what the tutorials for the course should be like. What resulted was that one aspect of the tutorials required students to write about their ideas. For example, in Figure 1 we see the direction in the tutorial to 'Write a short report'. John's reason for this was that "they're not used to being asked questions like ... 'write a paragraph of 75 words about such and such' ... a really common response to that was to just write down all the relevant definitions in sequence, and not make any reference to what they were actually asked for." Clearly focussing on ideas and language, devoting time to experiments and reports, comes at a cost. John expressed how "I've sacrificed tutorial time that would normally be spent doing hand calculations ... I've told the students, 'Well, actually you can do that in your own time. There's a consultancy session where you can go and get help if you're stuck. But I want to use the tutorial to do something extra." When he reflected on the value of the tutorials John observed that the students found it hard to express themselves mathematically in written language. However, in spite of their struggles at times, they were attending the tutorials in greater numbers than previous years and were more active participants.

> They're certainly behaving very differently from last year's class. For a start, even though there's no compulsion on them showing up to tutorials, I've got I'd say two thirds of the class actually showing up to the tutorials, whereas last year we were lucky if we got a third of them coming along ... the talking in the tutorial is definitely different as well.

Overall the students who were interviewed were often positive about the tutorials.

> S2 I guess formal reports is pretty good ... talking about it with someone else is actually really helpful, so it depends, I would probably keep that ... we had to write reports on certain questions ... and I think that was a really good way of learning the definitions and applying them.

S4 The tutorial system I thought was exceptional. Because ... we had tutorials once every two weeks and even though ... you really had to think about them. So I kind of developed the ideas quite a lot in my head.

S5 The tutorials were quite helpful because you go through and it says what did you learn, and you learn something by doing it.

Formal world thinking often begins with definitions of objects (Vinner, 1991) and the students were given a list of definitions from the start of the course; they were not expected to learn them but to talk and write about them. In fact they could take an A4 sheet into the examination with data, including definitions, written on it. The students' responses to this approach showed that they understood the importance of the language and the need to be able to talk about the ideas.

S6 Yeah, to the subject, they [definitions] are quite important, cause much of that area, linear algebra can't be described without actually understanding and knowing those terms.

S8 Yep, yep, it was made pretty clear to us that these were terms that we were going to need to know, in and out, and we were going to be able to have to use them in conversation ... So it was yeah made pretty clear that they were going to be very, very important ... They [the definitions] were definitely taught.

Visualisation

One of the cornerstones of John's approach to teaching was the value of visual imagery, in terms of encouraging mental imagery through the use of both physical objects and pictures. This is related to the embodied world of Tall's framework, which involves iconic and enactive actions. In the lectures John employed a combination of embodied, iconic and enactive, physical ideas with props, as well as pictures, to get across the ideas. He also values being able to make links between the representations. In his reflections on the lectures he indicated the value of pictures to him personally "I think I have always liked a good picture, although I don't remember any pictures being used when I learned linear algebra as a student—it was just lots of calculations, usually row operations." However, he is conscious of the need " ... to strike a balance between what my colleagues want the students to know in later courses (usually technical stuff like 'how to do this type of calculation') and giving them ... pictures or 'what this all really means', or ... communication skills." Some of the physical, enactive demonstrations he used, and the fact that a picture was also drawn, were described in his lecture reflections:

> I assembled a solid picture of our problem with the rectangular piece of board as the subspace U, my red OHP pen standing on end to represent the given point v (at the top of the pen) so that the projection p that we seek is at the base of the pen. A picture version of the situation was drawn too.
> So I waved my board and a pointer, and then drew a picture, illustrating that if our plane W went through the origin, then the plane and (a suitably positioned) normal line U were both subspaces of 3-space, and that vectors chosen, one from each subspace, were always perpendicular.

However, the pictures were sometimes used to show mathematical relationships, as seen in this example, which refers to the picture in Figure 2:

> I decided to remind them of our earlier picture of the action of a 2x2 matrix A [see Figure 2]. We see now that what was called the 'range of the transformation given by A' is actually col(A) and what was called the 'solution to Ax=0' is actually null(A). The other

line in that diagram is not a subspace as it does not go through the origin (or zero vector). But our third subspace, row(A) can be pictured in this diagram too ... vectors in row(A) are all perpendicular to vectors in null(A), so we can add a line representing row(A) to the domain part of the diagram, perpendicular to the line representing null(A).

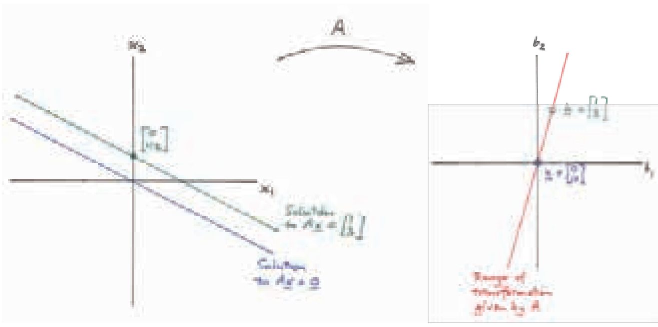

Figure 2. A picture of the effects of a linear transformation.

The interviews with the students showed that they valued the imagery, both enactive and pictorial, that John had incorporated into his explanations.

S1 We did lots and lots of drawings about taking the vector away and when he was describing linearly independent he did some quite good visualisations as well ... He actually got like two sticks or whatever and dropped them and said they're not parallel these are linearly independent.

S4 Yeah John did a lot, yeah he had a lot of using rulers and pencils. I really enjoyed it. I thought it was a fantastically taught class ... he just had a very visual emphasis in the class and really helpful.

S9 Yeah definitely, it definitely helped seeing the pictures. If I'd just read that, I wouldn't have got any kind of a grasp ... I might have been able to do some of it [without the pictures], but I definitely wouldn't have been able to do it as well.

In the interviews students talked about some of the geometric images they used to understand constructs: "Yeah linear combination ... I visualise that parallelogram when you add the vectors." (S3); "The span of two, one of the independent vectors no matter what space it's going to be a plane so if that's going to be a plane ... using a multiple of each of the spanning vectors you can get to any point in that sub space." (S4); and "So linear combination of a couple of vectors is going to span out the plane unless they're along the same line in which case they're dependent so just spans out that line." (S7).

The outcomes

Of course innovation in one's teaching does not necessarily imply that it has value for student understanding. Hence we have to ask the question, did it help understanding? John's answer to this includes the statement:

> You've got to remember, there were almost 170 students in the class ... I did feel that they had a better grasp of things than they've had in previous years ... There were some really pleasing ones, and on the exam I had that question where, I gave them three vectors and they had to talk about them, basically write a little story about them, using all the words that we learnt, and there were some really nice answers to that.

Certainly the student evaluation of the course supported the view that the students liked the approach taken. The scores provided by the 101 respondents were: Q1 The classes were well organised 4.7(/5); Q2 The lecturer was able to communicate ideas and information clearly 4.5; Q3 The lecturer stimulated my interest in the subject 4.1; Q4 The lecturer's attitude towards students was good, 4.7; and Q5 Overall, the lecturer is an effective teacher, 4.6. In addition, some of the comments made in the open question on the evaluation were often extremely positive. Examples include: "Best lecturer I ever had in Engineering" (S1); "One of the best lecturers in [name] university. Asset!!" (S8); " ... this course is probably one of the best taught classes I've had" (S20); "Excellent lecturer—Interesting presentations ... Examples and physical representations useful" (S37); and "I think he has perfected the art of teaching. His teaching style matches my learning style. Keep up what he is doing and students will do well." (S44).

Was this positive view of the course borne out by the assessment results? In the mid-semester test, in addition to standard skills, a conceptual question was included to examine the students' ability to relate mathematical thinking across the embodied, symbolic and formal worlds. Question 3(a) had two parts as seen in Figure 3. The mean score on this question was 6.55 out of 12, and the students did significantly worse on this question than on question 1, which comprised standard procedures ($mean_{Q1}$=60.4%, $mean_{Q3}$=54.6%, t=2.98, p<0.005). However, given the testing nature of some of question 3 this is a reasonable result. In part (a)(i) of the question they were required to interpret the symbolic-algebra equation **u** = 2**v** + 3**w** in an embodied-process manner by drawing a diagram. 67% correctly drew either a parallelogram or a triangle to represent the vectors and a further 27% were partly correct.

(a) Suppose that **u**, **v**, and **w** are nonzero vectors in $\mathbb{R}^3$ such that $\mathbf{u} = 2\mathbf{v} + 3\mathbf{w}$. i. Draw a diagram to illustrate the relationship between **u**, **v**, and **w.** ii. Use the appropriate technical terms from linear algebra to describe the relationship between **u**, **v**, and **w.**

(c) Suppose that **u**, **v** are linearly independent vectors in $\mathbb{R}^3$. i. Give a geometric description of the span of **u** and **v.** ii. Which of the following sets of vectors could be a basis for $\mathbb{R}^3$? (α) **u**, **v**, **u** + 2**v**. (β) **u**, **v**, **u**□**v**. (γ) **u**, **v**, **u** + 2**v**, **u**□**v**. [Formatting changed]

Figure 3. Two parts of the test question 3.

Part (a)(ii) then asked them to use technical terms to describe the relationship between **u**, **v** and **w**. Of the 35 students who got full marks on this part, 5 students mentioned only one concept, namely linear combination, 26 students mentioned 2 concepts (and 18 of these spoke of linear combination and span), and 4 students mentioned 3 concepts (linear combination, span, and linear dependence). Typical examples of comments from students in these three groups were: "**u** is a linear combination of **v** and **w**"; **u** belongs to the span of **v** and **w**"; and "**u**, **v** and **w** are linearly dependent". Part c(i) examined whether students could relate the definition of span of two linearly independent vectors to an embodied process. Tewnty-seven were able to say that the span was a plane in $\mathbb{R}^3$, but only 10 gained full marks by going on to say that both **u** and **v** would lie in the plane. Only seven of the students drew a picture for this part. For c(ii) the students needed to understand the definition of basis and then be able to test whether the sets of vectors satisfied the conditions that the set must a) be a minimum spanning (or generating) set and b) comprise linearly independent vectors, testing the relationship between the formal world definition of basis and symbolic-algebra object thinking. Only one drew a picture, showing that embodied thinking was not to the forefront on this question, and two used matrices to assist them, showing an absence of symbolic-matrix thinking. Some excellent thinking and reasoning targeted the key properties (see Figure 4).

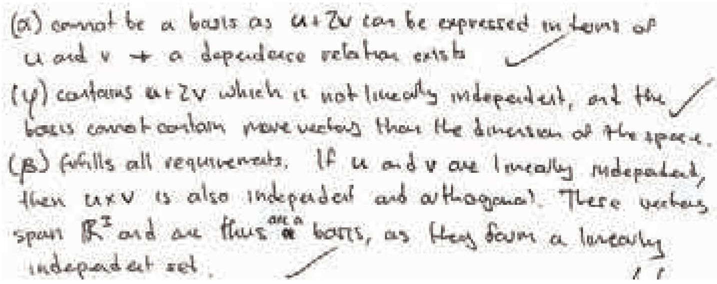

Figure 4. An example of clear application of the properties of basis.

It was also interesting that many used informal ways of thinking about the required properties. Examples of this included the vectors in (γ) being rejected because the set is ‘not efficient’ or there is a ‘redundant’ vector. In this latter case they were echoing John’s language, since in his lecture commentary he used it repeatedly, and wrote down the working definition of basis he gave them as: “A basis for a subspace is a set of vectors which span the subspace and in which there are no redundant vectors”, avoiding use of the term linear independence.

The final examination was a traditional one covering the whole course, with the required test of skills, such as using the Gram-Schmidt process to find an orthogonal basis for the column space of a given 4x3 matrix (Q2(b)). However, it also included questions such as “3(b) Give a geometric description of the following situation: **a, b, c** are linearly independent vectors in $\mathbb{R}^3$.” and 3(c): “Consider the vectors **u** = (1, 0, 0), **v** = (0, 2, 0), **w** = (3, 4, 0) [given as columns]. Write a short paragraph about **u**, **v** and **w**. Your paragraph should be at most 75 words long, but should include as many as possible of the following technical terms from Linear Algebra: *basis, dimension, dependence relation, linear combination, linearly dependent, linearly independent, span, subspace*.” Once again students found such questions harder than the more algorithmic questions. The average mark for Q2(b) was 2.6/3 whereas the averages for Q3(b), (c) were 0.8/2 and 3.8/6. However, a significant number of students gave fully correct answers (e.g., 22 out of 162 students got 6/6 for Q3(c)). The distribution of the final examination marks showed a mean mark of 32.7 out of 50, with 12.3% above 40 and a pass rate of 89.5%. These results compare favourably with previous years’, so the students were not disadvantaged in traditional understanding by the course presentation.

Conclusion

In this study we have looked at student reactions to a particular style of delivery for a second year course in linear algebra, and at the effect this style may have had on the students’ learning. Features of the delivery were the emphasis on language, visualisation and experimentation using technology. Experimentation was structured into fortnightly tutorial sessions, visualisation was encouraged through use of models and pictures in lectures, and language was emphasised in report writing. Students were generally positive about all these features of the course. Talking to fellow students during the experiments, knowing the correct technical language and actually using it in written reports, having to think about the material—all these things were reported as having helped them to learn. This reaction may not seem very surprising, but it is perhaps little unusual coming from a course in linear algebra. Students were also positive about the use of visualisation. For some, memories of boards and marker pens being manipulated conjured up the process of finding projections, while for others, sticks being dropped on the floor remained vivid reminders of linear independence. The assessment results show that many students found these ideas quite challenging. Most students performed better on routine algorithmic tasks in the test and exam, than they did on tasks exploring links between the geometric, symbolic and formal views of linear algebra. This is hardly surprising, of course, as algorithms can be applied without understanding, whereas the other tasks require making links between different representations of the concepts. A pleasing feature of the study was the number of students who succeeded in writing coherent prose that linked the various concepts of linear algebra, to each other in the formal world, and to concrete visualisations in the embodied world. On reflection the lecturer feels quite pleased with the result of his pedagogical experiment, and the supportive community of inquiry worked well too. A good number of students in a second year class learned how to express themselves in the language of linear algebra without loss of skills. This isn't to suggest that we have solved the problem of how to teach linear algebra, since some concepts, such as linear independence and basis, seem harder to learn than others. There lies the continuing challenge.

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CHALLENGING AND EXTENDING A STUDENT TEACHER'S CONCEPTS OF FRACTIONS USING AN ELASTIC STRIP

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This study investigated the fraction content knowledge of a student teacher, and his ability to use that knowledge in a novel situation through use of a scaled elastic strip. Data indicated that using the elastic strip was effective in challenging and enriching the participant's knowledge of ordering fractions. The results suggest that use of the elastic strip could assist student teachers to develop their understanding of fractional concepts.

Introduction

This paper reports a study exploring the use of an elastic strip (Figure 1), which can be viewed as a flexible number line and acts as a manipulative for supporting and extending fraction concepts. It reports on a teaching experiment that investigated the effectiveness of the elastic strip to challenge and give support to developing the content knowledge and pedagogical content knowledge of pre-service primary teacher.

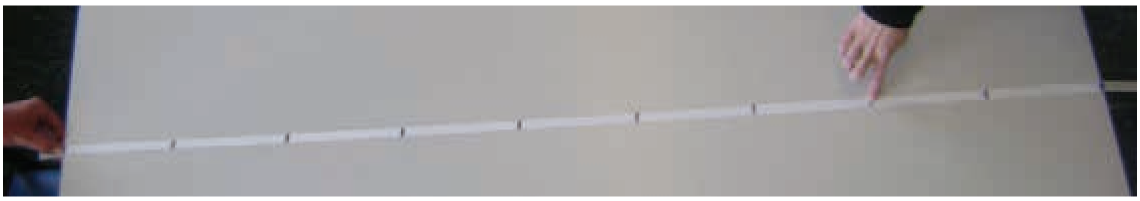

Figure 1. Elastic strip divided into 10 intervals being used to find $^7/_9$ of length of a table.

Primary teachers' knowledge about fractions

Many studies have shown that a high proportion of primary teachers lack sufficient content knowledge and Mathematical Knowledge for Teaching (MKT) (Hill, 2010) to teach the fraction concepts of primary mathematics effectively (e.g., Leinhardt & Smith, 1985; Ward, 2010). In a study in which 53 New Zealand primary teachers self-assessed their knowledge for teaching fractions, 27% rated themselves as very weak or weak (Ward & Thomas, 2007). In a study of the fraction MKT of 78 New Zealand teachers from years 1 to 9 Ward (2010) found 85% of teachers correctly ordered fractions $^3/_5$, $^1/_3$, and $^4/_8$, however just 30% were able to describe how they could support students to order these fractions using a conceptual approach.

Primary school-based studies of fraction learning

In an analysis of the strategies school students used when solving fraction tasks, Smith (1995) found that the competent performers used a rich range of approaches, well matched to specific tasks. Weaker students tended to use a narrower range of taught strategies performed in an algorithmic manner. In an Australian study, 323 grade 6 children were required to explain the reason for their selection of the larger of a pair of fractions. *Benchmarking*, whereby fractions are ordered by considering the relationship of each one to common benchmarks such as 0, ½ and 1, and use of *residual strategies* whereby fractions just less than 1 are compared by consideration of the difference of each from 1, were strategies that demonstrated good number sense that were used effectively. Neither of these strategies was familiar to many of the teachers of these children, indicating that they were likely to have been developed by individual children (Clarke & Roche, 2009).

Results from empirical studies have suggested that the teaching of fractions in primary school should be guided by the following:

- an increase in emphasis on the meaning of rational numbers rather than on calculation procedures (Charalambous & Pitta-Pantazi, 2007; Clarke & Roche, 2009; Moss & Case, 1999);
- making the process of constructing fraction equivalence more explicit in a range of fractional situations (Ni, 2001);
- explicit sharing of benchmarking (Clarke & Roche, 2009; Moss & Case, 1999) which, for example, supports the ordering of $^{3}/_{7}$ and $^{11}/_{20}$ by comparing them both with ½;
- a decrease in using pie graphs as a representation of fractions, and an increase in using other forms of visual representation (Moss & Case, 1999);
- building on children's self-developed solution strategies (Moss & Case, 1999);
- careful definition of numerator and denominator so that the improper fractions fit naturally within the definition (Clarke & Roche, 2009);
- explicit sharing of residual thinking which, for example, allows reasoning such as $^{7}/_{8}$ is greater than $^{4}/_{5}$ by comparing the amount by which each is less than one (Clarke & Roche, 2009); and
- increased emphasis on estimation and approximation when representing and operating with rational numbers, in order to develop number sense (Clarke & Roche, 2009).

Transforming primary mathematics teaching to meet these recommendations requires analysis of approaches to teaching that can support such instruction.

Models for teaching fraction concepts

There is a range of models commonly used to support fraction instruction, for example, sets of discrete objects, number lines, double number lines, and area models such as circles and rectangles. The selection of the most effective models for use in instruction is paramount (Cramer & Wyberg, 2009). An important feature of effective instruction is the explicit discussion of the attribute on which the model is based, such as relative length for linear models, relative area for two-dimensional models, and relative number in the set model (Steinle & Price, 2008).

Number lines are commonly used for fraction instruction. Effective use of number lines requires the learner to co-ordinate information provided pictorially by the marked line together with the numbers which give information about scale (Bright, Behr, Post, & Wachsmuth, 1988). Bright et al. (1988) suggested that using multiple number lines, partitioned in different ways but all showing the same fraction, would assist learners to construct richer understandings of number lines. In a similar way, Abels (1991) used a calibrated elastic strip as a tool for supporting the introduction to calculating percentage change. This tool is similar to the tool used in the current study (Figure 1).

Description of the elastic strips

The elastic strips used in the *teaching experiment* (Steffe, 1991) have graduated scales with equal intervals. The initial scale used in the study was about one metre in length and marked off in ten intervals (Figure 2). The elastic used to make the strips was able to be stretched to approximately double its un-stretched length. When the strips were used to find fractions of lengths, the physical restriction imposed by the limits on the elasticity necessitated the use of equivalent fractions to complete some tasks.

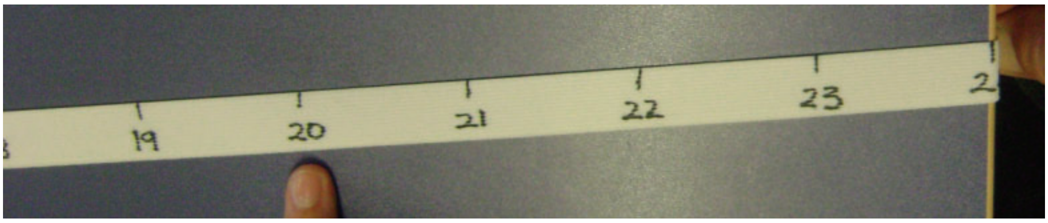

Figure 2. Elastic strip being used to find $^{5}/_{6}$ of the length of a table using the equivalent fraction $^{20}/_{24}$.

Method

The research model used was a teaching experiment in which the researcher held both participatory and data collection roles (Steffe, 1991). This paper reports on the teaching experiment conducted with one student teacher, Greg, held two months into his one year primary teacher education programme. Greg volunteered to participate as he believed his own knowledge of fractions was weak.

Greg completed a written questionnaire (Figure 3), and then participated in the teaching experiment which was informed by the results of the questionnaire. Initially Greg was questioned about his answers to the questionnaire, then shown how to use a ten-segmented elastic strip. He was then asked find points that were $^{7}/_{10}$, $^{7}/_{9}$, and $^{2}/_{3}$ of the length of the table. Intentionally, the strip was not sufficiently elastic to stretch across the table using just three segments. Other elastic strips were then introduced which had been graduated and numbered into 20 and 25 segments respectively and the tasks from the questionnaire were investigated using the strips. The physical nature of the task required Greg to give instructions to the researcher to act as partner in carrying out the tasks. When appropriate he was asked to support the instructions and actions with reasoning.

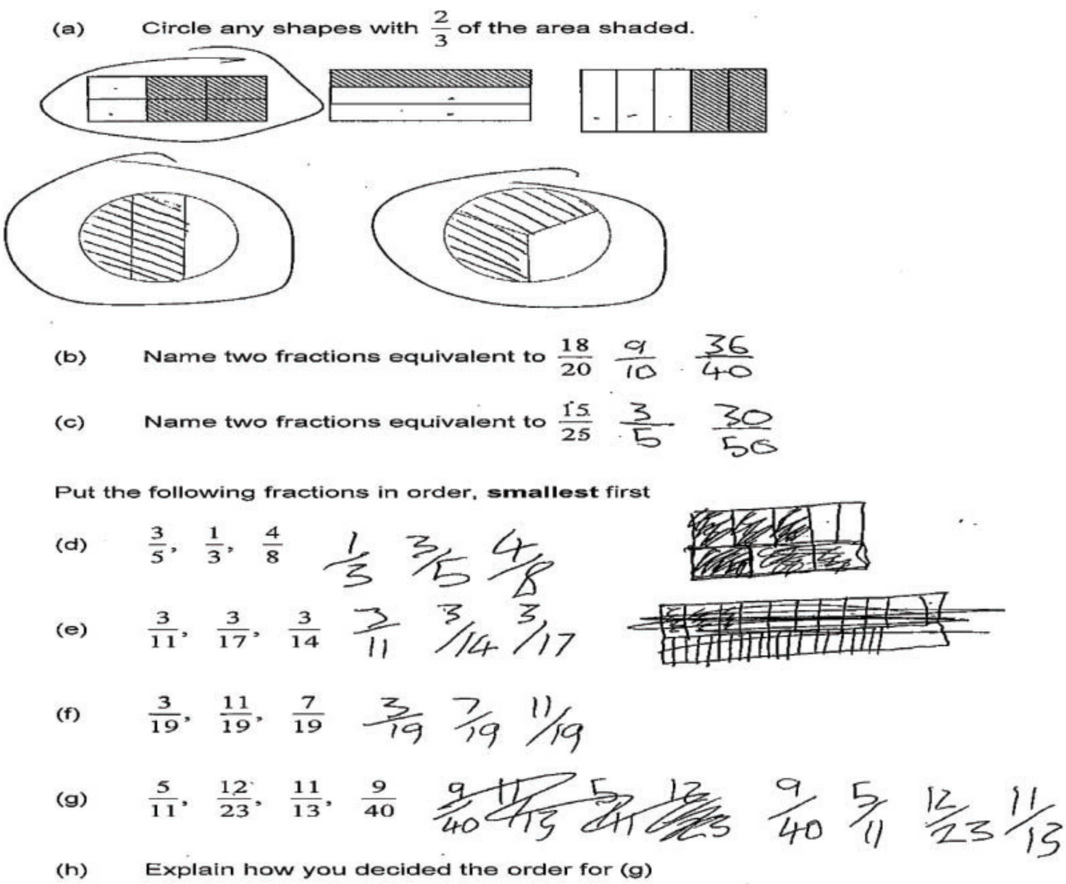

Figure 3. Greg's answers to the questionnaire.

Results

Themes emerging from the data (equivalence, ordering, and benchmarks) are presented below in order to illustrate ways in which the student teacher's knowledge was challenged and extended through the initial questionnaire and the interview.

Equivalence

Greg correctly calculated equivalent fractions in questionnaire parts (b) and (c). At the start of the teaching experiment Greg needed to find a suitable equivalent fraction for ${}^{2}/_{3}$. He quickly found ${}^{4}/_{8}$ and ${}^{8}/_{12}$, but neither of them was suitable to use with the ten-segmented strip. He had some difficulty realising that he could multiply both terms in the fraction by 3 to create the equivalent fraction ${}^{6}/_{9}$. From then on he found equivalent fractions fairly comfortably, however he was hampered at times by his lack of certainty with multiplication and division facts.

Ordering fractions with the same numerator but different denominators

Greg incorrectly ordered the fractions ${}^{3}/_{11}$, ${}^{3}/_{14}$, and ${}^{3}/_{17}$ as going from smallest to largest in question (e) (Figure 3). He explained his ordering:

Greg	I was thinking 3 pieces shaded out of 11, 3 out of 14, and 3 out of 17. But I think the order might be reversed. I really struggle with that one.

Later the question was revisited with the aid of the strip. Initially ${}^{3}/_{11}$ of the length of the table was found.

Researcher — Now we are going to find $^{3}/_{14}$. Before we do it, what is going to happen to each of the pieces?

Greg — It's going to stretch forward. So it is going to go (hand gesture indicating that the fraction was larger).

After the three fractions had been found using the strip, Greg proposed a rule:

Greg — So it is actually the other way round from my answer. I am just wondering when we have got the same number on top. Is that a general rule that you could follow, if you have the same number on top, and the denominator is bigger, the smaller the value?

After discussion about the ordering of unit fractions and then fractions with the same denominator, Greg was asked if the elastic strip had helped his thinking.

Greg — It just totally changed my way of thinking about fractions. It's a visual for me that I like to see.

Ordering fractions and benchmarking

Greg's incorrectly ordered the fractions from questionnaire part (d) (Figure 3) as $^{1}/_{3}$, $^{3}/_{5}$, $^{4}/_{8}$, and explained:

Greg — I tried to draw pictures to help me work it out. $^{1}/_{3}$ is quite easy to visualise. I just see one piece shaded out of three. The same with 3 out of 5, so I thought $^{1}/_{3}$ is smaller, $^{3}/_{5}$ is getting more pieces so if I have 3 pieces out of 5 shaded, I can see more pieces being shaded with less left over. And $^{4}/_{8}$ is $^{1}/_{2}$. So you are getting $^{1}/_{2}$ of something, and that is the biggest.

Researcher — Are you happy with that order?

Greg — I think it is wrong. Maybe I might change it to $^{1}/_{3}$, ½, $^{3}/_{5}$. I feel I should change it, I feel that $^{3}/_{5}$ is more than $^{1}/_{2}$, but I am not confident. This is where I really struggle.

After using the elastic strip to correctly order these fractions, Greg again commented that using the strip had helped his understanding.

In the questionnaire, Greg answered question (g) correctly as $^{9}/_{40}$, $^{5}/_{11}$, $^{12}/_{23}$, and $^{11}/_{13}$, but he was not confident about his answer.

Greg — $^{9}/_{40}$ is the smallest seems a ridiculous amount shaded out of 40. Littlest amount I could think of. Seems small to me. $^{5}/_{11}$, that is quite close to ½. $^{12}/_{23}$ is quite close to ½. $^{5}/_{11}$ and $^{12}/_{23}$ and seems almost the same: both close to ½. $^{11}/_{13}$ is seems quite close to $^{3}/_{4}$. There is a lot more shaded out of that proportion. If we did that stretchy thing we might actually be quite close.

Greg used the strip to locate each of the fractions in this set. He recognised that $^{9}/_{40}$ was approximately ¼, and after prompting to compare $^{9}/_{40}$ to $^{10}/_{40}$, recognised $^{9}/_{40}$ was just less than ¼. Similarly the strip was used to find $^{5}/_{11}$, and it was pointed out that 5.5 out of 11 would be ½, so $^{5}/_{11}$ is just less than $^{1}/_{2}$.

Researcher — These two ($^{5}/_{11}$ and $^{12}/_{23}$) were pretty close and you said they were both about $^{1}/_{2}$. Tell me about $^{12}/_{23}$.

Greg — $^{12}/_{24}$ would be ½. Half of 23 is 12.5, sorry 11.5. So $^{12}/_{23}$ is slightly more than ½.
So it is more than this one ($^{5}/_{11}$) because it is slightly less than ½. So if I was to work this out again knowing this now, I could do it. Half of

> 11 it is 5.5. Half of 23 it is 11.5, and so I could compare them to a half and see that one was slightly more than half and one was just less than half.

This idea of benchmarks was discussed and consolidated by considering $^{11}/_{13}$. Greg was asked to order a similar set of fractions and then asked if he had learnt anything from the session.

> G Yes I have. I have got more of an understanding of fractions and how they work. I'd love to take this strip into an exam and sit down and stretch it out. Now I have some kind of visual measurement in my brain that I can see; I can see that is close to 1, that is close to 0, that is more than a quarter, or less than quarter. Those are now my measuring blocks. 0, ¼, ½, ¾, and 1. I can work within those boundaries to work out what the answer might be. I might borrow it now and go home and practise it so I re-illustrate it in my head again.

Discussion

In the questionnaire Greg was able to use a fraction to generate equivalent fractions. His lack of certainty about the relative size of fractions casts doubt on whether this skill was securely linked to the idea of size of fractions. After the investigation of fractions with the same numerator, Greg's tentative proposal about a rule for ordering them is encouraging. Greg found the fraction ordering tasks in the questionnaire challenging, and the diagrams that he drew to support his working (Figure 3) appear unhelpful. He had recognised $^{5}/_{11}$ and $^{11}/_{23}$ as were being about ½, but was not confident in his ordering of those two numbers. The dialogue suggests that the use of the strips had assisted him to see how benchmarks could be used to find the approximate magnitude of numbers. This development was physically supported by the strips, however later in the teaching experiment Greg reported that he was developing and using mental images to assist his ordering of fractions.

Conclusions

The study showed that the fraction strip has the potential to assist learners in consolidating and reinforcing the images of the number line. The results support previous research (e.g., Leinhardt & Smith, 1985; Ward, 2010; Ward & Thomas, 2007) showing that some student teachers have significant gaps in their fractional content knowledge, casting doubt on their ability to effectively teach these concepts in primary classrooms. These concerns are illustrated in the vignettes from the interview with Greg who appears to need ongoing support to help him develop understanding of the key ideas of primary school fraction knowledge. For Greg, use of the elastic strip was an effective activity to challenge, consolidate, and extend his fraction thinking. Specifically the teaching experiment addressed fraction concepts of equivalence (Ni, 2001), representation of fractions using a linear model (Moss & Case, 1999), and benchmarking (Clarke & Roche, 2009; Moss & Case, 1999). The novel and the physical nature of the activity made recalling rote routines less likely, and communicating in order to complete the task required Greg to re-engineer his knowledge of fractions.

When using elastic strips, care is required to keep the learner's attention on the size of the unit. It is essential that the teaching does not stop with the use of the strip, but rather starts with it and moves on to developing images of the strip, and then to the key fraction ideas.

There have been calls for the teaching of primary school mathematics to have a greater focus on concepts (Charalambous & Pitta-Pantazi, 2007; Clarke & Roche, 2009; Moss & Case, 1999). A significant number of teachers have weak knowledge of fractional concepts (Ward, 2010; Ward & Thomas, 2007). Programmes that enhance the MKT of student teachers in fractional concepts need to be considered as one way to lift the teaching skill in this area. The use of the elastic strip offers the potential to challenge and extend the knowledge of fractions of student teachers.

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MĀORI MEDIUM CHILDREN'S VIEWS ABOUT LEARNING MATHEMATICS: POSSIBILITIES FOR FUTURE DIRECTIONS

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Pre-European traditional Māori education in New Zealand was integrated and holistic. With Western influence many Maori children struggled to achieve at school. Māori medium education based on retaining Māori values, language and culture therefore emerged to provide an alternative avenue for education. A key element in this initiative is to increase children's engagement with, and learning of, mathematics. Views from 61 Year 5-8 children in Māori medium contexts have been sought to provide insights about their mathematics education. This paper discusses some of these views and raises possibilities for future directions to support the momentum of this positive initiative.

Introduction

E kore au e ngaro he kākano i ruia mai i Rangiātea.
(I will never be lost, the seed which was sown from Rangiātea).

In traditional Māori society, education was oral, thematic and holistic (Barton & Fairhall, 1995; Riini & Riini, 1993). Māori children enjoyed the support of a variety of community members to fulfil their potential for learning (Hemara, 2000). A Māori world-view synthesized links between people, their activities and the environment. Shared meanings and understandings were integral to the learning process. This thinking is still prevalent in many parts of Maori society (Hemara, 2000; Pere, 1994).

Mathematics for Māori included a "very good system of numeration … doubtless quite elaborate enough for their purposes" (Best & Hongi, 2002). They also developed systems of measurement and geometrical concepts to support their needs and innovations. A strong oral tradition meant that an emphasis on the development of mental strategies as well as physical skills was expected for solving problems in a range of contexts. This often included mathematical thinking which assisted Māori in adapting to various environments by integrating mathematics ideas and tools within everyday practices such as making waka (canoes), constructing whare (houses), gardening and rongoa (medicine) (Ohia, 2002).

When Māori children began to participate in a Western form of schooling, there came a change in the ethos of education and the learning environment for them. The values promoted in that curriculum (including mathematics) supported the dominant culture and contributed to many Māori children's general underachievement in formal education settings (Bishop, 1988; Knight, 1994). D'Ambrosio (2001) argues that mathematics education has a responsibility to include ways to "reaffirm, and in some instances restore, the cultural dignity of children" (p.308). It is important for children from minority groups to appreciate that they possess a long and rich mathematical heritage and that they can be mathematically capable (Zaslavsky, 1998). Furthermore, Penetito (2010) suggests that successful education for Māori should begin with what it means to "be" Māori. Accordingly, alternative spaces such as Kura Kaupapa Māori (KKM) have been established as an educational option to educate Māori children through the medium of their own language and culture (Smith, 1991). This includes their mathematics education.

The official philosophical basis for guiding learning and teaching in KKM is Te Aho Matua o ngā Kura Kaupapa Māori (Ministry of Education, 2008a). A key element within this document is the notion of actively promoting a close relationship between kura (school) and the community which is considered important for supporting children's learning in mathematics (Anthony & Walshaw, 2007). While there are special teachers employed at the school, education for children enrolled at kura is a communal responsibility (Ministry of Education, 2008c). Participation of whānau (family) is expected and valued. However, involving Māori whanau in their children's formal education requires much consideration. Genuine engagement and discussion with them about children learning mathematics is not easy or simple as many have a background of unsuccessful achievement in mathematics education themselves (Meaney & Fairhall, 2003).

Current education literature portrays mathematics as a dynamic entity that is constructed by learners themselves (Dossey, 1992; Mason, 2008). Cotton (2004) states for example that mathematics is about supporting people to develop a range of mathematics ideas in order to make sense of their world and thereby control the complexities within it. Article 29 in the *UN Convention on the Rights of the Child* espouses that children have a right for education to be directed to the development of mental abilities to their fullest potential (Munn, 2005). To this end children should be encouraged to develop a variety of mental strategies and choose the most appropriate for the situation or problem they are engaged in (Suggate, Davis & Goulding, 2006).

National assessments have indicated progress by Māori children in recent years regarding mental computation. This research notes however, that children in these contexts need to develop greater problem solving strategies (Crooks & Flockton, 2006). To advance mathematical thinking the development of efficient multiplicative strategies is necessary (Young-Loveridge, 2008). Higgins (2005) suggests that the use of concrete materials can help promote such mathematical thinking and discussion. Using equipment (accompanied by appropriate discourse) can have a positive effect on Māori children's experiences in mathematics learning situations (Holt, 2001). Te Poutama Tau (Numeracy Development Projects) that has been implemented in some KKM, endorses this stance (Higgins, 2005).

Twenty-first century mathematics education promotes the use of tools such as digital technologies. Neal, Barr, Barrett and Irwin (2007) suggest that e-learning (learning supported by or facilitated by ICT) can provide a vehicle to achieve Māori aspirations in education. Use of appropriate technology can influence students' engagement with mathematical tasks and help them understand mathematics in alternative ways (Calder, 2009). The curriculum document for KKM, Te Marautanga o Aotearoa (Ministry of Education, 2008b) encourages the use of ICT to support children's learning. Recent initiatives by the Ministry of Education have endorsed this policy by providing for example, digital learning objects to support mathematics learning through te reo Māori (the Māori language).

Children are major stakeholders in their education. Their voices can alert educators to unique perspectives on mathematics learning (Averill & Clarke, 2006; Hāwera & Taylor, 2010; O'Shea, 2009). This paper seeks to acknowledge the importance of children's perspectives regarding their mathematics education. Analysis of these perspectives can only contribute to understanding their mathematics experiences and constructing possibilities for future directions.

Methodology

This paper focuses on the responses of 61 year 5-8 Māori children in four KKM. Data were gathered as part of a larger study. Most children had participated in Te Poutama Tau, the Māori medium equivalent of the Numeracy Development Projects for several years prior to the study.

Schools were asked to nominate year 5–8 children from across a range of mathematics levels. Children were interviewed individually for about 30 minutes in te reo Māori or English (their choice) in a quiet place away from the classroom. They were told that the interviewer was interested in finding out about their thoughts regarding the nature of mathematics and their learning of pāngarau/mathematics. Data regarding five questions are discussed in this paper to illuminate the thinking about a range of ideas that these children held about their mathematics learning experiences. These are:

- Ki ōu whakaaro, he aha tēnei mea te pāngarau?
  (What do you think mathematics is about?)
- Kei te kainga ētehi tāngata hei āwhina i a koe ki te ako pāngarau? Pēhea to rātou āwhina?
  (Are there people at home who help you to learn mathematics? How do they help?)
- I ā koe e mahi ana āu mahi pāngarau, ka whakamahia e koe ētehi taputapu pērā i te pirepire, te porotiti rānei? Ka whakamahia mo te aha?
  (When you do mathematics, do you use equipment like beads and counters? What are they used for?)
- Pēhea nga rorohiko? Ka whakamahia ēnei mo te ako pāngarau?
  (What about computers? Do you use these when learning mathematics?)
  Of the 61 children interviewed, 44 were Year 7 and 8 (12-13year olds) who were also asked to solve a problem involving multiplication:
- E hanga motokā ana te kamupene o Hera. E 4 ngā wīra mo ia motokā. E hia katoa ngā wīra mo te 17 motokā?
  (Hera has a car manufacturing company. She needs 4 wheels for each car. How many wheels does she need for 17 cars?)

Further analysis and discussion of this data and other questions children were asked can be found in Hāwera, Taylor, Young-Loveridge & Sharma (2007) and Hāwera & Taylor (2008, 2009, 2010).

Results and Discussion

Children's thoughts about the nature of mathematics

Table 1. Children think mathematics is about.

Number and operations	Don't know No reply	Learning and stimulating the mind	Links to a context	Everything	School work	Problem solving
18	17	12	8	3	2	1

When asked what they thought mathematics was about the children's responses varied with two categories dominating. Eighteen children indicated that mathematics was about number and operations while 17 did not offer a view about the nature of mathematics at all. This is of interest given that the pre-2008 national curriculum documents for English and Māori medium promoted the notion that children should learn a spectrum of mathematics ideas within a range of meaningful contexts, yet this intent has not been reflected in their responses. Only eleven of the 61 children suggested that mathematics might be connected to a situation outside the school arena.

For those children who do not have a view of the nature of mathematics, Presmeg (2002) suggests that they may find it difficult to recognize and appreciate the links between the mathematics ideas learned at school and those embedded in everyday practices in their communities. This absence of connection can compromise their ability to capitalise on the potential of the mathematics learning experiences available to them. Mathematics educators need to consider how learning for children in kura can be more closely integrated with those issues, activities, values and principles espoused and promoted within their communities. This may help these children to develop and articulate a view of mathematics and to make further connections with the mathematical ideas embedded in their lives and culture as Māori. There is an onus on education in New Zealand to support Māori children to achieve well mathematically while reaffirming the value of mathematics in their cultural milieu.

People at home whom children think help them learn mathematics

When children were asked about support at home for learning mathematics, 58 out of the 61 children responded in the affirmative. This support included mothers, fathers, grandparents, siblings, as well as uncles and aunties.

Table 2. Children's views about how people at home help them to learn mathematics.

Offer strategies	Not sure how	Ask questions	In various ways	Teach me mathematics ideas	No help
18	11	10	10	9	3

It is pleasing to note that 80% of the children reported that support for learning mathematics was available to them at home. This involved receiving advice about strategies to use, being asked questions, clarification of particular mathematics ideas and practise with number operations. However, 11 out of 58 children (almost 20%), were not sure how their families at home supported them with their mathematics learning. The involvement of whānau, hapū and iwi in Māori education is crucial for learners' success (Penetito, 2010). If children do not have a clear view of the role their families can play in helping them to learn mathematics, their expectations of any support from family members may be overlooked. Clarifying roles and expectations between members of families as well as between kura and families could assist children to learn mathematics. Families clearly have a powerful influence on children's learning (Anthony & Walshaw, 2007; Ministry of Education, 2008c). Graham (2003) suggests that kanohi ki te kanohi (face to face) interactions are critical for establishing reciprocal relationships between Māori and kura to engender change to current practices. It is incumbent on mathematics educators to be more creative in facilitating situations where this can occur.

Using equipment for learning mathematics

The children's responses to use of equipment are shown in Table 3. It is reassuring to note that 60 out of 61 children thought that using equipment could help people to learn mathematics. However, more than half said that they did not use it themselves (see Table 4). The major reason that children gave for not using equipment for mathematics learning was a perception that they did not need or want it. Some thought that the use of equipment would encourage an unnecessary reliance on this practice when older, while others considered it a necessity to visualise or become an abstract thinker as soon as possible in order to advance their mathematical thinking. Other children viewed the use of equipment as time-consuming and therefore an unproductive part of their mathematical learning sessions.

Table 3. Children's responses to using equipment.

Use equipment	Use equipment sometimes	Don't know	Don't use equipment
16	9	1	35

Table 4. Reasons for not using equipment.

Didn't need any	Didn't want any	Were not offered any
19	12	4

Mathematics education in New Zealand has for some time strongly encouraged the practice of using equipment in learning sessions particularly when new ideas are being introduced (Higgins, 2005; Ministry of Education, 2008c). It seems that the purpose and potential of using equipment could be made more explicit to some children so that they can avail themselves of opportunities for scaffolding their learning. Limited use of equipment can have consequences for learners and impact on their mathematical reasoning and subsequent understanding (Anthony & Walshaw, 2007; Higgins, 2005; Young-Loveridge, 2008). Future directions could include using equipment to focus on the exploration of ideas so that children recognise its value as media for understanding and learning mathematics.

Table 5. Use of computers for mathematics learning.

Yes	No
13	48

The adoption of innovative equipment for survival is not a new idea for Māori (Ohia, 2002). However, children's responses regarding equipment indicated that there was very little use of ICT (including calculators) to support their learning in mathematics. There did not appear to be any planned, systematic use of ICT in classroom sessions. The digital tools that Māori (like others) are able to access outside the school environment seem to be absent from their cache of mathematics learning tools at school. Recent government initiatives (which include digital learning objects in te reo Māori) potentially offer Māori medium children a further avenue for exploring mathematics. Possibilities for future learning need to emphasise a greater use of ICT to facilitate the learning of mathematics ideas (Calder, 2009; Ministry of Education, 2008b). KKM may need further support to develop ways to embrace digital technologies for enhancing mathematics programmes.

Children's strategies for solving a multiplication question

Data shows that 29 out of the 44 Year 7-8 children were able to solve the multiplication question correctly using a range of strategies. These strategies included:

- (SPVP) is the Standard place value partitioning strategy e.g., 4x17= (4x10)+(4x7)=40+28 = 68
- (DF) is the Derived fact strategy e.g., 4x17=(4x10)+(4x5)+(4x2)= 40+20+8=68.
- (TD) is the Times doubling strategy e.g., 4x17=(2x17)+(2x17)=34+34=68
- (TT) is the Times twice strategy e.g., 4x17=(2x17)x2=34x2=68
- (DD) is the Double double strategy e.g., 4x17=(17+17)+(17+17)=34+34=68
- (C4) is the Counting up in fours strategy e.g., (4,8,12,16, …, 68)
- (ALG) is a traditionally-taught written procedure.

Table 6. Strategies used for the multiplication task.

Kura	Number of year 7-8 children	SPVP	DF	TT	DD	TD	C4	ALG	No attempt made or strategy offered
1	6	0	0	0	0	1	0	1 (1W)	4
2	5	1	1	1	1	1	0	0	0
3	12	6	0	0	0	1	0	2	3
4	21	4 (1W)	4 (1W)	0	0	1	5 (1W)	6 (3W)	1
Total	44	11	5	1	1	4	5	9	8

(nW) indicates the number of incorrect solutions

Developing efficient mental strategies for solving problems has become a focus in mathematics education in New Zealand and other parts of the world in recent years. It is pleasing that some of these children demonstrated a range of strategies for solving a multiplication question. It remains a concern however that almost half were not able to begin the problem or they used what might be considered an inefficient mental strategy (counting on in 4's or an algorithm). Multiplicative thinking is crucial for proficiency in problem solving (Young-Loveridge, 2008). Avenues must be carefully considered to ensure Māori children develop the strategies needed to support appropriate mathematical thinking. Māori have a right to be in a position of solving problems efficiently so they can fully participate in the challenges that arise in today's world.

Conclusion

Research with children educated in Māori medium settings is limited. While this study involves only 61 children, they are from four different kura. If we are serious about maximizing opportunities for children in Māori medium to learn mathematics, their views must be considered. This data indicates that many children have well established views about the nature of mathematics, the support they have at home, the value of equipment and ICT and the strategies they might use for solving multiplication problems. The insights gained from considering these children's responses suggest to us that a more focused strategy is required in Māori medium education to seek out multiple ways of:

- helping children develop a broader view about the nature of mathematics;
- ensuring that utilizing equipment including ICT, is an integral part of mathematics programmes;
- exploring and maximizing whānau involvement in children's mathematics learning;
- supporting more children to become efficient multiplicative thinkers;
- connecting children's learning experiences with the mathematics in their community; and
- enhancing education so that children in Māori medium can succeed mathematically as Māori.

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HOW INCLUSIVE IS YEAR 12 MATHEMATICS?

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This paper draws from a longitudinal study of student achievement in Melbourne's northern suburbs. It examines Year 12 students' attitudes to mathematics, their experience of the mathematics classroom, their views of teachers and their expectations of success. Despite a differentiated Year 12 mathematics curriculum, there is evidence of inequity in students' experience of mathematics. Perceptions of mathematics classrooms and mathematics teachers, and expectations of success, vary according to subject, gender and social background. Implications for pedagogical and curriculum reform are discussed.

Introduction

Mathematics is not a level playing field. Achievement gaps based on socioeconomic status (SES) are evident in early primary school and increase in magnitude throughout the school journey. Recent achievement data show this to be the case nationally (ACARA, 2009) and within the Northern Metropolitan Region (NMR), which is one of the poorest regions in Victoria. In 2007 the NMR's literacy and numeracy achievement levels were among the lowest in the state (Department of Education and Early Childhood Development (DEECD), 2009). The region also had the lowest VCE study scores in Victoria and the lowest rate of transfer to university (Helme, Teese & Lamb, 2009).

Helme, Teese & Lamb (2009) reported that, within the region, there were marked social gaps in Year 7 achievement and subsequent achievement in Year 9. Most low achievers from poorer backgrounds did not improve their position relative to the average student during lower secondary education. For example, in mathematics 80 % of low achievers from the lowest SES band remained low achievers. The higher the social level of students, the greater the chance of the weakest learners improving—in addition to the lesser likelihood of low achievement in the first place.

In recognition of low student achievement levels, the NMR embarked on a major school improvement campaign in 2008 whose key initiative is the *Achievement Improvement Zones* (AiZ) project. The AiZ aims to improve literacy and numeracy levels across all schools in the region, focusing on leadership development, professional learning in numeracy and literacy, ongoing coaching and training and support from the region, department and education experts (DEECD, 2009).

This paper draws from a broader longitudinal study funded by DEECD that is evaluating the impact of strategies to lift achievement within the NMR, the Raising Achievement in Public Schools (RAPS) project. The project is synthesising data from a range of sources: NAPLAN data, student and teacher surveys and teacher interviews. Years 3, 5, 7 and 9 NAPLAN data will be monitored for a period of five years, with a focus on SES differences in achievement as students progress from one stage to the next.

Teacher surveys have been conducted in a sample of 10 primary and 10 secondary schools in the region to obtain a teacher perspective on the challenges of lifting student achievement and the strategies needed to achieve progress. Focus groups with teachers and leadership teams were undertaken in 2011 and a second wave of interviews is planned for 2013 to allow teachers to reflect on progress. Surveys have been completed with students in Years 3, 5, 7, 9, and 12 in the 20 sample schools to obtain a student perspective on learning and achievement. The focus of this paper is on students' experiences of mathematics in Year 12 (final year) and asks how inclusive it is.

Background and research questions

In Victoria, almost all young people complete school within the framework of the Victorian Certificate of Education (VCE). Mathematics in the VCE is optional, and is designed to accommodate a broad range of student skills, interests and abilities through the provision of a set of subjects that forms a hierarchy of difficulty. Further Mathematics is the least difficult and Specialist Mathematics is the most advanced, with Mathematical Methods occupying an intermediate position. Students enrolled in Specialist Mathematics must also be concurrently enrolled in Mathematical Methods. Students enrolled in the Victorian Certificate of Applied Learning (VCAL), a non-academic alternative to VCE, can undertake a VCE mathematics study or a VCAL numeracy skills unit, which is a more practical subject with a vocational emphasis.

Gender differences are characteristic of enrolment patterns in Year 12 mathematics. Enrolment rates derived from VCAA (2010) figures indicate that participation in Further Mathematics is much the same for males and females, but that girls are less likely than boys to select Mathematical Methods (26.0% compared with 37.4%) and even less likely to enrol in Specialist Mathematics, where the enrolment rate of boys is more than double that of girls (13% compared with 6.2%).

SES differences in VCE mathematics participation are also evident. Helme and Lamb (2007) examined 2005 enrolment data and found that enrolment levels in Further Mathematics were relatively similar in all SES quintiles, but in Mathematical Methods and Specialist Mathematics enrolment levels were substantially higher in higher SES quintiles. Yeoh and Lancaster (2010) also found a social gradient in the distribution of enrolments in Mathematical Methods and Specialist Mathematics, but not in Further Mathematics. SES differences also characterise mathematics enrolment in the NMR. Enrolment rates in Mathematical Methods are almost twice as high in the highest SES quintile as in the lowest, and for Specialist Mathematics this ratio exceeds 2:1 (Helme, Teese & Lamb, 2009).

Outcomes in Year 12 mathematics also demonstrate a SES pattern. Yeoh and Lancaster (2010) found a social gradient in study scores in all three mathematics studies. Helme, Teese and Lamb (2009) found that Further Mathematics students in the highest SES quintile in NMR were more than twice as likely than their counterparts in the lowest SES quintile to achieve a study score above the defined average of 30. Similar patterns were evident in Mathematical Methods. For example, more than half the Mathematical Methods students in the lowest quintile of SES achieved scores below 24 (53 per cent), compared with 18 per cent of those in the highest SES quintile.

There is also evidence of inequality in students' experience of Year 12 mathematics (Teese, 2000). Teese, Lamb, Helme & Houghton (2006) found substantial differences in students' views of mathematics between different schools and between different mathematics subjects. Mathematical Methods students were more satisfied with teachers, the classroom environment and the subject itself than Further Mathematics students. Furthermore, girls perceived mathematics as less interesting, more difficult and less relevant to the real world than boys did.

These findings warrant further investigation. If mathematics is inclusive, as it claims to be, there should be broadly similar levels of satisfaction across the different study options. One would also hope that quality of classroom experience would be much the same, regardless of individual characteristics such as gender and socioeconomic status. This study set out to explore whether or not this is the case.

Methodology

The data for this paper were derived from a survey of Year 12 students in ten secondary schools in the NMR. The ten schools were selected on the basis of their intake-adjusted performance on a range of indicators such as attendance, academic achievement and post-school transitions. The sample comprised five schools with above expected performance, one school with expected performance and four schools with below expected performance. The schools were diverse in terms of size and the socioeconomic background of students.

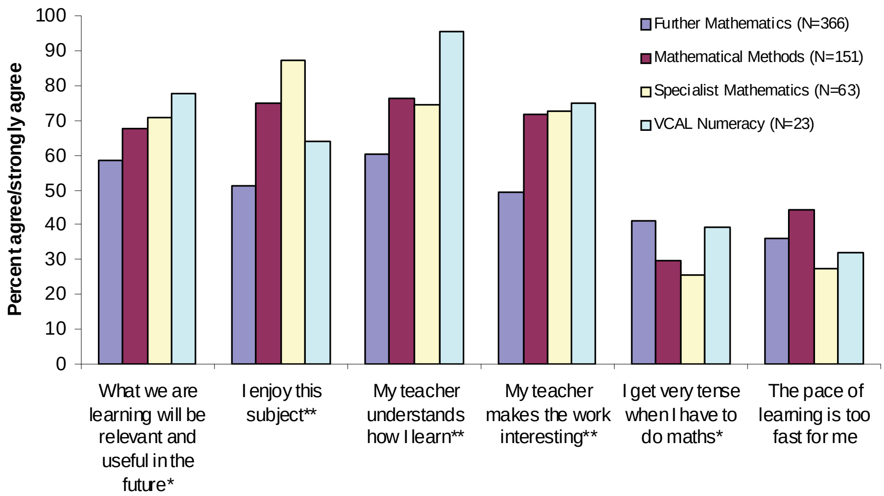
The Year 12 surveys were completed by consenting students during the last two weeks of Term three, 2010. Of a total enrolment of 1107 Year 12 students in the ten schools, 841 completed the survey, giving an overall completion rate of 76 per cent. Six-hundred and three students nominated a mathematics subject that they were currently enrolled in and responded to a set of questions about that subject. These questions canvassed their views of subject content and difficulty, quality of teaching, quality of classroom experience and their expectations of success. Students were also invited to contribute their ideas on how to improve mathematics and the way it is taught in their school.

Students were also asked to nominate, for each parent separately, their highest level of school completed and their highest post-school qualification, in order to establish a measure of SES. These data were combined into a single scale showing the qualification level of the parent with the highest qualification level. Students were then grouped into five categories according to parental qualifications. In the discussion that follows two contrasting socioeconomic groups were used: students with a parent who had obtained university qualifications and students for whom neither parent went beyond Year 11.

Results and Discussion

Is quality of classroom experience independent of the type of maths studied?

Figure 1 shows students' views of their current mathematics study, and indicates significant differences between subjects in students' perceptions of their classroom experience. VCAL numeracy students are more likely than other students to agree that their teacher understands how they learn and to view the subject as relevant to their future plans, suggesting a good match between subject content and pedagogy and the learning needs of students. Conversely, Further Mathematics students are the least likely of all students to agree that they enjoy the subject, that their teacher makes the work interesting or that their teacher understands how they learn.



Note: Levels of significance based on Chi-square tests: *p<0.5, **p<0.01

Figure 1. Students' views of current mathematics study, by subject.

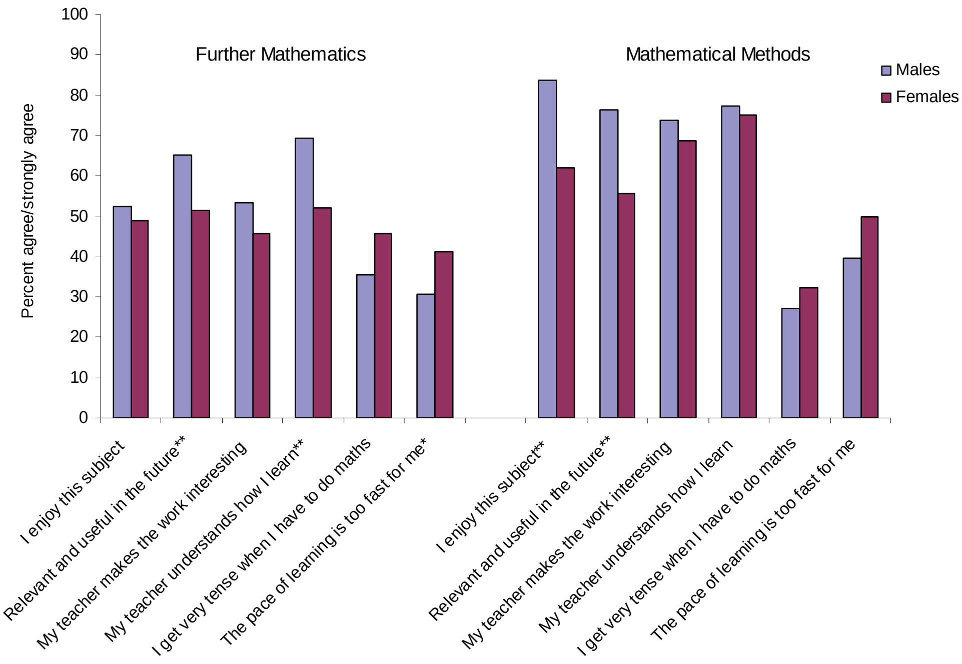
Can we conclude that there is a mismatch in Further Mathematics between curriculum content and teaching approaches, on the one hand, and the needs of learners, on the other? Is the relatively low level of student satisfaction in this study a reflection that Further Mathematics enrols many academically-weaker students who struggle to engage with mathematics and may have been pressured into taking "at least some maths"? They are certainly the least likely to endorse its connection with future life and career. If Further Maths is the path of least resistance for students who dislike mathematics, this places particular pressure both on the design of the study and on the nature of the teaching approaches that are required to engage students in successful learning.

The more academic options of Specialist Mathematics and Mathematical Methods are relatively strongly endorsed in terms of enjoyment and interest, suggesting a better match between the content and pedagogy of these subjects and the needs of their clientele.

Is quality of classroom experience independent of gender?

Figure 2 shows gender differences in students' mathematics classroom experience, for Further Mathematics and Mathematical Methods (numbers in Specialist Mathematics and VCAL Numeracy were too small for further analysis). Gender differences in favour of boys were evident for all items, but the magnitude of the differences varied.

Girls in both subjects were significantly less likely than boys to perceive mathematics as relevant and useful for the future. Female Further Mathematics students were significantly less likely than their male counterparts to agree that their teacher understands how they learn, and significantly more likely to report that the pace of learning is too fast. Female Mathematical Methods students were significantly less likely than their male counterparts to agree that they enjoy the subject. While gender differences in mathematics anxiety were not significant, the trend suggested here is consistent with the findings of previous studies (e.g., OECD, 2004).



Note: Levels of significance based on Chi-square tests: *p<0.5, **p<0.01

Figure 2. Students' views of current mathematics study, by gender.

Is quality of classroom experience independent of SES background?

Figure 3 displays quality of classroom experience in Further Mathematics and Mathematical Methods (Specialist Mathematics and VCAL Numeracy were excluded from the analysis due to small numbers).

While none of the differences was statistically significant, the data suggest some notable disparities between children of contrasting social backgrounds. In both subjects, the pace of learning appeared to be a greater problem for students from a less educated background, and in Mathematical Methods there were consistent differences in favour of children of university educated parents. These findings are consistent with Teese's (2000) view of the influence of family educational, economic and cultural capital on access to Mathematical Methods: the further up the hierarchy of cognitive demands, the greater the call made by the curriculum on cultural resources and the narrower the SES base from which these are available.

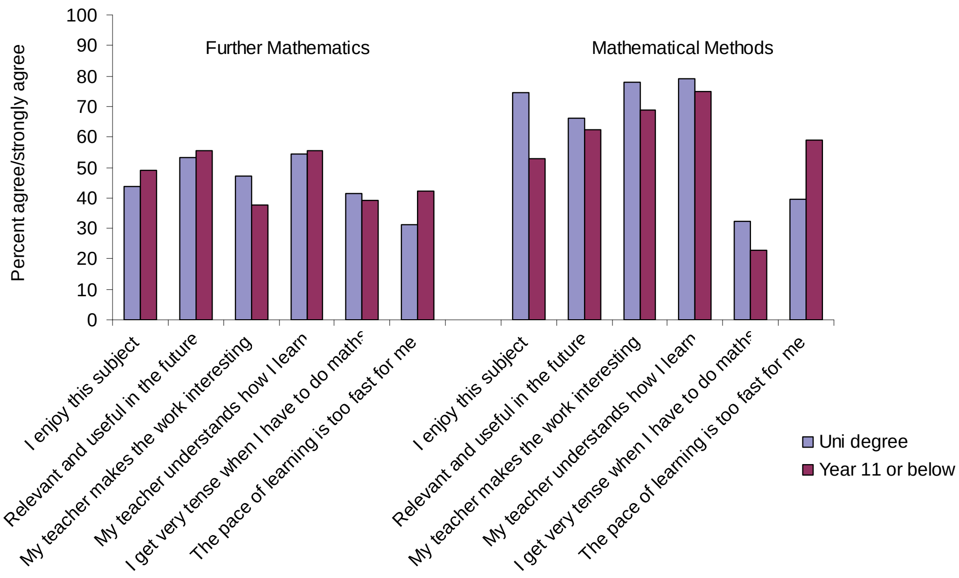


Figure 3. Students' views of current mathematics study, by SES background.

Are expectations of success equitably distributed?

As Figure 4 indicates, expectations of success varied among subjects. (VCAL was excluded from the analysis due to small numbers.) Further Mathematics students were the least confident of success, perhaps not surprising, given that many students taking this option have a history of low achievement, while Specialist Mathematics students were the most confident, with Mathematical Methods students occupying an intermediate position. In Further Mathematics there was no notable gender difference in students' expectations of success, but this was not the case for Mathematical Methods and Specialist Mathematics. Male Mathematical Methods students were more than three times more likely than their female counterparts to expect to do very well (29% compared with 9%). The corresponding percentages for Specialist Mathematics were 62% and 40%.

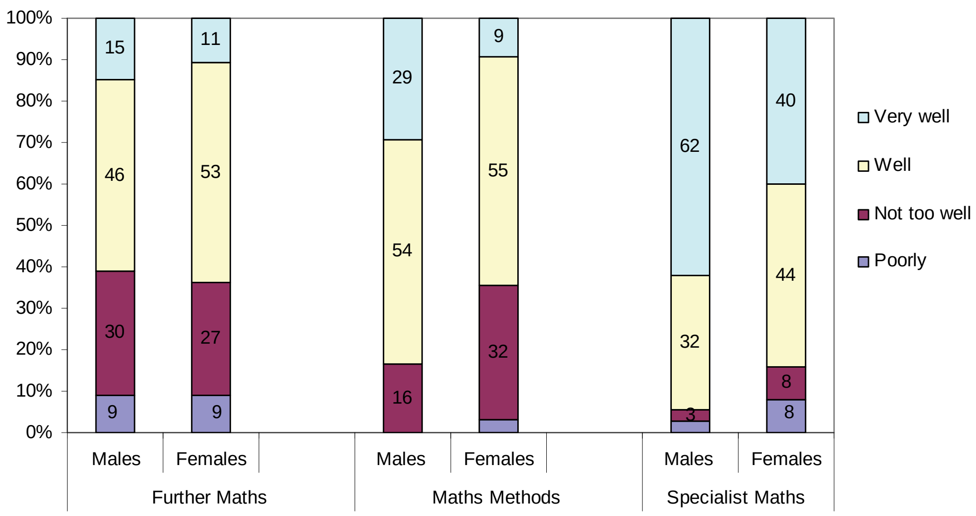


Figure 4. Students' expectations of success, by subject and gender

Given that girls take Methods less frequently than boys, one might expect that those who enrol in the subject have at least as much confidence as their male peers. But they do not, which suggests that there is still some way to go to improving gender equity, quite apart from the persistent gap in enrolment rates.

When data for all mathematics subjects were combined, students from more highly educated homes were more likely than students from less well educated homes to expect to do very well (25% compared with 14%). The compounding effect of being a boy from a well educated home background pushed this expectation to 33%, while that for a girl from a less well educated home reduced this expectation to only 10%.

These findings are consistent with the view that children of more highly educated parents have stronger social and cultural ties with the values and practices of senior secondary schooling, and therefore feel more confident of success. Conversely, children from families where there is no experience of completion of secondary education or of further education are more likely to struggle to adapt to the demands of the VCE (Teese, 2000; Teese, Lamb, & Helme, 2009).

Improving mathematics and the way it is taught

Students were asked to complete an open-ended item asking them to make suggestions for improving mathematics and the way it is taught. The findings of this aspect of the analysis will be reported separately as space does not permit a detailed discussion here. Suffice to say that the most frequently mentioned suggestion for improvement concerned quality of teaching. Students asked for teachers who could speak and explain more clearly, and adapt their explanations to individual needs. They also called for teaching methods that were less dependent on textbooks and were more interactive.

Other feedback included better teaching at earlier year-levels, more thorough preparation for Year 12, and a slower pace of teaching in Year 12. Students also asked for more enjoyable coursework and course content that has stronger links to real life situations. Students in all subjects expressed a desire for more “hands on” mathematics.

Student feedback suggests that some schools need to review curriculum, pedagogy and teacher allocation practices in earlier years. It also highlights the difficulties schools face in finding mathematics teachers who have the appropriate expertise. This is a particular issue in low SES schools in the region, which are more vulnerable to shortages of qualified mathematics teachers and high teacher turnover (Helme, Teese & Lamb, 2009).

These findings reveal a mismatch between what students say they need and what they get. While students appear to have a good understanding of their needs as learners, these often clash with externally imposed demands and constraints. Teachers must complete the coursework in the time available and prepare their students for final examinations. They cannot slow the pace, diverge from the set curriculum or amend the content to suit students’ needs and interests. This issue has been previously identified in low SES schools, where the need to accommodate student diversity is the greatest (Helme, Lamb, & Teese, 2009; Teese, Lamb, & Helme, 2009).

Conclusion

Despite a differentiated Year 12 curriculum, students undertaking different Year 12 Mathematics subjects in the NMR do not enjoy the same quality of experience. Perceptions of mathematics classrooms, mathematics teachers and expectation of success vary according to the mathematics subject students are enrolled in.

This study also shows that children of tertiary educated parents are more connected to mathematics and more confident of success, and therefore better placed to achieve good results and enter tertiary education. These students are also more likely to attend larger schools that do not experience the serious staffing issues confronted by smaller, poorer schools in the NMR. It is noteworthy that four of the ten schools in this study are no longer able to offer Specialist Mathematics classes.

Despite decades of research in gender differences and strategies for making mathematics content and pedagogy more responsive to the needs of girls, this study reveals there is still more to be done.

Student feedback obtained in this study provides some signposts for action, which include more engaging pedagogy, better ways of preparing students for VCE and a stronger emphasis on addressing individual learning needs.

Inequities in the experiences and outcomes of Year 12 mathematics students cannot, however, be overcome simply by teachers in low SES schools continuing to struggle to adapt to the demands of the VCE. School systems must also heed student feedback, reinvigorate the curriculum and provide enough trained teachers to ensure that all students have the opportunity to engage deeply with mathematics. What better time than now to review curriculum and pedagogy, in the current context of the development of the Australian mathematics curriculum? Otherwise, too many students will remain marginalised from mathematics and the rewards conferred by curriculum and pedagogy will continue to be inequitably distributed.

Acknowledgements

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CHALLENGING TRADITIONAL SEQUENCE OF TEACHING INTRODUCTORY CALCULUS

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Despite considerable research with students of calculus, rate and hence derivative, remain troublesome concepts to teach and learn. The demonstrated lack of conceptual understanding of introductory calculus limits its usefulness in related areas. Since rate is such a troublesome concept this study trialled reversing the usual presentation of introductory calculus to begin with area and integration, rather than rate and derivative. Two groups of first year tertiary students taking introductory calculus were selected to trial the effect of changing the sequence; a control group and a group which followed the reversed sequence. Two-sample t-tests undertaken in Minitab on the examination results indicate there is no significant difference between the examination results of the two groups. These results indicate that changing the sequence of delivery was not detrimental to the development of conceptual understanding of introductory calculus.

Background

Rate is an important mathematical concept that is often poorly understood by many people. It is a complicated concept comprising many interwoven ideas (see Figure 1) such as: change in a variable resulting from a change in a different variable; the ratio of two numeric, measurable quantities; constant and variable rate; and average and instantaneous rate. It expresses the change in the dependent variable resulting from a unit change in the independent variable, and involves the ideas of change in a quantity; co-ordination of two quantities; and the simultaneous covariation of the quantities (Thompson, 1994).

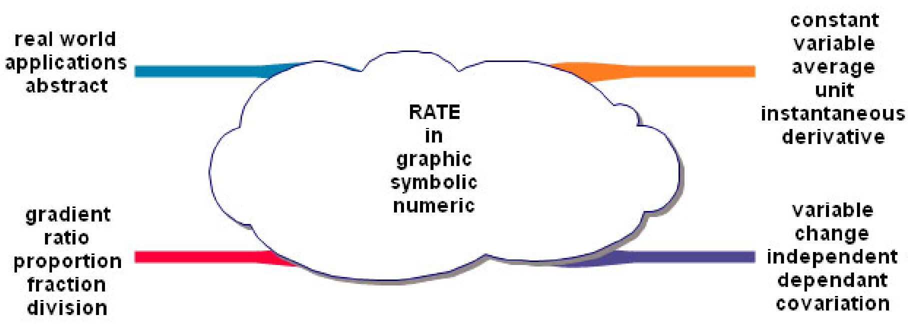


Figure 1: Complexity of the mathematical concept of rate.

Rate is strongly connected to other mathematical concepts, such as ratio; proportion; fraction; division; gradient; and derivative. In addition, rate may be seen as a purely abstract mathematical notion or embedded in the understanding of real-world applications. Rate is considered constant if the way in which the quantities change in relation to each other remains the same and variable if it differs. For example, speed may express a constant relationship between the distance and time, whilst this relationship varies when speed is changing.

Despite considerable research with students of calculus, rate remains a troublesome concept to teach and learn (see for example Orton, 1983; Ubuz, 2007). Calculus students‘ difficulties with rate manifest in many forms. One of the most significant of these is the confusion between the rate and the extensive quantities that constitute it (Rowland & Jovanoski, 2004; Thompson, 1994), for example understanding of speed as a distance. Thompson, reporting on the understanding of rate of his study of nineteen advanced mathematics tertiary students, describes students‘ confusion between the notions of change and rate of change and also confusion between amount and change in amount, for example when discussing the manner in which the volume of an inverted cone changed with height. Rowland and Jovanoski‘s study of the understanding of rate of fifty-nine first year science students with previous experience with calculus, found that many students confused amount and rate, for example they report one student‘s response that the constant term in a differential equation ―represents the amount of drug going into the patient‘s body" or the constant term in a differential equation ―is the initial amount of drug in the body" (p. 511). They suggest this confusion often resulted from a reliance on constant rate ideas not valid in the differential equations presented. Their findings indicate that the confusion between amount and rate, noted by Thompson (1994) still persists.

Other difficulties with understanding of rate include: confusion relating to symbols and their use as variables (White & Mitchelmore, 1996); lacking awareness of the relationship between slope, rate and the first derivative (Porzio, 1997); misunderstandings related to average and instantaneous rate (Hassan & Mitchelmore, 2006); related-rates problems in speed (Billings & Kladerman, 2000); and geometric contexts (Martin, 2000). White and Mitchelmore (1996) warn that symbolic manipulation may limit students‘ understanding to algebraic symbols and routine procedures. Hassan and Mitchelmore (2006) report on their study of fourteen Australian senior secondary students‘ understanding of average and instantaneous rate, suggesting that the students‘ previous introduction to calculus had not influenced the students‘ understanding of average rate or instantaneous rate. They emphasise the importance of a sound understanding of average and instantaneous rate before teaching more advanced concepts. Bezuidenhout‘s (1998) study involved five hundred and twenty-three South African first year calculus students and investigated these students‘ understanding of rate. He claims that the main confusion about rate involves the ―relations between the concepts ‗average rate of change‘, ‗average value of a continuous function‘ and ‗arithmetic mean‘"(p. 397). Similarly, Oliveros (1999) states that rate was often seen as a numerical operation, similar to the treatment of rate in early secondary years, rather than a relationship between quantities.

Since the 1980s concern has been expressed regarding the difficulties some calculus students have with the concept of rate. The older research is cited to emphasise the persistence of these difficulties. Orton (1983) reports on the understanding of derivative held by one hundred and ten undergraduates in an introductory calculus course and states that these students showed some fundamental misconceptions such as confusion between: rate of a straight line versus rate of a curve; rate at a point versus rate over an interval; and the derivative at a point versus the point‗s y-coordinate. More recently, Rasmussen and King (2000) observe that the confusion between rate of a straight line versus rate of a curve, noted by Orton many years earlier, was still evident in the initial understandings of rate brought by students to their project. Similarly, Hassan, Mohamed, and Mitchelmore (2000) report that the students in their study had difficulty linking tangents to rate and derivatives. The twenty-seven tertiary Maldivian students in Hassan et al.‗s study relied on formulae and rules. These students found it difficult to visualise the changing tangent as a point moves along a curve and were confused about the difference between average and instantaneous rate.

The demonstrated lack of conceptual understanding of introductory calculus limits its usefulness in related science applications. Lopez-Gay, Martinez-Torregrosa, Gras-Marti and Torregrosa (2002), in their study of 103 high school physics teachers and analysis of 38 Physics text books, stress the importance of students‗ understanding of differential calculus in understanding physics. They claim that physics students do not understand the use of calculus in simple real-world problems and have difficulty in applying it with autonomy. This suggests that the value of calculus to other fields of study is undermined by students‗ lack of conceptual understanding of it.

Despite the implementation of many innovations, such as the use of technology, designed to improve the outcomes of calculus courses, calculus students‗ lack of understanding of the fundamental ideas of change and rate persist (Hassan et al. 2000; Rasmussen & King, 2000; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Rowland & Jovanoski, 2004; Coe, 2007; Ubuz, 2007). Since these fundamental ideas of change and rate provide an important underpinning of derivative, researchers continue to persevere to find effective innovations. One such innovation is reported in this paper.

―[T]oday the concept of the derivative is usually presented first in calculus courses, with the notion of the integral coming later" (Boyer, 1970, p. 69). Currently, the usual sequence (Anton, Bivens & Davis, 2005 a recent tertiary introductory calculus text used in Australia) of introducing calculus involves limits, differentiation then integration where students are presented with a formal, abstract definition of limits and limit laws; a formal, abstract definition and rule ($dy/dx = \lim_{h\to 0}(f(x+h) - f(x))/h$) for differentiation; and integration viewed as anti-differentiation, with applications to area. In Anton et al. (2005), the first example of rate in their chapter on derivatives is the velocity of a moving body, such as a car or a ball. They emphasise this particular rate with a detailed discussion of displacement, velocity, average velocity, and instantaneous velocity. It is only after this detailed discussion that other examples of rate are mentioned, for example ―the rate at which the length of a metal rod changes with temperature" (p. 153). This is followed by a definition for the slope of a linear function as ―[a] 1-unit increase in *x* always produces an *m*-unit change in *y*" which is illustrated with the metal rod example. The unit rate is emphasised by the diagram seen in Figure 2. Variable rate is introduced through reference to the symbolic and graphic representations of the general function $y = f(x)$, again emphasising a unit rate approach (see Figure 2). The usual approach to the introduction of the concept of derivative assumes a sound understanding of rate and illustrates the derivative as the gradient of the tangent to the curve at a point, then moves quickly to emphasise symbolic manipulation.

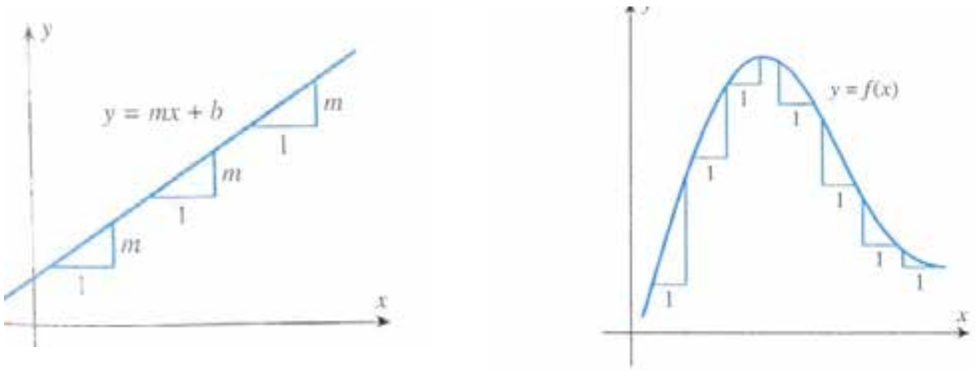


Figure 2. Rate diagrams from Anton et al., 2005, pp.153–154.

For example, in Garner et al. (2006), a mathematics text with an introduction to calculus used in some Victorian schools, the symbolic representations of a function are manipulated to establish a symbolic expression for instantaneous rate by taking the limit of the average rate. Some students become competent in this manipulation and can accurately produce the symbolic representation of the derivative (delos Santos & Thomas, 2005), but may not appreciate its meaning and connection to other mathematical concepts studied in earlier years.

Since rate is such a troublesome concept which affects the conceptual understanding of derivative, it is proposed that reversing the usual presentation of introductory calculus to begin with area and integration rather than rate and derivative, may improve students' conceptual understanding of this important area of mathematics. Doorman and van Maanen (2008) suggest that in ‗history we do not see the regular textbook approach from limits to differential quotient, from methods for differentiation to methods for integration, and finally the main theorem of calculus" (p. 10). This view is supported by Boyer (1970) who asserts ‗[t]hose textbooks that reverse the roles and place the integral before the derivative in a sense have history on their side, inasmuch as integration preceded differentiation by about two thousand years" (p. 69). This pilot study explores the effect of changing the usual sequence currently used to introduce calculus to begin with area and integration before rates and differentiation.

Method

Many mathematics students are introduced to calculus in secondary school; however, it was decided to trial the alternate sequence with tertiary students because of the high-stakes nature of the examinations of subjects undertaken in senior secondary school when calculus is usually introduced. It was anticipated that schools would be reluctant to participate unless they could be re-assured that their students would not be disadvantaged by undertaking the alternate sequence.

Two groups of first year tertiary students taking introductory calculus were selected to trial the effect of changing the traditional sequence described above. One group followed the traditional sequence and the other group followed a sequence beginning with integration. Both of these groups consisted of mathematics majors; human movement students intending to teach physical education and mathematics; and education students intending to teach mathematics. Whilst many students had been introduced to calculus at school, about one third of students had no previous experience of calculus.

Both groups were taught by the author—ensuring, as far as possible, a comparative delivery of the material. The learning experiences for both groups emphasised the support of a hand-held computer algebra system (CAS) and strong real-world connections. Delivery of content involved careful treatment of rate and extensive numerical integration stressing conceptual understanding as well as application of rules for differentiation and integration. At all stages of the delivery, strong, explicit connection to students' prior knowledge was attempted (Hiebert & Carpenter, 1992).

The concept of rate was explored in numeric, graphic and symbolic representations with instances of both constant and variable rate. The notion of average rate which built on students' understanding of linear functions, and hence constant rate, was employed to demonstrate a way of quantifying variable rate. The notion of instantaneous rate was developed by considering the average rates resulting from smaller and smaller intervals of the independent variable, that is, an informal treatment of the limit of average rate. The term instantaneous rate was eventually re-named derivative and symbolic manipulations undertaken to find the derivative from first principles using $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ noting that the fraction part is really just average rate. In this way, the concept of derivative was explicitly connected to the concept of rate. A consideration of limits was also employed in the introduction to numerical integration.

The concept of area was explored by considering a circle divided up into sectors (see Figure 1a) and finding the sum of the areas of the triangles bounded by two radii and a chord (the area shaded in Figure 1a). The number of sectors was increased and the table seen in Figure 1b completed, leading to a discussion of the relationship of this limiting process to the formula for the area of a circle. This exercise provided the link to students' prior knowledge for numerical integration. A feature of the introduction to the Fundamental Theorem of Calculus (FTC) was the explicit connection between numerical integration and differentiation from first principles (see Figure 1c).

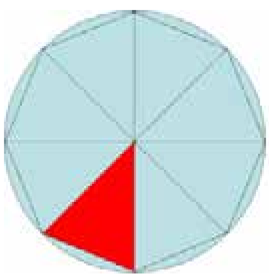
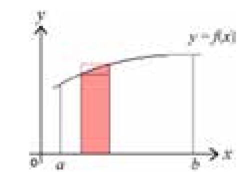


Figure 1a. Introduction to numerical integration.

No. of Triangles	Angle at centre	Total area
8	π/4	$2.82843r^2$
16	π/8	$3.06147r^2$
32	π/16	$3.12145r^2$
64		
128		

Figure 1b. Demonstration of limiting process leading to formula.



$$h \times f(x) \le A(x+h) - A(x) \le h \times f(x+h)$$

$$\text{As } h \to 0, \quad f(x) \le \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} \le f(x)$$

$$\Rightarrow \quad f(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h}$$

Figure 1c. Development of FTC.

The results of the end of semester exam were compared using a two sample t-test. The exam included a mix of CAS-supported questions (see Figure 2, below) and questions which required conceptual understanding of the concepts of integration and differentiation (see Figure 3) and represented 50% of the assessment for the unit.

Your backyard pool is kidney shaped and its width can be modelled as a function of its length (x) using the rule $w(x) = -\frac{1}{324}x^4 + \frac{1}{9}x^3 - \frac{25}{18}x^2 + 7x, 0 < x < 18$ (a) Use a numerical method to the total area of the pool by finding the area under this graph, for example divide the interval up into four subintervals and add up the areas. (b) Check your approximation using a theoretical method.	Sketch one graph which satisfies all of the following conditions; $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$, $f''(x) > 0$ on $(-\infty, 0)$ and $(12,\infty)$, $f''(x) < 0$ on $(0, 6)$ and $(6, 12)$ $\lim_{x\to 6} f(x) = -\infty$ $\lim_{x\to\infty} f(x) = 0$
Figure 2. Example of CAS-supported question.	*Figure 3. Example of question requiring conceptual understanding of integration & differentiation.*

Results and discussion

Figure 4 shows the results of a two-sample t-test undertaken in Minitab on the examination results of the two groups. The p-value of 0.809 indicates that there is no significant difference between the examination results of the two groups at the 95% confidence level. This suggests that changing the sequence of delivery of introductory calculus had no effect on the overall performance of the students.

Descriptive Statistics: group 1 - trad, group 2 - alt

```
Variable Mean StDev Minimum Q1 Median Q3 Maximum
group 1 - trad 64.19 22.84 14.29 50.89 68.45 79.76 98.81
group 2 - alt 62.76 19.13 20.67 46.33 60.00 79.67 96.00
```

Two-Sample T-Test and CI: group 1 - trad, group 2 - alt

```
Two-sample T for group 1 - trad vs group 2 - al
Difference = mu (group 1 - trad) - mu (group 2 - alt)
Estimate for difference: 1.43
T-Test of difference = 0 (vs not =): T-Value = 0.24 P-Value = 0.809 DF = 44
```

Figure 4. Minitab printout of 2 sample t-test on examination results.

It was thought that greater insight into the results might be gained by separating the results for differentiation questions from integration questions. Figure 5 shows the results of a two-sample t-test undertaken in Minitab considering questions where an understanding of differentiation was required. The p-value of 0.857, seen in Figure 5, indicates that there is no significant difference between the examination results on the questions relating to differentiation of the two groups.

Descriptive Statistics: group 1 - trad-diff, group 2 - alt-diff

```
Variable Mean StDev Minimum Q1 Median Q3 Maximum
group 1 - trad-diff 65.07 22.44 16.67 50.42 65.83 84.58 98.33
group 2 -alt - diff 63.98 20.64 22.22 47.78 65.56 81.11 94.44
```

Two-Sample T-Test and CI: group 1 - trad-diff, group 2 - alt-diff

```
Two-sample T for group 1 - trad-diff vs group 2 -alt - diff
Difference = mu (group 1 - trad-diff) - mu (group 2 -alt - diff)
Estimate for difference: 1.08
T-Test of difference = 0 (vs not =): T-Value = 0.18 P-Value = 0.857 DF = 47
```

Figure 5. Minitab printout of 2 sample t-test on differentiation questions.

Figure 6 shows the results of a two-sample t-test undertaken in Minitab on the examination results of the two groups but only considering questions where an understanding of integration was required. The p-value of 0.064, seen in Figure 6, indicates that there is no significant difference between the examination results on the questions relating to integration of the two groups.

Descriptive Statistics: group 1 - trad-int, group 2 - alt-int

```
Variable Mean StDev Minimum Q1 Median Q3 Maximum
group 1 - trad-int 61.98 28.40 0.00 41.67 66.67 83.33 100.00
group 2 -alt - int 47.84 25.15 0.00 28.13 50.00 68.75 93.75
```

Two-Sample T-Test and CI: group 1 - trad-int, group 2 - alt-int

```
Two-sample T for group 1 - trad-int vs group 2 -alt - int
Difference = mu (group 1 - trad-int) - mu (group 2 -alt - int)
Estimate for difference: 14.13
T-Test of difference = 0 (vs not =): T-Value = 1.90 P-Value = 0.064 DF = 46
```

Figure 6. Minitab printout of 2 sample t-test on integration questions.

Conclusion

These results indicate that changing the sequence of delivery of introductory calculus was not detrimental to the development of conceptual understanding of introductory calculus for this group of tertiary students. However, many of the students in both groups had previously been introduced to calculus at school, so it is unclear how much this school-based introduction influenced their level of conceptual understanding. It may be that the alternate delivery trialled here with these tertiary students has a different effect when utilised at the school level. Perhaps this sequence of delivery may be more effective when all students are first introduced to calculus. This pilot study was undertaken to gauge whether the alternate sequence seriously disadvantaged students. This was an important consideration as calculus is included in the subjects taken by secondary school students in their high stakes examinations for university entrance. It was necessary to establish this before attempting to trial the alternate sequence at the secondary school level. Further research will be necessary to explore the efficacy of the alternate sequence for introductory calculus at the school level.

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GENDER DIFFERENCES IN NAPLAN MATHEMATICS PERFORMANCE

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The results of the National Assessment Program – Literacy and Numeracy [NAPLAN] tests of Australian students in 2008, 2009, and 2010 were analysed by individual question responses (percentage correct) of females and males. The analysis of Grade 3 and Grade 9 data demonstrate that a decline in the achievement of females is evident and these gender differences become larger as students progress through their schooling.

Introduction

National Assessment Program – Literacy and Numeracy [NAPLAN] testing commenced in 2008. As part of the program, students in Grades 3, 5, 7 and 9 are simultaneously tested using national tests in Reading, Writing, Language Conventions (Spelling, Grammar and Punctuation) and Numeracy (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010a). As explained by ACARA (2010b), the tests broadly reflect aspects of literacy and numeracy in all States and Territories. The types of test formats and questions are chosen so that they are familiar to teachers and students across Australia. The Victorian Curriculum and Assessment Authority [VCAA] (2010a) affirms that "questions for NAPLAN tests are developed with reference to the nationally agreed Statements of Learning which reflect the core elements of the curriculum documents used in the different states and territories" (p. 2)

Results of the National Assessment Program – Literacy and Numeracy, in 2008, 2009 and 2010, the Program for International Student Assessment [PISA], in 2006 and 2009, and the Trends in International Mathematics and Science Study [TIMMS], in 2007, raise significant concerns with respect to the mathematical achievement of Australian females. The PISA is an initiative of the Organisation for Economic Co-operation and Development [OECD] in Paris and reports on indicators in education (Thomson & DeBortoli, 2008). Similarly, the goal of TIMMS is to provide comparative information on educational achievement across countries to improve teaching and learning in mathematics and science (Thompson, Wernert, Underwood & Nicholas, 2008).

Thomson and DeBortoli"s (2008) analysis of the 2006 PISA results showed that Australian males performed significantly better in mathematics than Australian females. This was not the case in PISA 2003 where, although males achieved a mean score higher than females, the difference was not statistically significant (Thomson & DeBortoli, 2008). Thomson, De Bortoli, Nicholas, Hillman and Buckley"s (2010) analysis of the 2009 PISA data showed that Australian males scored ten points higher, on average, than Australian females in the mathematical literacy assessment. They go on to say that was a significant difference. Thomson et al."s (2008) analysis of TIMMS data presents similar findings. They found that nationally, at the Grade 8 level, there was a substantial and significant gender difference in favour of males. Supporting the findings of the analysis of PISA data (Thomson & DeBortoli, 2006), Thomson et al. (2008) found that this significant gender difference in favour of males, that had not previously been seen in 2003 or 1995, appeared to be due to a significant decline in the average score for females over the 1995-2007 time span. This change in achievement should raise alarm; the fact that there were significant declines in the scores of female students that had not been observed for over a decade suggests that the current education system may not be effectively providing girls with equal opportunities for success in mathematics and so is particularly relevant with respect to the current analysis.

The results of the analyses of data from PISA in 2006 and 2009, and TIMMS in 2006 all point towards a considerable decline in the mathematics achievement of females that has not been observed for approximately a decade. The aim of the present analysis is to determine whether a similar decline in females" mathematics performance is evident in NAPLAN data for all grade levels taking the test (3, 5, 7, and 9) and for the three years 2008 to 2010.

Methodology

For the current analysis, the percentage correct achieved by both females and males on each individual NAPLAN question (provided by VCAA) was utilised. The percentage of correct responses by gender and grade level was compared from each test from 2008, 2009 and 2010. The data from each grade level was then grouped according to its mathematical dimension as described in the Victorian Essential Learning Standards [VELS] (VCAA, 2010b). The four mathematical dimensions of the VELS are: Number; Space; Structure; and Measurement, Chance and Data. The percentage difference in each question, by dimension, was then grouped into intervals of 5 percentage points. Questions for which there was no difference in percentage correct by females and males (ie. 0% difference) have been removed from the 0-4% group and reported separately; and the interval of 1-4% used.

Results and discussion

The results of the NAPLAN tests conducted in 2008, 2009 and 2010 for Grades 3 and 9 were analysed in order to investigate what gender differences exist between the performance of students from the highest and lowest grade levels undertaking NAPLAN testing. These tests consisted of between 31 and 35 questions from each of the Mathematics Dimensions as described in the VELS (with the exception of the Grade 3 tests in 2008, which did not include Structure questions). Grade 9 students undertake two NAPLAN tests; one in which they are able to use a calculator and one in which calculator use is not permitted. The number (% in brackets) of each question type is presented in Table 1.

Table 1. NAPLAN question types by dimension: Grade 3 and 9, 2008-10.

Grade		2008	2009	2010
		n	n	n
Grade 3	N[1]	14 (40%)	11 (31%)	13 (37%)
	M[2]	11 (31%)	10 (29%)	10 (29%)
	Sp[3]	10 (29%)	10 (29%)	10 (29%)
	St[4]	0	4 (11%)	2 (5%)
Total		35 (100%)	35 (100%)	35 (100%)
Grade 9 No Calculator	N	8 (25%)	9 (29%)	9 (28%)
	M	8 (25%)	8 (26%)	10 (31%)
	Sp	8 (25%)	7 (23%)	7 (22%)
	St	8 (25%)	7 (23%)	6 (19%)
Total		32 (100%)	31 (100%)	32 (100%)
Grade 9 Calculator	N	8 (25%)	8 (26%)	10 (31%)
	M	7 (22%)	8 (26%)	9 (28%)
	Sp	9 (28%)	8 (26%)	7 (22%)
	St	8 (25%)	7 (22%)	6 (19%)
Total		32 (100%)	31 (100%)	32 (100%)

[1] Number dimension
[2] Measurement, Chance and Data dimension
[3] Space dimension
[4] Structure dimension

The proportions of questions from each dimension remained relatively constant from 2008 to 2010 although there was an increase in the number of Measurement, Chance and Data [MCD] questions in the Grade 9 No Calculator test with a corresponding decrease in the number of Structure questions. Similarly, in the Grade 9 Calculator test, there was an increase in the number of Number and MCD questions in conjunction with a decrease in the number of Structure and Space questions.

When considering all NAPLAN questions (all years and all grade levels) in which a difference existed in the percentage of females and males who correctly answered a particular question, there were fewer questions in which females performed better than males, with males outperforming females at all grade levels. The differences in favour of males were also much larger than those favouring females. Table 2 shows these performance differences for Grades 3 and 9. A percentage difference of 1-4% for females indicates that a higher percentage of females than males answered this question correctly. It must be noted that although there were several questions for which females scored 5-9% better, there was no larger difference than 7% across all years and grade levels; that is, there were a few questions for which 7% more females than males answered correctly but this was the largest difference found for all years and grade levels.

Table 2. Number of questions with differences in percentage correct by grade level and gender; Grades 3 and 9.

		Girls				Boys				Total
Percentage difference	0*	1-4	5-9	10-14	≥15	1-4	5-9	10-14	≥15	
Grade 3 2008	5	10	2	0	0	7	10	1	0	35
Grade 3 2009	4	7	3	0	0	16	4	1	0	35
Grade 3 2010	3	9	3	0	0	15	5	0	0	35
Grade 9 No Calculator 2008	2	4	1	0	0	12	7	5	1	32
Grade 9 No Calculator 2009	5	6	0	0	0	13	6	1	0	31
Grade 9 No Calculator 2010	3	6	0	0	0	9	12	1	1	32
Grade 9 Calculator 2008	3	8	1	0	0	14	5	1	0	32
Grade 9 Calculator 2009	3	7	2	0	0	11	8	0	0	31
Grade 9 Calculator 2010	5	4	1	0	0	13	9	0	0	32

* Questions for which females and males performed equally well.

NAPLAN 2008–2010 Grade 3

When comparing the performance of Grade 3 students, Tables 3 and 4 show that there were no differences in the performance of females and males for 5 questions in 2008, 4 questions in 2009 and 3 questions in 2010. As the number of test questions in each year remained constant at 35, these results represent a small decline in the number of questions in which females and males performed equally well.

Table 3. Number and type of questions with percentage differences in favour of females - Grade 3.

Dimension	Number					Measurement, Chance and Data					Space					Structure				
% difference	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15
Grade 3 2008	2	1	1	0	0	2	2	1	0	0	1	7	0	0	0	-	-	-	-	-
Grade 3 2009	2	1	1	0	0	1	3	2	0	0	1	2	0	0	0	0	1	0	0	0
Grade 3 2010	1	4	0	0	0	1	1	2	0	0	1	3	1	0	0	0	1	0	0	0

Table 4. Number and type of questions with percentage difference in favour of males - Grade 3.

Dimension	Number					Measurement, Chance and Data					Space					Structure				
% difference	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15
Grade 3 2008	2	4	5	1	0	2	2	4	0	0	1	1	1	0	0	-	-	-	-	-
Grade 3 2009	2	7	0	0	0	1	3	1	0	0	1	5	2	0	0	0	1	1	1	0
Grade 3 2010	1	6	2	0	0	1	4	2	0	0	1	5	0	0	0	0	0	1	0	0

Grade 3 females appeared to do well in Space questions. In 2008, 7 of 12 Space questions were answered correctly by a higher percentage of females than males. This result was not repeated in 2008 or 2009. Instead, males outperformed females in Space questions in these years. Of the questions for which a higher percentage of females than males answered correctly, the difference in percentage was no greater than 7%. In contrast, there were two questions in which the difference favouring males was greater than 10%. Differences in performance, although small, are already obvious at the Grade 3 level, with males outperforming females in all dimensions.

NAPLAN 2008 – 2010 Grade 9 (No calculator)

In the Grade 9 No Calculator test results it can be seen that there were very few questions with no percentage difference in the performance of females and males. Tables 5 and 6 show that far fewer questions were more likely to be answered correctly by females than by males. Females outperformed males by more than 5% in only one MCD question. Males outperformed females for most questions, especially those from the Number and MCD dimensions.

Table 5. Number and type of questions with percentage differences in favour of females: Grade 9 No Calculator.

Dimension	Number					Measurement, Chance and Data					Space					Structure				
% difference	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15
Grade 9 No Calculator 2008	1	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	3	0	0	0
Grade 9 No Calculator 2009	0	0	0	0	0	0	2	0	0	0	3	1	0	0	0	2	3	0	0	0
Grade 9 No Calculator 2010	1	2	0	0	0	0	2	0	0	0	0	1	0	0	0	2	1	0	0	0

Table 6. Number and type of questions with percentage difference in favour of males: Grade 9 No Calculator.

Dimension	Number					Measurement, Chance and Data					Space					Structure				
% difference	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15
Grade 9 No Calculator 2008	1	2	2	2	0	1	1	2	2	1	0	4	3	1	0	0	5	0	0	0
Grade 9 No Calculator 2009	0	6	2	1	0	0	4	2	0	0	3	2	1	0	0	2	1	1	0	0
Grade 9 No Calculator 2010	1	4	2	0	0	0	2	5	1	0	0	2	4	0	0	2	1	1	0	1

In two questions, (one MCD and one Structure), males dramatically outperformed females. In these questions, a difference of greater than 15% existed in the percentages of males and females who correctly answered. These results can be contrasted to the results from Grade 3 students. In Grade 3, a much higher number of questions were more likely to be answered correctly by females than can be observed in the Grade 9 data, demonstrating the decline in achievement of females as they progress through their schooling.

NAPLAN 2008 – 2010 Grade 9 (Calculator)

Again, in the Grade 9 Calculator test results, there were few questions for which there was no difference in performance by gender, with results being similar to those from the Grade 9 No Calculator data. As shown in Tables 7 and 8, questions for which a higher percentage of females than males answered correctly comprise a greater proportion than those from the Grade 9 No Calculator data. Females appear to do best in Structure type questions. Interestingly, there was a reduction in the number of Structure questions from 2008 to 2010. It is clear that in Grade 9, both males and females do better on the NAPLAN Calculator tests than on the Non Calculator tests, but males dramatically outperform females in both test types.

Table 7. Number and type of questions with percentage differences in favour of females: Grade 9 Calculator.

Dimension	Number					Measurement, Chance and Data					Space					Structure				
% difference	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15
Grade 9 Calculator 2008	1	1	0	0	0	2	1	0	0	0	0	1	0	0	0	0	5	1	0	0
Grade 9 Calculator 2009	0	2	0	0	0	2	1	1	0	0	0	1	0	0	0	1	3	1	0	0
Grade 9 Calculator 2010	0	2	1	0	0	0	0	0	0	0	4	0	0	0	0	1	2	0	0	0

Table 8. Number and type of questions with percentage differences in favour of males: Grade 9 Calculator.

Dimension	Number					Measurement, Chance and Data					Space					Structure				
% difference	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15	0	1-4	5-9	10-14	≥15
Grade 9 Calculator 2008	1	5	1	0	0	2	1	2	1	0	0	6	2	0	0	0	2	0	0	0
Grade 9 Calculator 2009	0	3	3	0	0	2	0	4	0	0	0	6	1	0	0	1	2	0	0	0
Grade 9 Calculator 2010	0	2	5	0	0	0	5	4	0	0	4	3	0	0	0	1	3	0	0	0

Conclusion

The results of this analysis of NAPLAN data support the findings of Thomson and DeBortoli (2006), Thompson, Wernert, Underwood and Nicholas (2008) and Thomson, De Bortoli, Nicholas, Hillman and Buckley (2010). It is evident that the mathematics achievement of females in Australia is on a decline and it appears that gender differences in favour of males become larger as students progress through their schooling. As stated by Forgasz (2008), „gendered patterns from the past are still evident in the context of contemporary mathematics education in Australia" (p. 13). The issue of gender differences in mathematics achievement needs to be brought to the forefront of educational research before such differences become even more pronounced.

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MAKING A DIFFERENCE FOR INDIGENOUS CHILDREN

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The *Make It Count* project aims to provide better mathematical outcomes for Indigenous children, and, to that end, the Swan Valley cluster identified various initiatives. This paper reports on a research project that investigated those initiatives and resultant changes in practice. First, a modified First Steps in Mathematics professional learning program was provided for Education Assistants and Aboriginal and Islander Education Officers to upgrade their mathematical and pedagogical content knowledge. Second, elements of best practice in teaching Indigenous children were investigated. It is apparent that genuine professional learning communities have begun to develop in the wake of the professional learning and that there are clear directions for pedagogical practice that may lead to improved student attendance and engagement.

Introduction

During 2009 and 2010, the Swan Valley Cluster in the *Make It Count* project identified a number of factors having an impact on the mathematical learning of Indigenous children (e.g., Hurst & Sparrow, 2010). It was felt that, unless these factors were addressed, it would likely be difficult to achieve improved numeracy outcomes and that even if such issues could be addressed, the desired improvement in numeracy outcomes may not be immediately evident. Among other things, the professional learning of teaching and support staff was seen as an important factor, as was the raising of expectations for student learning, and these points are the focus of this paper.

First, during 2010, a program of professional learning based on *First Steps in Mathematics* was implemented for teaching and support staff. Second, information was gathered about the practice of the most effective teachers of Indigenous children. This paper reports on the research conducted into these two aspects of the Swan Valley Cluster Plan.

Background

Professional learning and professional learning communities

According to Bolam et al. (2005), professional learning communities (PLCs) have the "capacity to promote and sustain the learning of all professionals in the school community with the collective purpose of enhancing pupil learning" (p. 4). They include a joint responsibility for student learning, team planning, collaboration across roles, and the involvement of all staff, including support staff, in a range of professional learning. Mutual trust and respect are evident. These features sit well with the mantra of *Make It Count*, the aims of which include working "with all components of the school community—students, teachers and paraprofessional staff, school leaders, parents and parent groups, and the wider community" (AAMT, 2009, p. 1).

Anthony and Walshaw (2009), in discussing the importance of teacher knowledge as a component of effective pedagogy, note how "the development of teacher knowledge is greatly enhanced by efforts within the wider educational community", and also that "teachers can learn a great deal by working together with a group of supportive mathematics colleagues" (p. 26). Such a group of supportive colleagues might well include paraprofessionals such as teacher assistants and aides.

Mathematical and pedagogical content knowledge

Shulman (1986) identified several categories of content knowledge required for effective teaching. Subject matter content knowledge that constitutes "the amount and organisation of knowledge per se in the mind of the teacher" (p. 9) is much more than just 'knowing the facts' about mathematics, but includes why certain facts are as they are and the ability of teachers to explain such things to students. More recently, these ideas have been developed by others including Rowland. In a similar way, Rowland (2005) coined the term 'knowledge quartet' to describe different categories of knowledge, these being foundation, transformation, connection, and contingency. The first category, 'foundation', is much more than simply factual knowledge in that "the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way" (p. 4). In other words, it is the key content knowledge of mathematics that underpins the pedagogical knowledge and decisions that follow. It seems reasonable to suggest that strong pedagogical knowledge and skills will be more likely to emerge if teachers and paraprofessionals have strong 'subject matter knowledge' (Schulman) or 'foundation' knowledge (Rowland).

Cultural competency, awareness, and support

Perso (2003) considered the situation faced by Indigenous children in Australian schools as being similar to being in a different culture, a major reason being that most teachers lack sufficient awareness of Indigenous cultures in order to provide the support needed by those children. Further to this, Perso (2009) developed a *Pedagogical Framework for Cultural Competence*, in which she underlined the need for teachers to develop "cultural competence" in order to "demonstrate behaviours and attitudes that engage, build and maintain relationships with Aboriginal and Torres Strait Islander peoples"
(p. 1). Within the framework, Perso nominated eight categories, including pedagogies related to questioning, contextualized learning, and most significantly, classroom relationships and the identification of expectations. Indeed, Perso noted that the successful learning for Indigenous students "depends to a great extent on the personal relationship of trust and rapport established between teacher and student" (2009, p. 5).

These sentiments are further echoed by Anthony and Walshaw (2009) in discussing the notion of 'an ethic of care' as one of ten components of effective pedagogies in mathematics. They note that students learn best in a harmonious and caring environment in which they feel safe, and where expectations are clear. Indeed, "teachers can help create such an environment by respecting and valuing the mathematics and cultures that students bring to the classroom" (p. 7).

Methodology

A qualitative study was established to consider the two broad areas of content knowledge of teachers and paraprofessionals and links to the development of professional learning communities, as well as the practice of effective teachers of Indigenous children. Specifically, the study sought to answer the following questions.

- To what extent did the provision of professional learning for Education Assistants (EAs) and Aboriginal and Islander Education Officers (AIEOs) assist in the development of professional learning communities?
- What attitudes and strategies are evident in the practice of effective teachers of Indigenous children?

The study was based on two particular initiatives taken within the Swan Valley Cluster. The first was the cluster-wide provision of three half-days of professional learning in mathematics teaching for EAs and AIEOs, conducted by a qualified *First Steps in Mathematics* facilitator. Semi-structured interviews were conducted following the professional learning and included initial questions such as 'What you have learned and how it has helped you assist children?' and 'In what ways has your classroom role changed as a result of the professional learning?' Data were generated from the semi-structured interviews with principals, teachers, EAs, and AIEOs, and this provided a measure of triangulation as perspectives from a range of people in different schools were sought. In all cases, the discussion that took place following the initial questioning diverged from that point and enabled the generation of rich data from which emerged some strong themes.

The second initiative was a plan adopted by Swan View Primary School where the Indigenous cohort was concentrated in four classes taught by teachers identified as being culturally sensitive and empathetic towards Indigenous children. Again, semi-structured interviews were conducted with principals, teachers, EAs, and AIEOs. These interviews were audio-recorded and transcribed to determine emergent themes. Staff members involved in the Swan View plan provided the initial focus for interviews but those from the other cluster schools were also interviewed in order to gain a wider perspective.

A teacher questionnaire about strategies employed with Indigenous children and attendance records of Indigenous children were used to generate further data. In the data analysis stage, to identify emergent themes, interviewee replies and comments were analysed for key words and then grouped with similar ideas to form categories. The frequency of occurrence of each main idea was recorded for each of the interviewee types (principal, teacher, EA/AIEO). It is important to note that the following discussion draws upon comments made by interviewees that may support more than one of the themes.

Results and discussion

Professional learning and professional learning communities

The four themes that emerged in relation to the first research question are represented in Table 1.

Table 1. Themes related to outcomes of professional learning.

Theme	Frequency
Improved levels of confidence	34
Development of a professional team approach	34
Improved content knowledge and pedagogical knowledge	27
Improved engagement with children	15

Within the first theme, Improved levels of confidence, responses from all participant types were recorded for the key ideas of "being more proactive" (32%) and "having a greater level of confidence" (29%). Other responses related to "having an improved status in the school" (12%), "teachers having more confidence in me" (8%), and "feeling more independent in knowing what to do" (8%). Comments included:

> That's because we've been shown the direction and with the confidence we've got because we've done the PDs, we can keep going with it and not have to look for help and direction all the time. (Lesley—EA).

As well, this was stated with regard to the confidence of the aides following the professional learning:

> They're not just lackeys that don't know anything; these are now women of confidence, of knowledge, of capacities that they didn't know they had before. (Megan—Principal)

Indeed, the increase in confidence seems to be reflected in the teachers as well as in their aides, as shown by the following:

> I think the teacher has confidence in you because *you* [her emphasis] know what you're doing and that makes you feel good because the teacher can say 'You take [children] over there and do 'blah blah blah' with them', and that makes you as a person feel good. (Kelly, EA)

Development of a professional team approach

Within this theme, the key ideas noted were "more a part of a team approach" (32%), "being better able to support the teacher" (18%), "better able to watch the teacher and learn/know" (18%), "more comfortable and at ease with the role" (15%), and "able to step in when the teacher is occupied" (15%). These key ideas were typified by comments such as the following:

> You're more comfortable because you actually know what she [the teacher] is trying to do now ... you know where she's headed with it and we're all on that same level—there's no miscommunication ... because we've been shown the direction and with the confidence we've got because we've done the PDs, we can keep going with it and not have to look for help and direction all the time. (Jenny, EA)

Similarly, this anecdote was provided:

> You pick out one child who's not getting it and you're able to go over and say 'What about if it you about it like this?' and the teacher hears and says, 'Oh yeah, can you tell everybody about that?' and so you get up and tell the whole class. (Christy, AIEO)

The previous theme described the development of a greater feeling of confidence, as a result of a change of status. This latter point was echoed in this theme as well, with comments such as:

> You're not just the assistant doing the gluing and cutting out and things like that—you're actually helping with the actual learning. (Hayley, EA)

> Nobody treats us like we're *just* [her emphasis] teacher assistants. (Shona, EA)

Improved content knowledge and pedagogical knowledge

The key ideas within this theme were predominantly one of two—"better content knowledge/teaching knowledge" (88%) and "aware of other ways of teaching maths" (47%). The first key idea was noted in interviews with all participants. In describing her aide's role, one teacher noted:

> She did say that it [P/learning] helped her to understand what I was doing in the classroom. She did comment that 'Oh I know why you do that now'. It's been helpful for her to understand what we do as teachers. (Irene, Teacher)

> I know where the teacher is coming from, whereas if I hadn't had the PD, I'd really be in my 'forty years back maths thinking' and wondering what she is doing, but now I do understand. (Annie, EA).

> Basically, we're the children aren't we? We've had to go back and re-learn, and a lot of the procedures we learned [in the P/learning] have really helped, so when I see the teacher do it on the board, I think that's the way for them to do it—you're understanding it better; it's breaking it right down. (Rose, EA)

Improved engagement with children

Almost all of the responses within this theme were categorised as "being better able to recognise needs and help children" (80%). Again, this response was noted from all types of interviewees and it is summed up in the following statements.

When asked about being better able to recognise children's needs, one aide said

> Yeah, I know what part's missing. I can say 'We need to go back a step further because they haven't got this bit down'. (Shona, EA)

Also, the link with increased confidence and status is clear from this comment:

> They [EAs and AIEOs] are more effective on the ground with the kids because the teachers know that they know what to do. (Megan, Principal)

Similarly, the link to increased knowledge is made here:

> You write down which child is better at whatever ... so you relate that to the teacher that says 'This child is good at ...' and that's what I do, you know—that one doesn't know this bit much and this one doesn't know that bit much'. (Donna, AIEO)

In summarising responses about the benefits of the professional learning, the four themes (shortened to confidence, team, knowledge, and children) are encapsulated in the following comment:

> Teachers have got so many different children with so many different abilities that they're just so happy to have someone in there that they can go, 'This group are all at that level

> so you can work with them and I'll work with the rest. They're quite happy to do that because it gives the children better opportunities (Christy, AIEO).

Effective teachers of Indigenous children

As with the first research question, themes emerged from key word analysis of interviewee replies and comments. The four themes are represented in Table 2. Other themes emerged but, for brevity, the discussion here is restricted to four. As was stated earlier, it was considered by the cluster schools that broad underpinning factors supporting teacher effectiveness needed to be addressed before it could be hoped to specifically improve numeracy outcomes.

Table 2: Themes related to effective teachers of Indigenous children.

Theme	Frequency
Building a better relationship through empathy, connection	40
Displaying cultural sensitivity and awareness	37
Support mechanisms	33
Clarity of goals and expectations	27

Building a better relationship through empathy and connection

The key ideas within this theme were "connection" (30%), "empathy/warmth" (28%) and "trust/relationships" (25%). When discussing the question of why they are considered effective teachers of Indigenous children, teachers Dan and Teresa modestly stated that they didn't feel that they did anything different but evidence suggests otherwise. Indeed, their principal, in response to the same question stated:

> This may sound a bit fey but these teachers [Dan and Teresa] have an authentic presence in the life of the child. It is the bottom line for me—It's like, "I see the God in you looking at the God in me" ... that's the key. It's the recognition that you are a presence in my life. You might think that's a bit over the top but that's where it is—it's a sense of spirituality with the children, a sense of communion, but the bit I can't stress enough is the presence. (Megan, Principal)

Dan noted that he and his colleague, Teresa, often spent a lot of time talking to their Indigenous children, dealing with social and emotional issues during their recess breaks or planning time and putting 'school stuff' to one side. Teresa noted that they both had strong interest in "out of school issues" like sport, adding:

> A lot of Indigenous kids connect with that and that works well. I think we're approachable in that sense. (Teresa, Teacher)

Dan's comment provided further insight into what they do:

> The key thing is that you've got to have empathy with them. You've got to trust them, then they'll want to come and talk to you and then you're half way there. If you don't have that, then it's "I'm not interested, see you later". It's a critical thing—be open, let them talk to you, talk back with them. (Dan, Teacher)

Further to the building of a strong relationship, teachers agreed that if the teacher-student relationship is robust, then the teacher can ask students about almost anything:

> If you don't have that relationship, then it's like "Why are you asking me that question?" (Max, Teacher)

> They probably don't feel like they have to answer you, if there is no rapport" (Linda, Teacher).

Regarding the idea of 'making connections', a number of EAs and AIEOs noted that effective teachers:

> … have empathy for and connected with Indigenous children ... who approach them for help because they provide it willingly. (Annie, EA)

This was furthered expressed as follows:

> If you come at them [Indigenous children] on a personal level and you gain one little bit of information so you can ask them how the dog is, or something so that they know you're concerned about them ... connect, yeah. Once you've got that connection, a bit of respect, you're away. (Shona, EA).

Finally, this conversation further underpins the notion of caring and connecting:

> I think that showing that you care... that's it, you're part of the family ... I rouse them all the time [laughed] (Donna, AIEO).

> They love her because even after she's roused them, they know that she still loves them (Jean, Teacher).

> Consistency ... consistency. (Donna, AIEO)

Displaying cultural sensitivity and awareness

The key ideas within this theme were 'being culturally aware and in tune' (frequency 19/33), "public shaming and failure" (8) and "sense of equity" (7). It was noted by a principal that the most effective teachers are aware of the effect of impoverishment on the Indigenous children and that:

> They work on the premise that for equity of access to the learning program, it doesn't mean giving every child the same; it means giving them what they need. (Megan, Principal)

It was also noted that the awareness is two way, in that Indigenous children are very much aware of who is supportive of them.

> Indigenous kids just know. They come to a Wadgula school with built in crap detectors. They can tell, just the way you look, you smile, you touch. They learn to reach out to people like Dan, Teresa, and Beth. These are wonderful teachers who have a clear, authentic and defined presence in the lives of the children. (Megan, Principal).

Other teachers and aides noted the importance of being aware of what children had experienced the previous night at home, or coming to school, as these events had a profound impact on their in-school performance. Specifically regarding the notion of public shaming and displaying learning in public, several participants noted the importance of teachers being aware of this, but that it is something that can be overcome, as noted by the following comment:

> It all depends on the teacher, how they've modelled it, how they've developed those children, what kind of relationship they have. We have encouraged them to do risk-taking and things like that, whereas some teachers *will* put them on the spot and they *will* clam up. (Irene, Teacher)

Support mechanisms

The main key ideas within this theme were “role models/mentors” (33%) “support” (30%), and “success” (24%). Dan’s principal, Megan, noted that teachers like him were successful because they are good role models—important as Indigenous children are “watchers of people more than anything”. Dan agreed that having appropriate role models was crucial for the children:

> You’ve got to be really careful with what you do, what you say, because someone is watching all the time. (Dan, Teacher).

Dan also noted that the concentrating of Indigenous children into four classes had made a difference as it had given them a sense of camaraderie, enabling them to compete with one another at appropriate levels, without ‘being blown out of the water’. Beth agreed with this, noting the following:

> Most of my Indigenous children have come along in leaps and bounds this year so I think [having them grouped together] is really working. They associate with each other culturally, and they’re more comfortable. They’re more relaxed and you can target their needs ... they don’t have to go outside at lunchtime and prove themselves because they know where each other is in the pecking order, being in the same class (Beth, Teacher).

Specific strategies employed by teachers to provide support included extensive use of peer tutoring and providing ample opportunities for the Indigenous children to succeed.

Clarity of goals and expectations

The key ideas within this theme were simply “clear expectations” (44%), “realistic goals” (26%), and “consistency” (19%). Most participants differentiated between academic goals and behavioural goals noting that the former had to be realistic and linked to success, while the latter needed to be clear and firm. Many comments echoed the Cluster Plan regarding the raising of expectations, encapsulated in the following:

> Show that you’re a strong person ... they don’t respond well to weakness. If you go back on what you said you were going to do, you lose so much ground. You have to be very firm, fair, honest, follow through ... they like to know what your expectations are and that they can live up to them as well. (Leanne, Teacher)

Teaching strategies

The interviews highlighted a number of strategies used successfully with Indigenous children, and these data were also generated from the questionnaire. Significant were factors like cultural awareness, teacher student relationships, and support mechanisms. A common discussion point was the use of hands on materials and peer tutoring.

> It’s no good giving them book stuff, [they’ve] got to be doing it, touching things ... we use a lot of peer work with them. (Dan and Teresa, Teachers)

> They like to make things, build things ... hands-on manipulatives ... they like somebody working with them, but not pushing at them all the time with questions ... it comes in through general discussion and talking, but not direct questioning - they shy away. (Beth, Teacher).

Other teachers noted that their Indigenous children needed time at the beginning of lessons to play with equipment because many had lacked this sort of experience in their home background.

Table 3 contains a summary of questionnaire responses describing effective classroom strategies.

Table 3: Effective practice with Indigenous children.

Identify individual weaknesses and plan for teaching based on learning sequences.
Revise, re-teach, and use continuous reinforcement of key ideas to account for short attention span.
Use oral discussions and drawing to communicate ideas.
Use game playing to teach key concepts
Use rhyme, rhythm and movement in real life contexts
Use hands-on resources and manipulatives.
Use natural resources such as sea shells and familiar resources such as dice and cards.

Conclusion

It is apparent from the evidence presented that the mathematics professional learning for EAs and AIEOs contributed to the development of professional learning communities. As well, it is apparent that effective teachers of Indigenous children have particular qualities and use particular strategies that develop and enhance supportive and empathetic teacher-student relationships, and which will hopefully lead to improved numeracy outcomes for Indigenous children. The Swan Valley Cluster will build on the initiatives described here including an extension of the professional learning program for aides during 2011 and 2012. It is hoped that the anecdotal evidence provided in this paper will be translated into statistical gains in numeracy outcomes reflected in NAPLAN and other assessments during 2011–12.

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IMPLEMENTING A MATHEMATICAL THINKING ASSESSMENT FRAMEWORK: CROSS CULTURAL PERSPECTIVES

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Assessment systems should place more emphasis on the thinking process, not academic achievement alone. This study focuses on comparing Australian and Malaysian teachers' views on the practicality of implementing the Mathematical Thinking Assessment (MaTA) Framework. It involved eight mathematics teachers from Australia and Malaysia. All teachers implemented the MaTA Framework in their schools to assess students' mathematical thinking using a Performance assessment to elicit students' thinking processes during problem-solving. They also used a Metacognition Rating Scale, a Mathematical Dispositions Rating, and a Mathematical Thinking Scoring Rubric. Teachers were interviewed and their views towards implementing the MaTA Framework were reported in this study.

Introduction

The Trends in International Mathematics and Science Study (TIMSS) for Australian and Malaysian Grade 8 students' recorded a gradual decline over the years 2003 and 2007 with average scores of 505, 496, 508, and 474 respectively (Gonzales, Williams, Jocelyn, Roey, Kastberg, & Brenwald, 2008). These results suggest that the Australian and Malaysian Grade 8 students were more inclined to apply basic mathematical concepts than organizing their thinking effectively from the information given. These performances have roused, to a certain degree, national concern about the quality of mathematics education in both countries' education systems.

Therefore, we argue that the Mathematics Curriculum should give priority to fostering students' abilities to think and to organize information, as well as possessing procedural knowledge in solving problems (Ginsburg, Jacobs & Lopez, 1993). With the intention to foster this goal, the Mathematical Thinking Assessment (MaTA) Framework was developed. It aims to assess students' mathematical thinking performance in a holistic domain, which includes mathematical knowledge, mental operations and mathematical disposition. This paper focuses on comparing Australian and Malaysian teachers' perspectives particularly on the practicality of implementation the MaTA Framework in their respective countries.

Mathematical thinking

Mathematical thinking is usually referred to indirectly in the mathematics curricula produced by Australia and Malaysia as an important “process” to foster success in mathematical problem solving. In Australia, for example, “Working mathematically” is one of the important goals in mathematics (Stacey, 2005). Working mathematically is a process strand that comprises investigating, conjecturing, using problem solving strategies, applying and verifying, using mathematical language, and working in context (Australian Education Council, 1994). In Malaysia, the words “think mathematically” are contained in the aims of mathematics curriculum, which is “to develop individuals who are able to think mathematically ...” (Ministry of Education Malaysia, 2005, p. 2). It focuses on cultivating students who are able to possess mathematical content knowledge, and who learn effectively and responsibly in mathematical problem-solving and decision making.

The mathematics curricula in both countries seem to define mathematical thinking somewhat differently. This is to be expected because a well defined meaning or explanation of mathematical thinking has yet to be developed (Lutfiyya, 1998; Cai, 2002). As a result, there is no detailed description of the words “mathematical thinking” in most national mathematics curriculum documents (Isoda, 2006). As such, different perspectives on mathematical thinking are evoked. Mason, Burton and Stacey (1982), for example, defined mathematical thinking as a dynamic process enabling one to increase the complexity of ideas able to handle, and consequently expand understanding. Katagiri (2004) defined mathematical thinking as the ability to think and to make judgments independently while solving mathematics problems. Alternatively, Schoenfeld (1992) proposed five important aspects of cognition involved in mathematical thinking and problem solving: (a) knowledge base; (b) problem solving strategies; (c) monitoring and control; (d) beliefs and affects; and (e) practices (p. 348). His use of mathematical thinking is thus much more grounded in the process of its being used and what the problem solver brings to that process. More recently, Wood, Williams and McNeal (2006) defined mathematical thinking as the mental activity involved in the abstraction and generalization of mathematical ideas, adding further dimensions to the idea.

However, all the above definitions are not totally dissimilar. They seem to highlight three major domains of mathematical thinking: (a) mathematical knowledge; (b) mental operations; and (c) dispositions. This categorization was supported by the model of Component of Thinking proposed by Beyer (1988). Mathematical knowledge refers to mathematical concepts and ideas that one has acquired or learnt, while mental operations can be considered as cognitive activities that need to be performed when thinking (Beyer, 1988). As for thinking dispositions, these refer to a tendency or predilection to think in certain ways under certain circumstances (Siegel, 1999). Examples of relevant dispositions include reasonableness, thinking alertness and open-mindedness, as well as beliefs and affects.

In line with the above, it is proposed that mathematical thinking be characterized as including the following aspects:

1. It is a knowledge-dependent activity;
2. It involves the manipulation of mental skills and strategies;
3. It shows the awareness and control of one’s thinking such as metacognition; and

4. It is highly influenced by the dispositions, beliefs, or attitudes of the student.

Based on the foregoing, this study will take *mathematical thinking* to be mental operations that are supported by mathematical knowledge and by certain kinds of dispositions toward the attainment of solutions to mathematics problems.

Mathematical Thinking Assessment framework

The Mathematical Thinking Assessment (MaTA) Framework consists of four components: (a) a Performance assessment; (b) a Metacognition Rating Scale; (c) a Mathematical Dispositions Rating Scale; and (d) a Mathematical Thinking Scoring Rubric. The MaTA Framework is intended to be implemented by teachers with the aim of assessing students' mathematical thinking. The *Performance assessment* component is administered by the classroom teacher to assess students' mathematical knowledge and skills (conceptual, procedural, strategies and skills) while solving particular mathematical problems in one or more content areas that have been the focus of classroom instruction. The *Metacognition rating scale* is used, also by the teacher, to elicit students' cognition awareness, such as monitoring and regulation, during problem solving process. The *Mathematical dispositions rating scale* is used by the teacher to indicate students' predisposition toward learning of mathematics. Finally, the *Mathematical thinking scoring rubric* is used to score and grade students' mathematical thinking according to the domains defined in this study.

Teacher's perceptions

Even though performance assessment promises more fruitful feedback on students' learning progress, the use of this assessment has declined in tests in United States of America (Parke & Lane, 2007). One of the major reasons is because implementing performance assessment is time consuming (Ryan, 2006; Linn & Miller, 2005; McKee & Lucas, 2005) compared to standardized testing. With current teaching workloads, administration duties and class sizes, it is argued that it is not cost-effective for teachers to invest so much time in these aspects of assessment.

Difficulty in implementing performance assessment is another reason why it has proved less popular. According to Baker (1997), performance assessment is difficult and expensive to develop. Mckee and Lucas (2005) also stressed that it is difficult at the beginning. Teachers need new knowledge and skills to implement performance assessment (Stiggins, 1995; Adi Badiozaman Tuah, 2006; Buhagiar & Murphy, 2008). Hence, extensive training is needed for teachers on how to administer performance assessment in the classroom (Aschbacher, 1992).

Methodology

Participants

A total of eight secondary school mathematics teachers, four each from Australia and Malaysia were selected to participate in the study. All the selected teachers have at least five years of teaching experience in Mathematics. Through their teaching experiences, they were believed to be able to implement the MaTA Framework according to its guidelines in their respective schools.

Procedures and data analysis

All the selected secondary school mathematics teachers were briefed and guided by one of the researchers on how to use the MaTA Framework to assess students' mathematical thinking performances. This was conducted in a one-on-one basis where the researcher met the teachers regularly prior to and during the data collection processes. The following summarizes how teachers could be expected to implement the MaTA Framework in their home school context.

Step 1: Designing performance assessment

Based on the procedures or guidelines provided in the MaTA Framework, the teachers designed the performance tasks (i.e. test items or questions) and then administered these to their students. During the assessment, teachers encouraged students to use appropriate approaches to perform the tasks, such as explaining and justifying the answers obtained in their solutions, as required by the MaTA Framework. Usually, this was achieved by including specific prompts in questions, such as asking students to explain their thinking or to justify their solutions.

Step 2: Scoring students' performance

By referring to the scoring criteria and scoring guide for each of the domains in Mathematical Thinking Scoring Rubric, namely conceptual knowledge, procedural knowledge, thinking strategies and thinking skills, teachers were able score their students' levels of performances respectively based on their written solutions. After scoring students' written solutions, the teachers then used the Metacognition Rating Scale to rate students' levels of metacognition based on teachers' classroom observations. Similarly, the levels of performances for students' mathematical dispositions could be determined through a Mathematical Dispositions Rating Scale.

Step 3: Reporting students' mathematics performance

After scoring students' written solutions and rating their metacognition and mathematical dispositions, students' levels of performances for each domain were summarized into a standard report, entitled Teacher's Report on Student's Mathematical Thinking Performance. This report contained band scores and comments from the teacher for each domain of mathematical thinking. This report could then be given to students as feedback on each of the three areas indicating the quality of their performances, based on their written solutions and on their teacher's classroom observations.

Finally, all the teachers involved were interviewed for between 30 minutes to 60 minutes. The interviews allowed the teachers to justify their views concerning the practicality of implementing the MaTA Framework in their respective schools.

Findings and discussions

The findings reveal that teachers from Australia and Malaysia responded positively toward the impact of the MaTA Framework on the teaching and learning in the classroom. One of the Australian teachers commented that the band score provided under the MaTA Framework was consistent with and added value to the score given in the school's current approach to assessment,

> From the teacher perspective, I like the fact that I can compare my mark with another scoring rubric [the MaTA Framework]. I can recognize what I was giving … we were roughly the same. I wasn't being too lenient or too harsh, which is always nice. (Teacher 1/Australia)

On top of this, the MaTA Framework was perceived as able to promote students' thinking through solving and justification of solution, as evidenced by the following teacher,

> Because it encourages students to endeavour the answer…hence this helps the students to answer mathematics problem. For higher level mathematics problems, we are not going to encounter [problem like] one plus zero equal to one, we have to explain a lot. This is what I mean, the impact is great. Because we train the kids to think, endeavour to think! (Teacher 4/Malaysia)

The guideline provided in the MaTA Framework was able to help teachers to grade the students' solution in a systematic and homogeneous way. This helps to ensure consistency of grading and fairness to the students who were being assessed, as illustrated by one of the Australian teacher:

> It's very concrete, very detailed and very specific and therefore it would allow for large amount of consistency across (students and grades)." (Teacher 2/Australia)

Even though teachers responded positively towards the MaTA Framework, there were negative views expressed as well. Eight major aspects concerning the practicality of implementing the MaTA Framework were identified. However, this paper only presents three of them: time limitations, inadequate knowledge, and students' limited English proficiency.

Time limitations

All teachers involved in the study commented that scoring and reporting of students' performances in each domain of mathematical thinking were time consuming. However, teachers from Australia seemed to look at these constraints from wider points of view, such as needed professional development and changing school assessment culture. One teacher argued that "time that is required to implement this versus the amount of benefits that would be achieved, it's not a linear relationship". When he was asked to further elaborated, he said:

> If we could perhaps get really used to it and could become more time efficient, but it requires certain amount of professional learning and change in culture across the whole school, or say among all Maths teachers. It has to be something accepted by all Maths teachers and adopted across the whole country. It would require a fair amount of professional learning to be able to use the method. (Teacher 2/Australia)

By contrast, the Malaysian teachers were inclined to focus on the drawbacks, such as heavy workload, pressure of covering the syllabus and needing to keep the students on track, as illustrated by the following teacher.

> Again…(it) is the time factor. Do we have the time to do it? Now teachers are much overloaded, they have still got to do their report books, and they have still got to do the mark sheets … Even though he is a subject teacher, but the subject teacher could be a form teacher for another class and so on and so forth. So it is extra work for the teacher and then you have to score them individually … question by question. It is time consuming. That is one of…I think the major factor…time which we don't really have.

> Everyone is trying to finish the syllabus, trying to do a lot of revision so that [students] can pass the exam with a 7 or 8 grade. (Teacher 1/Malaysia)

This finding was consistent with what was found by Ryan (2006), Linn and Miller (2005), McKee and Lucas (2005) and Parke and Lane (2007) that longer time was needed to implement performance assessment compared to other types of assessment. Nevertheless, the teachers admitted that this type of scoring and reporting could become easier once they were familiar with the terms or keywords used in the scoring and reporting. As teachers commented,

> It's not complicated, it's quite simple to use. As I said it just takes a while to fill up, once you have marked the actual assessment tasks yourself. (Teacher 1/Australia)

> Once you are familiar, it should be quite easy. (Teacher 3/Malaysia).

This result was again in line with McKee and Lucas (2005) who claimed that performance assessment tends to be more difficult at the early stage of implementation.

Inadequate knowledge and skills

The teachers were familiar with traditional forms of assessment where scoring focuses only on the final answer produced by the students. Therefore, when the teachers were asked to focus more on assessing students' thinking process, they found it more difficult to give fair and appropriate scores to students' performance based on the scoring rubric. The Australian teachers admitted that they had inadequate knowledge and skill to implement the MaTA Framework. They agreed that this inadequacy could be remedied by teacher re-training. As one of the teachers said, "Not many teachers would be confident to be able to handle this type of assessment [the MaTA Framework] accurately. It requires a different teacher training approach for the present school teachers" (Teacher 3/Australia). Malaysian teachers also admitted this inadequacy, but they preferred self-directed learning to introduce themselves to this type assessment. They asked whether there was module provided to guide them through this assessment:

> Normally when we try to create something that is very new, …the Malaysian way is they want something to look at first, to go through first, they want something as examples, as a guideline or reference for them. And from there…I cannot say they want to copy or something, but normally they will follow exactly from there. (Teacher 1/Malaysia)

These responses reflected that the teacher professional development in Australia tends to be more structured, with any implementation of new education policies requiring systematic dissemination and training. However, the situation in Malaysia is quite different where only few experienced teachers get selected for such training; and they are then expected to give "in house" training to other teachers from different schools at a later date. Very often, important information dissipates during the sharing process. Worse still, some teachers are not asked to attend any training due to tight budgets. As a result, many teachers have to learn the new education policies for themselves based on guidelines or modules provided by the Malaysian Ministry of Education.

Limited English proficiency by students

Even though Australia is an English speaking country, the teachers involved gave a surprising remark by saying that limited English proficiency was one of the major drawbacks that caused some students to perform poorly under the MaTA framework. One of the teachers gave the following example which was happening in his class:

> We have a student this year in year 12, he is of Chinese background, his English language is very poor ... he is so frightened of choosing subjects that will affect his marks based on his poor English. He actually chooses to do Further Maths, Maths Methods, and Specialist Maths in Year 12. (Teacher 1/Australia)

Because of students' limited English proficiency, some mathematics teachers focused on teaching mathematical skills, with less emphasis on solving mathematical problems, as commented by one of the teachers:

> Yeah, that's [English proficiency] is critical, one of the reasons why we do mostly skill based teaching in this school is because we've got an extremely high (number of) non-English speaking background students, 80% of them. (Teacher 3/Australia)

Hence, students who were poor in English were not keen on being assessed using the MaTA Framework. This was similar to the Malaysian context where students preferred standardized-testing, such as tests with multiple-choice questions (Hwa, 2010). One of the Malaysian teachers said that,

> They have the idea but they don't know how to explain it, how to write their idea ... because some of our students' English is not good" (Teacher 1/Malaysia).

As a result, students from non-English speaking backgrounds were expected to perform poorly in MaTA Framework. Students who were struggling in mastering English were rarely comfortable with being asked to write justifications of their solutions.

Conclusion

We found that the MaTA Framework provides sufficient information in guiding secondary school mathematics teachers from Australia and Malaysia to assess students' mathematical thinking. The guidelines proposed were effective in enabling the mathematics teachers involved to implement the MaTA Framework in their schools. However, the data also reveal that the MaTA Framework was seen by teachers as lacking in simplicity and ease of use compared to traditional forms of testing. Much more time was needed to prepare performance assessment by using the MaTA Framework. Besides being time consuming, factors such as inadequate knowledge and skills by teachers, and limited English proficiency among students were seen also to affect the practicality of the MaTA Framework in the school context.

These views were expressed in somewhat different ways by Australian and Malaysian teachers towards implementing the MaTA Framework. Hence, in order to give greater attention to the assessment of mathematical thinking, increasing teachers' exposure to the key ideas of a framework such as MaTA, and consequent teacher training through workshops or seminars are necessary. These should increase the quality of the performance assessment, but also foster greater consistency in scoring and reporting of students' mathematical thinking.

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LANGUAGE-RELATED MISCONCEPTIONS IN THE STUDY OF LIMITS

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This paper reports on the language-related misconceptions of a group of post-secondary students when working on problems involving the limit of a real-valued function at a single point. In this qualitative study, 50 post-secondary students took a test and participated in a survey, from which 10 were interviewed after the test. The data revealed several misconceptions held by the post-secondary students about the limit concept that were related to the issue of language. Such language-related misconceptions resulted from incorrect *internal representations* and the *inability to reify* the limit.

Analysis is the most important area in mathematics, where students have to learn concepts that are linked to the notion of limit of a function at a point. From the understanding of the limit concept, other fundamental concepts like continuity, differentiability and integrability are all established. Hence, the limit concept underscores almost every branch of mathematical analysis and can be studied in various settings. As Huillet (2005) stated:

> The limit concept can be studied in many different settings: geometrical (area and volumes), numerical (sequences, decimals and real numbers, series), cinematic (instantaneous velocity and acceleration), functional (maximum and minimum problems), graphical (tangent line, asymptotes, sketching the graph of a function), formal ($\varepsilon - \delta$ definition), topological (topological definition, concept of neighbourhood), linguistic (link between natural and symbolic languages of limits), algebraic (limits calculations). Each of these settings underscores a specific feature of the limit concept. (p. 172)

Historically, the idea of limits resonated since the Greek era, around 600 BC. The Greeks however focused on results and the idea of limits was used only intuitively. In the 17th Century calculus, the notion of limits came to the fore through the works of Newton and Leibniz. However, the idea of calculus rested on weak foundations. It was only about 150 years later that the rigorous definition of the limit was constructed through the works of Cauchy and Weierstrass.

The term *limit* used in this paper means the limit of a function at a point, unless otherwise stated. In other words, $\lim_{x \to a} f(x) = L$ means, for every ($\forall$) $\varepsilon > 0$, there exists ($\exists$) a $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$ and $x \in$ domain of f. It is to be noted that the limit of a function is defined to exist, if the left hand and right hand limits both exist and are equal. As can be seen from the definition above, grasping the idea of limit requires students to decode its meaning from a relatively complex symbolic statement.

Educators and students face the hard transition that is necessary to leap from the routine to the non-routine aspects of mathematics where limits are first encountered. This transition is hard because the limit concept represents a concept that requires advanced mathematical thinking processes (see Dreyfus, 1991, pp. 35–36). Such advanced thinking processes calls for the assimilation of three key characteristics. These are, *generalization* (to derive from particulars), *synthesizing* (process of merging into a single picture), and *abstracting* (transition from the concrete to the abstract). Students require new methods to assist them in making the transition from the secondary to post-secondary level mathematics.

Given the complex nature of the definition of the limit of a function at a point, it is not surprising that some students develop misconceptions about limits. Thus, it is fundamental for educators to investigate what the misconceptions are and why the misconceptions occur. A study on the misconceptions arising from limits may provide reflection into how curriculum should be designed and how teaching of limits should be carried out.

This paper will address two research questions.

1. What kinds of language-related misconceptions are there when students study limits?
2. Why do such misconceptions occur?

The term *misconception* as used in this study refers to the reason which constitutes the basis over which an error is made, with reference to the individual student's perspective. "The misconception which forms the basis of the observed error may lie in the child's conceptual knowledge or knowledge store or in the strategies which are developed in order to handle the problems under study" (Booth, 1983, p. 32). Another definition of misconception as noted by Ferrini-Mundy and Lauten (1993), describes misconceptions as *non-traditional student views* (see p. 156).

Literature review

The language issue in the study of limits has been investigated previously. Monaghan (1991) stated that misconceptions in learning limits arise as a result of the language used in limit terminology. He reported ambiguities that arose primarily from four phrases, namely: *approaches, tends to, converges,* and *limits.* These terms were cause for ambiguity because students formed their own interpretations from the four terms mentioned. Students used speed limit to rephrase limit when asked to replace the word limit within another context. The word *limit* was seen as a boundary which could not be exceeded. The word *converges* was construed in the context of lines converging to lines, but not to numbers. The terms *approaches* and *tends to* seemed to give the students the impression that the limit is a dynamic concept. Accordingly, it can be said that all these language misconceptions stemmed from the everyday meanings of the word being used in limits. Monaghan added that: "Students should be led to explore and discuss their conceptions and to realize how everyday meanings of mathematical phrases can direct them into fallacious interpretation" (p. 24).

Monaghan (1991) also studied the consequences the use of language has on teaching mathematics. When English phrases like *tends to* and *approaches* are used in mathematics, the terms have a different meaning in mathematics. The mathematical language is a precise one in contrast to the spoken English language, and in addition, students seem to attach their own meanings to things. In fact, according to Quine (1968) understanding things involves interpreting meanings. Such interpretation also entails translation. Quine argued that this translation is relative to each individual and thus indeterminate. Hence, students who attach meanings to concepts may simply be interpreting what they translate the meaning to be. Consequently, meanings are subject to the students' ontological relativity.

Davis and Vinner (1986) argued that misconceptions will continue to proliferate as long as the word *limit* is used too early in the calculus syllabus. They proposed that using the word *associated number* - a neutral phrase in place of limit, at least at the onset of a calculus course might help. For example, for the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ instead of asking for the limit of the sequence, one could ask for its *associated number*. Swenton (2006) argued that many difficulties that occur in the study of limits are caused by inadequate mathematical language: "we argue that a large number of the difficulties, both specific and general, that occur in the instruction of limits stem from the lack of a mathematical *language* that properly addresses the fundamental nature of limits conceptually, computationally and logically" (p. 643). Swenton proposed using *near-numbers* as a language for limits (see p. 644 for details).

On the other hand, Schwarzenberger and Tall (1978) added that the technical language used is a matter that may create conflicts in learning. Schwarzenberger and Tall divided conflicts into conscious and subconscious ones. These authors also queried that when we say 'make the *n*th partial sums as close to *s* (its limit) as we please, by making *n* sufficiently large', what precisely is the meaning of that? Ambiguity arises as to 'how large is large' and 'how close is close'. A cognitive subconscious conflict arises here as the term *close* means near but not coincident. Hence, a misinterpretation may occur; namely, the *n*th partial sums can be very near to *s*, but never equal to it. Skemp (1986) added that, "we should never use convenient but loose phrases such as 'as small as we like' " (p. 67).

Epp (1999) contributed to the issue of language in mathematics, by bringing in the aspect of quantification. Some students, she argued, get confused between the everyday usage of certain words and their technical meanings in mathematics. Students are not able to spontaneously read into the truth or falsity of universal and existential statements. Evidence showed that the conditional statement 'If A, then B' is misconstrued as being equivalent to the converse; 'If B, then A'. Epp highlighted that textbooks have a part to play in the formation of this errors. Williams and Irving (2002) discussed about the discourse people subscribe to at different times. They contended that different language registers allow for the construction of different universes of meaning (see p. 209). Hence, English and mathematics domains are mutually exclusive when it comes to certain terms involving limits where meanings have to be defined precisely.

The disadvantages of relating mathematical language to the English spoken language have been highlighted. However, the aforementioned relation can be advantageous as well, when first presenting the limit concept. Gass (2006) used the word *approaches* on purpose instead of *converges* to establish a familiarity between the mathematical and English languages. “I prefer ‘approaches’ rather than ‘converges’, because it maintains a familiar-setting language while we take on the challenge of limit definitions and limit proofs” (p. 148).

Methodology

This paper reports on data collected from a larger qualitative study involving 50 post-secondary students about the students’ misconceptions in the study of limits. These students had taken an introductory course in calculus that was taught during their first year in a private university in Singapore. A test comprising of 3 items each with several parts on the limits of a function at a single point was administered to the students. The functions included covered continuous, piece-wise continuous and discontinuous functions. The functions were also represented differently; for example, some were represented by formula while others by tables or graphs. Rational, modulus and floor functions were among the functions included on the test.

The data responses were collated and coded according to the type of error made. Subsequently an interview schedule (comprising of 4 items) was designed based on the errors observed on the test and 10 of the students were interviewed. The 10 students interviewed were chosen based on the different kinds of errors they made. In particular, only the students who made language-related errors will be the focus in this paper. The emphasis accordingly lies on the reasons why students faced language difficulties.

A survey was also carried out based on the interview data to probe further into the validity of some of the language misconceptions.

Results and discussion

The data collected from the participants included: test scores, test scripts, a survey and interviews. In discussing the two research questions that follow, this paper will focus primarily on the results from the test scripts and interview data. Due to space consraints, only a few cases are highlighted here.

What kinds of language misconceptions are there when students study limits?

The language error appeared on a number of items of the main test. In particular, some of the respondents used terms such as *indeterminate, non-applicable, indefinite, undefined* and *does not exist*. In this study, some responses included regarding $\frac{0}{0}$ as a limit that *does not exist, is undefined, is indeterminate,* or *is non-applicable*. In addition, some students also wrote *indefined*. The response *indeterminate* was partially correct, however it had to be simplified into a limit that could be determined; often, this simplification was not performed and *indeterminate* was left as the final answer. Considering some of the students in the interview sample, Brooke and Ivy are low-ability students while Joseph a middle-ability one. Brooke during the interview described the limit as infinite. “Ah, infinite, does not exist at all”. Ivy thought that infinity was similar to indeterminate. Joseph on the other hand regarded the term *does not exist* to be similar to *indeterminate* and *undefined*. It is evident that the students in this study possess varying interpretations of the words *infinite, does not exist, indeterminate, indefined,* and *non-applicable*. In the literature, Monaghan (1991) uncovered that the terms: *approaches, tends to, limit,* and *converges* were all taken to mean different things because of the physical connotation each term entailed. For example, limit meant a boundary that could not be exceeded. Thus, the language issue is a factor that led to incorrect limit values. The students are not aware of when to use the words *infinite, does not exist,* and *indeterminate*. Brooke for example thought that *infinite* meant the same thing as *does not exist*. Responses from some students showed that the term *approaches* was misconstrued as synonymous in meaning to *approaching from the left hand side*. The following interview extract shows that some students assumed that it was acceptable to attach their own meaning to certain terms (in this case approaches means approaches from the left). During the interview, a particular student Mary was asked for the limit as x approaches 5 for a function defined by: $f(x)=1$ for $x>5$ and $f(x)=-1$ for $x<5$. Mary responded as follows.

I:	What about for this graph, the limit as x goes to 5?
Mary:	As x goes to 5, it will, as x goes to 5, negative 1.
I:	Why negative 1?
Mary:	Because I view it in such a way that approaches means this way you see.
I:	Meaning comes from the left?
Mary:	Ah, normally when you say its approaching something means it's from a smaller number.

It is evident from the response of Mary above that the notion of 'smaller to bigger' factors into the meaning of the term *approaches*. Hence entailing inherently the concept of ordinal numbers which is then associated with the limit.

Why do such misconceptions occur?

The language of limits needs to be clarified. It is possible that confusion is caused because, in real numbers sometimes people write $\frac{a}{0}=$undefined and sometimes $\frac{a}{0}=\infty$ for $a\neq 0$.When this real number theoretical fact is imported into the limits domain, the misconception that '$\infty=$undefined' is formed. Regarding limits, '∞' represents unboundedness and *does not exist* means that the left and right hand limits are different. David, a high-ability student responded on item 1(v) of the main test (see Figure 1) that the limit was 0 and this was similar to saying that the limit does not exist. David's statement clearly shows that, the notion of a limit that does not exist is not well understood.

Item 1(v): Find: $\lim_{x\to 0}\lfloor x\rfloor$ v) 0 ∴ it does not exist

Figure 1. Response from David on item 1(v).

Schwarzenberger and Tall (1978) argued that technical language used can create conflicts in learning and some of these conflicts are subconscious ones. Hence, this sample demonstrated learning conflicts that stem from language in real numbers being regarded as similar to language in limits. While Monaghan (1991) showed that terms like *tends to* and *converges* are taken to have different meanings, the data in this study show that the value zero is assumed to be synonymous to 'the limit does not exist'.

On the other hand, Goldin (1998) claimed that certain words may bring out images in the minds of the students. "For instance, words and phrases not only have grammatical and syntactic structure; they evoke non-verbal images" (p. 144). Some students in this study have linked certain words to represent certain objects; such as linking the phrase *undefined* to the indeterminate form $\frac{0}{0}$. The phrase undefined has possibly evoked the image of representations of fractions where the denominator is zero.

The difficulty encountered with terms such as *indeterminate, non-applicable*, *indefinite*, *undefined,* and *does not exist* can also be analysed through a learning metaphor highlighted by Sfard (1998) which is explained as follows. Sfard claimed that the Acquisition Metaphor (AM) involves acquiring knowledge based solely from the individual standpoint. The focus of AM with respect to learning has the individual as the emphasis. Accordingly, the respective students could have acquired knowledge of the terms such as *indeterminate*, by acquiring the knowledge on their own. The misconception that *indeterminate* has the same meaning as *undefined*, is a misconception that might have been constructed by the students themselves. The acquisition of knowledge as analysed through the AM, takes place only between the student and the terms. There is no facilitation to correct or check student understanding. Hence in the absence of teacher intervention, it is likely that language-related misconceptions may manifest.

On the other hand, the students who did not have language-related misconceptions with terms such as *indeterminate*, could have learnt or acquired knowledge of terms by reaffirming their acquisition with some external authority. The acquisition where learning is checked can be analysed through another learning metaphor put forward by Sfard (1998); namely, the Participation Metaphor (PM). The PM explains that knowledge acquisition occurs through participation with the mathematical community. Students who had their learning facilitated probably did so using textbooks or inquiring with their teachers. Such participatory learning could account for students who were able to distinguish the differences between the terms *indeterminate* and so on. Essentially, two different learning approaches yielded two different learning outcomes. The AM: where learning takes place individually versus the PM: where learning involves student participation and interaction with others.

Moru (2009) argued that responses from her interviews with 15 first-year undergraduate mathematics students included thinking of $\frac{0}{0}$ as 1, 0 or ∞. The responses were accompanied by statements like 'it's undefined' or 'the limit does not exist' (see p. 441). A particular student of Moru's, S126, claimed that $\frac{0}{0}$ does not exist and $\frac{0}{0}$ is ∞ because anything divided by 0 is ∞ (see p. 441). S126 thus thought that '∞' meant the same thing as 'does not exist'. The language factor persists in the response by S126 since he or she attached similar meanings between ∞ and 'does not exist'. Moru claimed that *generalization* is the epistemological obstacle accounting for the misconception. Accordingly, it is possible that the students in the present study have made generalizations of their own when they surmised that classifying '$\frac{0}{0}$' as *undefined* or as *does not exist* means the same thing.

The data revealed that some students thought the limit value was an approximation. For example, responses showed that the limit was a range or an estimate. Using Dubinsky and McDonald's (2001) theory (see Dubinsky & McDonald, 2001, p. 277) of Action, Process, Object, and Schema (APOS), it can be argued that students who were able to find the limit value successfully made the transition from the Process stage to the Object stage. Successful transition refers to the sequence of approximations (the Process) evolving into finally the limit value (the Object). In other words, successful transition calls for *reification* to occur. The students who left the limit as an approximation regarded the Process and Object stages as similar. Thus, students who were able to go one step further to the Object stage were successful in the computation of the limit. Those students who left their answers as an approximation could have done so because of language factors. For example, terms like 'tends to' or 'approaches' as Monaghan (1991) argued were regarded as having different meanings. The limit in this sample is seen to be an approximation possibly because of words such as 'approaches', which carries the connotation of never being reached. Hence, approximations seem reasonable if students subscribe to the English meaning of 'approaches' in contrast to the mathematical meaning. Consequently if language precedes mathematics, then the limit is not *reified*.

Conclusion

The key reasons for the language misconceptions are a *lack of proficiency in the English language* and the inability to *reify* the limit as an object may have contributed to misconceptions. Other factors accounting for the misconceptions include *strong internalized behaviour* and *knowledge acquisition* through individual and non-participatory modes of learning. Reasons such as *generalization* (a particular epistemological obstacle) and following *normal behaviour* in computing limits subject to everyday meanings of the English language, may have also contributed to misconceptions. It should be noted that certain presentations (e.g., use of colloquial language) in textbooks are also responsible for the formation of misconceptions (see Kajander & Lovric, 2009, p. 175).

Accordingly, the planning of courses on limits will have to consider carefully the role of language in the study of limits. The terms highlighted above have to be explored in greater detail with students. Clear distinctions have to made about the meanings attached to terms such as *indeterminate*, *undefined*, *does not exist,* and so on. While in any research an attempt is made for the study to cover as large a scope as possible, limitations would nonetheless exist. In this investigation, the results from the data collected may not necessarily be generalized to a wider population because of the specific nature of the sample in this study. Language-related misconceptions derived from this sample may not be similar to those arising out of other samples of students with different mathematical backgrounds. Looking forward, it is therefore timely to recommend a study on language-related misconceptions on limits to be conducted on students with different samples with varying mathematical backgrounds.

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EARLY YEARS SWIMMING AS NEW SITES FOR EARLY MATHEMATICAL LEARNING

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Australia is a country that has a strong interest in water and swimming with most of the population living within one hour of a body of water. Significant numbers of parents take their under-5s to swimming lessons. Anecdotally, the swim industry believes that swimming enhances many aspects of young children's growth and learning. This paper explores the ways in which the swim environment for under-5s offers significant opportunities for learning many mathematical concepts that have transferability to school contexts. However, with the costs of swimming lessons being high, questions are posed regarding equity and the potential of swimming to further advantage the already advantaged.

In this paper we discuss the opportunities for learning mathematics that are afforded by early-years swimming lessons. The data are drawn from a very large study that focuses on the broader benefits of swimming lessons for under five year old children. While working through this large data set, it was clear that the early years swimming environment was rich with mathematical concepts and language, and also included a range of structures that helped children prepare for school in general.

Initially, the importance of prior-to-school mathematical experiences are briefly discussed, before the study is contextualised by outlining the significance of swimming in Australia. Then, after briefly outlining the details of the study, we use the data to show how in simple ways children who experience early years swimming lessons can be advantaged in their preparation for school and more specifically, the learning of mathematics.

Mathematical experiences prior to school

Research into early childhood mathematics learning has received increased attention in the last decade (Perry, Young-Loveridge, Dockett & Doig, 2008). The importance of this area was emphasised by the Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT/ECA) in their *Position Paper on Early Childhood Mathematics* (2006, p. 2):

> The Australian Association of Mathematics Teachers and Early Childhood Australia believe that all children in their early childhood years are capable of accessing powerful mathematical ideas that are both relevant to their current lives and form a critical

> foundation for their future mathematical and other learning. Children should be given the opportunity to access these ideas through high quality child-centred activities in their homes, communities, prior-to-school settings and schools.

There is a general consensus amongst researchers that, prior to school, children are able to experience and engage with meaningful mathematical ideas and concepts. Through this they are able to begin to develop a mathematical foundation, and this enhances their future learning in mathematics (Clarke, Clarke, & Cheeseman, 2006; Kilpatrick, Swafford, & Findell, 2001). Therefore, it is important that children are able to access mathematical concepts and ideas through a range of activities prior to school settings (Perry & Dockett, 2005).

Research on prior-to-school mathematical experiences has focussed on both educational settings (e.g., pre-school centres) (e.g., Fox, 2005) and informal everyday settings (e.g., Clarke & Robins, 2004). In informal settings the activities are often more playful and not explicitly focussed on the learning of mathematical ideas, but they are nevertheless replete with mathematical concepts that can be experienced and understanding developed (Clarke, et al., 2006; Goos & Jolly, 1994). For example, in observing toddlers playing outside, Lee (2010) noted activity related to a range of mathematical categories including (in descending frequency): spatial concepts, number, measurement, patterning and shape. She noted that not only were the children engaged in activities that were mathematically rich, but that they were also using and learning mathematical ideas and problem solving. In this project we are also examining activity that is not specifically focussed on mathematical development, but rather we are examining the swim context to see what opportunities it affords for the incidental and informal learning of mathematical ideas.

Swimming in Australia

In a country where 90% of the population lives near the ocean, water is a significant recreational activity and part of the Australian identity. With families having pools and living near water (both salt and fresh water), having a nation that is competent in the water is critical to the national health as well as being a strong part of the national identity. Accidental death by drowning is the leading cause of death among children between 1 and 4 years of age, and 80% of all child drownings occur with the under 5 age group (this was 229 children in 1999 and 2003) (Australian Bureau of Statistics, 2006). Similar figures exist for New Zealand (Kypri, Chalmers, Langley, & Wright, 2000) and the United Kingdom (Sibert et al., 2002). However, and sadly, for every death, there are approximately eight near deaths which often result in brain damage and other permanent impairments. Therefore, swimming and water confidence need to be a critical part of children‘s learning.

To reduce the incidence of young children accidentally drowning, many swim programs have been developed for the under five year olds. Langendorfer (1990) argued that there was considerable anecdotal evidence to suggest that such programs not only improve children‘s water safety, but they may also enhance the development of young children, although there was little empirical evidence to show any links. That said, aside from the immediate benefits of ‗drown-proofing' young children, it would seem that early swimming programs offer considerable other benefits to children.

Broader benefits of early years swimming

The instructional environment that is a feature of the pedagogy of the learn-to-swim program may offer new opportunities for young children to be exposed to a pedagogic discourse (Bernstein, 1990) that would not otherwise be available to them, particularly for those children whose home environment does not align with the school environment. By exposing young children to this pedagogic interaction, their early habitus (Bourdieu, 1981) may be shaped by this environment, thus predisposing them to engage within this form of interaction in ways that prepare them better for school. As such, this interaction may create new forms of habitus for all participants. Participating in environments where the pedagogic discourse is a feature of interactions may well help enhance the linguistic capital of participants in terms of coming to understand the pedagogic relay. As Zevenbergen (2000) has argued, being successful in school discourses is as much about ‗cracking the code' of the pedagogic discourse as it is about intellectual concepts and processes.

The instructional discourse of the learn-to-swim programs creates opportunities for young children to build their intellectual capital of which linguistic capital is a key element. For example, as Zevenbergen (2001) has noted in mathematics, students from disadvantaged families, particularly working-class families and Indigenous families are less likely to use the formal register of school. As such, many of these children come to school with an impoverished language in comparison with their middle-class peers. She has noted that many of the linguistic terms common to early mathematics (e.g., colour, shape, number) may not be a feature of some families' out-of-school language. The instructional discourse of the learn-to-swim programs fosters many of these terms—"get the red ball"—so that the children have greater opportunities to learn the school discourse. In this way, there is every chance that the students may have greater success in schools due to their exposure to the patterns of signification (concepts/language) within the learn-to-swim program that augers well with school knowledge. Through the instructional discourse, students will be exposed to rich iterations of language, thus offering potential to extend their linguistic capital. The swim environment may thus add to the students' repertoire of skills and dispositions that ultimately may position them favourably for schooling.

The study

The data reported here are drawn from a much larger project that seeks to identify the possible ways that early years swimming adds capital to young children. In this paper, we draw on the data around mathematical concepts. The swim environment, because of its three-dimensionality, offers rich language that resonates well with the language of school mathematics. We contend that exposing young children to the early years swimming environment offers new potential for learning many aspects of mathematical language and concepts in an environment that is different from most other learning contexts. The research question posed for this paper is: "In what ways, if any, does the early-years swim environment offer potential for learning mathematical language and concepts?"

Framework

This project draws heavily on the work of Bourdieu to frame the project. Often progress in young children is described within developmental frameworks where there are identified stages of development with characteristic features. Walkerdine (1984) has argued that often the theory shapes the practice, which, in turn, results in the predicted child behaviour, therefore confirming stages of development as if they are natural orders. Walkerdine has been foundational in challenging the status quo. In this project, we see the environment as a critical factor in shaping children‘s learning and dispositions. To this end, the work of Bourdieu has been useful in theorising the ways in which practice, in this case—swimming, is instrumental in shaping the learning outcomes of young children.

Sample

A total of 45 swim schools across Australia are participating in the lesson observation component of the larger study. As most lessons for under-5s are conducted in the mornings, there are usually six sessions offered each day—30 mins for each lesson between 9 am and 12 noon. During each session there can be as few as one class, and as many as eight classes, depending on the size of the swim school. This is the general pattern across the swim schools, although some swim schools offer lessons on the weekend and/or after school.

Data collection

What is reported in this paper is drawn from a much larger project where we are using multiple methods of data collection including surveys, interviews, observations, and formal testing. This paper draws from the observational data component of the project, and some supporting data is also drawn from the interview transcripts. While the larger project will have, over time, a large data base of observational videos, this paper draws from one case—a 30 minute lesson with five 3-year-old boys. The rationale for this process was based on the depth of analysis of the video to enable identification of possible forms of mathematics learning being made possible through the environment.

In focusing our observations for this aspect of the project, we were seeking to identify potential learning outcomes. As such, the focus was to identify possibilities for learning, rather than the strength or frequency of such potential learnings. This aspect will be developed later in the project once a more robust database of terms and categories has been established from the data.

Observations

At least two researchers visited each site to observe the swimming lessons. The observers videotaped aspects of the lessons and took detailed field notes, including noting examples of language used in the instructional discourse of the teachers. The observations included a classes taught by a range of teachers from relatively new to very experienced instructors, and spanned the entire range of pre-school age groups. The observed classes also included lessons where the parents/caregivers were in the water with the children and ones where they sat poolside. Immediately after each site visit, the researchers involved set aside time to review the experience and the data collected, and during these discussions key features of the data were noted and marked for future analysis. The researchers also discussed the key mathematical concepts they observed, and together they negotiated a shared account of each site visit. For this paper, aspects of pedagogical discourses have been drawn out from the observational and interview data. Indeed, we have tried to use the ‗common' and ‗everyday' aspects of the data to underpin and exemplify the points raised in the remainder of the paper.

Findings

As noted previously, in this paper we are only focussing on the data related to opportunities to learn mathematics, hence the results outlined below are limited to this aspect. The findings are discussed as they are presented in turn.

Pedagogic discourse in learning mathematics

Mathematical learning occurs via the pedagogic discourse. This discourse is one that has particular regulatory rules and protocols that are part of the discourse. Students are exposed to the discourse as they inserted into the teaching/learning environment (Zevenbergen, Mousley, & Sullivan, 2004). As the swim environment is one where there is a high emphasis on safety, teachers work in small classes and are focused on ensuring all children are engaged with the lesson.

In the following extract, the teacher is relaying a number of important aspects of the teaching/learning environment. Here he is directing the student about where to commence, but also explaining the importance of waiting until it is the student's turn to undertake an activity. By waiting, the teacher is then able to work with, and assess, the student's behaviour and undertake any necessary corrections. The importance of being able to take turns is embedded in the interactions.

Teacher: Jack[1], go back to the wall, start from the wall and wait your turn, Buddy.

This was also noted in an interview:

Teacher: You need to have eyes in the back of your head. As soon as you hear a splash or yell, your immediate reaction is to see what has happened, to see if it is one of your kids. You don't get much a chance if they fall over: you have to make sure you know where each of them are at any point in time.

In observing lessons, schools had various ways of managing safety and learning. The structure of the pools allowed children to sit on long underwater benches so that while they were in the water they were not submerged. This ledge was also useful for babies who crawled and hence would not have their faces below the water. Children were taught to line up by having illustrations, such as feet, under the water and they would follow the foot prints. Teachers would elicit instructions, such as ―kick, kick, kick, kick, stop" and then pause, waiting for the children to stop kicking their feet in the water. These simple patterns of interaction are similar to those found in school so, from a very early age, young children were being exposed to the instructional discourses that would induct them into similar practices that they would find in the school context.

Within a Bourdieuian framework what can be seen is that the swim environment is adding new forms of knowing to children that, in turn, is internalised into their habitus. These new dispositions to the learning environment will position them more favourably with teachers as they display these new learnings. The displays of learning that can be

[1] Pseudonymns are used in this paper to protect the identity of participants and sites.

observed in the children need to align with the practices valued in the field if the child is to be seen as displaying valued knowledge. Such displays, in turn, can then be exchanged for other rewards in the learning environment. In the swim environment, these are often certificates that acknowledge what has been learned and, as a consequence, progression into a different class. While the swim environment primarily focuses on skill development leading towards independent swimming, what is of value is the incidental learning that can be readily observed. For us, there were many practices that created potential for mathematics learning that would prepare students for their mathematics learning but also support them in their transition into formal schooling. It is this aspect of the swim environment that is the focus of the remainder of this paper.

Mathematical discourse

Throughout the observation there were many times when the teacher used mathematical language and ideas in their instructions. In this section we will present some of those aspects, and all the extracts are instructions given by the teacher while taking the lesson with the 3-year-old boys.

The terms used in mathematics lessons relate to aspects of the mathematics curriculum including number (one, two three), to measurement (big, fast, slow), to space in the areas of geometry (circle, straight, line, edge) and positions (up, down, underneath, side-by-side, together, backwards, edge). For example:

T: After <u>one-two-three</u>, we are going to push <u>off</u> with our hands like a rocket.

T: I need to see really <u>big</u> arms, <u>big</u> and <u>slow.</u>

T: Clinton, can you follow the <u>big</u> <u>line</u> <u>on</u> the roof" [points to the line painted on the ceiling]?

T: Okay watch me, I am going to have my hands <u>on</u> the <u>edge</u>, toes <u>on</u> the wall, head <u>backwards</u>, looking up at the <u>line</u> <u>on</u> the roof. Watching me, push <u>off</u> the wall, eyes <u>up</u>, glide, like a ferry boat [teacher demonstrates]. Alex, hair <u>in</u> the water <u>first,</u> and push and glide. Hold your body nice and <u>straight</u> and <u>long</u>.

As can be seen from the few examples above, these routine instructions—instructions that could be heard constantly throughout all the lessons, are rich with mathematical language and concepts. It is important to note that as many of these instructions were given, they were accompanied by gestures or signals to reinforce the spoken words.

Also, the use of ‗little' words (e.g., prepositions, adverbs) are important as these often have important meanings in mathematics (e.g., off, up, out). Being exposed to mathematical discourses where prepositions have been used is integral to learning, and it has been found that when students are not able to grasp the use of prepositions, there is considerable scope for error (Zevenbergen, Hyde, & Power, 2001). Below are a few examples:

T: Sitting <u>on</u> the edge of the pool, rockets <u>up</u> in the air, now on <u>one</u>, <u>two</u>, <u>three</u>, slide <u>in</u> and push <u>off</u> the wall swimming <u>out</u> to me using <u>big</u> arms.

T: Alex, put your rockets <u>up</u> like Benjamin. No, not hands <u>side-by-side</u>, hands one <u>on</u> <u>top</u> of the other. Keep your toes <u>underneath</u> the water, nice <u>long</u> legs, no spaghetti legs, nice <u>long</u> <u>straight</u> legs.

T: Climbing up <u>out</u> of the pool, using your muscles, tummies, hands and <u>one</u> knee. Standing <u>on</u> the edge now. Now, using your hand making a <u>circle</u> with your arm. Going <u>up</u> <u>past</u> your ear and <u>around</u> to your leg, <u>up</u> <u>past</u> your ear and <u>around</u> <u>down</u> to your leg. <u>Big</u> circles, I want <u>big</u> <u>straight</u> arms [teacher manipulates the child's arm to demonstrate].

The quotations above are representative of the common instructional dialogue of the swim lessons observed. Indeed, the few included above only came from one observation, and certainly there were many more that could have been included from the lesson. Thus, it can be seen that the swimming lessons are replete with mathematical terms and concepts, and the important ‗little words', and they are experienced in meaningful context. These mathematical ideas include the concepts identified by Lee (2010), and others that are particularly relevant in the water context. Furthermore, it was clear during the observations that the children understood the mathematical terms, and they demonstrated their understanding by performing the appropriate action or behaviour.

Concluding comments

What we propose is that the swim environment offers considerable potential to add new forms of capital to early learners. In this case, there is ample evidence to suggest that the swim environment can add or enhance the mathematical vocabulary of young learners and which, in turn, we trust would enhance mathematical understandings. In Bourdieu's exchange economy, the learnings that young children gain through their exposure to the pedagogic discourse and the mathematical discourse may create new forms of knowing and acting in the social and academic worlds. Their transition into formal schooling may be enhanced through the exposure to these discourses and as a consequence they are better prepared to interact in the formal discourses and discursive practices of schooling.

Of course, the children who are offered these opportunities are generally from the middle/upper class, so in further iterations of the study we hope to open the swim school experience to children who do not usually have access. Also, we will be working with some of the children involved in the swimming lessons to see if the mathematical understandings evident in the pool transfer to other contexts (i.e., outside the pool). If, as we suspect, the mathematical learning advantages are clear, then it will provide impetus to make these sorts of experiences available for all young Australians. This is particularly the case because the learning benefits add to the already significant gains for the children in terms of the obvious safety and physical development gains.

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DIGITAL GAMES: CREATING NEW OPPORTUNITIES FOR MATHEMATICS LEARNING

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Drawing on the work of James Gee in literacy, we apply his contemporary approach to not only the knowledge systems of mathematics but also the processes by which school mathematics can be learned through the digital games environment. Using a number of games, and young people working these games, we propose that there are novel ways to learn not only many of the concepts that are integral to school mathematics, but such concepts can be learned in ways that are deep. The games environment offers an engaging environment that is substantially different from that experienced in formal school settings. We suggest that many of the principles that underpin the games environment may create new opportunities for teaching and learning that will (re)engage learners and learning of school mathematics.

Most social and education commentators concur that the digital age has impacted significantly on society, education, and young people. Terms such as millenials, digital natives, Gen Y, and nexters are just a few that have been coined to try to capture the very different generations that are emerging into schools and work and whose lives have been significantly shaped by a plethora of digital technologies. While there is a significant literature on this generation in terms of the values and their impact on the workplace, less is documented in terms of learners and learning, particularly in mathematics.

In the area of literacy, Gee has been a leader in reshaping thinking about this generation in terms of their learning preferences, which have been shaped by their exposure to digital games, but also his critical analysis of the learning environment per se. In his comprehensive analysis of the learning principles that underpin the games environment, Gee suggests that many of these principles are missing from school learning environments. These design principles raise pedagogical issues for teachers and systems. Not only do these principles engage learners for extended periods of time in the games themselves, they are also creating environments that stimulate the gamer, create genuine scaffolding to enable the gamer to successfully transition through the game, and create opportunities for success. Sadly, many of these characteristics are absent from school pedagogy. Controversially, Gee proposes that for many of the digital natives entering contemporary classrooms, immersion in these games environments has created new opportunities for new forms of learning. While there may be some space for challenging this position through arguing that games are generally played by adolescents, in a study of preschool children (3-5 year olds) (Zevenbergen & Logan, 2008) found that more 95% of preschool children had access to computers, generally outside the preschool setting. These computers were providing opportunities for engaging in literacy and numeracy activities that were not possible for previous generations. For example, not having fine motor skills to create letters is no longer an obstacle in the computer environment where children were able to recognise letters and create them via the keyboard. Parents also reported that many of the children accessed games on the computer—approximately 80% of the children playing games, with some differences in genders (with more boys playing the games than girls).

Gee (2003) has proposed a number of principles that underpin the games environment that work to engage gamers in the game but through these principles they learn to play the game and experience success. Some of the more poignant principles for learning environments include:

- *Active learning principle:* All aspects of the game environment are established to encourage active and critical learning;
- *Committed learning principle:* Gamers participate in extended engagement so that they feel commitment to the game in a world that they find compelling;
- *Achievement principle:* There are intrinsic rewards for each level of the game to recognise the achievements of the gamer;
- *Amplification of input principle:* For little input, the gamer receives considerable outputs
- *"Regime of competence" principle:* The gamer operates within, but at the outer edge, of his/her level of competence so that there is both safety and challenge;
- *Multiple routes principle:* There are many ways to solve the game, each of which caters for the strengths and interests of the gamer;
- *Intuitive knowledge principle:* Knowledge built up through playing the game is valued and honoured among participants;
- *Discovery principle:* Very little overt knowledge is given, as most knowledge is acquired through experimentation and discovery;
- *Concentrated sample principle:* Early in the game, gamers experience a concentration of signs that they practise, developing proficiencies for later in the game; and
- *Transfer principle:* Support is given to practise skills and knowledge that are then transferred to later problems.

In this environment, learning is seen as hard but fair since gamers can see that while unable to pass a particular skill that they will come to learn it and it will be needed for later in the game. Learning is within the confines of the game, so failure is not public but is intrinsic to the learning process.

When it is clear that young people learn in this type of environment, and engage in it for considerable amounts of time, educators may need to pose critical questions regarding school learning environments and their relevance to digital natives.

New forms of mathematics learning

Lowrie (2005) noted in his numerous studies of young children engaging with Pokemon, that this environment created significant different opportunities for engaging with spatial concepts. Unlike traditional teaching of spatial mapping where the map is represented in a two-dimensional space and contained to the page, the Pokemon environment creates opportunities to ―go beyond the seen" to visualise the worlds beyond what is observed on the screen. He found that children created images of the world beyond what they could observe. Being able to create such images was essential to the success of the game since it enabled players to plan their pathways through the world/s being represented in the game. Furthermore, Lowrie found that in this environment, the players also had to create three-dimensional visualisations so more complex representations were essential. This visualisation process is significantly more complex and demanding than the activities that are typically part of the school mathematics experience for 8 year olds.

At the other end of schooling, Jorgensen (Zevenbergen) has studied the numeracy practices in contemporary workplaces as undertaken by a range of young workers. In these studies she reported that there are emerging new numeracies that have been shaped by radically different workplaces than in the past. These workplaces have been significantly shaped by new technologies. For example, in the Jorgensen (2010) study of retail assistants, the technologies of shopping are quite different from old practices. In contemporary shopping, the cash register not only tallies prices but also acts as a stock control device. Many older people bemoan the numeracy of young people but many of the practices of contemporary retail demand that operators undertake particular processes so that human error is eliminated. For example, in many restaurants, costs are not entered but the register has the menu items listed so that the operator enters foodstuffs rather than prices. This may seem to ‗downskill' workers but the owner-operator needs to have information to enable more profitable ways of working. By knowing sales of the period of day or week or month, owners are better able to plan their sales to create more sales of popular products and to eliminate wastage. Thus, not only has the nature of work been radically altered by technologies, but young people approach their work in different ways.

It is in this changing context, where new forms of numeracy are being created by these environments, that we sought to better understand the potential for learners of mathematics. The conservative practices that sometimes appear to be immovable in school mathematics (Gutierrez, 1998) may be enhanced by better understanding the potential of the games environment to create engaging contexts for learning many mathematical concepts and processes. To this end, we posed the question: What learnings are made possible for primary school children through the use of digital games?

Method

The method we adopted in this research is unique as it is constrained by a number of key factors. First is the researchers' limited knowledge (and skills) in the gaming environment. For researchers who are one or two generations removed from the game environment, we are not immersed in the culture of gaming and the ways in which the environment shapes actions and beliefs, so this limits our capacity to engage with the games environment in a naturalistic way. Conversely, gamers are able to intuit how games work, and are highly competent with games consoles. Second, it is not possible for the gamer to play the game while simultaneously eliciting his/her thinking. The game becomes an environment that absorbs the learning; making it difficult to engage with the complexities of the game while talking about strategies, actions, and justifications at the same time.

For the remainder of the paper we have adopted the protocol of referring to the person who provided the rich description of the game as the ‗gamer' and the person who would play the game as the ‗player'. For the purposes of this paper, we engaged a serious gamer to work through two games (one of which is the basis of this paper) and to provide a running commentary on how to work through the game. This included how to play the game, what to expect, where to cheat, moves to make, and so forth, so that the research team could gain an understanding of the game, its challenges, and how the game is played. In seeking a person external to the mathematics education community to provide the description of the game, we were cognisant of the need to not look for the mathematics in the game. This has been a common approach in non-school settings where mathematics teachers/researchers have worked in such sites to try to uncover the hidden mathematics. This is a strong criticism of the ethnomathematics tradition. We sought to not fall into this trap since our objective was to identify the principles for learning alongside any forms of mathematics (new or old) that may be learned through participating in the games environment. By using an external to the field, this bias was less likely to occur.

The gamer used a Nintendo DS to play the game *The Legend of Zelda: Phantom Hourglass* (Nintendo, 2008). This is a game suitable for general audiences and hence suitable for primary school children. In another part of this project where we have sought to identify the types of games played by primary-aged students in terms of most popular games and the time spent playing games, we found that this game was one of the most popular among this age group. Below is an example of the text provided by the gamer.

> When you play for the first time, tap on the *NEW GAME* file to create your own save game. You will need to keep saving your game as you progress as *The Legend of Zelda* is a rather large adventure game.
>
> You will then need to *ENTER A NAME* for your character. Once you have done this, hit the OK icon on the bottom right. It will also ask you if you are left or right handed so choose the appropriate one. (You can always come back to your Save game to continue your quest after turning the console off. Just select the save game you created when loading the game back up.)

The instructions are very clear in terms of the moves and what the player needs to do to navigate through the game. The gamer also explained the various moves that were possible and how these can be used for different purposes in the game. For example:

> *Swinging your sword:* Early in the game you will find a sword to use against monsters to help you progress through the levels. When you see a monster just simply *tap on it* and your character will attack it and deal damage to it. You can also slash the long grass that you see in the environment by *sliding your stylus in a downward direction* or in a *left to right direction* to stab things. There is also a ―spin" attack that is useful if you have multiple enemies surrounding you. That will deal damage to all of them: just simply *draw a large circle* around your character to do this.

Collectively, his descriptions provided a clear description for novice players (and researchers) on how the game was organised and the skills and tools that would be needed to progress through the game.

He also offered a rich description of how the game screen could be used to play the game. Much like Pokemon, the games screen on *The Legend of Zelda: Phantom Hourglass* provided partial representations of the worlds through which the game would be played. Also, this game had a large screen display which was the playing area as well as a smaller, full scale map so that the gamer could create a sense of where s/he was in the overall landscape.

> There are two screens at your disposal in this game. The *top screen* will mainly display your map with your character icon to tell you where you are on the map and other information like huts and caves you can venture into, while the *bottom screen* (touch screen) is your main *movement and interaction* tool. [Also] The bottom screen will show you how much health you have. There are the 3 Heart icons on the top left hand corner of the bottom screen. These will deplete if you take damage from monsters during the game.♥♥♥
>
> There is also a green Rupee icon underneath the hearts which tells you basically how much money you have. Rupees are the currency in the game to let you buy useful items from merchants that you will come across. ♦
>
> You can drag your MAP SCREEN (top screen) down to the bottom touch screen by pressing the **B BUTTON** or the **DOWN DIRECTION** on the control pad at any time to jot down some notes on your map so you can remember where important landmarks are or if you need to write puzzle solutions down.

Collectively these descriptions provided insights into the multiple sources of data that will be processed by the gamer as s/he moves through the game levels. Many variables must be considered simultaneously. The gamer provided 23 pages of instructions and explanations to explain to the research team how the game was played. The background information finished at page 6 and herein the game commenced.

Gee's learning principles

It is not our intention to analyse the game in terms of Gee‘s learning principles. However, we do want to acknowledge their presence in the game. As Lowrie‘s (2005) work illustrated, the game environment offers a rich site for developing spatial skills. Similarly, in *The Legend of Zelda: Phantom Hourglass* the player must navigate through a range of landscapes, creating a mental map of where s/he has been, and as s/he moves through the game, create richer memories of these worlds in order to better work through them. There are not explicit instructions given to the gamer as to how to navigate through these worlds but, as he illustrates explicitly, this is common in the games environment (see bolded text below). But as the player moves back and forth through sites/locations, a strong memory of the path is being created. This process allows the player to develop a familiarity with the spatial layout of the game. In so doing, it creates a visual memory of the spatial representation of the landscape.

> You should now leave the hut and try to walk up the pathway to the left of the hut and you should see a sign that tells you that it‘s dangerous to walk any further (by tapping on the sign) because of the monsters ahead. When you have tried heading NORTH Ceila

> tells you that it‘s too dangerous so you must go back to speak with Oshus/Grandpa in his hut.
>
> Once that is done you may now leave the hut and venture into the cave to the right hand side of the hut. There will be a barrel blocking the cave so just tap on it to pick it up and tap away from your character to throw it. The path is clear now.
>
> ***NOTE: This type of information gathering is widely used throughout the game so some backtracking will be required!**

Mathematics and the games environment

Number

An example of this potential methodological flaw would be possible in the following extract where the player must know the number of trees in order to open a cave. This is a very low level of mathematics that we do not see as enriching mathematical understandings of young people. Hence, we do not see this aspect of the games environment as creating new possibilities.

> When you have walked up to the closed door Ceila will say that you must write down how many palm trees there are on the beach which is SOUTH of the cave when you exit. Simply walk around the beach area and count the PALM TREES only. If counted correctly you should have counted **7** of them.
>
> You may now head back to the cave and tap on the sign next to the door to write down the number of palm trees you counted to unlock the door. Just write the number **7** on the screen and the door should magically open! Proceed up into the middle of the room and tap on the treasure chest in the middle of the room and you will be given **OSHUS`S SWORD**. This will help you defend yourself against any enemies you come across. You may now leave the cave and proceed up NORTH.

In terms of identifying new possibilities for learning mathematics and new forms of mathematical thinking, the use of the gamer helped to bridge the chasm between school mathematics and new mathematics.

Mapping

In line with Lowrie‘s work, we see that in this game also, players need to develop good mental maps of the worlds through which need to move. Moving back and forth through worlds enables this sense of space to be developed. As the gamer noted in the earlier sections of the paper, this is a strategy that is commonly used. It also helps us to explain the learning principles upon which the game is based. The novelty of repetition where the player must move back and forth between sites helps to build a robust mental image. However, the repetition is quite different from the repetition found in many of the drill exercises used in mathematics teaching. In this environment, the drill is masked by a motive to learn a new skill or information that will be used for another level or situation.

> Now you must exit the dungeon and make your way back to the port in town. Head towards the ship and some more story/explanation will follow. You must then solve a little puzzle when the **Sea Chart** is on the bottom screen. You will need to rub your stylus on the bottom right hand side island until you can see a picture of a symbol. Oshus will tell you that this is the *ISLE OF EMBER*. That will be your first destination when you set sail.

> To set sail somewhere you must start *drawing a line* from where the feather quill is on your map to another point on the map. Your first destination is the isle of ember so draw a line from the feather to the ANCHOR symbol on the isle of ember. **Once you do this tap on GO!**

This moving around, travelling back over previous paths is a strong feature of the game and highlights how the player comes to create mental images. In the extract below, the player has reached the end point for a level and now must return to the start point to enable him/her to move into the next level.

> Isle of Ember - Mercay Island.
> Return to Astrid's house. Linebeck will be there, and Astrid will give you a Power Gem. After that, return to the boat to leave the island. Return to Mercay Island. Once back on Mercay, make your way back to the Temple of the Ocean King.

Using previous learnings from other levels is common in games design as it is essential to the success of the game. In ―The Legend of Zelda" this principle is at work. Players learn skills in a previous level that enables them to pass through a different level. In the following extract, this principle can be observed. However, what can be seen in the extract is the complex spatial knowledge that is needed to move through this one temple.

> This will—as you can guess from the last temple—extinguish the flames in the southwest corner. Get over there and run through to the northwest corner, then hang right and **step on the small floor switch** near the three-block barrier. This lowers the blue door in the corner, so go back over there and grab the Small Key. With the key, run over to the northeast corner and use it on the locked door, then go downstairs.

The directions that the gamer has provided highlight the possibilities for developing rich spatial understandings that extend beyond what is generally possible in the mathematics classrooms. Moreover, the processes through which these understandings are made possible are quite different from those used in most mathematics teaching situations.

Conclusion

What we have sought to illustrate in this paper is twofold. First, we draw on the pedagogical principles identified by Gee at the start of this paper. While we can observe that there is considerable repetition in the games environment, it is often with purpose and function. The player must be able to build skills and understandings that will enable him/her to move through the current and subsequent levels. The scaffolding provided by this repetition enables the development of new learnings that will empower the player to move forward through the game. The repetitions are neither boring nor lacking in purpose.

What is more important, however, is the acknowledgement that the games environment not only scaffolds learning, but in so doing, enables new understandings to be developed. In this game, the movement through various worlds requires the player to build complex spatial maps that go beyond what is seen. Learners must create images that extend beyond the immediate screen, and often as three-dimensional representations. This is far more complex than the maps currently used in curriculum offerings for age groups. This challenges orthodoxies that shape curriculum offerings for young learners. How these complex understandings are made possible through the games environment represents a considerable challenge to mathematics education—not only in the terms of what is learnt, but also how it is learnt.

As the gamer has illustrated in his description of the game, there is a complexity in developing the mental image of the spatial world/s represented in the games environment. As learners engage with the game, there is considerable scope for them to learn.

Acknowledgement

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LEARNING EXPERIENCES OF SINGAPORE'S LOW ATTAINERS IN PRIMARY MATHEMATICS

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This paper explores the learning experiences of 346 year four low attainers in mathematics from Singapore. The pupils were interviewed about their learning experiences related to mathematics lessons in school. An innovative method, using pictures as stimulus, was adopted to engage pupils to talk about their lessons. From the interview data it is apparent that there was a mismatch between how pupils were taught and preferred to be taught. Almost all the pupils experienced teacher-led whole class instruction during their mathematics lessons. A study of three teacher-led whole class instruction lessons showed that these lessons were not unique but had some commonalities. The mathematical tasks used during instruction were routine and repetitive. Teachers also did not stimulate the development of pupils‘ metacognition.

Background

In this paper the term ―low attainers" refers to pupils who attain very much less in mathematics when compared to their contemporaries (Haylock, 1991) in the mainstream primary school. The use of this term does not make any judgment about the reasons for low attainment in mathematics. Low attainment in mathematics has been found to be a result of not a single factor but of the interplay of subject related difficulties, specific intellectual/behavioural characteristics of the pupils and pedagogical shortcomings (Haylock, 1991). The research reported in this paper is part of a larger research study that explores the factors related to low attainment of primary pupils in Singapore (Kaur & Sudarshan, 2010).

The research question

The research question that is addressed in this paper is one of the larger study‘s six research questions. The question is ―What are the learning experiences of low attaining mathematics pupils in school?"

Review of literature

The review of literature in this paper is specific to the learning experiences of low attaining mathematics pupils. According to Reusser (2000) there is sufficient evidence in research on mathematics learning and teaching that most observed failures and substandard performances are due to deficiencies in the teaching and learning environments rather than genetic factors. In his review of theoretical and empirical research from a cognitive instructional perspective, Reusser contends that an effective teaching environment positively impacts students' mathematics attainment levels regardless of grade levels or mathematical ability. His perspective of an ‗effective teaching environment' for low attainers centres around adaptivity and empathy in teaching. He recommends the use of micro adaptation—moment-to-moment decisions of teachers aimed at tailoring instruction to the needs of different learners.

Direct structured instruction has also been found to be effective with students having difficulties in mathematics (Harris, Miller, & Mercer, 1995; Jitendra, & Hoff, 1996; Van Luit, 1994; Wilson, Majsterek, & Simmons, 1996). Direct instruction is systematic explicit instruction which is teacher-led (Jones, Wilson, & Bhojwani, 1997) and generally follows a fixed pattern of actions (Archer & Isaacson, 1989). Kroesbergen and Van Luit (2002) detail a typical direct instruction lesson as having three phases. In the opening phase the students' attention is gained, previous lessons are reviewed and the goals of the lesson are stated. In the main part of the lesson the teacher demonstrates how a particular task can be solved, following which the students and teacher work together on a few more similar tasks. When the students appear to have sufficient understanding of the tasks they are given new tasks to practise independently. The teacher monitors the students during such practice and provides feedback on completed tasks.

Cardelle-Elawar (1995) found that low achieving students showed metacognitive potential when stimulated by explicit individualized instruction and recommends that: i) special consideration should be given to each individual student's uniqueness, strengths and weaknesses; ii) these students need a supportive atmosphere in which errors and mistakes are considered a source of learning and not an occasion for punishment; iii) these students need more structure in the classroom; and iv) these students warrant a great deal of interaction between teacher and student. According to Watson (2001), these students are also able to make shifts in their thinking from the superficial features of mathematical tasks to forms of mathematical thought. She cites a specific example where students were able to shift from seeing fractions as congruent shapes to seeing fractions as quantities using the idea of areas to make the link. Watson asserts that low attainment is not the result of an inability to think but the lack of structured work that promotes higher order thinking among low attainers. Zohar and Dori (2003) also found that low achieving students can gain from teaching and learning processes that are designed to foster higher order thinking skills. They suggest that teachers should encourage students of all levels to engage in tasks that involve higher order thinking skills.

Methodology

Subjects

The subjects of the study are 346 year four pupils from nine primary schools in Singapore who qualified for participation. They were nominated by their respective schools, had parental consent for participation and took the mathematics benchmark tests of the study.

Instruments

Specific to the research question addressed in this paper, only instruments used to collect data from pupils and teachers on learning experiences of the pupils will be presented.

Pupils' interviews

From our interactions with the pupils and their teachers in the project we found that pupils lacked the language to talk about their learning experiences and about the actions of their teachers during mathematics lessons. So, to facilitate pupils' talk about their learning experiences, we adopted an idea from child psychiatry about using pictures as stimuli for interviews (Angold, 1976). We also found the pupils rather reserved in their oral communication with us when they were in a one-to-one interview setting. This observation led us to adopt a group interview format for our study. Pupils were interviewed in groups of four to five persons. During the interviews the researchers of the study used four pictures of mathematics lessons to stimulate talk amongst the pupils about how their mathematics teachers usually taught them in class and what their preferences for learning mathematics were. The four pictures shown in Figure 1 were used for the interviews.

Picture A: Teacher-led whole class instruction	Picture B: Group work (pupils working on tasks without manipulatives)	Picture C: Individual working on task with manipulatives	Picture D: Group work (pupils working on tasks with manipulatives)

Figure 1: Pictures of mathematics lessons.

The prompts used for the interviews belonged to three categories, mathematics lessons, homework, and self. In this paper we only focus on the prompts related to mathematics lessons. The four pictures A, B, C and D were put on the table around which the pupils and interviewer sat for the interview session. The following prompts were used to engage pupils in talking about their mathematics lessons at school.

- Prompt 1: Which picture shows the way your mathematics teacher usually teaches you in class?
- Prompt 2: Which class do you want to be in? Why?
- Prompt 3: Which class don't you want to be in? Why?

Lesson observations

Nine schools participated in the project. The lesson of one teacher per school who welcomed the researchers to his/her class was observed. In one of the schools, two teachers volunteered and therefore a total of ten lessons were observed. The teachers taught mathematics to pupils participating in the project. Our lesson observations were guided by the following main analytical questions that resulted from our theoretical framework. The questions are:

AQ1. What was the instructional sequence of the lesson like?
AQ2. Did the teacher tailor instruction to meet the needs of different learners?
AQ3. What were the characteristics of mathematical tasks used in the lesson?
AQ4. Was the classroom learning environment a supportive one?
If so, how did the teacher nurture such an environment?

Data and findings

In this section we first present the data and findings of the interviews according to the three interview prompts chronologically. Next we present our analysis of three of the ten lessons that were observed as part of the project. We have selected these lessons as they typify teacher-led whole-class instruction which almost all pupils experienced during mathematics lessons.

Interview data and findings

Ninety-eight percent of the pupils interviewed said that their teachers always used teacher-led whole class instruction during mathematics lessons. Figure 2a shows the preference of the pupils with regard to the type of instruction they desired. The highest percentage (40%) of the pupils said that they preferred to work in groups on mathematical tasks with manipulatives during mathematics lessons.

Figure 2b shows the preference of the pupils with regards to the type of instruction they disliked. More than half of the pupils (62%) preferred not to be in a class where they would have to work by themselves on a task with manipulatives.

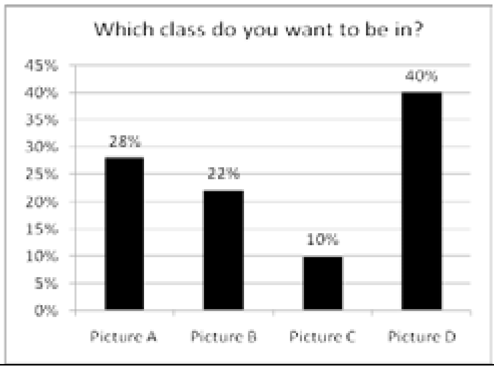

Figure 2a. Responses to prompt 2

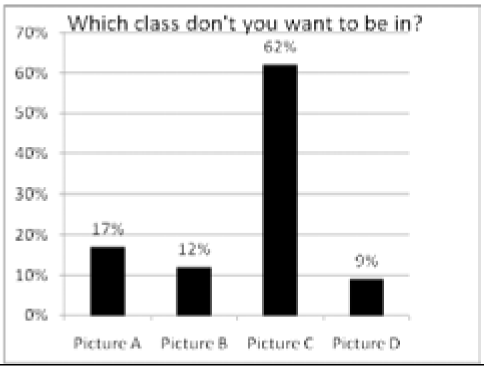

Figure 2b. Responses to prompt 3

From Table 1 below, it is apparent that pupils found: a) interacting with peers a fun and good way to learn; and b) the “hands-on” experience gratifying and meaningful in learning. It is also apparent from Table 2 that pupils: a) lacked the confidence to attempt tasks without the support of peers and teachers; and b) felt bored and lonely working by themselves.

Table 1. Sample of pupils"responses to the „Why" of "Which class do you want to be in?"

Sample responses for Picture D - Group work (pupils working on tasks with manipulatives):
S038: ―Discuss with a group, tell each other the answer and find out which is the correct answer".
S057: ―When we do the activity, we can feel that maths is fun".
S105: ―More fun to work with a group and can discuss with friends if I am not sure".
S234: ―I feel happy when I can do things to help me understand and improve".
S537: ―We can see how things happen and touch things".

Table 2. Sample of pupils"responses to the „Why" of "Which class don't you want to be in?"

Sample responses for preference Picture C – Individual working on task with manipulatives:
S074: ―Work alone may not know how to do and then do the wrong thing".
S084: ―It is boring and lonely when doing by ourselves".
S197: ―If alone cannot study well, cannot ask anybody about the activity".
S453: ―Scared if I don't understand what teacher wants".
S528: ―Don't want to do things alone. With other people we can do better".

Analysis of lessons observed

Each lesson was observed by at least two researchers. Following the observation, a reflection of the lesson was guided by the analytical questions that provided the theoretical lens for analysis. The main aspects of the lessons on which the researchers concurred are presented in Table 3.

Discussion and concluding remarks

From the interview data of the pupils in the study, it is apparent that 98% of pupils are taught mathematics in classrooms where teacher-led whole class instruction is the norm. But, teacher-led whole class instruction was the preference of only 28% of the pupils in the study. Forty percent of the pupils preferred to work in groups on mathematical tasks with the help of manipulatives. They found interacting with peers a fun and good way to learn, and the ―hands-on" experience gratifying and meaningful in learning. From the above findings the apparent mismatch between how teachers teach these pupils and how these pupils would like to be taught in mathematics lessons may partially explain the low attainment in mathematics of these pupils. This finding reinforces that of Reusser (2000) that most observed failures and substandard performances are due to deficiencies in the teaching and learning environments rather than genetic factors.

The three lessons observed depicted teacher-led whole class instruction. All had three phases but there was variation between corresponding phases across the lessons (see Table 3). Although all teachers stated the goal of their lesson, only Teacher A reviewed the last lesson before embarking on the present one. In the main phase, although all the teachers demonstrated how to solve particular tasks, only Teacher A went on to do more tasks similar to the particular ones with inputs from pupils, before setting them new tasks to work on individually during the consolidation phase.

Table 3. Analysis of the three teacher-led whole class instruction lessons.

Analytical Question	Teacher A (School 3) Topic: Time [duration]	Teacher B (School 7) Topic: Symmetry	Teacher C (School 5) Topic: Tessellation
AQ1	i) Introductory phase: review of past lesson and use of real life contexts to arouse pupils' interest ii) Main phase: development of concept and application of knowledge (adequate examples worked on the board with inputs from pupils) iii) Consolidation phase: pupils worked individually on new tasks, while teacher provided between desk instruction and feedback on completed tasks.	i) Introductory phase: mentioned that the past lesson completed the topic time. Stated the goal of the present lesson. ii) Main phase: demonstration of concept using manipulatives, video clips and cut-outs of alphabets, followed by ―hands-on" work by pupils in groups—identifying the lines of symmetry of the alphabet. iii) Consolidation phase: pupils worked individually on similar tasks without assistance from teacher or peers.	i) Introductory phase: mentioned that lesson was on a new topic—tessellations. ii) Main phase: demonstration of the concept of tessellation via examples and non-examples. Pupils worked in groups with unit shapes to make tessellated patterns. Pupils showed the class their patterns and teacher encouraged peer evaluation. iii) Consolidation phase: pupils worked in pairs and again were given unit shapes to make tessellated patterns.
AQ2	No apparent attempt	No apparent attempt	No apparent attempt
AQ3	Routine and repetitive.	Routine and repetitive.	Routine and repetitive.
AQ4	Supportive. Encouraged pupils to ask questions, welcomed mistakes and praised pupils for participation.	Supportive. Encouraged pupils to talk to peers about their work, welcomed mistakes and praised pupils for completing their work on time.	Supportive. Encouraged pupils to comment on their peers answers and praised pupils for their attempts.

While pupils were working on the new tasks, Teacher A provided between-desk instruction and feedback on completed tasks. However, for Teachers B and C during the main phase, pupils did tasks similar to those the teachers had demonstrated but in groups. Following this Teacher B assigned pupils individual work on similar tasks devoid of any assistance from peers or teacher, while Teacher C got pupils to do pair work on tasks similar to those they did during group work. From the instructional sequences of the three teachers, it is apparent that the lesson of Teacher A is similar to that advocated by Kroesbergen and Van Luit (2002). Hence it may be said that although almost all the pupils were experiencing teacher-led whole class instruction during their lessons, the variation between the types of such instruction may not be addressing the needs of the low attainers. Furthermore, teachers made no attempt to tailor their instruction to meet the needs of different learners.

The tasks used by the teachers were routine and repetitive, and it appears that teachers made no attempt to engage pupils in higher order thinking. This practice is at odds with the findings of Watson (2001) and Zohar and Dori (2003) who found that low attaining pupils are capable of making shifts in their thinking and improving in their mathematics attainment when challenged with higher order thinking tasks. In the three classrooms, the learning environments were conducive, teachers were welcoming of mistakes, praising pupils for good effort, encouraging pupils to ask questions and engage in peer evaluation. However, the main focus was the use of correct procedures to solve mathematical tasks. Errors made by pupils were not used as springboards for reflection. Also questions asked by pupils were not exploited to engage the class in critical thinking. Hence it may be said that although the learning environment could have stimulated the metacognitive potential of the pupils it was not harnessed. This was yet another setback as Cardelle-Elawar (1995) found that low achieving pupils benefited from metacognitive training.

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MATHEMATICAL IDENTITY, LEADERSHIP, AND PROFESSIONAL DEVELOPMENT: HIDDEN INFLUENCES THAT AFFECT MATHEMATICAL PRACTICES

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The New Zealand Government recently introduced National Standards in response to concerns about levels of student achievement in mathematics and literacy, and significant investment has been made in the *Numeracy Development Project*. Principals are responsible for improving teacher practice but most principals were educated in contrasting pedagogies to that of the NDP and have more language arts than mathematics strength. This qualitative case study compared two diverse, primary-sector principals, chosen for contrasting mathematical backgrounds and leadership of mathematical professional development. The results illustrate that a school principal is an influential „cog‟ in the mathematics professional development process and that their direct participation in mathematics professional development is required for effective leadership of mathematics. It provides evidence about the mediating influence of leadership in mathematics professional development and learning in schools. Practical implications for improving classroom practice in a distributive leadership environment will be discussed.

Introduction

Principals of primary sector schools are less involved in professional development for mathematics than other subject areas. Many current principals did not learn school mathematics through a constructivist pedagogy and many may not have taught in a classroom using constructivism. Spillane (2005) unmasked substantial differences between subject areas in terms of leadership. In mathematics-related leadership routines, fewer leaders were involved and they rarely contributed, whereas direct principal involvement was more prevalent in literacy routines. Timperley, Wilson, Barrer and Fung (2007), in their meta-analysis of research on teacher professional learning and development, highlighted a lower profile for school leaders in professional development in mathematics than in any other curriculum area, “It may be that what was being asked was as challenging for leaders as for the participating teachers” (p. 75). To compound the problem, Nelson and Sassi (2005) state that the knowledge that principals hold, in terms of mathematics education, will be reflected in how they approach the mathematical content and pedagogical improvement needs of their staff.

This paper explores how principals identify effective classroom practice and professional development needs if their own mathematical identity, knowledge, and presence are weak.

Background

The role of the school leader

In the past 20 years, schools have changed in terms of governance and management (e.g., New Zealand‟s Tomorrow‟s Schools) and subsequent changes in leadership have been necessary. The move requires school leaders to empower others whilst staying in touch with „best‟ practices and assisting staff in working towards improvement in professional practice. Many writers support the concept of distributive leadership (e.g., Leithwood, Harris & Hopkins, 2008), and principals have been encouraged to embrace this paradigm as a means of sharing the demands of the heavy workload imposed by self-management. The core of building pedagogical capacity in schools lies in how the principal focuses on the development of teachers‟ knowledge and skills (Fullan, 2002). According to Robinson, Hohepa, and Lloyd, (2009), the leadership dimension that produces the largest effect size on student achievement is where the school leader is participating in, and promoting, formal and informal opportunities for teacher learning and development as leaders or as learners, or both. Extending current knowledge *in mathematics* is important in light of the dramatic changes effected from cognitive psychology and mathematics education research. Spillane (2000) asks if it is reasonable to expect leaders "to develop rich conceptions of mathematics" (p. 169) and concludes that it is reasonable because of their role in selecting and organising professional development for teachers.

A key element in educational leadership is for principals to intentionally enter classrooms to gather information and support teachers on an ongoing basis (Williams, 1996). Classroom visits should be designed specifically to assess the degree of professional development implementation. Fink and Resnick (2001) identify that effective principals "are in teachers‟ classrooms every day" (p. 606).

The leader-middle management-teaching team relationship

When shared leadership is incorporated over an existing hierarchical structure it may lead to challenges when viewed from the perspective of Anderson‟s (2004) „model of leadership reciprocity‟. Anderson describes three situations: the „contested model‟, the „buffered model‟, and the „interactive model‟.

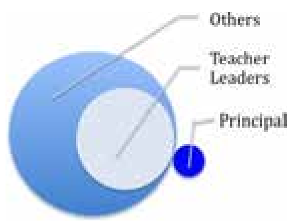

Figure 1. Contested Model

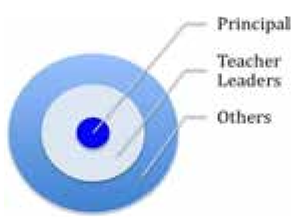

Figure 2. Buffered Model

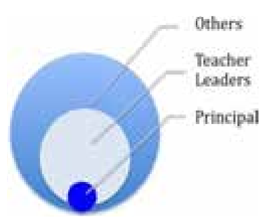

Figure 3. Interactive Model

In the "Contested model‟, shown in Figure 1, the principal stands „out of the loop‟ usually in formal leadership roles and perhaps in opposition to teacher leaders. Figure 2 shows the „Buffered model‟, where access to, and influence upon, others is mediated through the teacher leaders. The final of the three models, shown in Figure 3, is the „Interactive model‟ where the principal, teacher leaders, and other members of the school community share accessibility equally and communicate freely.

Mathematical Identity of teachers and principals

Grootenboer and Zevenbergen (2008) define identity as "how individuals know and name themselves … and how an individual is recognised and looked upon by others" (p. 243). The cultural or psychological interactions that affect a person"s method of relating to mathematics are termed „mathematical identity". According to Grootenboer and Zevenbergen, the teacher"s role is to facilitate the development of students" mathematical identities by bridging students and subject, enabling a positive relationship with mathematics. A negative aspect of mathematical identity is mathematics anxiety, and Hembree (1990) showed that pre-service primary teachers had higher levels of mathematics anxiety than any other major on US university campuses, whilst it is estimated that more than half of all Australian primary teachers have negative feelings about mathematics (Carroll, 2005). Weak teacher mathematical identities must be addressed through appropriate leadership and sustained professional development.

Effective professional development in mathematics

Research suggests that content, rather than context, of learning is the most influential factor in determining whether professional development in mathematics will result in improved student achievement (Timperley et al., 2007). Professional development of 14 hours or less showed no effect on teachers" learning. The largest effect involved programs offering 30 to 100 hours spread out over 6 to 12 months (Darling-Hammond & Richardson, 2009). Research shows that there is a link between improving mathematical identity and engagement with professional development activities (e.g., Millett, Brown, & Askew, 2007). Higgins and Parsons (2009) characterise professional development that encourages change in mathematics instructional practice as having a focus on subject matter knowledge, an understanding of how students learn the subject matter, and how to convey content in meaningful ways.

Data collection and analysis

This paper reports a case study of two New Zealand primary schools; referred to as School A and School B with the respective principals referred to as Principal A and Principal B. It examined the principals" mathematical identities and leadership, the teachers" mathematical identities, and the professional development offered in mathematics pedagogy. The two schools were selected because they showed clearly contrasting mathematical histories of their principals, different approaches to professional development, and different outcomes on the mathematical identities of the teachers. Both schools had similar student ethnic compositions and socio-economic locations. Around 60% of the total teaching staff of each school responded to the survey and volunteers were then interviewed in a semi-structured format.

A qualitative approach was selected as the most appropriate method of obtaining data. A survey design was supplemented by interviews to explore primary principals" and teachers" perceptions of the provision of mathematics professional development in view of the inherent mathematical identities and the surrounding issues. The survey included an adapted Mathematics Value Inventory, or MVI, (Luttrell et al., 2010) whereby questions were posed about attitudes towards mathematics and responses made on a five-point scale ranging from strongly disagree to strongly agree. The statements included indicators of either negative or positive beliefs about the value of mathematics. The data were analysed by the researcher using a grounded approach of identifying codes, categories, and themes that were then used in conjunction with dialogue and quotes from participants.

Findings

The principals' and teachers' mathematical identities

Principal A did not formally study mathematics beyond secondary school nor completed any further qualification that included a mathematics or mathematics education component. Principal A described a history as a mathematics student and reflected negatively on their learning experiences in mathematics, "I am one of those kids who didn't get it at school and I know what that feels like." Principal A did not connect to mathematics as a school student because, in a psychological sense, the education received failed to relationally bridge student and subject as described by Grootenboer and Zevenbergen (2008). A teacher education programme started to change the perception of their mathematical identity by changing elements of identity (such as a new path for their life history and improved affective qualities and cognitive dimensions for mathematics). Using the Mathematics Value Inventory (MVI), Principal A showed that they currently hold positive feelings towards mathematics. The repaired mathematical identity, combined with negative childhood memories, gave Principal A an increased understanding into students who struggle to understand mathematical concepts, "I can say to the staff that it's not that they are not trying: they don't get it." The MVI showed that 90% of teacher respondents at School A held positive feelings towards mathematics despite prior mathematics anxiety for some participants. A teacher explained, "I was actually frightened at the thought of learning maths well enough to teach it. Coming here and getting the training has only made me more enthusiastic for maths."

Principal B had formally studied Level 1 mathematics as a university undergraduate for a non-mathematical degree, with no further qualifications that included mathematics or had a mathematics education component. Principal B had a strong mathematical identity as a school student and as an adult. They held mathematics to have a high value and recognised the importance of mathematics as a subject area and as a life skill. Principal B had not taken part in Numeracy Project professional development because, the Principal stated, "we couldn't see the need". Up to 44% of teacher responses at School B showed negative feelings and an explicit lack of value held for mathematics.

Professional development related to mathematics

At interview, evidence was provided that Principal A consulted with and considered the needs of individual teachers, pedagogical practices were observed and weaknesses identified, student achievement data was noted, and governmental initiatives were taken into account before goals for professional development were established. The school focus on mathematics (often with external facilitators) was reflected whereby 100% of teacher respondents had undertaken mathematical professional development in school. All newly employed teachers received an intensive introduction to the Numeracy Project and then joined the whole school professional development programme.

School B"s professional development programme was in direct contrast to School A"s. Only 8% of teachers at School B had undertaken mathematics professional development. Literacy was the most common curriculum for professional development followed by ICT, inquiry, and English as a second or other language (ESOL). Mathematics and music professional development followed these subjects and science and Physical Education tied for last place. Teachers from School B stated that the school participated in too many initiatives and this resulted in a lack of focus for professional development programmes. They voiced concern at the apparent lack of links to best practice and classroom visits. Despite the teachers in School B, overall, holding higher mathematics qualifications than those held by the teachers in School A, they indicated a lack of content knowledge. "There are teachers at our school who don't know higher than 3A. How are we catering to those top kids when the teachers don't know it?" School B"s teachers described the professional development in mathematics offered in the school as inadequate to meet their needs. One teacher stated, "I can"t even remember doing maths PD. I don't think we are helping people that don't have strength in maths enough."

The principals' promotion of, and participation in, professional development in mathematics

Principal A was a consistent participant in mathematical professional development through personal attendance at professional development activities, working alongside the mathematics curriculum team, and in staff meetings where mathematics professional development was provided. Ninety percent of School A"s teachers believed that Principal A promoted access to mathematics professional development well or better.

Almost half of School B"s teachers recorded that the principal never participated in professional development in mathematics whilst 42% stated that this occurred once each year or less. Teachers outlined how they considered that Principal B"s participation was inadequate in professional development and staff meetings. Thirty-one percent of School B teachers responded that they needed more professional development to maintain their skill and 46% said they did not receive any mathematics professional development at all.

The principals' leadership of mathematics

Principal A described their leadership of mathematics as distributive, adding "We have a maths team of teachers. I am involved in the team and decision-making and in setting a budget that will allow for the gains made to be sustained." Principal A assumed an interactive role of leadership reciprocity (Anderson, 2005) through attendance at the meetings and direct involvement in the professional development. Principal A merged internal professional development with external opportunities, including the provision of externally facilitated workshops twice each term. Resources were provided to allow attendance at these workshops and the principal managed the resources. Curriculum team meetings focused on how best to implement new learning and recommended the resources required. The principal attended these meetings. Staff received information through regular staff meetings where the curriculum team shared learning opportunities.

Principal B also espoused their preferred style of leadership as distributive, but not all School B teachers saw Principal B"s version of shared leadership as being ideal, especially in mathematics. The teachers outlined how they felt that the leadership team no longer led curriculum. One teacher said, “The whole curriculum focus has been lost from a senior management point of view”. Yet Principal B acknowledged that principals “need to engage in learning with the person to whom it is delegated so that you have a shared understanding.” However, Principal B also outlined a view that principals need not understand the content of mathematics: “I don't think a principal has to be mathematically orientated but has to understand that mathematics is one of the foundation skills”.

Discussion and conclusion

Mathematical identity of principals

Common sense might suggest that a principal with a strong mathematical identity would identify more strongly with the subject and focus more on the provision of professional development specific to mathematics. My research found a stronger relationship between a formerly weak mathematical identity that had been addressed and the promotion of, and participation in, mathematical professional development than between a strong but unaddressed mathematical identity and promotion of, and participation in, mathematics professional development.

Principal A‟s position on the Mathematics Value Inventory (Luttrell et al., 2010) indicated that they had successfully repaired their mathematical identity and they have shown enthusiasm for professional development in mathematics content and pedagogy. This created a clear purpose for mathematics leadership, simply described by Principal A as being that a child in their care “would not suffer a belief that they were mathematically useless”. In contrast, Principal B described having a lifelong comfort with mathematics, indicating a positive mathematical identity from an early age, and having studied mathematics content at university. The MVI scale indicated that Principal B held mathematics to have a high value despite a lack of professional development in the subject and a fading knowledge base.

The mathematical identity of teachers

Teachers at School A benefited from stronger mathematical identities than teachers at School B. Only 10% of School A teachers indicated any negative feelings towards mathematics on some statements, contrasting sharply with School B‟s teachers where 28% indicated negative feelings towards mathematics.

Those with a weak mathematical identity at School A started to address that identity immediately upon joining the school. Teachers stated that the focused approach gave them stronger content and pedagogical knowledge. That the entire school participated in mathematics professional development encouraged sharing and reflection, identified by Alton-Lee (2003) as a vital ingredient in effective professional development, and this strengthened their mathematical identity. In contrast, School B teachers had less developed mathematical identities and demonstrated having more negative feelings for mathematics. Some School B teachers lacked confidence in teaching mathematics and some perceived that they did not have the content knowledge required to teach at the level expected of senior primary students. In spite of this, a need for teachers to participate in mathematics professional development was not identified. The lack of targeted professional development opportunities in School B indicated that the mathematical identities of School B‟s teachers were being neglected.

Educational leadership of mathematics in practice

Principal A promoted, and participated in, professional development in mathematics, and conducted classroom visits to evaluate mathematical teaching practice. This task had priority over other administrative demands. It appeared that Principal B set other priorities above attending staff meetings for professional development in mathematics and did not do classroom visits for the purpose of evaluating the teaching and learning of mathematics. By not participating in professional development in mathematics, Principal B was not involved at a level where they could influence classroom practice or objectively identify teacher and student needs. Principal B reflected Spillane‟s (2005) findings of a lower leadership profile and different leadership routines applied to mathematics when compared to other curriculum areas.

Distributive leadership of mathematics in the case study schools

Both principals demonstrated contrasting positions on Anderson‟s Model of Leadership Reciprocity (2004). Principal A was firmly positioned within the Interactive model. Accessibility between the Principal, the lead teacher of mathematics, and other teachers was shared equally and communication between the constituent members was free flowing. In addition to highly visible participation in mathematical professional development, Principal A reinforced the role of educational leader in mathematics by promoting staff access to mathematical professional development and providing the necessary resourcing to ensure that access. Principal B was positioned in Anderson‟s Buffered model. Principal B stated that they did provide the resources to enable professional development, but the absence of direction towards mathematics and a lack of participation in mathematics professional development meant that the support was without structure or priority. Statements made by the teachers indicated a declining participation by the principal in mathematics professional development or no participation at all. By not interacting with the wider teaching body in a reciprocal manner, Principal B was considered to be more concerned with the study of pedagogical improvement rather than the practices that would lead to the improvement itself.

Principal participation in professional development

Nelson and Sassi (2005) raise the question of how much and what kind of knowledge is sufficient in order for principals to be able to make effective decisions regarding instruction. Principals who have not undertaken ongoing professional development in mathematics may not recognise, nor provide for, excellence in mathematics teaching and learning. Principal A improved both their own knowledge and their influence on teacher practices by sharing in the learning about mathematics alongside the teaching staff and being available for regular discussions with staff on mathematics. Nelson and Sassi tell us that the nature of a principal‟s mathematics knowledge affects their appreciation of mathematics instruction, and it was apparent that Principal A was well enough informed to have been able to make decisions regarding instruction.

From Principal B‟s perspective, they „shared‟ the leadership. However, it was clear that the teachers did not see the principal as the leader or as a member of a leadership team for mathematics. It was also apparent that the teachers had undertaken little mathematical professional development. The lack of professional development activities offered in mathematics and the absence of classroom visits and discourse about mathematics are evidence that Principal B‟s hands-off approach to mathematics led to a limited appreciation of how and when to help initiate change in mathematics. Identification of the „expected outcome – observed implementation‟ gap of professional development is critical for a principal if resources are going to be effectively targeted. Principal B was not in a position to measure the gap. As such, there was no awareness of the need to address the weak mathematical identities of the teachers at School B.

Principal B‟s lack of strategic approach to professional development was shown in the wide and discrepant range of professional development activities undertaken. School B‟s professional development was done with the intent of providing personalised professional development for each teacher but the identification of pedagogical needs was made by the individual teacher themselves and not on data or observation. Therefore, if the teacher‟s mathematical identity was so low that this cognitive effect turned into a behavioural response of avoidance of participation in mathematical professional development, as described by Richardson and Suinn (1972), then there was no mechanism to ensure that students were achieving a good mathematical education.

School A‟s ongoing concentration of professional development towards mathematics ensured that the teachers at School A complied with Darling-Hammond and Richardson‟s (2009) research that the largest effects come from programs offering 30 to 100 hours spread out over 6 to 12 months. Alternatively, School B‟s „smorgasbord‟ approach to professional development meant that each teacher received less than 14 hours of mathematics professional development, the level at which Darling-Hammond and Richardson stated that there would be no effect on teacher learning.

The influence of the principal on the mathematical identity of teachers

Principal A positively influenced the mathematical identity of their teachers through the promotion of a formally structured, long-term mathematics professional development plan based on the Numeracy Project. The Principal‟s direct involvement, where they asked questions to deepen their own understanding and encourage others to seek clarification, demonstrated to the teachers that a mathematical identity may be improved by critical evaluation and reflection. The teachers from School B expressed concern that there was no educational leadership of mathematics and that there was a lack of professional development offered to improve their ability. The mathematical identity of teachers at School B suffered and they exhibited lower value perceptions of mathematics and higher mathematical anxiety than the teachers at School A.

Implications

As primary sector teachers have a higher anxiety towards mathematics than any other curriculum area, addressing the teachers‟ mathematical identity through sustained professional development should be a priority in all primary schools. It is the responsibility of the principal to attend to quality professional development, and to interact in the design process, if consistently high quality mathematical practice is to be attained within their schools. This will only be achieved through taking an interactive position in leadership reciprocity and not by arm‟s-length management of resource provision. Principals need to be aware that they need personal professional development in order to attend to their mathematical identity and to make decisions concerning effective practice in mathematics teaching and learning. Principals need to understand the benefits of „standing alongside" their teachers as they undertake professional development in mathematics; learning the same content at the same rate as the teachers. Principals should reflect on the difference between distributive leadership and exclusion of themselves from areas in which they should be involved and informed. They may also gain information to identify where capacity lies or is absent in mathematics.

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REFORM IN MATHEMATICS: THE PRINCIPAL'S ZONE OF PROMOTED ACTION

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This study draws on Valsiner's (1997) extension of Vygotsky's (1978) theory on Zone of Proximal Development that had been further extended by those interested in teacher professional development (e.g., Blanton, Westbrook & Carter, 2005; Millett & Bibby, 2004). These theories were used to guide the constant comparative analysis of interview data collected during a case study involving all teachers and leadership in one primary state school in Queensland. Through this analysis it became apparent that the principal created a Zone of Free Movement and a Zone of Promoted Action that limited teachers' meaning making of reform in mathematics. Being alert to these actions is important if we are to truly understand how change in mathematics reform may be being implemented in schools.

Literature review

Concerns for the implementation of mandated curriculum reform has been consistently identified in the research literature (Fullan, 2001), with reform in mathematics no exception (e.g., Smylie & Perry, 2005). Handal and Herrington (2003) highlight this argument by specifically listing research on mathematics reforms where findings indicate that the implementation of innovations have failed. Researchers explain the high failure rate of reform in mathematics by pointing to the teacher, noting a number of inhibitors e.g., the majority of primary teachers are women who may not have pursued higher mathematics study and that teachers do not have the depth and breadth of content knowledge to successfully implement curriculum reform in mathematics (Ball & Bass, 2003; Ma, 1999; White, Mitchelmore, Branca, & Maxon, 2005). Furthermore, it is argued that teachers' beliefs and attitudes about teaching mathematics are often formed from their own school experience that reflected a traditional style of teaching (Brosnan, Edwards, & Erickson, 1996). Emulating this traditional style of teaching is said to give the teacher a sense of security and control. It provides the insecure teacher with a comfortable teaching environment; they are, therefore, reluctant to relinquish control of the lesson to their students as is expected with the current belief in co-constructive pedagogy in mathematics (Davis, 1990; Schoenfeld, 2000). Recognising these inhibitors to curriculum change in mathematics, there is a call in the literature for teachers to 'unlearn' their own school experiences in order to be open to change (Ball & Bass, 2003).

Considering how this unlearning would be achieved, literature encompassing educational change, teacher professionalism and professional development were reviewed (Lamb, 2010). Here it was found that emergent theories in each of these areas coalesced around the concept of the professional learning community (PLC) described in terms of collaborative relationships, shared vision and values, and the active promotion of learning. As a consequence, the literature presents strong support for situating curriculum reform within the context of a professional learning community (Stoll, Bolam, McMahon, Wallace, & Thomas, 2006). Moreover, this literature also alerts us to the impediments in developing a professional learning community: inappropriate structures, inadequate social capital and sustainability, as well as inhibitors to successful professional learning (Smeed, Kinmber, Millwater, & Ehrich, 2009).

As a way to interpret how the PLC engages with reform in mathematics, researchers (e.g., Millett & Bibby, 2004) have utilised Valsiner's (1987) extension of Vygotsky's (1978) theory on Zone of Proximal Development (ZPD). In his theory, Valsiner locates a further two zones, which he called the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA). The ZFM is set by the adult and defines what action the child is allowed to undertake and the thinking to which the child is exposed. The ZPA is the tangible range of actions that the adult promotes in an effort to influence the child's behaviour. The interaction of these zones is such that the action that is promoted must be allowed and therefore the ZPA is within the ZFM. However, the ZPD can only be stimulated if it lies within the ZPA, while greater potential for development may exist outside the promoted action. This theory is represented in Figure 1.

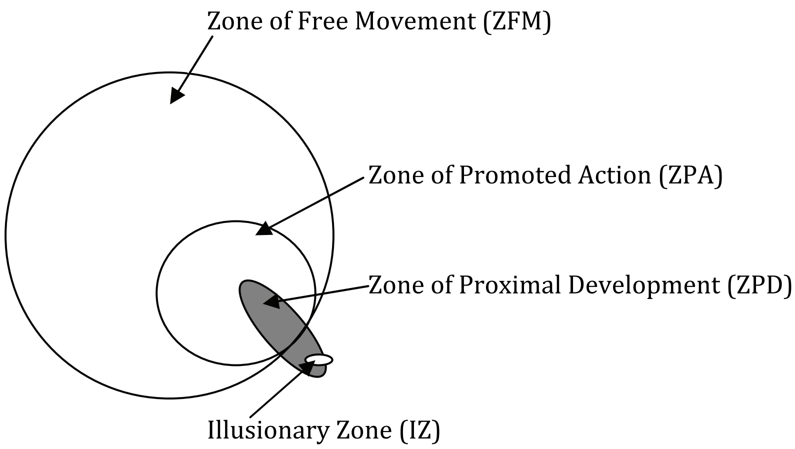

Figure 1. Model of interaction of the ZFM, ZPA, IZ and ZPD (adapted from Blanton, Westbrook and Carter, 2005, p. 8).

Blanton, Berenson and Norwood (2001) used this theory to analyse experienced teachers' responses to professional development arguing that the "ZPD is affected by the intellectual quality and developmental appropriateness of interactions with a more knowing other" (p. 5). Further developing this theory, Blanton, Westbrook and Carter, (2005) identified an Illusionary Zone (IZ) of promoted action as a zone of permissibility that the teacher appears to establish through behaviours and routines used in instruction but in reality does not allow. IZ was distinct from the ZPA in that ZPA should be contained within the ZFM (one can only promote that which is at least allowed) while IZ was that which appeared to be promoted but in fact was not allowed.

This theory is also utilised by Millett and Bibby (2004) where they argue that the "situation" (p. 3) in each school PLC will be different depending on the ZFM, ZPA, and the possibility of IZ, leading to different responses and therefore different outcomes for mathematics reform. They provide a model (see Figure 2) for analysing the local context of reform in mathematics.

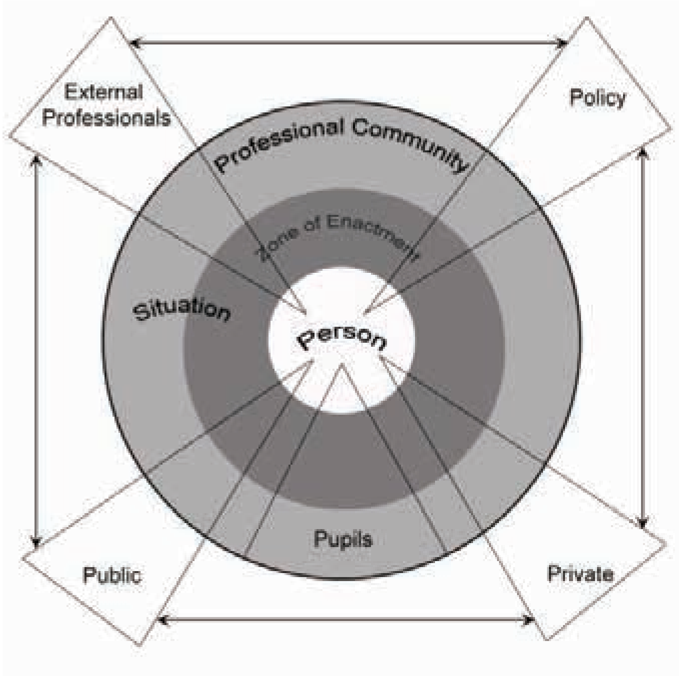

Figure 2. Theoretical model for analysing the context of reform in mathematics (Millett & Bibby, 2004, p. 3)

Here, Millet and Bibby (2004) focus on developing an understanding of a teacher's response to sources of support in the reform in mathematics by placing the teacher at the centre of the model. To be considered here is the teacher's "personal agency beliefs" and "beliefs about self-efficacy … and academic self esteem" (p. 5). In short, this theory seeks to understand the teachers' capacity to change by examining the context and culture of the teachers' working environment which includes the school's professional learning community and pupils. They term this environment the "situation" (p. 3).

Discussing their model in this way, Millett and Bibby then draw attention to the the *zone of enactment* (p. 4) for each teacher within the overall situation. This is an "an area of potential for professional development, the space in which the individual makes sense of reform or change initiatives in essentially a social process" (p. 1). In their view, the process of implementing curriculum change begins with one or more external factors (e.g. external professionals, policy, public and private). In the first instance, the personal and professional characteristics of the teacher will influence this decision whether to accept or reject this demand. However Millet and Bibby also argue that the zone of enactment is a "social construct" (p. 4) and, as such, will be influenced by interactions within the situation. If these interactions include "rich deliberations" that were "grounded in practice and supported by resources, curriculum change [is] more likely to be operationalised" (p. 4). Millett and Bibby (2004) argue that this interpretation of the actions within the school environment respects the position that each school's response can be different even though external factors remain the same.

Each of these external factors directly impacts the teacher's professional development as indicated by the arrows coming through to the 'person' in Figure 2. "External professionals" encompass all those professionals from outside the school who can support teacher change through professional development. This is often backed by what they term the "public" and includes commercial sources of support that are not a part of the government support system. For Millett and Bibby (2004), external "policy" refers to what they call an "avalanche" (p. 9) of policies while "public" refers to those who are outside the school but are none the less interested in education such as parents and the media.

In short, this model assumes that the various sources of support that operate within the situation will either stimulate a teacher's "zone of enactment" and lead to reform in mathematics or permit inaction and the ultimate failure of the intended reform. Developing this thought, Millett, Brown, and Askew (2004) later identify four conditions necessary for the development of the PLC and the realisation of the teachers' zones of enactment: time, talk, expertise, and motivation. They argued that it was essential for teachers to be given time to talk, to engage with the expertise of others for the motivation of teaching mathematics to develop.

This paper considers the role of the principal in providing these essential elements for successful reform in mathematics by further adapting Blanton et al (2005) adaptation of Valsiner's (1997) lens of the ZFM, ZPA and IZ from looking at teacher actions with students and considering Millett and Bibby's (2004) model by looking at the principal's actions within a PLC. The research question asks, can these models provide insights into how principals' actions impact teacher efforts at reform in mathematics?

Research methodology

Consistent with the research question, this case study focussed on the perspectives of key personnel involved in the implementation of the reform mathematics syllabus at Riverview Primary School (pseudonym), a state school in South East Queensland. The leadership team led by the Principal and all 26 classroom teachers (Years 1-7) participated in this case study.

This study was informed by symbolic interactionism (Blumer, 1998). As methodology, symbolic interactionism requires the adoption of two distinct stages within the study: "exploration" and "inspection" (Blumer, 1998, p. 40). The exploration stage allows the researcher to construct meaning about "what's going on around here" (Charon, 2007, p. 194), as well as to identify issues for further investigation during the inspection stage. Reported here is the qualitative data collected that included individual and group semi-structured interviews during both stages of the study. In the first stage the school's Principal and Head of Curriculum (HOC) were interviewed. From the analysis of this interview data a number of issues then led the investigation into the inspection stage of the study. Here, focus group interviews were conducted with each year level of teachers. This was followed by further clarification of the issues with individual interviews involving each members of the leadership team.

This two-stage data collection process was supported by a three step iterative process of data analysis termed as first, second and third order interpretation (Neuman, 2007). The first-order interpretation is from the perspective of the participants being studied. The second-order interpretation stems from the perspective of the researcher, and involves eliciting the underlying coherence or sense of meaning in the data. Third-order interpretation involves the researcher assigning general theoretical significance to the data.

Results

First order interpretation

During the exploration stage of the study the Principal was interviewed to get a sense of ‘What’s going on around here. He outlined that the teachers had been given time to talk, were provided with PD and resources, and he believed that the teachers were motivated about reform, stating that the school PLC had committed to a shared vision of implementing reform across all curriculum areas and as a consequence the teachers are encouraged to embrace reform in mathematics. He described supports that had been provided to achieve this vision.

> I have built structures in the school, having year level coordinators who ... have had things explained to them, not just about maths but about all curriculum areas … Every other week is the year level specific curriculum meetings … We have now established another set of release times for teachers with advanced organisers for moderation. …This is not just for the purpose of having results clarified …We have them all annotated and they can see them on the web as samples. So that is part of the planning processes across the whole school, and obviously those planning processes will be even more developed with the mathematics as well.

When the principal was asked about PD he stated:

> ... what we’ve done with outside consultants is say, right, we know what we are going to do in this school, we have our curriculum journey mapped out, if we’re looking sensibly at mathematics at the moment; we have to acknowledge that’s what we’re doing... We’re looking at mathematics, what do we need? … I have PD money, the amount of that PD money is determined by us and our School Council, based upon our identifications of immediate needs and projections for planning.

These comments raised questions about how effective the teachers believe these structures and PD are for actually supporting planning in mathematics at the year level? Do these structures dovetail with whole school planning structures to support the school vision? The analysis of these data led the investigation into the inspection stage of the study where focus group interviews were conducted with all 26 classroom teachers.

When the teachers were asked about the supports provided for reform in mathematics at the year level and the whole school level a completely different perspective on the actions of the principal was presented.

> [At year level meetings] We normally sort of just keep track of what everyone’s doing. Then we see if anyone needs assistance with their teaching, and what problems they’ve had with any students.

and

> When we have these moderation days or we have planning days, we go through all the writing tasks and the science etc.; it’s really supposed to be the units we’ve already been planning with the HOC [Head of Curriculum], we really don’t give maths much thought.

During the discussion with the teachers, it became apparent that a reform textbook had been introduced to the school, with no further professional support provided as one teacher explains:

> It was just kind of, you know, as we introduced [the textbook] and all the rest of it – investigations were kind of encouraged, but there wasn't any guide as far as kind of training and how to do it.

The teachers explained that following a bad year on state testing the decision was made to move away from the reform textbook and instead, adopt a school program from another school. This program was a list of content to be taught each year, semester and term.

> So we were told, you have to submit to the new program… this is our program, follow it. It's not as prescriptive as [the textbook] but I think I'm getting a bit lost.

Moreover, the teachers expressed frustration at having to implement the investigative pedagogy of the new syllabus with little guidance. The perception being that the lack of PD prevented them from appropriately implementing the new mathematics syllabus.

> I think one of the problems right across the board is there hasn't been any attention given to developing skills for teachers to do investigations and develop investigative thinking.

As a consequence the teachers expressed doubt that they were engaging in mathematics reform with one teacher saying:

> I think you still keep failing, even though you try and take on board a lot of the new stuff. You still fall back on your strengths and what you know how to teach well, and you do try to incorporate the new things or the styles of teaching or content as well, but, you know it's hard to shake 12 years of teaching a particular way - maths in a particular way - you can't just all of a sudden change.

When these views were presented to the Principal he denied that there was a problem with change, or that there was conflict between his views of support for the teachers reform efforts and the teachers' perception of support. He said:

> The teachers here are very, very good. We go to a lot of trouble here pulling units together …We hear them saying, 'Oh, there is too much to do'… I have no drama with what we are doing. If we left it to the laissez-faire system we would have anarchy.

The Principal goes on to explain that for mathematics, the teachers were given the school program to support them with implementing the *Essential Learnings* (Queensland Studies Authority, 2007) that superseded the Mathematics Year 1-10 Syllabus (Queensland Studies Authority, 2004):

> The program must be followed. It is getting tighter and tighter. The Essential Learnings makes that quite clear ...There's nowhere to deviate ... This is what you have got to do boys and girls so just do it. How you do it, is up to you. We can sit and whinge and carry on but the reality is you've got to do it.

Second order interpretation

This principal like most principals is under pressure to implement a range of reforms and to demonstrate that the students in the school can achieve at state/national averages or better. In attempting to achieve these goals, priorities must be set. The priorities for this school PLC was to have year level and whole school planning in literacy and science as well as professional development in these key learning areas. In this way, the principal had provided opportunities for reform. However, reform in mathematics was initially limited to the provision of a textbook that was later removed and replaced with a school program that contained a list of content to be covered.

Third order interpretation

It is useful to interpret the principal's actions through the lens of the ZFM, ZPA and the IZ to consider the impacts on the PLC. It is clear from the principal and the teachers that a range of new structures within the school had been established to support curriculum reform. In this way, the ZFM for reform had been set. However, the topics for discussion at these meetings did not include mathematics reform, instead English and science were the focus at these meetings. Therefore, the ZPA did not include mathematics reform. The principal would argue that he had indeed promoted reform in mathematics by providing the textbook and then the school program. Yet for the teachers, the school program was merely a list of content that permitted the teachers to teach in the way they had traditionally taught. It can be argued that the principal's actions for reform were occurring in the IZ where he was not attuned to the teachers calls for support with reform in mathematics and as a result continued to convince himself that the teachers were implementing reform. These actions had indeed prevented reform in mathematics.

Conclusions

This research suggests that Valsiner's (1997) zone theory can be used to interpret the Principal's actions in constraining or promoting teacher action and thinking. The ZFM in this case study represents constraints and affordances as directed by the principal in relation to reform in mathematics and could be considered to include opportunities for whole school and year level planning as well as PD. The ZPA represents the actual opportunities that the Principal provided for teachers to engage with mathematics reform in the form of time to talk and to access expertise through PD so as to promote motivation about the reform within the school's PLC. In this case study the distinction needs to be made about generalised reform across the key learning areas in this primary school and reform in mathematics specifically. Here the IZ includes mathematics reform as it was not actually promoted by the principal, unlike English and science reform. For the interplay of the ZFM and the ZPA to impact on the ZPD, and lead to enhance content and pedagogy knowledge of the reform, teachers must perceive that the reform is being promoted by the principal i.e., within his ZPA. For this to be achieved the teachers would need to perceive that the principal is providing sufficient support for them to engage with the reform, and that, once these zones have aligned, the reform does in fact respect teachers' knowledge and beliefs about the teaching and learning of mathematics. In this case study, the concerns of the teachers remained unheard by the principal as he believed opportunities for reform had been provided. This study concludes that by developing a greater understanding of the principal's ZPA, and the possible existence of the IZ, better opportunities to align the ZFM and the ZPA will see enhanced potential for teacher ZPD.

The findings from this study suggest that Valsiner's (1997) zone theory can be applied to leadership contexts as a way to support understanding why some reform measures are successful while others are not. It is also recommended that this theory can be used by school leaders for self-reflection and critique about the level of effectiveness in supporting teachers through reform in mathematics.

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PRESERVICE TEACHERS LEARNING MATHEMATICS FROM THE INTERNET

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Although preservice primary teachers' limited mathematical knowledge has been well documented, little research has been conducted on programs to improve it. We report on first-year, teacher education students' use of recommended internet resources on different mathematics topics. Our findings suggest that many of our preservice teachers had not previously used internet resources for learning, except to do research. They also saw mathematics learning as occurring only when they are taught by a teacher and so internet resources were of limited value. Ultimately these beliefs, if left unchanged, will have an impact on their teaching of mathematics to primary school students.

Background

In this paper, we evaluate a resource for supporting preservice teachers' learning of primary school mathematics. It was a CD with links to websites and references to the textbook on the different topics covered in a test that preservice teachers had to pass in their first mathematics education subject. Although some students made use of the CD, many expected to have a teacher teach them how to answer each question. This research has implications not just for improving preservice teachers' mathematical knowledge but also for their pedagogical understandings about how mathematics is learnt.

In recent years, many preservice primary teachers' (PPTs) limited mathematical knowledge has been acknowledged. In Australia, the Senate Standing Committee on Workplace Relations and Education (2007) stated "early tests of numeracy conducted by education faculties showed that a very large proportion of [teacher education] students cannot do grade 5 maths because they never learned a lot of maths at school" (p. 58-59). Consequently, government regulatory agencies such as the Office for Standards in Education [OFSTED] (1994) in the UK and more locally the NSW Institute of Teachers (2006) instituted mathematical requirements for entry into primary teacher education courses.

However, requirements for PPTs to have high school mathematics qualifications may not provide them with the necessary knowledge to teach primary school mathematics (Tobias & Itter, 2007; Goulding, Rowland, & Barber, 2002). At Charles Sturt University (CSU), a concern about ensuring that PPTs had appropriate mathematics for primary school teaching resulted in an assessment being a mathematics test in the first mathematics education subject. However, research from 2008 suggested that studying for this test confirmed PPTs valuing of procedural rather than conceptual mathematical understandings (Meaney & Lange, 2010). The test reinforced the PPTs' identities as students rather than becoming teachers, so learning was focused on passing the test rather than on being able to provide learning opportunities for children. Subsequently, the test was changed in 2009. At the beginning of semester, PPTs were given 50 short-answer questions based on children's responses to mathematical problems, inspired by Hill, Schilling, and Ball (2004). About two thirds of the way through the semester, they had to respond to ten of these questions in a formal test environment. PPTs who failed this test had a second chance to pass, several months later after they paid a $100 fee. If they failed the second test or did not take up this option, they had to re-enrol in the subject the following year.

Morris (2001) had found that the fail rates of PPTs, taking an audit of their mathematical understandings, had reduced significantly in the year when they had been provided with a specimen paper at the beginning of their course. Given that we knew from our previous research that many students had very spiky knowledge background (Meaney & Lange, 2010), in that they knew some things but did not know others, we anticipated that giving the students the set of 50 potential questions would help them to tailor their learning to the areas that they identified as being difficult. Self-auditing of mathematical knowledge for a similar test had been used in other research (Goulding et al., 2002; Corcoran, 2005). Nevertheless, we were also aware that preservice students could be overly optimistic about their mathematical capabilities, simply because they were unaware of the misconceptions that they had (Morris, 2001). Therefore, lectures were provided for one hour a week that specifically covered the topics in the test but did not provide answers to the specific questions. In a similar vein to Ryan and McCrae (2006), we wanted students to use the test questions as a catalyst for learning the mathematics that they would be required to teach, rather than seeing the test questions as the only mathematical knowledge that they would need.

Resourcing preservice teachers learning

We knew that for some students, the lectures would not be sufficient to overcome the gaps in their mathematical knowledge. Consequently in 2010, following a model similar to that of Lin (2010, see Figure 1), we collated websites for the relevant mathematics topics, where the emphasis was on the mathematical concepts, rather than on simply learning how to use a procedure. The websites varied from providing text-based material on children's common misconceptions, to videos on YouTube about operations with fractions, to simulations/animations to games. Most links had been trialled in the lectures for this subject in 2009. Links to these websites were sorted according to topic, provided with references to the relevant section of the textbook and organised as a website on a CD (see Figure 2, next page). Every student was provided with a CD.

We decided to present support material in this way for several reasons. As part of a wider initiative to support PPTs gaining knowledge of how to integrate ICT into their classrooms, all first-year, primary teacher education subjects at CSU included ICT requirements, including engaging with a range of web resources. Although some students arrive at university with significantly more ICT skills than their lecturers (Barnes, Marateo, & Ferris, 2007), it has been found that graduating teachers feel apprehensive about using ICT in teaching (Lin, 2008).

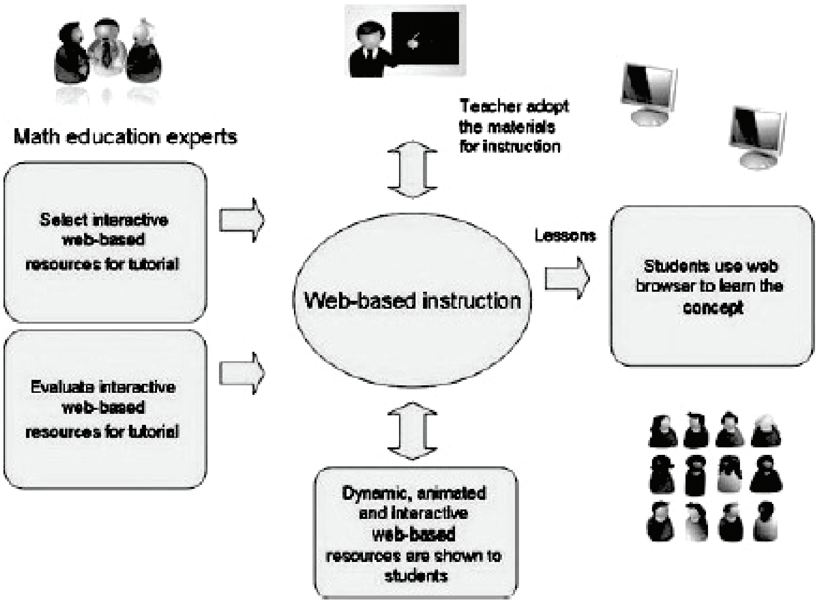

Figure 1. Web-based instruction model from Lin (2010, p. 63).

Recent research suggested that mathematics teachers, when they used the internet, predominantly used it for planning and made limited use of learning activities designed for students (Moore-Russo, Viglietti, & Bateman, 2009) resulting in a well-funded initiative being implemented in Australian universities to improve graduating teachers' "effective and innovative use of ICT in education to improve student learning" (Australian Learning and Teaching Council [ALTC] & Australian Council of Deans of Education [ACDE], 2011, p. 4).

Figure 2. Front page of the CD resource.

There was also some evidence that suggested that having preservice teachers engage with ICT not only made them more enthusiastic about teaching with ICT (Lin, 2008) but also was more valuable for supporting their learning than traditional lectures (Lin, 2010). Although Lin's studies were done with small numbers of students, the findings were encouraging. However, almost no other studies have investigated PPTs' use of web-based resources for improving mathematical knowledge. It was important to us to discover how the CSU PPTs used the CD and how we could improve it so that it was of most benefit to our students. Therefore, we wanted to find out what supported and what hindered PPTs' use of the CD to improve their mathematical knowledge.

Methodology

In this paper, we concentrate on information provided by the PPTs. At the beginning of the semester 190 PPTs completed a survey about their previous use of web-based resources as well as their preferred way of learning for the test. The PPTs were also asked to keep a diary of web links from the CD that they used and how useful they found them. Only twelve preservice teachers, across the two campuses, handed in these diaries after the test was graded and the marks handed out. Of these, only five recorded that they used the CD more than once. Following the test, focus group interviews were held with a small sample of students. These interviews were not carried out by lecturers.

Results and discussion

In this section, we summarise the results from the data. A chi-square test on the numerical data suggested that previous experiences of using web-based resources of the two cohorts were different. This suggests that even within CSU, PPTs bring a large variety of experiences to their university studies. Consequently in Table 1, we have provided numbers of responses from student teachers from both campuses (Bath for Bathurst, or WW for West Wyalong). Our data collection methods did not enable us to match the surveys to the PPTs involved in the interviews so the differences between the cohorts is not explored. Although 190 surveys were collected, some questions were not answered by everyone.

The results show that most PPTs had some previous experience with using web-based resources. Nevertheless, at both campuses at least 15 percent of PPTs had never used these resources. One comment from the survey exemplified some PPTs' lack of experience with computers. It came in relation to a question on whether they would use the CD to study for the test:

> If it works in a DVD player because I don't really know what else it will work in. (Bath)

Answers to the open-ended questions also suggested that many preservice teachers felt that the internet was expensive and computers were difficult to use. In responding to a question on what they did not like about using web-based resources, PPTs wrote:

> Money to download, speed of net. I don't like reading off screen. (Bath)
> Involves you to be around computers etc which are not always easy to access. (WW)

Table 1. Preservice teachers' previous use of web-based resources.

Have you used web-based resources for learning before?	*Yes*		*No*					
	WW	*Bath*	*WW*	*Bath*				
	86	65	15	18				
a. If yes, how often did you use these resources?	All the time		Quite a lot		Occasionally		Very Rarely	
	WW	*Bath*	*WW*	*Bath*	*WW*	*Bath*	*WW*	*Bath*
	13	4	32	15	33	33	10	17
b. What sort of resources did you use?	Video from sites such as You Tube		Text-based material such as explanations		Animations of experiments, etc.		Games for testing skills	
	WW	*Bath*	*WW*	*Bath*	*WW*	*Bath*	*WW*	*Bath*
	38	31	79	62	17	12	18	36

Note: Numbers represent PPTs.

At both campuses, the most common previous use of web-base resources was reading text-based explanations. As well, about a third of preservice teachers had used videos from the web and at the Bathurst campus, a similar proportion had used games. A relatively small proportion had used animations. These results suggest that even if preservice teachers have had experience of web-based resources then it was likely that these resources were conceptualised as being similar to book resources. Thus, learning through web-based resources was conceived as finding and reading information that was laid out in a step-by-step manner. Comments about the advantages of using web-based resources reflected these beliefs.

> They step things out for you. (Bath)
> There is such a wide variety of resources you can nearly always find relevant information. (WW)

At times, this wealth of information was perceived as being overwhelming and difficult to handle because they could not judge its validity.

> Get lost. Too much info. Not easy to direct around Internet. (WW)

By providing them with the web-links and a short description of what was on each website, we had hoped to overcome the difficulties of students finding appropriate sites.

Although some PPTs felt that web-based resources allowed them to learn at their own pace, there were differences of opinion in whether websites provided the interactivity that many viewed as important for their learning. Comments about not getting individual help from websites came almost exclusively from Bathurst students.

> Can access them at home, and can target the area I feel I need to work on. (Bath)
> Because it will be like having someone there, I think I will use it all the time. Need a lot of help in maths. (WW)
> If you don't understand, you can't get instant support. (Bath)

However, for some PPTs, their nervousness about having to sit a mathematics test meant that any help was welcome.

> All the help I can get. Maths is one of my weakest subjects. (Bath)

In answering a question about the support that they needed for studying for the Mastery Test, some PPTs at both campuses valued support from lecturers or other PPTs. More comments of this kind came from Wagga Wagga based PPTs.

> Teacher support, when I get a clear explanation of a problem. (Bath)
> One-on-one, go through questions in-depth and show me how to get the answer, group work. (WW)

In responses about the support they needed, previous experiences of learning for mathematics tests was evident in how they felt they should learn for this test. Although the preservice teachers had been provided with the 50 questions, 27 PPTs at Bathurst and 11 at Wagga Wagga requested practice question and previous test papers. The test was the focus for preservice teachers and at the beginning of the semester, they did not see the need to know material because they would be teachers.

> Need to go over the exam; have someone explain so I understand; write notes over it. (Bath)
> Explaining in depth how these questions really need to be answered. (WW)

Although almost all students in the initial survey stated they would at least try out the CD, in fact very few PPTs used it. Of the comments in the diaries, most indicated that PPTs looked for information on specific topics. The following comment about the Maths for Kids website illustrates this:

> Didn't understand long division. So helped lots. (WW)

For one preservice teacher, there were indications that she saw herself as learning because she was to become a teacher, not just to pass a test.

> Helpful in my understanding & a way of teaching it, too. (Bath)

After the test, the focus group interviews suggested that more PPTs could see links to being a teacher and this may well have been connected to another assessment where they had to work with primary school children.

> I used that [the CD] for one bit of it, for decimals and how to explain that, but, we actually used it for another assignment, the next assignment, for a problem solving exercise. (WW)

However, the availability of a computer and ICT skills continued to be a reason given by students for not using the CD.

> I was sweating a bit just to get a pass, I took the CD home and had a look and had a look at the links on it and I said you've got to be kidding me, I had very little amount of internet usage at home, so I wasn't going to go through all this at home. (Bath)

For some preservice teachers the CD was useless because it was not how they expected to learn.

> Like even having a teacher sitting there drawing on the whiteboard or something and showing, and then break it down into stages, because that's the way I learn, when it's broken down, if it's just given to me and I see there's the answer and you've got to figure

it out, I'm completely lost, but if it's broken down into different stages and I can see the logic, oh okay, so if I was given a similar one, I can probably work that out too. (Bath)

Even after completing the test and commenting on the importance of explaining the mathematical concepts to their potential students, these preservice teachers still seemed to see mathematics learning as involving an expert, such as a teacher, showing a novice, such as a student, how to do the problem by breaking it down into stages, in other words as procedural understandings of mathematics. In commenting on the test, one student said:

> I think it was worse than an HSC [High School Certificate] exam because teachers prepare you for HSC. Well the HSC you've got questions and teachers like they work over the whole broad and we weren't told really how they wanted us to explain it. (WW)

These comments, and there were many, showed not just how hard it is to ensure that preservice teachers value conceptual understandings of mathematics (Meaney & Lange, 2010) but that procedural understanding ingrains in them a particular view of what learning involves. This has implications for how they see their role as a teacher and what they expect from the students who will learn from them:

> Could someone just come up with some sort of handbook or even a CD or something, that says, okay, if you are teaching fractions, this is what you say, this is what you write on the board? (Bath)

Although these PPTs had two further semesters of mathematics education subjects to complete their degrees, there is an issue about overturning these ideas about mathematics and how it is taught. If this is not recognised and addressed specifically, it is likely that even with our best intentions, these preservice teachers will teach procedural mathematics from the whiteboard once they have graduated.

Conclusion

From this research, it is clear that PPTs arrive at university with a range of different ICT experiences, few of which seem to make them inclined to use ICT in their own learning. Partly, this has to do with their beliefs about how they learn mathematics.

In recent years, the idea of pedagogical content knowledge (Shulman, 1986) has enabled PPTs' lack of content knowledge to be researched in isolation from pedagogical knowledge. If pedagogical knowledge is mentioned then it is in relationship to mathematically-competent PPTs gaining the most from their mathematics pedagogical subjects (Capraro, Capraro, Parker, Kulm, & Raulerson, 2005). Our research suggests that this separation is unhelpful. Unless, we as teacher educators recognise that preservice teachers expect to learn in the way that their previous experiences suggest mathematics must be learnt, then we will struggle to convince them of the need for conceptual understanding and that it needs to be gained through active engagement. Like the mismatch that Skemp (1976) described between some teachers and their students on what 'understanding' was, we will continue to work together in our mathematics education subjects from non-intersecting views about what mathematics is and how it can be learnt.

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THE PUBLIC'S VIEWS ON GENDER AND THE LEARNING OF MATHEMATICS: DOES AGE MATTER?[1]

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In this study we build on Leder and Forgasz's (2010) examination of the public's perceptions about the learning of mathematics at school and its role in determining males' and females' career preferences. Data were gathered at 12 different sites throughout Victoria and via an innovative recruitment tool, the social network site Facebook. The latter provided a unique opportunity to target a wider audience across Australia. Our finding that younger respondents (under 40) were more likely than those over 40 to question girls' aptitude for mathematics is of concern.

Background to the study

Leder and Forgasz (2010) argued that "attempts to measure directly the general public's views about mathematics, its teaching and its impact on careers are rare" (p. 329), and noted that 20 years had passed since the *Maths Multiplies Your Choices* media campaign, aimed at encouraging parents to consider their daughters' careers, had been conducted.

The findings in this paper build on the small study reported by Leder and Forgasz (2010), as more data have been gathered. The focus here is on exploring age-related, rather than gender-related, differences in respondents' views.

Age as a variable of interest stemmed from trends suggesting that gender equity considerations may be less troubling to younger Australians than to those who lived through the struggles to achieve equity in the latter part of the twentieth century. In an interview on the eve of International Women's day (ABC, 2007), Sarah Maddison argued that having once been a leader in establishing gender equity, Australia had slid backwards, and many gains achieved were now undone.

Recent changes in generational differences in views on equity issues have been reported. In summarising results from several studies, Powlishta (2002, p. 169) claimed that "attitudes become more egalitarian with age" and that in their attributions of characteristics to males, females, or both/neither, "adults were less stereotyped in their attitudes than were children". According to Farley and Haaga (2000), however, while younger people are generally more liberal than their grandparents, in the US "young

[1] We thank Glenda Jackson for gathering the raw data reported in this paper, Hazel Tan for her help in setting up the Facebook survey, and Monash University for the financial support provided.

people have become more conservative, as has the rest of America" (p. 133). This trend is also evident in Australia. In their recent examination of repeated cross-sectional surveys (1986–2005) of Australians' beliefs about family roles and men's and women's work, van Egmond, Baxter, Buchler, and Western (2010, p.162) reported that:

> … on most of the issues examined here, Australian men and women have become increasingly more egalitarian in their views about gender arrangements. But the story is not so straightforward. The trends have taken a different direction since the mid-1990s … Over the last 10 years attitudes to gender arrangements have shifted and the trend toward liberalization has slowed markedly and possibly stalled.

The study

In this study we explore whether age-related differences are found in respondents' views on gender issues associated with mathematics learning.

Participants were given a brief summary of the study's aims as part of the *Explanatory Statement* required for obtaining ethics approval. Core elements are captured in the excerpt below:

> We have stopped you in the street to invite you to be a participant in our research study. ...We are conducting this research … to determine the views of the general public about girls and boys and the learning of mathematics. We believe that it is as important to know the views of the public as well as knowing what government and educational authorities believe.

Data were gathered from 12 different heavy foot-traffic sites throughout Victoria. To reach an even more diverse group, participants were also solicited via Facebook. The Facebook survey contained the same core items used in the face-to-face survey. Thus our data base comprised 13 different sites[2].

To ensure maximum participation, we limited the survey to 15 core items. These covered the learning of mathematics at school, perceived changes in the delivery of school mathematics, facility with calculators and computers, and aspects of careers.

Aims

In this paper we focus on items concerned with respondents' beliefs and their expectations of parents and teachers – significant figures in the learning environment of students – about the learning of mathematics. Whether responses differed by participants' age was of particular interest. Items relevant for this paper are listed in a later section.

Method

About four hours (morning or afternoon) were spent at each site to gather the face-to-face data. This yielded around 50 completed surveys per site, exceeding the minimum number considered adequate for data to be analysed using chi-square tests (Muijs, 2004). The procedures followed were described in some detail in Leder and Forgasz (2010) and are not repeated here. Instead, we focus on the routes followed in gathering the Facebook data.

[2] Respondents from various countries participated in the Facebook component. In this paper we restrict the sample to those who indicated they were residents of Australia.

Consultation with the University's Human Ethics Committee revealed that Facebook has rights to data collected from any applications, including surveys, created within Facebook. To avoid possible privacy and ethical issues (Hull, Lipford, & Latulipe, 2010) the questionnaire used in the larger study was duplicated as an online survey using SurveyMonkey (http://www.surveymonkey.com). A link was created to it from the advertisement placed on Facebook. Briefly, the procedure used (described in detail in Forgasz, Leder & Tan, 2011) was:

1. Set up a Facebook account.
2. Design a 110 x 80 pixel image for the advertisement.
3. Produce a destination URL when participants clicked on the advertisement.
4. Create a name for the advertising campaign and text.
5. Decide the target population: individuals aged over 18[3].
6. Select a daily budget.
7. Determine pricing: i. price per click willing to be paid (varied between 60 and 80 cents) and ii. daily budget (we settled on $60).
8. Decide the length of the campaign.
9. Provide additional information e.g., currency to be used and payment method.

A copy of the Facebook advertisement is shown in Figure 1.

Figure 1. The Facebook advertisement

Instrument

Our discussion is limited to responses to the following questions:

1. Should students study mathematics when it is no longer compulsory?
2. Who are better at mathematics, girls or boys?
3. Who do parents believe are better at mathematics, girls or boys?
4. Who do teachers believe are better at mathematics, girls or boys?
5. Do you think that studying mathematics is important for getting a job?
6. Is it more important for girls or boys to study mathematics?
7. Who are better at using calculators, girls or boys?

These items required simple responses: "yes", "no", "don't know"; or "boys", "girls", "the same", "unsure". All participants were invited to explain their answers. The comments reported in this paper are from those recruited via Facebook, to offer greater

[3] Ethics approval conditions influenced the decision to restrict the sample to participants over the age of 18.

insights into the beliefs of this group. Individuals also provided background information, including their age which was subsequently categorized as younger group (under 40) or older group (40 or older).

Sample

The sample comprised 689 (615 face-to-face, 74 Facebook) respondents. Of these, 327 were males and 362 were females; 361 were under 40 and 328 were 40 or older. As comparisons between the responses from face-to-face and Facebook respondents to the survey questions of interest revealed no statistically significant differences, the data were pooled for the analyses discussed in this paper.

Results

Response rate for Facebook participants

During the Facebook data collection period, we focused on Australia for five days[4]. There were 2,004,460 impressions, that is, the advertisement was shown just over two million times. These yielded 339 clicks on the advertisement and 62 (18%) respondents to the survey. This response rate is within the limits for mail surveys (between 10% and 50%) reported by McBurney and White (2004) in their comparison of response rates for different methods of survey administration.

Findings for the questions

The frequencies (and percentages) of responses to the seven survey items listed above are shown in Table 1.

Table 1. Frequency and percentage responses to survey items.

Item	**Yes**	**No**	**Don't know**	
Should students study mathematics when it is no longer compulsory?	436 64.3%	160 23.6%	82 12.1%	
Do you think that studying mathematics is important for getting a job?	523 77.8%	82 12.2%	67 10.0%	
	Boys	**Girls**	**Same**	**Unsure**
Who are better at mathematics, girls or boys?	149 22.2%	93 13.8%	263 39.1%	167 24.9%
Who do parents believe are better at mathematics, girls or boys?	156 23.2%	90 13.4%	161 24.0%	265 39.4%
Who do teachers believe are better at mathematics, girls or boys?	79 11.8%	85 12.6%	192 28.6%	316 47.0%
Is it more important for girls or boys to study mathematics?	24 3.6%	9 1.3%	610 90.8%	29 4.3%
Who are better at using calculators, girls or boys?	87 13.0%	53 7.9%	378 56.5%	151 22.6%

[4] Prior to this time, participants resident in many different countries, as well as a small number of respondents from Australia, responded to the advertisement. In this paper only data gathered from respondents living in Australia are considered.

In Table 2 the frequencies (and percentages) of responses by age group are shown. Chi-square tests were used to determine if the frequency distributions of the responses by age group were statistically significantly different; the outcomes of the chi-square tests are also provided.

Table 2. Frequency and percentage responses to survey items by age group, and χ^2 significance levels.

	Under 40				**40 plus**				**Sig. level**
Item	Yes	No	Don't know		Yes	No	Don't know		
Should students study mathematics when it is no longer compulsory?	223 63.7%	89 25.4%	38 10.9%		213 64.9%	71 21.6%	44 13.4%		ns
Do you think that studying mathematics is important for getting a job?	250 72.5%	52 15.1%	43 12.5%		273 83.5%	30 9.2%	24 7.3%		p<.01
	Boys	**Girls**	**Same**	**Unsure**	**Boys**	**Girls**	**Same**	**Unsure**	
Who are better at mathematics, girls or boys?	89 25.8%	33 9.6%	161 46.7%	62 18.0%	60 18.3%	60 18.3%	102 31.2%	105 32.1%	p<.001
Who do parents believe are better at mathematics, girls or boys?	78 22.6%	44 12.8%	98 28.4%	125 36.2%	78 23.9%	46 14.1%	63 19.3%	140 42.8%	p<.05
Who do teachers believe are better at mathematics, girls or boys?	46 13.3%	42 12.2%	119 34.5%	138 40.0%	33 10.1%	43 13.1%	73 22.3%	178 54.4%	p<.001
Is it more important for girls or boys to study mathematics?	11 3.2%	4 1.2%	316 91.6%	14 4.1%	13 4.0%	5 1.5%	294 89.9%	15 4.6%	ns
Who are better at using calculators, girls or boys?	56 16.3%	20 5.8%	215 62.7%	52 15.2%	31 9.5%	33 10.1%	163 50.0%	99 30.4%	p<.001

In the subsequent discussion of the findings, all data referred to can be found in Tables 1 and 2.

Should students study mathematics when it is no longer compulsory?

Almost two-thirds (436: 64.3%) of those responding answered this question affirmatively, fewer (160: 23.6%) disagreed, and the rest (82: 12.1%) were equivocal. A chi-square test revealed no statistically significant differences by respondent age. Reasons given for the need to continue studying mathematics included:

> Develops analytical skills and rigorous thought. Gives a deeper understanding in sciences and other areas of thought, such as computer science and economics. Suitable also for those interested in philosophy and the pure arts, such as music (specifically, musical

composition). ...Mathematics is the underlying basis to understanding the modern world, on either a physical or social level (for example, in economics). (younger)

Maths is important. But it would be too easy to not do it. A person will always take the easy way out if possible. (older)

Explanations from those who disagreed included:

Some people don't need it. You don't need maths to become an author. My dad tells me that he has learned algebra, yet he thinks he never has used it in his life and he is a taxi driver. My mum is a bit crazy over maths, keeps pushing for me to study it ... Why learn some maths, like velocity if you won't need it for your career? (younger)

Depends on the level – my attitude is that some math should be compulsory up to year 5. After that there would be little point in forcing students to do something they don't like or have been failing at. (older)

Who are better at mathematics, girls or boys?

Approximately one-third (263: 39.1%) of the sample thought boys and girls were equally good at mathematics; a quarter (167: 24.9%) was unsure. Of the remainder, 149 (22.2%) thought boys were better; fewer believed girls were better (93: 13.8%). A chi-square test revealed a statistically significant difference in replies by respondent age (χ^2 = 37.335, p<.001, df = 3. Effect size (φ) =.24). Almost twice as many older (60: 18.3%) than younger (33: 9.6%) respondents thought girls would be better. Although more younger (161: 46.7%) than older (102: 31.2%) respondents thought there would be no difference between girls and boys, more younger (89: 25.8%) than older (60: 18.3%) respondents also thought boys would be better.

Reasons given included:

Same: I have known both girls and boys that are equally good at mathematics. The boys tend to use it more and appear to pursue it, but girls can be equally as good. (younger)

Don’t know: Girls tend to do better in the earlier school years but boys do better later so by year 6 boys are generally ahead. However some of my female friends have been physics lecturers so... (older)

Boys: I've met both males and females who are good at math. Though I have only met males who are exceptional at mathematics. (younger)

Girls: Boys tend to have worse concentration than girls (younger)

Who do parents believe are better at mathematics: girls or boys?

Most respondents were unsure (265: 39.4%) or thought that parents believed that there would be no difference (161: 24%). Almost a quarter (156: 23.2%) thought parents assumed boys would be better, while 90 (13.4%) thought parents considered girls were better. A chi-square test revealed a statistically significant difference in answers by respondent age (χ^2 = 8.026, p<.05, df = 3. Effect size (φ) =.11). Older participants were less likely to believe that parents would rate them the same (older, 63: 19.3%; younger, 98: 28.4%) but were more likely to be unsure (older, 140: 42.8%; younger, 125: 36.2%). Those answering “boys” or “girls” were more likely to provide explanations:

Boys get more encouragement and positive reinforcement for achievement in maths. (younger)

I find this slightly discriminating... but anyway...probably guys. (younger)

In my experience parents are more willing to think boys are better, even if subconsciously. (younger)

Boys: Girls tend to be quiet achievers. (older)

Who do teachers believe are better at mathematics, girls or boys?

The response pattern to this item was similar to the question about parents' views. Most were unsure what teachers believed (316: 47%) or thought that teachers would consider girls and boys to be equally good at mathematics (192: 28.6%). The remainder was evenly divided whether they believed teachers thought boys (79: 11.8%) or girls (85: 12.6%) were better. A chi-square test revealed a statistically significant response difference by respondent age ($\chi^2 = 17.766$, $p<.001$, df = 3. Effect size (φ) =.16). Older respondents were most likely to indicate they were unsure (178: 54.4%), while many younger respondents (138: 40%) thought teachers believed that girls and boys were equally good at mathematics. Few provided reasons for their answers, with several re-iterating their explanations to the question about parents.

Do you think studying mathematics is important for getting a job?

A clear majority (523: 77.8%) answered affirmatively. The remainder disagreed (82: 12.2%) or were ambivalent (67: 10%). A chi-square test revealed a statistically significant difference in replies by respondent age ($\chi^2 = 11.828$, $p<.005$, df = 2. Effect size (φ) =.13). Older participants were more likely to believe mathematics was important for getting a job (older, 273: 83.5%; younger, 250: 72.5%) and less likely to be uncertain (older, 24: 7.3%; younger, 43: 12.5%).

Elaborations on the answer given included:

Creative reasoning is encouraged in mathematics. A mathematical background assists a person to problem-solve on a conceptual rather than specific level, an inherently valuable trait to many professions. Employers recognise the ability to think laterally, logically and creatively, while developing conceptual and innovative solutions to particular problems; mathematics trains the mind to do this. It is, therefore, inherently valuable to attaining a job in the field of choice. (younger)

Clearly depends on the job - but in almost every walk of life a better understanding of the processes/machinery/product is enhanced by a better understanding and almost inevitably some maths is required for that. Even fine arts and music. (older)

Is it more important for girls or boys to study mathematics?

Almost all (younger, 316: 91.6%; older, 294: 89.9%) considered it equally important for boys and girls to study mathematics.

There is becoming less of a gap between "male" jobs and "female" jobs. There is no reason that it's more important for one gender than another. (younger)

They are quite likely to be going for the same jobs so need the same skills (older)

Just under half (29: 4.3%) of the remaining 10% were equivocal. Of the rest, a slightly higher proportion of respondents considered mathematics more important for boys

(24: 3.6%) than for girls (9: 1.3%). A chi-square test revealed no statistically significant response differences by respondent age.

Who are better at using calculators, girls or boys?

Over half the respondents (378: 56.5%) thought there would be no difference. A chi-square test revealed a statistically significant response difference by respondent age ($\chi^2 = 31.744$, $p<.001$, df = 3. Effect size (φ) =.22). More older respondents nominated girls as the better group (younger, 20: 5.8%; older, 33: 10.1%), and more younger respondents nominated boys (younger, 56: 16.3%; older, 31: 9.5%).

Comments included:

All gen-y kids are very tech savvy, this is not restricted to one gender. (younger)

I suspect they are both the same. (older)

Boys: Calculator = Machine (somewhat) = Boys will operate it better. (younger)

Discussion

Of the seven (out of 15) survey questions examined in this paper, no statistically significant differences by respondent age were found on two. About two-thirds of respondents indicated that pursuing mathematics beyond the compulsory period was important. Almost all believed that mathematics was equally important for girls and boys. These are heartening findings.

A statistically significant difference by respondent age was found on items regarding the importance of mathematics for getting jobs. The older cohort agreed more strongly than the younger respondents who were less certain.

The four remaining items with statistically significant differences by respondent age involved gender-related beliefs. The older group was more convinced that teachers and parents would consider boys and girls to be equally good at mathematics, and was less equivocal than the younger cohort. Compared to older respondents, the younger cohort was more likely to consider boys to be better than girls at mathematics and also better with calculators. Collectively these findings suggest that while those under 40 believed that parents and teachers were likely to be more egalitarian, they themselves hold more strongly than those in the older group to the traditional gender-stereotyped view that boys are more suited to and successful in mathematics than girls. These data imply a backwards slide in Australians' views of gender equity in mathematics. Whether the findings of this study link to the small but consistent gender gap favouring boys in NAPLAN results and the increasing gender gap in Australian results in TIMSS over time (see Leder & Forgasz, 2010) needs to be explored further.

Reflections at the time of the celebration of the 100th International Women's Day indicate that this "backward slide" is not unique to issues linked to the learning of mathematics. As noted by Cox (2011, p. 13), "[T]he F-word [Feminism] is not very popular these days with many younger women, who feel it does not relate to their lives. Newman (2011) questioned whether schools have a role to play with respect to gender issues in schooling. He claimed that "[A]fter more than a century of struggle, feminists say gender inequality is alive and well… The extent to which gender-related themes are incorporated in lessons remains at teachers' discretion… (and) the reality remains that there is no unit of study in the (Victorian) state secondary curriculum devoted to gender issues" (p. 14). In the past, interventions had a place in raising awareness of gender issues in mathematics and science learning but over time funding dried up. The extent to which pre-service teacher education programs now incorporate gender-related issues in their curricula varies from institution to institution. The findings of the present study suggest that some action is again needed to alert the teaching profession and the general community that differences remain in perceptions of boys' and girls' mathematics capabilities and future career potential.

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EFFECTS OF USING HISTORY OF MATHEMATICS ON JUNIOR COLLEGE STUDENTS' ATTITUDES AND ACHIEVEMENT

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Many researchers advocate the use of history of mathematics in education. However, empirical research in this area is scarce. To address this issue, a quasi-experiment is used to investigate the relationship between the use of history of mathematics, and the attitudes and achievement of junior college students in Singapore. Multivariate analysis of covariance and analysis of covariance, with pre-test scores as covariates, and post-test scores as dependent variables, suggest that history of mathematics can improve students‟ attitudes and achievement in mathematics.

Introduction

The benefits of using history of mathematics in education have been widely discussed by researchers globally. A list of these benefits are summarised in Fauvel and Van Maanen (2000). Many researchers have suggested that the use of history of mathematics in education are related to positive students‟ learning outcomes (Calinger, 1996; Fauvel, 1991) and supported its inclusion in national curricula (Fauvel & Van Maanen, 2000). However in practice, history of mathematics is rarely used in schools (Fried, 2001), as teachers cannot commit the time to prepare the relevant teaching materials (Fauvel, 1991; Fowler, 1991; Gulikers & Blom, 2001). To reap the benefits of using history in mathematics lessons, researchers must first convince education planners and teachers that the time spent in preparing these teaching materials can translate to better students‟ learning outcomes. One possible way is to provide empirical evidences that demonstrates positive relationships between the history of mathematics and desirable students‟ learning outcomes. However, such studies are either lacking or dated. Hence this study uses a quasi-experiment to investigate the relationship between the use of history of mathematics in classrooms, and students‟ attitudes and achievement in mathematics.

Literature Review

Lim and Chapman (in preparation, a) review definitions of history of mathematics used in education by various researchers and suggest that it should include (1) the use of anecdotes and biographies of mathematicians (Bidwell, 1993; Wilson & Chauvot, 2000); (2) the discussion of historical motivations for the development of content (Katz, 1993); and (3) the use of original materials from historical sources (Arcavi & Bruckheimer, 2000; Jahnke et al., 2000). This study adopted these definitions.

Theoretical literature review

Many researchers have argued that the use of history in mathematics lessons can lead to better attitudes and achievement in mathematics (e.g., Fauvel, 1991; Gulikers & Blom, 2001). Firstly, the use of anecdotes and biographies of mathematicians make lessons more interesting and dynamic (Perkins, 1991; Siu, 1997). Hence students should find their mathematics lessons more enjoyable when history is used.

In addition, students are more motivated to learn about mathematics if they are able to identify the important role that mathematics play in human culture through history (Tymoczko, 1994). In addition, students are able to appreciate the usefulness of mathematics in real life through history, as mathematical concepts are often developed to solve real-life problems in the past (Burton, 1998). Consequently, the use of history in lessons can improve students‟ perception about the value of mathematics to mankind and motivate them to learn mathematics.

Moreover, learning takes place more effectively when a learner retraces the key steps in the historical development of the subject (Gulikers & Blom, 2001). As mathematical concepts are often oversimplified in textbooks and by teachers (Freudenthal, 1991; Siegel & Borasi, 1994), students may not be able to understand these concepts which are often broken up into smaller parts and presented to them from an expert‟s viewpoint (Tall & Vinner, 1981). Showing students the historical development of mathematical concepts can help them to see the links between the broken parts and improve their understanding of these concepts (Furinghetti, 2000), which may translate to better achievement test scores.

Finally, teachers can better understand the common difficulties faced by current students by examining the errors and misconceptions of past mathematicians (Moru, Persens, Breiteig, & Ndalichako, 2008; Sierpinska, 1992). They can then take pre-emptive measures to ensure more effective learning (Sfard, 1994). At the same time, when students realise that it is common for mathematicians to commit errors and learn from their own or others‟ mistakes, they appreciate that collaboration and perseverance are necessary to derive mathematical concepts which they often feel are beyond their ability to derive or understand initially. This gives them the confidence to explore and participate in mathematical activities (Siu & Siu, 1979). Hence, the use of history in mathematics lessons is a feasible technique to improve students‟ attitudes and achievement.

Empirical Literature Review

Although many articles discuss about the benefits of using history of mathematics in education, few empirical studies on history of mathematics have been conducted (Gulikers & Blom, 2001). Amongst these studies, most suggest that history of mathematics leads to better student learning outcomes. For instance, a qualitative study by Ponza (1998) reported that Grade 7 students displayed better attitudes toward mathematics after they worked on a project about the life of a French mathematician. In another qualitative study (Dittrich, 1973), Grade 11 and 12 participants showed greater interest in mathematics after the researcher used biographies of mathematicians and historical sources in their lessons. No significant improvement in the mathematics achievement of these participants was observed as improving problem solving skills was not one of the foci of the course. These qualitative studies suggest a positive relationship between the use of history and students" attitudes toward mathematics. However, quantitative studies are necessary to further convince educationists of this relationship.

One of the earliest quantitative studies was conducted by McBride and Rollins (1977) who used original proofs by famous mathematicians in a college algebra course in the United States. ANCOVA, with initial attitudes scores as covariates, showed a significant improvement in the experimental group"s attitudes toward mathematics.

More recently, a quasi-experiment was conducted by Lit, Siu and Wong (2001) to examine the effect of using the historical development of the Pythagorean Theorem on the attitudes of Secondary 2 (Grade 8) students in Hong Kong. *T*-test results indicated a significant improvement in attitudes toward mathematics. However, the experimental group did not display a significant improvement in mathematics achievement. As the participants were assessed on different topics for pre-test and post-test, the achievement tests used in this study produced questionable results. In addition, *t*-test does not account for initial differences between the experimental and control groups. ANCOVA, with pre-test scores as covariates, may produce more meaningful results.

Ng (2006) investigated the effects of an Ancient Chinese Mathematics Enrichment Program (ACMEP) on the mathematics achievement of 414 secondary two (Grade 8) students in Singapore. ANCOVA, with achievement pre-test scores as covariates, showed that the experimental group performed significantly better than the control group. However, the results need to be interpreted with care as participants could choose whether or not to participate in the ACMEP. Students who opted to participate in the program might have better attitudes toward mathematics in the first place. Future experiments should remove this assignment bias to obtain more conclusive results.

Conscious of the dearth of studies on the use of history of mathematics, especially at the junior college and polytechnic level in Singapore, Ho (2008) conducted action research on the effects of using history of algebra on the attitudes of 102 polytechnic students. Participants in the experimental group performed better in two domains of attitudes, namely belief and perseverance, than their counterparts in the control group. As part of the study, Ho also developed a test to measure attitudes toward mathematics. However, the test had not been assessed for reliability and validity. Hence, the results of this study may be disputable.

In view of the weaknesses in the above-mentioned studies, this study aims to use a quasi-experiment to investigate the relationship between history of mathematics and students" attitudes and achievement in mathematics. This study hypothesizes that students who are introduced to history of mathematics in their lessons have better attitudes and achievement post-test scores than students who do not, after pre-test score differences are statistically controlled using multivariate analysis of covariance (MANCOVA) and analysis of covariance (ANCOVA).

Methodology

A quasi-experiment in which participants were selected from existing classes in a junior college was used in this study.

Participants

Four college year one (Grade 11) classes taught by the same teacher participated in this quasi-experiment. Two classes, one with class size 26 and the other 25, were assigned to the experimental (history of mathematics) group (total $n = 51$) and the other two classes, each with class size 26, were assigned to the control (no history of mathematics) group (total $n = 52$). The assignment of the classes to the two groups was done randomly. The participants had an average age of 17 years.

Research Design

Table 1 illustrates the design of this research. The use of history of mathematics is denoted by X. The same teacher went through a standard set of tutorial questions with both the experimental and control groups. To ensure that the same amount of time was spent with each group, the teacher went through the tutorial questions at a slower pace with the control group. All participants took three sets of achievement pre-tests and post-tests on three calculus topics: (1) techniques of differentiation, (2) applications of differentiation, and (3) integration. To minimize carry-over effects, the same set of attitudes tests was administered only twice to both groups of participants, once before the treatment and once after the last treatment session.

Table 1. Research design of the study.

Achievement	Control ($n = 52$)	O_1		P_1	O_2		P_2	O_3		P_3
	Experimental ($n = 51$)	O_1	X	P_1	O_2	X	P_2	O_3	X	P_3
Attitudes	Control ($n = 52$)	A_1								A_2
	Experimental ($n = 51$)	A_1	X			X			X	A_2

Legend:
X: Treatment (history of mathematics)
O_r: Achievement pre-test r, for $r = 1$, 2 and 3.
P_r: Achievement post-test on calculus topic r, for $r = 1$, 2 and 3.
A_1: Attitudes pre-test
A_2: Attitudes post-test

Variables

The independent variable in this study is the use of history of mathematics in lessons and the dependent variables are students" attitudes toward mathematics and achievement in mathematics. The confounding variables are pre-test differences between the experimental and control groups. Table 2 shows the variables that are held constant across both the experimental and control groups.

Treatment

The teaching package for the experimental group was aligned with the definitions of history of mathematics used in this study.[1] For instance, the dispute between Sir Isaac Newton and Gottfried Wilhelm Leibniz on who has precedence over the development of the fundamental theorem of calculus (Hall, 1980) was used as an anecdote to arouse students" curiosity and interest. The experimental group was also introduced to the

[1] Readers may contact the author for the complete teaching package on history of mathematics that was used with the experimental group.

motivation behind the development of mathematical concepts in the past, such as the development of differential calculus by Newton to solve physics problems on motion (Grabiner, 1983). Furthermore, original materials from historical sources such as the proof for Snell‟s law using differentiation was used to illustrate to students the use of calculus to prove a formula that they were asked to memorise without understanding when they were in lower grades.

Table 2. List of variables held constant.

Variables Held Constant	Remarks
Age and year of study	Same age group of 16 and 17 year old college year 1 students taking the General Certificate of Education Advanced Level (GCE „A‟ level) 9740 H2 mathematics examination administered by the University of Cambridge-London Examination Syndicate (UCLES).
Participating school	All participants were from the same junior college.
Number and duration of lessons	All participants attended 16 lectures and 22 tutorial sessions, conducted over a period of four months. Each lecture session and each tutorial session lasted one hour. History of mathematics was used in at least 40% of the tutorial time for the experimental group.
Teacher and teaching approach	All participants were taught by the same teacher using a teacher-centred approach.
Students‟ notes and tutorial questions	The same set of lecture notes and tutorial questions without any mention of history of mathematics was used for both groups and for all topics.

Instruments

Mathematics attitudes

The Attitudes Toward Mathematics Inventory (ATMI) (Tapia & Marsh, 2004) and the modified Academic Motivation Scale (AMS) (Lim & Chapman, in preparation-a) were used for both pre-test and post-test. The psychometric properties and factor structures of both instruments had been established using participants with similar profiles to the participants of this study by Lim and Chapman (in preparation-a, in preparation-b).

The ATMI consists of 40 items that measure general attitudes toward mathematics using four factors, namely enjoyment, general motivation, self-confidence and value, on a five-point Likert Scale that ranges from strongly disagree to strongly agree. On the other hand, the modified AMS is made up of 20 items that address the question "Why do you spend time to study mathematics?" It measures participants‟ motivation toward mathematics using five factors that exist on a self-determination continuum (Deci & Ryan, 1985). These five factors can be categorised from lower to higher level of self-determination, into amotivation, external regulation, introjection, identification and intrinsic motivation. Each factor consists of four items and is measured on a 5-point Likert scale that ranges from "does not correspond at all" to "corresponds exactly".

Mathematics achievement

Three sets of achievement tests which were equivalent in terms of difficulty level and content were constructed to measure mathematics achievement. The questions for the tests were modified from past years GCE „A‟ level questions. To assess the content validity of the achievement tests, a setter of the GCE „A‟ level 9740 H2 mathematics paper from UCLES, Dr. Lionel Elliot, was invited to review the test items and mark schemes, and to provide feedback on their validity. He also verified that the content and difficulty level of the three sets of tests were similar. All the participants" scripts were evaluated by the same teacher based on the mark schemes. To assess the alternative form reliability (Clark-Carter, 2004) of the tests, a pilot study was conducted on 73 participants who were similar in profiles to the participants of this study. High reliability over a one-month duration for all pre-tests and post-tests was reported (average $r = 0.87$, $p = 0.01$).

Results and Discussion

All results of this study were analysed using SPSS 19 (SPSS Inc., 2010). Table 2 presents the mean scores and standard deviations of the post-tests scores of the ATMI, the modified AMS and the three achievement post-tests. The achievement scores are given in percentages. Pre-test scores are used as covariates which are statistically controlled using MANCOVA and ANCOVA when comparing the post-test scores of the experimental and control groups. All the underlying assumptions of MANCOVA and ANCOVA are supported by the data.

Table 2. Descriptive statistics of ATMI, the modified AMS and achievement test scores.

		Control (*n* = 52)		**Experimental (*n* = 51)**	
	Factors	**Mean**	**SD**	**Mean**	**SD**
Attitudes Tests					
ATMI	Enjoyment	3.31	0.80	3.40	0.75
	Motivation	3.10	0.86	3.13	0.75
	Self-confidence	3.54	0.71	3.71	0.84
	Value	3.65	0.59	3.88	0.45
Modified AMS	Amotivation	1.82	0.89	1.65	0.84
	Intrinsic motivation	3.00	0.85	3.28	0.75
	Identification	3.20	0.72	3.42	0.73
	Introjection	2.72	0.85	3.14	0.99
	External regulation	2.81	0.67	2.90	0.99
Achievement Tests					
Test 1 (Techniques of Differentiation)		68.46	18.93	77.16	12.18
Test 2 (Applications of Differentiation)		53.01	17.90	62.55	14.69
Test 3 (Integration)		40.77	19.57	56.21	19.70

Results and discussion on mathematics attitudes

Except for amotivation which contains negatively-worded statements, the experimental group reported higher scores for all domains of attitudes in the ATMI and the modified AMS.

Results of ATMI

There is a significant effect of the treatment variable (history of mathematics versus no history of mathematics) on the combined dependent variables of the ATMI, $F(4, 94) = 2.70$, $p = 0.035$, partial $\eta^2 = 0.103$. Analysis of each dependent variable shows that the experimental group performs significantly better in terms of value, at a Bonferroni adjusted α level of 0.0125, $F(1, 97) = 6.75$, $p = 0.011$, partial $\eta^2 = 0.065$.

Results of the modified AMS

After reversing the scores for the amotivation items, the results of the modified AMS show a marginally significant effect of the treatment variable on the combined dependent variables, $F(5, 90) = 2.31$, $p = 0.051$, partial $\eta^2 = 0.031$. Analysis of each dependent variable shows that the experimental group performeds significantly better in terms of intrinsic motivation ($F(1, 94) = 4.94$, $p = 0.029$, partial $\eta^2 = 0.050$) and introjection ($F(1, 94) = 7.07$, $p = 0.009$, partial $\eta^2 = 0.070$), at a Bonferroni adjusted α level of 0.01.

Discussion

Burton (1998) proposes that history of mathematics allows students to see the usefulness of mathematics in real-life. This is supported by the results of this study as the experimental group performed significantly better than the control group in terms of value. The experimental group is also significantly more intrinsically motivated and introjected than the control group after the use of history. This finding is in line with the arguments of D‟Ambrosio (1995) who suggests that students are more motivated to learn mathematics if they explore mathematicians‟ cognitive activities in the past, as this allows them to appreciate the role of human minds in constructing mathematical knowledge.

Although the experimental group performed better in other domains of attitudes such as enjoyment and self-confidence as proposed by Perkins (1991) and Siu and Siu (1979), results are not significant. Future research with a bigger sample size may produce more conclusive results. In general, the positive results on attitudes after the use of history of mathematics are similar to those of previous empirical studies such as McBride and Rollins (1977) and Lit, Siu and Wong (2001). Although results are significant only for a few domains of attitudes, namely value, intrinsic motivation and introjection, these domains of attitudes are positively related to desirable student learning outcomes such as high academic achievement, low anxiety and low dropout rate from school (Gottfried, 1982; Vallerand et al., 1993). This implies that history of mathematics is beneficial to students and should be used by teachers in schools.

Results and discussion on mathematics achievement

The experimental group performed better in all three achievement post-tests. ANCOVA was conducted on each of the three achievement tests, with pre-test score on the same topic as covariate. There is a statistically significant effect of the treatment variable on Test 1 ($F(1, 100) = 9.72$, $p = 0.002$, partial $\eta^2 = 0.089$) and Test 3 ($F(1, 100) = 15.78$, $p = 0.001$, partial $\eta^2 = 0.136$).

This result affirms Furinghetti‟s (2000) suggestion that learning can be made more effective through the use of history. Interestingly, the achievement results by Lit, Siu and Wong (2001) are in contrast to the results of this study and Ng (2006)‟s study. As the study by Lit, Siu and Wong took place over only three weeks, while this study and Ng‟s study took place over four months and seven months respectively, time may be a necessary factor to observe better achievement test results after the use of history.

However, the improvement in achievement may be a result of better attitudes, rather than a direct consequence from the use of history of mathematics. Further research is required to investigate the effects of history of mathematics on academic achievement.

Conclusion

This study aims to convince educators about the benefits of using history of mathematics in schools through a quasi-experiment. Results show that there is a significant and positive relationship between history of mathematics and students‟ achievement and certain domains of attitudes that are in turn positively related to other desirable students‟ learning outcomes. Hence the use of history of mathematics in classrooms is highly recommended.

However, this study is not without limitations. Firstly, the experiment involved the teaching of calculus topics only. Secondly, the participants of this study came from only one junior college in Singapore. Further studies that involve other mathematical topics, and more participants from different schools and countries are necessary to generalise results. Qualitative data can also be collected through the administration of journals, interviews, and recordings of classroom activities to better understand how students‟ attitudes and achievement are affected by the use of history in their mathematics lessons.

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WE CAN ORDER BY ROTE BUT CAN'T PARTITION: WE DIDN'T LEARN A RULE

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Improving numeracy performance of all students across Victoria is a government priority. A key element of this initiative lies with tertiary institutions that are responsible for adequately preparing pre-service teachers for teaching primary mathematics. This paper examines data from a larger, longitudinal study of primary pre-service teachers' mathematical content knowledge and focuses on responses to fraction tasks by nine pre-service teachers in the study who are in the final year of their course. Two dimensions were used to categorise their responses. The majority of these pre-service teachers did not demonstrate a fluid and flexible knowledge of fractional numbers; half demonstrating a regression in their knowledge of this topic since the beginning of the course. These pre-service teachers will be challenged when working with students who have a wide range of numeracy experiences and abilities.

Introduction

The aim of the larger study is to identify when, what and how primary pre-service teachers' mathematical content knowledge (MCK) develops during their course as there have been few longitudinal studies completed on how teachers' mathematical knowledge changes over time (Ball, Bass, & Hill, 2004). Teachers need, use and develop their MCK to understand mathematical concepts and processes as they teach (Chick, Baker, Pham, & Cheng, 2006; Hill, Ball, & Schilling, 2008; Ma, 1999; Rowland, Turner, Thwaites, & Huckstep, 2009; Shulman, 1986). Two important dimensions teachers need to know in order to teach are foundation knowledge, that is the content primary mathematics' teachers must draw on, as well as an understanding of how to make connections within and between topics (Rowland et al., 2009). It is important to examine issues relating to the development of pre-service teachers' MCK in order to plan and improve pre-service teaching as a means for also improving school students' numeracy outcomes.

Background

Forms of teacher knowledge

Teachers require a range of knowledge to draw on when teaching. For the past two decades research on mathematics teaching has included a focus on the knowledge teachers‘ use and need for their craft of teaching (Grossman & McDonald, 2008). Shulman (1986) was one of the first to categorise the characteristics for distinguishing teacher knowledge: content knowledge, pedagogical content knowledge (PCK) and curricular knowledge. Since then scholars have continued to understand and build on this work (Chick, Baker et al., 2006; Hill et al., 2008; Ma, 1999; Rowland et al., 2009).

The focus of this study is MCK or content knowledge. Shulman (1986) described content knowledge as the amount and organisation of knowledge in the mind of the teacher. Later studies have expanded on Shulman‘s definition and unpacked its complexities. Ma‘s (1999) study of Chinese teachers identified a form of content knowledge as a thorough understanding of the mathematics, having breadth, depth, connectedness and thoroughness, referring to this as Profound Understanding of Fundamental Mathematics (PUFM). Demonstrating mathematical connections and fluency of concepts and procedures is a key feature of PUFM.

According to current literature, MCK includes three facets: common content knowledge (CCK), specialised content knowledge (SCK) and knowledge at the mathematical horizon (Ball, Thames, & Phelps, 2008; Hill et al., 2008). CCK is simply when someone is able to calculate an answer and correctly solve a mathematical problem whereas SCK is unique to teaching (Ball et al., 2004; Ball et al., 2008). Teachers use SCK for identifying a range of solutions and mathematical connections when working with students, planning lessons and evaluating students‘ work (Chick, Pham, & Baker, 2006; Schoenfeld & Kilpatrick, 2008). Advanced content knowledge is evident when the teacher demonstrates a broad understanding of the complexities of MCK, for example; how mathematical ideas connect to the mathematics they are teaching, demonstrating peripheral vision of the curriculum. When a teacher models this advanced knowledge they are said to have an understanding of knowledge at the mathematical horizon (Ball & Bass, 2009; Hill et al., 2008).

Knowledge of content matters for teaching and excellent teachers of mathematics demonstrate a sound, coherent knowledge of mathematics appropriate to the students they teach (Australian Association of Mathematics Teachers [AAMT], 2006; Ball et al., 2004; Ma, 1999; Schulman, 1986, 1987). However, the literature continues to report on teachers‘ and pre-service teachers‘ gaps and weaknesses relating to their MCK (Fennema & Franke, 1992; Goos, Smith, & Thornton, 2008; Rowland et al., 2009).

Newton‘s (2008) review of the literature reported studies of elementary (primary) pre-service teachers‘ fraction knowledge to be limited and studies had focussed mainly on division, for example Ma‘s study (1999). Her study of 85 American elementary pre-service teachers‘ included administrating a written test at the beginning and at the end of a semester-long course designed to increase their knowledge of fractions. Results showed an improved knowledge of the four operations with fractions in the post test responses, but there was little flexibility demonstrated in the methods used when solving problems in both the pre-test and post-test. Further studies, including longitudinal studies could contribute to this gap in research.

A framework for Mathematical Content Knowledge

Rowland et al.‘s (2009) *The Knowledge Quartet* was implemented when working with pre-service teachers (trainee teachers) as a tool for identification and discussion of four important dimensions for describing the types of MCK required to teach mathematics well: foundation, transformation, connection and contingency (see Table 1). Foundation is described as the knowledge possessed and the other three dimensions rely on conceptual connections for teaching.

Table 1. The codes of the Knowledge Quartet (Rowland et al., 2009, p. 29).

Foundation	Adheres to textbook Concentration on procedures Overt subject knowledge Use of terminology	Awareness of purpose Identifying errors Theoretical underpinning
Transformation	Choice of examples Demonstration	Choice of representation
Connection	Anticipation of complexity Making connections between procedures Recognition of conceptual appropriateness	Decisions about sequencing Making connections between concepts
Contingency	Deviation from agenda Use of opportunities	Responding to children‘s ideas

Methodology

The study and selecting participants

This paper reports on part of a larger study that includes a longitudinal qualitative component that explores the on-going learning of mathematics of 17 pre-service teachers in the various settings they encounter during the Bachelor of Education programme. This cohort is completing a Bachelor of Education Prep to Year 12 teaching course and will have qualifications to teach in primary and secondary schools. Their secondary qualification is aligned to particular discipline specialisations studied during the course and may or may not include mathematics. The 17 pre-service teachers had volunteered to participate in this longitudinal study.

For this study data collection involved qualitative analyses for nine pre-service teachers‘ responses to fraction items undertaken in the second-year of the course and again in the fourth-year of the course. Five of these pre-service teachers were not Mathematics majors and four were Mathematics majors. This study compared their results and responses to fraction items: one question selected from a Mathematical Competency, Skills and Knowledge Test (MCSKT) and responses to four items answered two years later during an individual interview. The remaining seven pre-service teachers were not selected for this study because they were studying at a different campus and had completed a different MCSKT.

Instruments

The pre-service teachers were given two questions, the first during second-year and the second at the end of fourth-year of their course. These fraction problems were selected to investigate their thinking used to solve two similar but different fraction problems; both were ordering tasks.

Second-year fraction test question. During the second year of the course all pre-service teachers (including pre-service teachers in the longitudinal study) completed a MCSKT to assess their mathematical knowledge of mainly number topics, for example fractions, decimals, percentage and ratio. The MCSKT consisted of 49 questions; pre-service teachers provided short answers using words or symbols and were encouraged to record their working out. All test items ranged in difficulty examining mainly procedural knowledge to a Year 8 standard. No calculator was permitted. For Question 19 pre-service teachers were asked to order a set of (fractional) numbers (0.42, two fifths, 4/9, 0.44 and one third) from least to greatest (Table 2).

Fourth-year fraction items. During the fourth year of their course the nine pre-service teachers in the longitudinal study completed a one-on-one interview with the researcher. In order to analyse their development of MCK of ordering and partitioning, they answered four items relating to common fractions. Each pre-service teacher was given three pairs of fractions and asked to identify the largest common fraction (Item 1: 3/5 and 2/3; Item 2: 3/5 and 3/4; Item 3: 3/5 and 5/8). For Item 4 pre-service teachers were asked to place common fractions (2/3, 3/4, 3/5 and 5/8) onto a number line. For each item they were asked to explain their reasoning.

Before commencing this interview the pre-service teachers were not aware they would have to demonstrate their MCK therefore were not given an opportunity to revise their knowledge of fractions. All interviews were digitally recorded and transcribed for later analysis.

Identifying methods and coding

Correct and incorrect responses for the second-year MCSKT Question 19 were entered into a spreadsheet. Table 2 summarises the responses of the nine pre-service teachers (their pseudonyms) in this study, listing their responses and an indication of whether the answer was correct (tick) or incorrect (cross), a description of the method used, and an indication to show whether the pre-service teacher was a mathematics major (tick) or not a mathematics major (cross).

A second spreadsheet was prepared for the fourth-year pre-service teacher data recording the four items, the number of correct and incorrect responses by Mathematics majors and non-Mathematics majors. For Items 1, 2 and 3 the method and the total number of pre-service teachers who used each method was coded: (known) fact, drew a linear (strip) model to compare the two fractions, converted to equivalent fractions in order to compare, converted to equivalent decimal and/or percentage to compare fractions, used number sense, or made a correct guess (Table 3). For Item 4, the number of correctly ordered responses was recorded for Mathematics majors and non-Mathematics majors. The number of pre-service teachers who demonstrated proportion when partitioning and placing the numbers onto the number line are also recorded in Table 3. Rowland and colleagues‘ (2009) qualitative framework (Table 1) was then used to code strategies and draw conclusions of pre-service teachers‘ MCK for the fraction items (Table 2 and Table 3). The aim was to compare the results of second-year and fourth-year data to identify foundation and or connections. Contingency and transformation were not used for this study as they linked to knowledge in action and what a teacher does during teaching.

Results and discussion

Table 2. Pre-service teachers' responses (N=9) to Question 19, Second-year MCSKT.

Name	Response	Answer	Method	Math Major
Lisa	One third, 0.42, 4/9, 0.399,	x	Converted to hundredths incorrectly	x

	two fifths			
Peter	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal	x
Michael	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal	x
Elizabeth	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal and used a proportional strategy	x
Con	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal	✓
Kerri	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal	✓
Janette	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal	x
Sean	One third, 0.399, two fifths, 0.42, 4/9	✓	Converted to decimal	✓
Shelly	One third, 0.399, two fifths, 4/9, 0.42	x	Unable to convert 4/9 to decimal; correctly converted others to decimals	✓

The responses of Question 19 from a second-year MCSKT indicated that most (7) of the nine pre-service teachers could order the numbers correctly: one third, 0.399, two fifths, 4/9 and 0.42 (Table 2). All correct responses showed some working out and recording, converting the fractions to decimals. This method concentrated on procedure and is foundation knowledge with some knowledge of the connections between the concepts of common fractions and decimal fractions. For example they demonstrated two fifths as equivalent to four tenths. There may have been further connected knowledge but Question 19 did not provide enough scope to identify this. Interviewing pre-service teachers after the test would have provided further probing of MCK.

Elizabeth‘s correct solution for Question 19 demonstrated a proportional strategy and was coded as demonstrating connection (Rowland et al., 2009). She was trying to make sense of the numbers so rather than converting 4/9 to 0.44 she recorded that 4/9 was ―just under 0.5” (Figure 1). She was most likely using half as a reference point and knew the other numbers were ―more than just under” one half. This example provided the most evidence of connected knowledge. However, it did not demonstrate if she knew how to change a common fraction (4/9) to a decimal fraction.

A range of experiences would have assisted pre-service teachers to prepare for their MCSKT contributing to the number of correct responses for Question 19. Before completing their MCSKT the pre-service teachers had just completed two education units during the second semester of the second year of their course. Both units focused on developing understanding of the primary mathematics curriculum as well as teaching and learning numeracy. All pre-service teachers had access to a sample MCSKT as a method of preparation for this assessment task. They also attended a primary school placement where they observed, participated and taught primary mathematics lessons. Similarly, they may have brought this knowledge to the course as foundation knowledge learnt during first year of the course or from their own mathematics education at primary or secondary school.

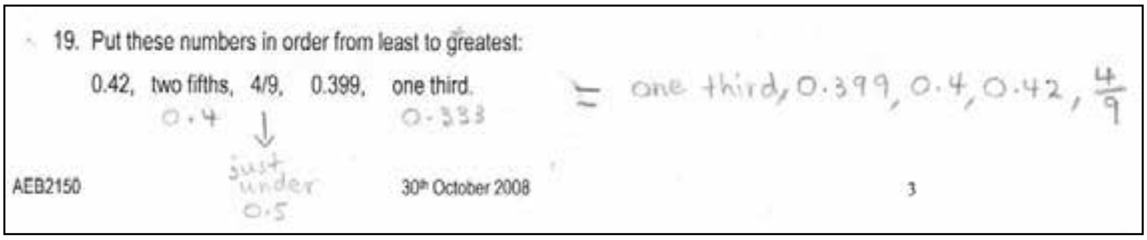

Figure 1. Elizabeth's response to MCSKT Question 19.

Shelly, a Mathematics major, was unable to convert 4/9 to a decimal fraction. There is evidence of working out which has been rubbed out. Her recording shows 1/9 = 0.102 which is incorrect. She may have drawn on procedural knowledge to convert two fifths to a decimal and used a known fact for one third but lacked procedural fluency by not knowing a method for converting 4/9 to $0.\dot{4}$. Demonstrating procedural fluency is knowing procedures to use and performing them flexibly, accurately, and efficiently (National Research Council, 2001).

Lisa was one of the other pre-service teachers from the sample who was unable to provide a correct response for Question 19 and incorrectly converted the numbers to, hundredths. For example she recorded 4/9 as 45/100 and one third as 23/100. It is difficult to identify her errors since she did not record her thinking.

For Item 1 and Item 2, all responses were correct: 2/3 (Item 1) and 3/4 (Item 2). About half the pre-service teachers knew this as a fact because they were able to record the answer using quick mental methods. During the interview they explained how they knew this by drawing a model or by comparing equivalent fractions, decimals and/or percentages. This response was coded as foundation, common content knowledge as they could demonstrate an accurate method for ordering common fractions and explain their thinking or procedure.

Table 3. Fourth year pre-service teachers' responses (N=9) to 4 fraction items[1].

Item	Question	Number of Correct Responses		Correct proportion (N=9)	Method (N=9)					
		Maths Majors (N=4)	Non-maths Majors (N=5)		Fact	Linear model	Equivalent fractions	Decimals and/or percentage	Number Sense	Correct Guessed
	Which one is larger, …									
1	3/5 or 2/3?	4	5		3	2	4			
2	3/5 or 3/4?	4	5		4	1	4		1	
3	3/5 or 5/8?	4	5		0	0	6	1	1	1
4	Record these fractions on a number line 3/5, 2/3, 3/4 ,5/8	2	2	3						

[1] Space does not permit the inclusion of the names of pre-service teachers.

Elizabeth drew a linear or strip model for all three items in order to shade and then compare these fractions. She did not elaborate on this method and lacked foundation knowledge and appeared to have forgotten any methods she had demonstrated two years earlier for Question 19 (Figure 2). She said, ―This is the easiest way of me thinking about this stuff… obviously if I could do it with a ruler it would be a lot more accurate." She correctly guessed the answer for 3/5 and 5/8 because the models looked similar. Again, she was not able to change a common fraction to a decimal fraction, as she had done two years earlier to compare fractions, or draw on her MCK to seek a suitable method.

For Item 1 and Item 2, four pre-service teachers chose to convert the fractions to equivalent fractions to compare their size. Janette used number sense by looking at the numerators and denominators. She selected 3/4 as larger than 3/5 and knew that quarters were larger than fifths. Her response was coded as ‗connection', making connections between concepts.

All answers to Item 3 were correct. The most common method used to solve Item 3 was demonstrated by six pre-service teachers. They drew on a rote procedure, making equivalent fractions to compare 3/5 and 5/8 as 24/40 and 25/40. They performed procedural knowledge using step by step procedures and thus demonstrated foundation knowledge.

To compare and order fractions, students should develop a range of strategies (Petit, Laird, & Marsden, 2010). Only one pre-service teacher had the confidence to use their knowledge of fractions, decimals and percentage that demonstrated extending fraction ideas. Con was a Mathematics major and estimated the correct answer using connections with rational numbers as well as number sense. For Item 3 he said, ―It is close. This [5/8] has to be more than point six [0.6] because one eighth is equal to more than ten percent. One eighth has to be bigger than ten percent. Four eighths is 50 percent or half or whatever and this [3/5] is sixty percent… so 50 plus more than ten percent is equal to 61 point 8 [61.8%]. I think it is point 888[0.888%] maybe something like that… I just know it is more than ten percent." This explanation drew on extended rational number knowledge by partitioning the fraction and breaking the problem down into steps that helped him justify the answer and demonstrate ‗connection'.

For Item 4, only four—or less than half the pre-service teachers in the sample (N=9)—were able to correctly place the fractions in order on the number line even though they all correctly compared pairs of these fractions. Of the four pre-service teachers who answer correctly, two were Mathematics majors and two were not. Peter who was not a Mathematics major did not place the numbers in proportion on the number line. This is Foundation knowledge.

Michael, Con and Shelly were able to record their fractions on the number line in order and in proportion, using other numbers such as zero, half and one as bench marks. Making conceptual connections with the number line and with other representations of fractional numbers is evidence of connection or rational number sense (Lamon, 2005).

Conclusion

Rowland's et al. (2009) framework was useful for identifying foundation knowledge when analysing the fraction question and items (tasks), as many pre-service teachers drew on known procedures in their responses. However, the items were closed questions. This was a disadvantage in the study when examining connection as most responses demonstrated one method of solution making it difficult to identify breadth and depth of pre-service teachers' MCK. Item 4 provided the best opportunity for providing evidence of connection as it was a multi-step problem involving known facts and procedures for comparing and ordering and partitioning and sense making strategies for representing common fractions with different denominators on a number line.

The results for the nine pre-service teachers MCK can be summarised as follows:

- One pre-service teacher lacked foundation knowledge in both second-year and fourth-year as she was unsuccessful with both ordering tasks; further investigation is needed to explore why she has not improved her MCK.
- Three pre-service teachers demonstrated success with Item 4 during fourth-year and demonstrated foundation and connection knowledge; they are maintaining and/or improving their MCK.
- Nearly half the sample, four pre-service teachers, could order the fractional numbers in second-year but could not order a similar set of common fractions correctly in fourth-year, demonstrating less foundation knowledge. This is a concern.
- One maths major demonstrated extended rational number knowledge. Other aspects of the longitudinal study will focus on how he transforms his knowledge and explains mathematics to primary students.

These results will be used to provide directions for probing more deeply into these pre-service teachers' MCK. They indicate that there is the need to engage pre-service teachers in tasks that promote understanding of specialised content knowledge to foster development of mathematical connections and not merely foundation knowledge. Tasks should be designed to assess their connection knowledge. Teaching experiences in various settings need to be designed to further connect knowledge and the other dimensions of specialised content knowledge they need to draw on when teaching. *The Knowledge Quartet* can be used as tool for supporting this development.

If the findings of this study are widespread and graduating teachers have gaps in their foundation knowledge and demonstrate narrow connected knowledge they will struggle when working with students within a classroom. Students have a wide range of numeracy experiences and abilities and their teachers need to draw on their MCK to build capacity in numeracy teaching and learning.

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ASSESSMENT OF SECONDARY STUDENTS' NUMBER STRATEGIES: THE DEVELOPMENT OF A WRITTEN NUMERACY ASSESSMENT TOOL

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This paper examines the piloting of a *Written Strategy Stage Assessment Tool* designed to identify students' ―global" strategy stages and provide formative data for teachers. The Year 9 cohorts from two schools were assessed at the end of the school year. Numeracy experts then interviewed a sample, to identify each student's strategy stage, using an oral assessment. Results from the written assessment gave relatively consistent measures of stages in terms of the criteria set and a relatively close match to national data. Comparison of the written and oral assessment results showed the stages identified by the two measures to be generally consistent.

Background

The extension of the New Zealand *Numeracy Development Projects* (NDP) from primary into secondary schools, as the *Secondary Numeracy Project* [SNP], led to a call for a written assessment tool to replace the NDP oral assessment tool (NumPA) as the main source of initial information about secondary students' mathematical knowledge and understanding. NumPA information is used to place students initially into stage related teaching groups, as promoted in NDP, and new written assessment information would need to be able to be used in the same way.

A written tool was seen as enabling the initial assessment of whole classes of secondary students to be carried out more ―efficiently". The use of oral individual student assessment, as occurred in primary schools, was seen as an unproductive use of teacher time given the larger numbers of students that secondary teachers had to deal with due to their taking multiple classes.

The use of oral assessment negates the challenge of inadequate student reading levels, potentially a major issue with primary school students, whereas reading levels were not seen as problematic with secondary school students. Thus, the larger number of students in secondary schools who need assessing per teacher and the expectation of adequate reading levels were the prime drivers for developing the *Written Strategy Stage Assessment Tool* (WSSAT).

An efficient written assessment tool would:

- reduce the amount of teacher time taken per student;
- allow a standardised (and objective) marking schedule; and
- give a written record of students' answers (without any working).

Most importantly it would give appropriate global strategy stages and useful formative information, reflecting that from NumPA.

The WSSAT, as can be attempted with any written test, was designed to meet the three criteria. The time to complete was aimed at about 30 to 40 minutes, which for classes of 30 (or whole school cohorts) would use significantly less teacher time than individual oral assessments. There was a standardised, straightforward marking schedule (correct/incorrect), enabling quick, consistent and objective marking—particularly by teachers new to SNP or even by a person with little knowledge of numeracy, such as a parent helper. The responses on a carefully formatted answer sheet allowed for quick marking as well as determination of the strategy stage, while providing a clear written record of the students‘ performance. Experience during the earlier trials and the pilot in this study have indicated that these efficiency aspects of the design have been achieved.

In addition, the availability of the written record for evaluation and moderation purposes reduces the potential for individual variability that can occur when conducting oral assessments. It may also enhance the accuracy of the assessment, particularly in the multiplicative and proportional domains, for which Thomas, Tagg, and Ward (2006) set the accuracy of various types of existing numeracy assessments at 76%. Thomas et al. also found that many secondary teachers ―rated students‘ strategy stages lower than the rating of the researchers, explaining their decisions in terms of consolidating students‘ understanding at an existing level" (p. 101): that is, they rated students at a lower stage than NumPA would have assigned if used as intended.

The extent to which the WSSAT achieves a reasonably ―useful" determination of students‘ strategy stages for the initial placement of students into teaching groups is the focus of this paper.

The nature and structure of the written strategy stage assessment tool

The WSSAT is closely aligned with the NDP strategy and knowledge frameworks, which are organised in three and four domains respectively, with a focus on the higher stages (5 to 8) dealing with part-whole thinking (Ministry of Education, 2008). It aims to identify a global or overall strategy stage (as does the NDP Global Strategy Stage [GLoSS] tool) rather than domain-specific strategy stages (as does NumPA), and uses some similar items. Other items consist of short-answer and multi-choice questions on place value and decimals (Brown, Hart & Dietmar, 1984; Hart, 1981) and on number sense (Lomas & Hughes, 2008, 2009; McIntosh, Reys, Reys, Bana, & Farrell, 1997).

The focus of the WSSAT is primarily global strategy stage identification rather than the identification of specific strategies. There are seven domain specific strategy items at each of four stages (5 to 8) assessing a range of strategies and further items assessing some knowledge aspects. The knowledge items are a mixture of prerequisite knowledge for, and knowledge directly related to, each stage. To achieve a global strategy stage, students need to correctly answer four of the domain specific strategy items for that stage. Four correct items are used to assign an overall (global) stage as they indicate the mastery of strategies, at a particular stage, in some but not necessarily all of the three strategy domains (addition and subtraction, multiplication and division, and proportions and ratios). That is, there is a range of evidence about strategy mastery that underpins the assignment of an overall strategy stage.

As with the NDP oral assessments, the highest strategy stage achieved is taken as that student‘s stage for teaching purposes. For example, if a student meets the criteria for stages 5, 6, and 7, they are classified as being at stage 7, or if they meet the criteria for stage 5 and not those for stage 6, but do meet the criteria for stage 7, they are classified as being at stage 7. Where the students do not meet the WSSAT criteria for stage 5 or higher, they are assigned to a category that covers stages 1–4.

The WSSAT as a written assessment relies on the written answers without any indication of the process (or strategy) used. This is different from NDP oral assessments, in which students give their answers orally and can be prompted to talk through the processes they used to arrive at their answers thus revealing the particular strategies used. Therefore, using oral assessment to assign strategy stages can relate more to process and particular strategies, whereas assigning strategy stages using the WSSAT is based solely on outcomes (Lomas & Hughes, 2008).

The WSSAT answer sheet has clear directions for the students to follow with space for answers only. It was formatted for ease of marking, giving both a quick indication of a student‘s strategy stage and more detailed formative data for planning and teaching purposes. Calculator use was not allowed, mental working was promoted, and writing (―working"), other than for recording answers, discouraged.

The range of items selected for each stage attempted to isolate and encapsulate some of the conceptual aspects and strategy elements relating to that stage as per the domains in the strategy section of the NDP Number Framework. For example, item 36, ‗Work out 5 sixtieths of three thousand six hundred‘ relates to stage 8 of the proportion and ratio domain, in which students are expected to use multiplication strategies to solve problems with fractions.

In addition, the nature of the WSSAT items was designed to reflect elements of students‘ understanding that might be present in a dialogue between teacher and student but would not always be evident in written work with an outcome focus. Thus, a key issue in the WSSAT was to ensure that items used ―forced" the student participants to use a particular process and restricted their use of any other approach. That is, the WSSAT attempts to minimise the number of items that could be answered procedurally or answered by less sophisticated strategies. For example, some of the written assessment items such as item 3, ‗7 tens + 20 =‘ and item 28, ‗Work out 5 sixths of 42‘ use combinations of numbers written as words and digits. Another example is ‗Eleven thousandths equals:‘ with four choices offered: ‗A. 0.0011 B. 0.011 C. 0.11 D. 11000‘. This format avoids the answer ‗11 over (divided by) 1000‘, which ―side steps" the issue of understanding decimal fraction notation. The aim is to expose the student‘s in-depth understanding by stating material in a way that both limits the use of procedural methods and requires more understanding of number structure. This approach was also seen as a way of keeping aspects of oral language use within a written format.

An example of an item that tries to force a particular strategy is item 14, ‗341 - ‗what‘ = 299‘. Here, 299 is close to a tidy number, and the most likely NDP solution consists of making the 299 up to 300 and increasing 341 to 342, resulting in 42 as the answer. The choice of these particular numbers lessens the possibility of students using a strategy such as doubling if numbers such as ‗51 - what = 26‘ were used. The extent to which this approach has worked will be determined in part by the alignment of the stages assigned by the written assessment to the national data and oral assessment.

The sets of items also have the potential to provide a more detailed and standardised diagnostic map of students' learning needs than an oral assessment. This potential is enhanced by the written format, which allows a student to attempt all the items, thus demonstrating any pieces of knowledge and strategy understandings that the student might have beyond the point where an oral assessment would stop.

Initial development of the WSSAT

The WSSAT was trialled twice during the initial development phase giving rise to the version examined here. The initial trial demonstrated sufficient internal consistency but the assigned stages did not match sufficiently the parallel oral assessment, the national curriculum level expectations, or, more specifically, the low-decile[1] data (Lomas & Hughes, 2008). On this basis, a number of changes were made to the organisation of the items within each part of the WSSAT (reflecting a stage), the positioning of items in each part, and the style of some items (Lomas & Hughes, 2008). The trial of the second version gave rise to further minor revisions to enhance consistency for parts C and D, that is, Stages 7 and 8 (Lomas & Hughes, 2009).

Method

The revised WSSAT was piloted with two schools, and a parallel oral assessment interview was conducted with a sample from one of the schools. Due to the site-specific nature of the data collected, this research is a form of case study. Thus, the data is unlikely to match the national data sets—particularly the national aspirational expectations—too closely, and care must be taken in generalising any findings.

The participants in the end-of-year pilot were drawn from two schools in major cities: a Year 9 cohort of a large, Auckland, decile 3 (low socio-economic environment) secondary school of mixed ethnic composition, excluding some special needs students, and the complete Year 9 cohort of a medium-sized Wellington, decile 6 (medium socio-economic environment) secondary school of mainly New Zealand European students.

The written assessment was only given to the students present on the particular day for each Year 9 cohort, while the oral assessment was later given to a subset (60 students) of the Year 9 Auckland students who had responded to the WSSAT. This sample was drawn from several classes from the four bands (strands/streams) into which the school organised their classes (see Table 1) at the schools convenience. This affected the nature of the sample, which is not representative because it was drawn equally from upper- and lower-band classes and included no students from the middle-band classes.

Table 1. Auckland school classes in bands (high to low), showing student roll numbers and the number of students participating in the WSSAT and oral assessment.

	Class Name (Auckland)												
	P9A1	P9A2	P9B1	P9B2	P9B3	P9B4	P9C1	P9C2	P9C3	P9C4	P9D1	P9D2	Total
Roll	33	32	33	33	33	32	25	20	27	25	28	27	348
WSSAT	26	26	28	24	25	22	25	20	23	16	21	24	280
Oral	10	20	–	–	–	–	–	–	19	11	–	–	60

[1] Deciles are measures of socio-economic status.

The oral assessment

The oral assessment research tool was an expanded form of the GLoSS (Lomas & Hughes, 2009) and used some of the GLoSS- and NumPA-type items, supplemented by other items that gave increased coverage of higher stages. As well as the questions being asked orally, a card with the question written on it was placed in front of the student as a reference (as is done with GLoSS and NumPA). The oral assessment was conducted by external interviewers who had expert knowledge of NDP, NumPA, and GLoSS.

Data collection

The WSSAT was conducted in each class‘s usual classroom setting, mostly under the supervision of the regular mathematics teacher, in the last few weeks of the fourth term. Standardised instructions were given explaining how teachers were to conduct the assessments (Lomas & Hughes, 2008), and all the answer sheets were marked by one of the research team to ensure consistency. Copies of the marked answer sheets were returned to the school for potential diagnostic/formative use by the school but what, if any, use was made of these is unknown.

The GLoSS-type oral assessment was conducted in the two days following the written assessment at the Auckland school.

Analysis

The results of the written assessments were first analysed for the internal consistency of the tool in identifying a student‘s stage, that is, whether a student assigned as being at Stage 6 had also been assigned as being at Stage 5, and so on. Then they were analysed against three other measures of achievement, one school-based and two based on nationally collected data from the NDP, which give measures of global, rather than domain-specific, strategy stages. The stages that the students achieved were compared with:

- the banding (where applicable) of the class they were in, to see whether this reflected the school‘s placement of students;
- the national, Year 9, low- or medium-decile stage distribution data from the NDP; and
- the national, Year 9, stage distribution data (the aspirational expectations).

The results of the written and oral assessments from the Auckland pilot school were compared to establish a relationship between these two forms of assessment. The oral assessment was assumed to be the more accurate and was taken as the baseline for the comparison due to its alignment with national data collection methods. This assumption was based on two main factors. Firstly, the oral assessment was an extension of the GLoSS and NumPA tool and thus was collecting some of the same data, although by experts rather than classroom teachers. Secondly, the extra questions were provided by a numeracy expert with an intimate knowledge of the development and use of both GLoSS and NumPA. This connection to these existing and ―proven" NDP assessment tools provided a basis for comparison of student results arising from the oral tools‘ use with the national data sets.

Results

The data for each cohort is analysed separately to allow direct comparison with the appropriate decile (socio-economic) level of Year 9 national data.

Internal consistency of WSSAT

All the Auckland students assigned as being at Stage 6 had also been assigned as being at Stage 5, while of the 84 students who could be assigned as being at Stage 7, 21 (one-quarter) had not achieved the criteria for Stage 6. Of these 21, 15 had missed the criteria by only one correct response. A further two students who achieved the criteria for being assigned as being at Stage 7 had not achieved the criteria for either Stages 5 or 6. For the 46 students assigned as being at Stage 8, ten (over one-quarter) had not achieved the criteria for Stage 7, and one further student had not achieved the criteria for either Stages 6 or 7. However, of the ten not achieving the criteria for Stage 7, four had missed by only one correct response.

All the Wellington students assigned as being at Stage 6 had also been assigned as being at Stage 5, while of the 39 students who could be assigned as being at Stage 7, eight (almost one-fifth) had not achieved the criteria for Stage 6. Of the eight, three had missed the criteria by only one correct response. For the 21 students assigned as being at Stage 8, only one had not achieved the criteria for Stage 7.

These data suggest that the WSSAT was largely internally consistent in assigning stages except at Stages 7 and 8, where a greater level of variation was evident, although less variation was evident in the medium-decile Wellington school data.

Conformity of WSSAT assigned stages with students banding into classes

The Wellington classes were not banded but the banding of classes in the Auckland school generally reflected the stages assigned by the WSSAT: classes in higher bands achieved more of the higher stages, and classes in lower bands achieved fewer of the higher stages (see Table 2). Additionally, in line with the internal consistency of the WSSAT the meeting of the criteria for particular stages also aligned with the banding of the classes, with fewer lower band students meeting the criteria for each stage.

Table 2. Auckland school classes in band order, showing the number of students participating and the number of students meeting the criteria for achieving a particular stage.

	Class Name (Auckland)												
	P9A1	P9A2	P9B1	P9B2	P9B3	P9B4	P9C1	P9C2	P9C3	P9C4	P9D1	P9D2	Total
No. of students	26	26	28	24	25	22	25	20	23	16	21	24	280
No. ass. Stage 5	26	26	28	23	25	22	25	15	22	15	19	22	268
No. ass. Stage 6	25	25	21	17	16	16	15	2	15	6	5	4	167
No. ass. Stage 7	23	20	11	13	14	5	5	5	11	7	2	4	120
No. ass. Stage 8	16	11	2	6	1	4	4	0	1	0	0	1	46

Comparison between oral and written assessments

The stages determined by the oral assessment of students closely matched the stage determined by the WSATT at Stages 7 and 8, but less so at other stages (see Table 3). A third of students achieving Stage 5 on the oral assessment achieved Stage 6 on the WSSAT, and two-thirds of students achieving Stage 6 on the oral assessment achieved Stage 7 on the WSSAT. However, there were no differences of more than one stage, unlike the initial trial data (Lomas & Hughes, 2008). This may reflect the more even spread across stages achieved by the revisions of the WSSAT.

Table 3. Stages assigned to Auckland students from WSSAT compared with the oral assessment.

	Number of students (n = 60)												
	2	1	-	9	5	-	5	9	-	7	1	3	18
Oral assessment stage	1–4	1–4	5	5	5	6	6	6	7	7	7	8	8
WSSAT stage	1–4	5	1–4	5	6	5	6	7	6	7	8	7	8

The level of alignment between the two sets of data is 61% with 41 students having the same stage for both assessment tools. This is not too dissimilar to the 75% accuracy given by Thomas et al. (2006) for secondary teachers' assigning of stages. Of the 32% of students with different stages, 16 (27% of the total) have higher WSSAT stages assigned and 3 (5% of the total) have lower. The higher WSSAT stages could be problematic with students being placed in inappropriate teaching groups—particularly in light of secondary teachers' assigning students lower groups for consolidation purposes (Thomas et al., 2006).

Comparison with New Zealand national aspirational expectations

The assigning of stages from the WSSAT for the two schools gave rise to a distribution reasonably similar to the data for both the medium- and low-decile schools respectively and to the national aspirational expectations (see Tables 4 and 5).

For the medium-decile Year 9 cohort used in the pilot, the areas of greatest disparity (around a 50% difference or more) with the medium-decile data for end-of-year Year 9 students were the higher number of students achieving at Stage 8 (19% compared with 10%) and the lower number of students achieving at Stages 1–4 (0% compared with 6%) (see Table 4).

Table 4. The percentages of stages assigned to Wellington students from the WSSAT, the medium-decile data, and national aspirational expectation data for Year 9 students (end of year).

	Stages				
	1–4	5	6	7	8
WSSAT: percentage of students (n = 113)	0	28	19	34	19
Medium-decile (averaged) percentage (Tagg & Thomas, 2008)[2]	6	22	32	30	10
National aspirational expectations percentage (Tagg & Thomas, 2007)	2	14	27	39	18

[2] The low- and medium-decile percentages are average figures derived from the respective percentage data for the additive, multiplicative, and proportional strategy domain percentage data.

The greatest area of disparity for the medium-decile Year 9 students compared with the Year 9 national aspirational expectations was the higher number of students achieving at Stage 5 (28% compared with 14%).

For the low-decile Year 9 cohort, the areas of greatest disparity (around a 50% difference or more) with the low-decile data for end-of-year Year 9 students were the higher number of students achieving at Stage 8 (16% compared with 5%) and the lower number of students achieving at Stages 1–4 (4% compared with 11%) (see Table 5). This may partly be explained by the exclusion of a group of lowest performing Year 9 students from the data collection process.

The area of greatest disparity for the low-decile Year 9 students with the Year 9 national aspirational expectations was the lower number of students achieving at Stage 5 (28% compared with 14%). This may partly reflect a difference between low-decile students and a national norm, although a similar disparity was evident in the comparison for the medium-decile data (see above).

Table 5. The percentage of stages assigned to Auckland students for each assessment tool, the low-decile results, and national aspirational expectation data for Year 9 students (end of year).

	Stages				
	1–4	5	6	7	8
WSSAT: percentage of students (n = 280)	4	28	23	30	16
Oral assessment: percentage of students (n = 60)	5	23	23	13	35
Low decile (averaged) percentage (Tagg & Thomas, 2008)[2]	11	29	33	22	5
National aspirational expectations percentage (Tagg & Thomas, 2007)	2	14	27	39	18

The oral assessment‘s assigning of stages to students is reasonably close to the national aspirational expectation percentages for all stages except those achieving at Stages 6 and 7. However, if the sample had been less skewed and included middle band students where more students achieved at Stages 6 and 7 there may have been a closer fit overall.

Discussion

A factor to consider in comparing the WSSAT and oral assessment stages with the national data sets is the degree to which the national data sets accurately represent the stages that the students at Year 9 can achieve. The national data is based primarily on aggregated teacher gathered data and its‘ accuracy may be variable. For example, secondary teachers‘ accuracy of 76% and their assigning of lower stages (Thomas et al., 2006) would suggest an underestimation of student performance overall, but possibly more so at Stages 7 and 8, in which the learning demands are greater. The possibility of such a trend is apparent in both the medium- and low-decile cohort data. For example, compared with the medium-decile data, there are three times the percentage of students achieving at Stage 8 and 50% more students achieving at Stage 7, but only about two-thirds the percentage figure of students achieving at Stage 6.

The WSSAT numeracy stages achieved by the Auckland students reflected their placement in ability banded class groups indicating that the WSSAT results paralleled other school based measures of students‘ mathematical performance used for student placement. Similarly, the WSSAT assigned stages for the two Year 9 cohorts stage distribution were reasonably close to the national aspiration expectation distribution, allowing for their decile levels, and to the low- and medium-decile distributions. These data indicate that the WSSAT items are measuring strategy (or possibly something that gives a parallel measure) to the extent of being able to assign students' global strategy stages with some accuracy. Thus, the written items appear to access elements of student strategy, with the written record allowing later access to explore responses to items designed to elicit specific strategies.

Conclusion

Overall, the WSSAT has reasonably high levels of internal consistency for Stages 5 -8 and could be used to assign students a global (numeracy) strategy stage. In addition, there is a reasonable congruence of the stages assigned by WSSAT, with both the low- and medium-decile school data, the national aspirational expectations, and with the oral assignment of NDP stages. Thus, the WSSAT could determine a student's global numeracy strategy stage with a sufficient degree of accuracy to allow for students' initial placement into stage appropriate teaching groups. In this sense, WSSAT appears to be a potentially useful tool in secondary schools, given its time efficiency. However, its potential and usefulness for teachers as a diagnostic and formative planning aid in working with their students needs to be explored.

WSSAT may also have uses with other groups where the reading level is adequate such as Year 7 and 8 (11 and 12 year old) primary students for the assigning of an initial global strategy stage, and with pre-service teacher education students and in-service teachers for professional development purposes in identifying and addressing deficiencies in their mathematical knowledge relevant to teaching.

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YOUNG CHILDREN'S REPRESENTATIONS OF THEIR DEVELOPING MEASUREMENT UNDERSTANDINGS

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This paper explores the development of young children's understandings about measurement, and the ways in which children represent these understandings. This paper presents a selection of data gathered during a three-year study that examined young children's engagements with measurement in prior-to-school and out-of-school contexts. In this present investigation, children's representations in the form of drawings and narratives are analysed in relation to a framework of emergent measurement. Initially, this paper considers the understandings about measurement which children are demonstrating in alignment with the framework, before offering a selection of data which represents a disruption to the framework and contests existing ideas about young children's measurement understandings.

Background

Clements and Stephan (2004) have suggested that understandings of measurement begin to develop in the prior-to-school years. Young children know that continuous attributes such as mass and length exist, although they may not be able to quantify or measure them accurately (Clements & Stephan, 2004). However, by about four or five years of age, most children begin to make progress in reasoning about and measuring quantities by overcoming perceptual cues and learning to use words that represent quantity of a certain attribute (Clements & Stephan, 2004). Children then learn to compare two objects directly and recognise equality or inequality (Boulton-Lewis, Wilss & Mutch, 1996). At this point, children are ready to learn to measure by connecting number to quantity (Clements & Stephan, 2004). Typically, students first learn to measure using informal units before progressing to the use of formal units. Although researchers debate the order of the development of these concepts and the ages at which they are developed, they tend to agree that these ideas form the foundation for measurement understanding (Stephan & Clements, 2003).

There are many developmental sequences for measurement learning presented in the literature, but most are similar in their progression from identification of the attribute and use of informal measurement through to the use of formal units. Three examples sequences of measurement learning are that of Clements and Stephan (2004); Piaget, Inhelder, and Szeminska (1960); and Board of Studies NSW (2002). These three perspectives on the development of children's measurement understandings bear noticeable similarities. Table 1 summarises the key points of each of the three developmental sequences.

Table 1. Measurement learning frameworks of Clements and Stephan (2004), Piaget et al. (1960), and Board of Studies NSW (2002).

Clements and Stephan (2004)	Piaget et al. (1960)	Board of Studies NSW (2002)
Awareness of continuous attributes, but unable to quantify or measure accurately.		
Use of words that represent quantity of an attribute, direct comparison, and recognition of equality or inequality.		Identifying the attribute and comparison.
	Not capable of measurement, construction of units is impossible.	Informal units.
Connect number to quantity, identify unit of measure, and measure through unit iteration.	Ability to use a common measure, use of unit iteration.	Formal units.
	Direct measurement is possible.	Applications and generalisations.

When we look at the summary of the measurement learning frameworks of Clements and Stephan (2004), Piaget et al. (1960), and Board of Studies NSW (2002), it could be said that the framework can effectively be divided into two levels, these being *emergent* measurement and *proficient* measurement. *Emergent* measurement encourages children to develop an understanding of measurement by using it for their own purposes, talking about their measurement ideas, representing measurement processes in ways which make sense to them, and becoming more aware of their own measurement thinking (Whitebread, 2005). By contrast, *proficient* measurement requires: comprehension of measurement concepts, operations and relations; skills in carrying out procedures flexibly, accurately, efficiently and appropriately; ability to formulate, represent and solve problems; and capacity for logical thought, reflection, explanation and justification (Kilpatrick, Swafford & Findell, 2001). This study explores children's emergent understandings and how these are leading into more proficient understandings.

Research design and methods

Participants

The data were collected at two schools in regional New South Wales. The participant children had just commenced their first year of formal schooling, known as Kindergarten in NSW. Children in NSW commence Kindergarten in late January. They "must start school by the time they are 6 years old but they may start in the year that they turn 5, provided their fifth birthday is before July 31 of that year. Hence, it is possible for a new Kindergarten class to contain children aged between 4 years 6 months and 6 years" (Perry & Dockett, 2005, p. 65).

Data collection

Data collection took place during February and March 2009, at the start of which the children had been at school for approximately two weeks. It was confirmed by all of the Kindergarten teachers that no formal teaching about measurement had taken place in the classroom up to this point in time, or throughout the data gathering period. The children were asked to complete a series of six drawing tasks relating to different measurement concepts, and provide a description of each drawing. This description was annotated on the drawing, and both the drawing and annotation were considered as a whole. The tasks were designed to progress from an open-ended exploration of what children themselves considered measurement, through to investigations of specific content areas and concepts. The tasks, and the measurement content and concepts they address, are outlined in Table 2.

Table 2. Drawing tasks and the measurement content and concepts they address.

Drawing task	Measurement content	Measurement concept
Task 1: Draw yourself measuring	Open-ended	Awareness of attributes
Task 2: Draw something tall and something short	Length	Comparison
Task 3: Draw something heavy and something light	Mass	Comparison
Task 4: Draw something hot and something cold	Temperature	Comparison
Task 5: Draw a ruler	Length	Unit iteration
Task 6: Draw a clock	Time	Unit iteration

Data analysis

Analysis in this study was based on the common elements across the representative developmental progressions shown in Table 1. It can be seen in Table 1 that notions of *attributes* and *comparisons* are common to both the Clements and Stephan (2004) and Board of Studies NSW (2002) progressions, while *units* are common to all three. Table 3 provides the resultant framework for analysis of the children's measurement content knowledge. Decisions were made as to which, if any, of these elements and descriptors were represented by each drawing and its description, and the data was coded accordingly.

Table 3. Framework for analysis of emergent measurement understandings.

Element	Descriptors
Attributes	Understanding that objects have attributes which can be measured.
Comparisons	Understanding that the key idea is to compare like attributes. Comparing objects directly. Multiple comparisons of objects.
Units	Recognition of units. Sequencing of units. Equal partitioning of units.

Results and discussion

The three elements of *attributes*, *comparisons*, and *units*—and their corresponding descriptors—form the basis of this discussion. Woven throughout the discussion are the descriptions of the drawings given by the children.

Awareness of measurement concepts and attributes

This section explores the children’s responses to Task 1, the “Draw yourself measuring” task, with discussion based on the “Attributes” descriptor of the measurement framework.

Understanding that objects have attributes which can be measured

With respect to Task 1, the drawings collectively represented the concepts of area, length, mass, and temperature. Length was the most commonly represented concept, followed by area, mass and temperature respectively. This was not surprising because length is the measurement concept most easily understood by young children due to it being the most concrete, visual measurement concept for children to perceive (Gifford, 2005). As a result, the process of length measurement is also the most tangible and direct measuring process for young children. Indeed, the majority of the children in this study described their drawings as them finding out how “tall” or “long” the object being measured was. For example, Imogen described her drawing as “I’m measuring a piece of paper. I’m getting a ruler. I’m finding out how long it is”.

In addition to this notion of using measurement to “find out” an object’s properties, some children described contextualised applications of measurement. These drawings were highly personalised, with rich accompanying narratives. For example, Zofi drew a picture of herself testing the temperature of the bath water in preparation for a bathing a baby: “I’m measuring the bath and the baby’s helping me. I’m checking the water”.

Despite the fact that the task asked children to “Draw *yourself* measuring”, most children showed measuring being carried out by others (usually their parents). For example, Caitlin drew “A brick wall and my Mum. She’s measuring it to find out how big it is, with a measuring tape”.

Ability to compare measurable attributes

The comparison descriptions of the emergent measurement framework are organised to reflect a progression in understanding about comparison. These comparison skills will be discussed in relation to children’s responses to Tasks 2, 3, and 4.

Understanding that the key idea is to compare like attributes

With regard to Task 2—”Draw something tall and something short”—almost all of the students were able to represent objects relating to the attribute of height and identify whether an object was “tall” or “short”. For example, Brodie drew “a short box and a tall box”, while Lachlan drew two cars, explaining “This car’s short. This one’s tall”.

When considering Task 3, the “Draw something heavy and something light” task, children were required to identify the attribute of mass by describing objects as either “heavy” or “light”. Most of the children were able to do so, such as Ella who drew “A big heavy bookshelf. A feather is light”, and Annabelle, who said “A leaf is light. A rock is heavy”.

Finally, Task 4 asked children to compare objects in relation to their temperature, describing the objects as either “hot” or “cold”. For example, Angel stated “The sun is hot. Snowballs are cold”, while Blake said “The pool is cold. Lava is hot”.

Comparing objects directly

As evidenced by the preceding examples, children demonstrate a basic understanding of comparison by applying dichotomous descriptors of an attribute to objects. However, moving beyond this simplistic understanding is the notion of *direct* comparison, and the use of more sophisticated comparative language. In the case of Task 2, direct comparison of height could be evidenced by the positioning of the objects along a common baseline. The majority of the children were able to represent objects in this manner and state which was the taller/shorter of the two, as did Sarah, who drew a tree next to a volcano and stated “the tree is taller”.

When considering Task 3, the direct comparison of mass was evidenced by the children’s descriptions of comparing the masses of objects by lifting them – a process known as *hefting*. For example, Blake explained “A cat is light and a motorbike is heavy. I tried to pick up my cat once and it was light. I couldn’t pick up my motorbike because it was too heavy”.

Similar to Task 3, direct comparison was evident in responses to Task 4 with the children describing “feeling” objects to compare their temperatures. For example, Jurre described his drawing as “This is a sun with lots of arms and it is melting lots of ice blocks. The sun is hot. The ice blocks are cold. I know that because I feeled them”.

Multiple comparisons of objects

When considering progression in understanding about comparison, at the most sophisticated level children demonstrate an ability to compare more than two objects. The three comparison tasks given to the children did not explicitly ask them to draw more than two objects, however many children in fact chose to do so.

Task 2 required children to make multiple comparisons of objects on the basis of height. It was expected that the children represent their chosen objects in order along a common baseline, identifying which was the tallest and/or shortest. For example, Chelsea drew four flowers in order of height, and in her description she identified which was the tallest and which was the shortest. Similarly, Caitlin drew three mermaids in order of height and identified which mermaid was the tallest.

When looking at the responses to Task 3, it is interesting to note that—unlike with Task 2—very few students extended the task to making multiple comparisons of objects. Some of these children drew multiple objects and classified each of them in regards to “heavy” or “light” descriptors, while others were able to use more sophisticated comparative language relating to the ordering of objects, for example Andrew, who drew a cat, a ladder and a person and stated “The ladder is the heaviest”.

Similar to the responses to Task 3, there were also very few responses to Task 4 which represented multiple comparisons of objects. The majority of these responses were limited to simply drawing several objects and classifying them as “hot” or “cold”, or in some cases, the additional terms of “very hot” and “very cold”. A small number of children showed a sense of ordering in their responses, such as Abby who explained “This person is hot. This person is hot and cold – autumny. This person is cold. They’re outside and they don’t know how to get home”.

Knowledge about unit structure and iteration

The units component of the emergent measurement framework consists of three descriptors which reflect a progression in understanding about units. This progression in understanding about units will be discussed in relation to children's responses to the two tasks focusing on units, Tasks 5 and 6.

Recognition of units

In relation to Task 5—"Draw a ruler"—at the most basic level of understanding the children were able to recognise the units on a ruler by making reference to either the numbers or the lines on a ruler. At this level, the children typically did not accurately represent the units and instead attempted to show the units by using dots or similar. Other children were able to describe their recognition of units but were unable to represent this in their drawing, such as Krystal who said "They have numbers. We can count them. I don't know how to make the numbers on a ruler".

Similarly with Task 6—"Draw a clock"—at the most basic level of understanding the children were able to describe either the numbers or the lines on the clock, but had some difficulty in representing these. For example, Mikayla explained "It has numbers but I'm not sure how to draw them", while Lilli used circles to represent the numbers on a clock, stating "It has numbers to see what the time is".

Sequencing of units

At the next level of understanding, the children showed evidence of units with a sense of sequencing, but not necessarily evenly partitioned or with numbers in the correct order. For example, in her response to the "Draw a ruler" task, Jade wrote numbers at one end of her ruler, stating "It has numbers on it to see how much it is. No, how *long* it is".

When looking at the responses to the "Draw a clock" task, the children typically attempted to write numbers around the outside of their clock face, but did not accurately represent the numbers 1 to 12 with even partitioning, or in some cases, continued numbering beyond 12. For example, Hannah wrote numbers halfway around her clock and stated "A clock has 10 numbers. The numbers are so you see what time it is", while Makaylee wrote the numbers 1 to 12, but upon discovering that they did not go all the way around the clock she continued the sequence of numbers, explaining "It has numbers that go all the way around. They don't stop, they have to go all the way around".

Equal partitioning of units

At the most sophisticated level of understanding, the children were able to demonstrate equal partitioning of units and represent this is in a spatially appropriate manner, i.e. along the full length of the ruler, or appropriately positioned around the clock face. Additionally, when the children included numbers as representations of units, typically the numbers were placed in the correct order. In response to Task 5, Blake chose to represent the units on a ruler by using equally partitioned lines, alternating between long and short lines as is often seen on a standard ruler, and he was able to explain that "The lines are for the numbers that tell how big the paper is".

With regard to Task 6, many children were able to represent a stereotypical clock face with some accuracy by evenly partitioning the numbers 1 to 12 around the clock face in a spatially appropriate manner. Many children also used lines to demarcate the units on a clock face, such as Blake who described his drawing as “This is a clock at my home. It doesn’t have numbers – it has little lines”.

Disruptions to the framework

As the previous data has shown, many of the children in this study were able to demonstrate understandings about measurement that aligned to the framework of understanding for young children. However, there was a significant body of data from both of the participant schools which represented “disruptions” to this framework – that is, the children were presenting knowledge and skills which were not in alignment with the expected development for children of their age.

One such disruption evident in the data was the integration of measurement concepts. Rather than understanding measurement concepts such as length, mass, etc in isolation, many of the children actually integrated concepts and in this way, used one concept to understand another. For example, in her response to the “Draw something heavy and something light” task, Annabelle used her understanding of area to contribute to her understanding of mass, explaining “Big blocks are heavy. Little blocks are light”. Lachlan blended both area and mass in his “heavy and light” drawing, explaining “When things are big, they are heavy. The bigger they are, the heavier they are”.

Another significant disruption was the children demonstrating an understanding of the measurement process at the start of school, far sooner than would generally be expected. When the children first completed the “Draw yourself measuring” task at the beginning of Kindergarten, it was evident from the children’s drawings and descriptions that despite the fact that the children had not engaged in any *formal* measurement learning experiences at this point in time, their personalised engagements with various people and in various contexts had contributed to an emerging understanding of the measurement process. To begin, many of the children showed an understanding of using direct comparison when measuring. Abby (Figure 1) compared the lengths of two pencils by placing them side by side with the ends aligned: “I am measuring two pencils. I put them beside each other to see which is the longest”.

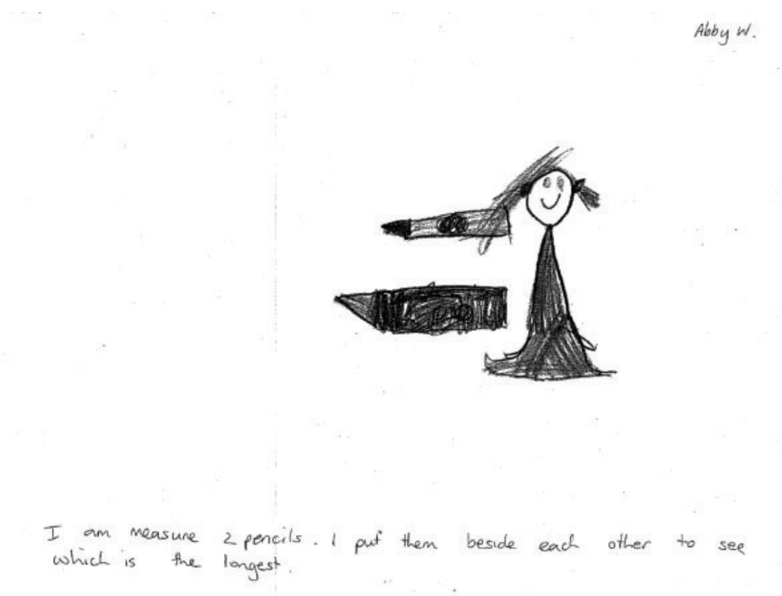

Figure 1. Abby’s drawing.

Similarly, Kody was able to compare himself to a big rock, explaining “This is me against a big rock. The rock is bigger”.

The children also showed more sophisticated understandings of the measurement process by moving beyond direct comparison to notions of using measuring tools. Most of the children made reference to a generic “measurer”, for example, Emily-Rose explained “I’m measuring a house with a measurer. You just put it on something and see how long it is”. However, some of the children described using standardised measuring tools including tape measurers and rulers, such as Chloe, who described her drawing as “I’m measuring a dog with a measuring tape to find out how big it is”. The children were also attempting to use notions of quantity and units at this early stage. In Chloe’s drawing, it can be seen that her measuring tape has been represented as a sequence of numbers, which are equally partitioned. Lara similarly showed a sequence of equally partitioned numbers on her measuring tape, used to measure a tree.

A number of students were able to show more sophisticated understandings of measuring than would be expected by attempting to identify a quantity in relation to formal units. For example, Kyra used her understanding of the measurement process to articulate a measurement of height: “I am measuring Mrs M. I use a pencil to draw a line against a measurer with a giraffe on it – at my house. She is 6 metres tall!”

Importantly, there were some children who could demonstrate how their measurements were reached. As shown in Figure 2, Jurre was able to line up the end of his measuring tape with the end of the car in order to determine a measurement of 15, while Brodie was able to articulate a process of counting units, saying that “There are 33 spaces around my car. I counted the spaces”.

Figure 2. Jurre’s drawing

Interesting applications of formal units included Willis who explained “I am measuring a clock tower with a long ruler. It is 16 kilometres”; and William, who described his drawing as “I am measuring a clock with a measurer. It is 12 megalitres”. While these formal units were, for the most part, used inappropriately, it is important to note that even at this early stage of schooling the children could see a need for formal units.

Conclusions and implications

Results in this study have shown that young children have highly sophisticated understandings of measurement. These understandings both align with, and challenge, extant frameworks for the development of measurement knowledge. Within an emergent measurement context, these children have shown understandings about the measureable attributes of objects, comparisons of attributes, and the application of units. With particular regard to units, it is important to note that the children show a remarkable awareness of a range of formal units, including some that they would not normally be expected to have an awareness of, i.e. megalitres. It is also important to highlight that children have individualised, idiosyncratic ways of understanding measurement concepts, such as using one attribute to understand another, i.e. comparing areas in order to compare masses. Of crucial significance is that these are the understandings which children have developed *for themselves* in prior-to-school and out-of-school contexts, and educators must recognise and build on these existing understandings so as to make the in-school measurement learning relevant and meaningful.

The notion of representing measurement understandings in a visual form has widespread classroom applications beyond simply as a data gathering technique. The drawing activities described in this paper could easily be adapted to classroom practice, and such an adaptation would allow teachers to both recognise and extend the understandings about measurement which children possess; assess children's understandings about particular measurement concepts; discover information about the contexts and experiences that influence children's developing understandings; and gain insight into the personalised ways in which children construct measurement understanding.

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FROM CLASSROOM TO CAMPUS: THE PERCEPTIONS OF MATHEMATICS AND PRIMARY TEACHERS ON THEIR TRANSITION FROM TEACHER TO TEACHER EDUCATOR

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This paper examines the experiences of teachers seconded as mathematics educators in the Faculty of Education at an Australian University. Data were collected from four participants who responded to a series of questions derived from an existing framework on the transition from teacher to teacher educator. The purpose of the study was to undertake a reflective exploration of the major challenges faced by seconded teachers from their perspectives, with the findings indicating that participants generally felt they lacked the requisite skills and knowledge required to teach in a tertiary environment. The study adds to the limited research in this area and has implications for tertiary providers and teacher educators.

Introduction

While extensive reference is made in the literature regarding the transition from pre-service teacher to beginning teacher, Badali and Housego (2000) claim that another very important transition is that from teacher to teacher educator. The two separate transitions, from pre-service teacher to teacher and then from teacher to teacher educator should be considered as a continuum of professional development (Badali & Housego, 2000).This appears, however, to be an under-researched area, as even within the widely researched realm of teacher education there has been relatively little written about teacher educators as an occupational group (Martinez, 2008). More specifically, research into the transition and experience of secondees from the role of teacher to teacher educator in the area of mathematics education appears to be uncharted territory. Such arrangements—often referred to as secondments—involve a teacher taking leave from a school temporarily to teach at the university in a full time capacity. Secondees are usually appointed for one or two years and in this context, undertake an 80% teaching load. At the time of writing, the author was in her second year of a seconded arrangement.

Review of the literature

There has been relatively little research about beginning teacher educators and even less about the experience of seconded teachers or seconded mathematics teachers. Some recent studies have used self study and reflective narrative to examine the transition process and experiences of new teacher educators (Swennen, Shagrir & Cooper, 2009;

Wood & Borg, 2010). These studies have recognised that whilst the transition from teacher to teacher educator can be rewarding it is also demanding and complex and requires supportive systems in place to facilitate the transition. In practice, however, it seems that new teacher educators are often given little guidance, with Swennen et al. (2009) suggesting that most new teacher educators organise their own induction. Zaslavsky (2008), who provides a rare example of research in this area directly related to mathematics teacher educators, claimed that teacher educators are essentially ‗self-made'. That is, new teacher educators make their own transition from their experience as a mathematics teacher and there are few explicit curricula for mathematics teacher education. Some research, (e.g., Adams & Rytmeister, 2000; Martinez, 2008; Swennen et al., 2009) has attempted to address this through the identification of several key activities that could help facilitate the successful transition from teacher to teacher educator. These activities include new teacher educators working collaboratively with experienced colleagues on developing teaching material, receiving feedback on their tertiary teaching from colleagues in a supportive learning environment and creating a support group for professional discussion and the exchange of ideas (Swennen et al., 2009). There is little research, however, on how these ideas are enacted in practice or whether or not they make a difference to the experiences of new teacher educators.

The work of new teacher educators demands professional knowledge of content and pedagogy skills beyond those required by the classroom teacher (Swennen et al., 2009). Although new teacher educators may be considered good classroom teachers, they may not possess the knowledge and skills to be effective teacher educators. The category of teacher knowledge of particular importance for the new teacher educator is pedagogical content knowledge (PCK). This category, as conceptualised in the seminal work of Shulman (1986), represents the blending of content and pedagogy, and is the category most likely to distinguish the understanding of the teacher from that of the student. While PCK has been the focus of many studies involving pre-service teachers and practising teachers, it appears that research into the type of PCK that might be required for a teacher educator is still to be carried out. Murray and Male (2005) address this to some extent, in that they distinguish between first and second order teaching, whereby the classroom teacher practises first order teaching and the teacher educator practises second order teaching. According to them, second order teaching requires more than the skills and knowledge to teach particular content; it also involves the knowledge and skills about the education of teachers, implying a certain level of PCK. Similarly, Peled and Hershkovitz (2004) suggest that whilst there are inherent similarities between the learning of teachers and teacher educators, the teacher educators have a greater responsibility for connecting theory with practice. Other research (e.g., Guilfoyle 1995) highlights the requirement by teacher educators to understand the culture of academia. As new teacher educators usually have limited experience in academia, they may have difficulty making sense of the complexity of the role. For example, new teacher educators can experience conflict between teaching and research responsibilities (Adams & Rytmeister, 2000; Martinez, 2008; Wood & Borg, 2010).

Martinez (2008) identifies a gap in the field of research in the area of teacher education by highlighting a need for more systematic research into the fundamental characteristics of teacher educators. Such characteristics include how the work of teacher educators is constructed, the competencies relevant to teaching about teaching and the necessary support and professional development required by teacher educators. He also identified six challenges for new teacher educators, including the changed nature of the learners (from school age students to adults), professional autonomy, institutional structures and size, work environment including technology, the modelling imperative (practising what we preach) and the research and promotion culture. Of particular relevance to this paper are the challenges he identified as being the shift from teaching school age children to adults, the need for self-management in an autonomous role and coming to terms with the research culture of university. Badali and Housego (2000) also constructed a framework for understanding the challenges involved with seconded arrangements, but they extended this to include the transition back into the classroom. Their framework identified seven phases of secondment: seeking the position, preparing for secondment, expressing self doubt, adjusting to tempo and workload and working with adult learners, looking for support, and returning to the school community. Both frameworks acknowledged the changed nature of the learners that the new teacher educator is required to teach, from school aged students to diverse adults in a non-compulsory environment. As Martinez‘s framework focused specifically on the transition from teacher to teacher educator, it was used as the main theoretical framework in the study discussed in this paper.

Methodology

Data collection and analysis

Using the framework devised by Martinez (2008) as a guide, a questionnaire was developed and administered to the participants via email. The items were as follows:

1. How did your past classroom teaching impact upon your tertiary teaching? (For example, were there any similarities or differences and what adaptations to your practice did you need to make when you began teaching at the university?)
2. How confident were you with the content that you were required to teach?
3. Comment on your experience of having to deliver other people‘s prepared material as a secondee.
4. List the top three challenges you faced as a classroom teacher making the transition to teacher educator.
5. What advice, if any, would you give a future secondee in light of your own experience?

The participants emailed their responses to the questionnaire to the author between five and ten days after receiving the questionnaire. Each participant answered all questions. The most detailed responses were given for the first item on the questionnaire. The responses provided for each of the five items were allocated to one or more of the six categories identified by Martinez.

The participants

The participants were four seconded teachers who were experienced classroom teachers. The author is included as one of the participants, with the other three participants being teaching colleagues of the author (one on a different campus). Jo, Ann, and the author were in the second year of their secondment, while Darryl had several years‘ experience as a secondee. (Names are pseudonyms.) Jo and Ann were both from a primary school teaching background whereas the author and Darryl were high school mathematics teachers. Darryl and the author taught only mathematics curriculum units whereas Jo and Ann were asked to teach into a range of discipline areas including literacy, numeracy, and generic Education units.

Results and discussion

This section presents the results and discussion, structured around the challenges identified by Martinez (2008).

New learners: Children to adults

Martinez (2008) acknowledges that a different set of skills and knowledge is required to teach a diverse range of adults in a non-compulsory educational setting, and that this marks a key transition challenge for new teacher educators. For example, while classroom management of challenging behaviour from children is less of an issue in the tertiary setting, dealing with older students in a fee paying non-compulsory setting ―can be daunting and harrowing" (Martinez, 2008, p. 39). The same concern was raised by the participants in this study, who commented on the management of adult learners, including establishing rapport and communicating expectations (such as mobile phone etiquette). Ann and Jo both indicated that they needed to make adaptations to their style of teaching at university, from the inquiry-based student centred style in the primary classroom, to a more teacher centred delivery of content. ―It was different using power points and having to be the leader more; less discovery work than primary school" (Jo).

Assessment emerged as a strong theme in terms of highlighting the differences between teaching children and adults. The process of assessment in the primary setting is often ongoing and formative, and influenced by opportunities for sustained interactions over a year long period. Assessment of assignments at university, however is summative and proved to be challenging for Ann in particular:

> As the more structured approach was not the main approach used in my primary/learning support teaching years, I find the assessment structures and practices frustrating sometimes. In the primary setting I was used to ongoing assessment of students that I taught daily, allowing me to get to know them well.

Jo, Ann, and the author saw the marking of assignments for students as an issue, made even more challenging by needing to cater for students studying in both online and face-to-face modes. The issue of providing pre-service teachers with feedback on assignments and how much time to spend marking each assignment were common themes within the responses given by the participants, with Jo indicating that she was challenged with ―knowing what to expect in assignments and to be able to give clear feedback" The author found the marking particularly challenging at first given that her experience with assessment as a high school mathematics teacher involved marking mathematical items rather than written assignments:

> The marking is so relentless and I found it a real challenge to mark a written assignment of 2000 words as I have been used to marking mathematics problems. Getting used to understanding referencing was a challenge.

It is not surprising that the assessment processes in a tertiary setting proved to be challenging in that most of the participants had not undertaken any further academic study since receiving their teaching qualifications, nor were they used to marking large volumes of material. While moderation processes assisted with establishing expectations, many participants still felt that they lacked the in-depth knowledge and skills required to critique tertiary writing.

While some contrasts between teaching children and adults were evident, interestingly the secondees indicated that they did not tend to change their approach to teaching mathematical content. In general, they found that the pre-service teachers lacked confidence with mathematics, had poor mathematical content knowledge and had similar learning needs to school age students. Ann noted the following:

> Not so much of a big change with the first year students because their needs are quite high and I found myself using the strategies I used as a primary teacher to get maths concepts across.

Darryl suggested that the pre-service teachers‘ lack of ―productive dispositions" towards mathematics presented a major challenge for developing their mathematical knowledge for teaching. This point was also raised by the author who indicated that some groups of pre-service teachers held the view that mathematics is about knowing formulae and following procedures and that they may not be receptive to developing a more relational understanding of mathematical concepts. Overall, the participants were surprised at the pre-service teachers‘ lack of content knowledge and the extra challenges this raised; essentially they were seconded to teach ‗how to teach mathematics‘ but instead found themselves teaching mathematics. This issue raises the question as to whether or not this challenge is unique to secondees teaching in this particular discipline area.

Autonomy

In her framework, Martinez (2008) refers to autonomy as the requirement for self-management by a new teacher educator. Items 2, 3, and 4 from the questionnaire were particularly useful in eliciting responses from the participants that were related to this category. Three secondees, for example, highlighted the contrast between the regimen of the school term and the more flexible working arrangements at the university. Jo, Ann, and author commented that this new autonomy made them feel more like trusted professionals. ―I have been included as part of the staff, and trusted to act professionally, such as being able to work from home" (Jo). There were, however, some challenges associated with professional autonomy including the issue of self management. Jo, Ann, and the author highlighted the need to learn to cope with the contrast between relatively ‗quiet‘ times and more hectic periods involving a combination of teaching, marking, study, and research. Similarly, Martinez claims that new teacher educators‘ autonomy is often accompanied by overwhelming workloads at particular times during the academic year.

Institutional structures and size

The institutional structures and size of a university are often vastly different to a school. According to Martinez (2008) studies have indicated that coming to terms with institutional complexities is a key challenge for new teacher educators. Contrary to this claim, responses from the participants in this study indicated that this was not a particular issue or challenge for them.

Work environment (including technology)

Martinez (2008) highlighted the increasing sophistication of the online learning environment and its implications for new teacher educators who may be required to teach in this mode. The participants in this study were expected to teach in a variety of modes, including online, and to make use of online communication systems. Three participants commented on the requirement to grasp the technology associated with online teaching; however, they were primarily concerned with the pedagogy of online teaching rather than the use of the technology. The participants' main concern in terms of teaching mathematics was the perceived difficulty in an online environment of replicating the types of practical experiences afforded in the face-to-face mode, which they were more comfortable with having come from the classroom. The following quote was part of Ann's response to the fourth item on the questionnaire:

> Online learning is new to me. I have only had one online (maths) unit to teach so far and I did not enjoy this compared to face-to-face delivery. In face-to-face tutorials I can give instant feedback in many ways: verbally, using a model or diagram and this is much more difficult through online delivery

The modelling imperative

Martinez (2008) suggests that the modelling imperative is the most challenging aspect of the transition of a new teacher educator. ―Practising what [we] preach" (Martinez, 2008, p. 42) involves a very high level of meta-cognition, as the teacher educator must be able to explain and justify their teaching practices. Jo and Ann generally felt confident with teaching content that was closely aligned with the content of the primary school curriculum, but often found the more theoretical aspects of the units challenging, as the following quote illustrates:

> In some areas I was confident—curriculum ones especially reasonably confident especially with first year pre-service teachers. Less confident in some where there is a huge theory base—I needed to read a lot and understand some of the particular theories again (Jo).

Darryl and the author expressed an initial lack of confidence with teaching about primary pedagogy, rather than the mathematics itself. Although both had a strong mathematics background, they lacked the experience of teaching in a primary setting and therefore felt their PCK, particularly in the area of primary mathematics, was inadequate. Darryl highlighted the challenges involved, particularly at the beginning of the secondment, with understanding the fundamental ideas underpinning a particular lecture or tutorial when it was written by someone else, usually an established academic. The author experienced similar concerns and had not anticipated the level of background reading and thought processing required to make sense of a pre-prepared lecture, as the following quote illustrates:

> I felt out of my depth. I had anticipated that teaching a primary mathematics curriculum unit would be difficult as I have a high school maths teaching background. Sometimes I appeared ill prepared because at times I resorted to reading the slide and the students can really see through that. The background reading and mental thought processing that has to happen to make sense of one slide is considerable.

In contrast, Jo and Ann commented that one of the strengths of being a secondee was that the pre-service teachers valued their ‗real world' classroom teaching. ―The students appreciate what you have to offer and are grateful for your school experience" (Jo). It is interesting to note that Jo's comment was made in relation to her teaching of a generic unit on assessment and reporting, rather than mathematics.

Research and promotion

A lot of the references to this category came from responses to the fourth item on the questionnaire. Although secondees may be encouraged to become involved in research, unlike academic appointments they are not required as part of their conditions to do so. Ann and Daryl expressed concerns about the pressure to undertake research and the negative impact this may have on their teaching. The extent to which all the participants perceived this to be a challenge varied.

> As a teaching only secondee, I have limited knowledge of this area. However it appears that there is great pressure on and a struggle for academics to meet research deadlines. I have decided not to go down this path as it would impinge on the quality of my teaching. (Ann)

This comment tends to suggest that Ann considers teaching and research to be mutually exclusive, and that it is not possible to excel at both. Darryl, too, felt strongly about what he perceived as conflicting interests between teaching and research:

> I had expected the university to be an intellectually vigorous institution focused on producing students who had the capacity to be effective teachers upon completion of their degree but I think the University is primarily concerned with research.

For others, however, the opportunity to become involved in the research culture was viewed as stimulating and highly motivating. ―At the start of the secondment, hearing academics talk about writing papers or journal articles seemed so new and foreign, whereas now it's part of working life at university and something I hope to aspire towards" (the author).

Summary

The data provided an insight into some of the key issues with becoming a new teacher educator as identified by the four secondees at a particular time during their secondment. Whilst many of the responses were able to be categorised into Martinez's (2008) framework of challenges, some appeared to be more relevant than others, and some additional themes arose. One of these relates to the skills and knowledge needed to teach teachers. Although the secondees were experienced and competent classroom teachers, they felt they did not have the skills and knowledge to effectively link theory with practice as required by the teacher educator. While their recent classroom experience was unexpectedly advantageous in assisting pre-service teachers with their mathematical content knowledge, they were all too aware of a lack of experience with teaching in a tertiary environment. Related to this were the additional challenges associated with assessment processes, teaching in an online environment, and not being part of the research culture. This raises the question of how best to support secondees with this knowledge acquisition given that ultimately this knowledge must be actively and personally constructed by the individual secondee. Another related theme to emerge was the conflicting combination of pre-service teachers' low mathematical skills proficiency and the secondees' developing knowledge for teaching in a tertiary setting. Although the secondees aimed to teach the pre-service teachers how to teach the mathematics, they first had to address the content itself because the pre-service teachers did not have the foundational content knowledge.

Conclusions

This study has added to the limited research on the transition from teacher to teacher educator in the area of mathematics education, through providing an account of four different perceptions of aspects of this transition. The findings indicate that Martinez‘s (2008) framework proved useful in interpreting participants‘ responses, but there are indications that the framework could be extended to include other aspects; a future study involving in-depth interviewing of secondees‘ experiences may help to inform this. The main concern or challenge raised by the participants related to their perceived lack of teacher-educator PCK. This raises the question of how best to support secondees with this knowledge acquisition, given that ultimately this knowledge must be actively and personally constructed by the individual secondee. An unexpected challenge, and perhaps one that was unique to mathematics education secondees, was the need to address the pre-service teachers‘ lack of mathematical content knowledge. While some participants viewed this as an opportunity to capitalise on their recent primary classroom teaching, others felt that their secondary teaching background impacted upon their ability to do this credibly.

While admittedly the study was limited in terms of sample size and data generation, it does raise some questions about the preparedness of secondees to teach pre-service teachers about teaching, particularly in the area of mathematics education. It was also interesting to note that at least two of the participants questioned the value of research, leading one to question whether or not the University‘s policy of not requiring secondees to research is counterproductive to the research culture it should be trying to foster. Overall, however, the participants were positive about their seconded experiences and the opportunities to link practice with theory; it is hoped that this account will help to inform tertiary providers about the need to foster secondees‘ transition from teacher to teacher educator.

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ENGAGING THE MIDDLE YEARS IN MATHEMATICS

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Student engagement in mathematics in the middle years is consistently reported to be a challenging problem. Yet, as this action research study shows, it is possible to engage students in meaningful mathematical learning with the use of relevant investigations. This project with 14 Year 8 and Year 9 mathematics teachers was structured around an action research model with teachers supported to refine capabilities and pedagogical processes to implement mathematical investigations that 'make sense'. Following implementation, teachers reported that students were more engaged compared to traditional mathematics lessons and that students recognised the value and application of mathematics, which in turn leads to greater engagement in mathematics.

Introduction

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. *Albert Einstein*

Student disengagement in mathematics is recognised to be a challenge for educators. Students continue to reject mathematics when they have a choice, particularly in the senior school years and at tertiary levels. "[M]athematics … is perceived to be 'hard', 'boring' and 'useless'" (Brown, Brown, & Biddy, 2008) and of little practical value and so many complete their formal mathematics education with poor mathematical identities. For instance, in 2009 only 7.5% of Queensland students studied both Mathematics B and C in Year 12 but to continue with mathematics it is necessary to study both (Queensland Studies Authority (QSA), 2010).

Making sense of mathematics

To mathematicians, mathematics is about making sense of the world and seeing the connections between mathematics and the world, and the connections between different areas of mathematics (Burton, 1995, 1998-1999). This idea of *making sense* of the world provides a possible avenue for increasing engagement in mathematics in the classroom. As Schoendfeld (1992, p18) notes, "[C]lassroom mathematics must mirror this sense of mathematics as a sense-making activity if students are to come to understand and use mathematics in meaningful ways". Furthermore, the availability of technology (calculators, computers etc) has eliminated the need for most pen-and-paper calculations (Battista, 1994) yet this is still the focus of many classrooms – teaching children to do things machines are good at which does not make sense to students. If computers and calculators were used to do the things they are good at, it would leave students with the tasks of problem formulation and interpretation of the calculative work of machines.

By providing students with investigations and problems we give them the opportunity to ‘do’ mathematics and to *make sense* of their world. The goal of these tasks is “for students to make sense of a real-world use of mathematics, to get them involved in ‘problem formulation, problem solving, and mathematical reasoning’” (Battista, 1994, p463). As students solve complex tasks they get opportunities to discuss mathematics, to “conjecture, test, and build arguments about a conjecture’s validity … and to be encouraged to explore, guess and even make errors “(Battista, 1994, p. 463).

For students to successfully work with these tasks, they may initially need a significant amount of thought to make sense of the task and/or mathematics before they can start mathematising. According to Romberg (1994), the steps involved in doing mathematics are:

- initially one needs to formulate the problem and to think about which variables are important and which relationships between variables matter and which do not;
- a model then needs to be determined which may be mathematical or physical;
- numbers can be substituted into the variables and numerical procedures used to find a solution of a numerical model. Alternatively students may use the physical model or act out the problem to find the solution; and finally
- the validity of the solution needs to be considered – does it make sense? What if I made a minor change here or there? This may necessitate going around either the whole cycle or part of the cycle again.

For the teacher, working with investigations and making sense of the world can be much harder than teaching factual information. A focus on pedagogy rather than content is a major shift that needs to occur. As Burkhardt (1988) explains, teachers:

- need to consider the different approaches taken by the students;
- need to decide when to support students with suggestions or questions that will help whilst still allowing the students to be responsible for finding their own solution and this is for each student or group of students in the class; and
- may be put in the potentially uncomfortable position of not knowing all the answers.

Mathematical investigations

Problems in the real world are ‘ill-structured’ and so it is necessary initially to formulate them in a well-structured way (Heylighten, 1988). Problem formulation is commonly carried out by the teacher, which leaves the student with the task of applying an appropriate algorithm which may be able to be calculated by machine. Taking problem formulation from students removes a key opportunity for students to engage in sense making using mathematics (Battista, 1994).

A good investigation has multiple entry points, allowing students to start at their own level and to design their own pathway (or pathways) through it. Indeed, investigations allow students to undertake activities and thinking that resemble that of the practice of mathematicians, and so they can be viewed as authentic mathematical tasks (Burton, 1998). In this way, investigations allow for the alignment of teaching, learning and assessment.

Boaler (2008) demonstrated that it is possible to engage students in deep mathematical learning using an investigative pedagogy, particularly those students who have been alienated by traditional approaches to mathematics education. Investigations are open-ended questions or problems that are set in a range of contexts. By using investigations that are directly related to the students' lives, mathematics becomes no longer 'useless'. To achieve this, teachers need to provide a socially supportive and intellectually challenging environment in the classroom (Fredricks, Blumenfeld & Paris, 2004) so that students are able to develop strong mathematical identities. When the task is relevant and meaningful most students enjoy a challenge (The Centre for Collaborative Education, 2000, cited in Hilton & Hilton, 2005).

Encouraging students to formulate problems is not easy, as the data from this small study reported in this paper indicates. The normal routines of teaching mathematics are not easily adapted to support a pedagogy that in many instances supports a mathematical activity where there is no one right answer. In this study we will explore the journey taken by a small number of teachers as they set out to facilitate the pedagogy of investigations to enable their students to *make sense* of mathematics.

The study

The aim of this action research project was to improve middle years student engagement in mathematics by employing investigations that *make sense*. A group of Year 8 and Year 9 mathematics teachers, comprising eleven female and three male teachers, volunteered for the project. There were six state schools each represented by two teachers, with two also sending their head of department.

The project was funded by Education Queensland with an overall desire to enhance pedagogical practices leading to an improvement in numeracy results across the region. An understanding about participation in the project was that there would be a commitment to support these teachers by allowing them flexibility in their work programs to trial some of the initiatives. Teachers were encouraged to attend in pairs with their mathematics head of department to enable a continuation of the conversations back at school. The project ran during the last term of 2010 and it is important to note that because of this timing it was difficult to maintain enthusiasm as teachers had end of year pressures. Fourteen teachers from six schools completed the professional development and three schools, including Schools A and B that are the focus of this paper, chose to replace their final assessment item with an investigation that lasted at least four weeks and included in-class teaching.

Action research process

Underpinning this study was an Action Research approach. Action research is a well-accepted methodology developed by Lewin in the 1940's that has recurring cycles of action and reflection (Dickens & Watkins, 2011). The model employed in this project is adapted from Kemmis and McTaggart (1988) and is summarised as follows:

- Plan – Priorities for action
- Act and Observe – Is it working? How do we know?
- Reflect – What are the problems?
- Revise plan – Review plan
- Act and observe – How is it going? How do we know?
- Reflect – Have we got it right?

The project was devised to facilitate participants to work through the model at least once.

Fundamental to action research in the *planning phase* is the provision of information and this was achieved through two whole day professional development sessions. Discussions initially centred on the needs and interests of adolescents, given the context of this project in middle years classrooms. This focus was planned to help participants with ways of devising pedagogical approaches to meet student needs. The focus then moved to practices for differentiated learning in the classroom. Mathematical investigations were then offered as a way to stimulate the interest of the students and cater to the diversity within a class.

In this *planning phase,* data were collected in the form of teacher and student questionnaires which were adapted from the work of Beswick, Watson, and Brown (2006). The questionnaires were constructed to collect data to determine both teachers' and students' confidence with mathematical concepts, their responses about mathematics and numeracy in everyday life and mathematics and numeracy in the classroom and the types of activities that were valued. The purpose of the questionnaire was to determine if there was a correlation between what the teachers reported and what their students thought. Teachers used this information to assist their planning.

Teacher participants were asked to provide comments about their experience in the action research study. The first data collected was purely to ascertain the reasons why the participants volunteered. The following verbatim comments provide an insight into the various reasons, which are consistent with the aims of the study:

- Engaging middle school students without working 60 hour weeks.
- Better engagement from my year 9s and teaching in a way that is less didactic / more student or interaction focussed.
- I would like to make changes to the current mathematics program to engage the lower achieving students.
- To get some ideas about investigations and reflection.
- I want to feel confident in my ability to teach maths in a way that is engaging and relevant to students.
- Better ways to engage the middle years teachers and in turn the middle years students (maths HoD).

Following the two day professional development participants devised plans for their chosen class to be implemented during the final term at school. They used the action research model described above as the basis for their planning. The researchers visited the teachers in their schools to provide support and made classroom observations of some classes. In this way, both the researchers and the teacher participants were engaged in the next phases of the action research project, that is recurrent *act and observe; reflect; revise plans* that took place. After these meetings one of the researchers wrote observations. At the end of term the teachers were encouraged to again *reflect*. Finally, they *planned* the next action cycle for implementation in 2011.

Results and discussion

The data reported in this paper came from the reflections of one of the researchers and the teachers in two of the schools, identified as School A and B. In school A the students were given the 'ill-structured problem' of designing a middle years area in the space outside their classroom which then consisted of concrete, grass and a few bench seats. In school B the two teachers developed a structured investigation where students investigated loans for a car purchase and compared simple and compound interest for an investment.

School A – An ill-structured investigation

School A was a very large P-12 school with a middle school structure consisting of Years 7, 8 and 9. The mathematics head of department was not actively involved in this research project. One of the teachers had expressed concern during the professional development session that mathematics at her school was "impossible". She complained that she had: no support; a low ability class who were not interested, badly behaved and couldn't cope with the work; and that she wasn't allowed to adapt the tests so that she could reduce the amount of content that she needed to get through. In her responses to the survey this teacher had agreed that quantitative literacy was just as necessary for efficient citizenship as being able to read and write. She decided to ask her class to design a middle years area in the space outside their classroom. The task which was to be used for assessment was left open and through class discussions she and her students planned how they would approach the task. The students were told at the beginning that the Principal would be invited to look at their final models for the middle years area and that possibly some of their ideas would be implemented, giving the project a sense of authenticity. The class initially discussed what to include in their designs and all ideas were noted.

The researchers joined the class when they went outside to measure the permanent fixtures e.g. chairs, concrete paths etc. The students were excited about the task and shared this with the researchers; telling the researchers what they were going to do and how they were going to do it. At this stage the students had formulated the problem; they knew they had to decide what they were going to include by surveying, they knew they had to measure, do a scale drawing and then could make their scale model. There were a number of students with poor measuring skills, for example not reading the metres on the tape measure only the centimetres and starting at 10cm as they thought the stiff part was something to hold onto. The teacher asked questions to enable the students to see their errors themselves. For example, when a group claimed the area was considerably wider than it was long she said, "Let's have a look at your measurements," pointing to a 23m length and a 57m width. "What do you think?" and then "What are you going to do about it?" when the student asked, "Do we have to do it again?" she replied, "What do you think?" With this type of questioning the teacher is forcing the students to think about what they are doing and to take responsibility for their learning.

Afterwards all of the students in the class went back into the classroom to add these extra measurements to their scale drawing. The students were talking about how to do the scale drawings, and how they had collected the measurements to remind themselves where they had measured from and to and what part of the diagram it was. Sometimes they stood up and looked out the window to check where things were outside. The students worked slowly but were interested in getting help from the teacher and researchers so that they could prepare the drawing. Students told the researchers that they had to do the basic scale drawing before they could include their own additions.

By the end of the term the students had constructed scale models of their designs. All had included extra seating, shade and bins and had included their own ideas such as handball courts, basketball rings and palm trees. The teacher's feedback was that there was much improved engagement and learning for most of the students and the hands on component was important. She observed that the task couldn't be too long as the students lost focus and that group work was difficult. In her class pairs were more successful than trios as when three students were working together one tended to sit back and let the others complete tasks. The teacher also observed that she had more opportunities to find out what the students actually did know as she moved around the room talking to the groups of students.

This teacher was enthusiastic about her experiences with the investigation and the difference this pedagogy had made to her class. Consequently she has taken on the role of Year 7 co-ordinator for 2011with the aim of getting all the Year 7 teachers together to plan investigations as part of the assessment and pedagogy for mathematics.

School B – A highly structured investigation

School B was a large secondary school (Years 8-12) with a similar clientele to school A. In this school teachers did not trust the students to bring their equipment to class so the students left their mathematics exercise books in the classroom to ensure they had them for every lesson. The two teachers who participated in the study wrote an investigation exploring loans and investments, with the support of their mathematics head of department who had not attended the professional development. The task included comparing two different methods of paying off a car and comparing simple and compound interest for an investment. In their response to the survey, the two teachers had disagreed with the statement that quantitative literacy was just as necessary for efficient citizenship as being able to read and write.

The students were given a choice of purchasing five cars or a motorbike. The highly structured investigation was outlined on a task sheet that included the price of each vehicle, the number of kilometres travelled and the weekly repayments if purchased through the dealer's finance company. The task sheet stepped out what needed to be done at each stage of the investigation and supplied the required formulas so that the students only had to insert the numbers into the equation and calculate the answer. This meant that this group of students missed the opportunity to formulate the problem and also make personal choices by allowing them to choose a car for themselves. When the researchers went into two different Year 9 classes with one of the teachers and a Year 9 class with the other teacher only one of the classes were actually working on their investigation the other classes were developing the mathematical skills and knowledge necessary for the next part of the investigation. The students were doing their own individual work, however there were a lot of discussions with their peers. These students couldn't really tell the researchers much about the task, and were not particularly interested but when asked did admit that it was a useful assignment as they would be buying a car once they left school.

When the two teachers from school B reflected on the task they reported that students enjoyed the buying the car task and came up with a variety of reasons for their choice of which car to purchase, with the comments "[K]ids loved the topic – very interested in cars, both the girls and boys. Surprisingly, kids came up with a variety of reasons for buying their cars, and didn't just go for the one that looked 'cool'". The teachers reflected that the students had taken ownership of the task and that the students appeared to enjoy using the spreadsheets, stating that "[T]eaching focus on kids taking ownership of task was very positive. Kids liked making spreadsheet and learning Excel tricks". The students reportedly, were not interested in saving for a house deposit. Teachers reported being concerned about the loss of interest towards the end of the task and the need to perhaps keep the task shorter, stating that "[T]ime pressure and the fact that this was the first time using the task made it difficult to get through the last few questions with the same enthusiasm as the first half of the task. Perhaps the task is too long?" The teachers also found it difficult to allow the students to take charge due to concerns about engagement, as noted in this comment "[I]t was difficult to allow kids to take charge of the task due to fear that they would get off-task/waste time". The teachers admitted that the task hadn't allowed the students enough opportunities to demonstrate their ability to reason mathematically.

Working through the reflection phase, teachers were planning to introduce more investigations into the following year's work program but were concerned about getting the balance between allowing students the opportunity for creativity whilst still being practical to mark, stating that the "[T]ask review needs to strike a better balance between including creativity in the task and making marking practicable".

Conclusion

By using an action research approach to facilitate the inclusion of investigations as a pedagogical practice, the teachers in this study reported that the students appeared to be more engaged with the investigation tasks and the learning compared to conventional mathematics lessons. By giving the teachers the knowledge and support to implement mathematics investigations in an informed way that included reflection and revision, these teachers have provided the opportunity for students to *make sense* of the value of and applications of mathematics which in turn has encouraged the students to participate and engage with the learning.

Perhaps because their teacher values quantitative literacy, the students at school A were given an ill-structured investigation which they formulated and worked through with the teacher. By contrast at school B, where the teachers did not put such an importance on quantitative literacy, the students were given a highly structured investigation that did allow for some choice but included the algorithms to use.

The biggest increase in engagement, as reported by the teachers and observed by the researchers, was with students in school A. This may be because they had more opportunity for ownership of the task as they were not just using given algorithms but had to formulate the problem themselves. This is something that needs further investigation and has implications for engaging adolescents with mathematics.

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BUILDING PRESERVICE TEACHER CAPACITY FOR EFFECTIVE MATHEMATICS TEACHING THROUGH PARTNERSHIPS WITH TEACHER EDUCATORS AND PRIMARY SCHOOL COMMUNITIES

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Calls have been made for teacher educators to innovate upon well-established teacher education programs. During 2010, a project was initiated that sought to study the impact that a school-university partnership had on building preservice teacher capacity for effective teaching of mathematics. Early findings suggest that a range of factors including observation of lecturers teaching mathematics lessons, and participation with teacher educators in lesson planning, team-teaching, and post-lesson reflections can be helpful in building capacity for effective mathematics teaching.

Introduction

It is generally agreed among classroom teachers and researchers that teaching is complex. Indeed, the complexities that the teaching of mathematics poses may be challenging not only for novice teachers but also for the more experienced (Kazemi, Franke, & Lampert, 2009). As teacher education plays a key role in supporting novice teachers for situations they may face in the classroom, it is important to reflect upon the nature of preservice teacher education and the opportunities that preservice teachers (PSTs) are given to develop their abilities to teach.

A recent government report (Hartsuyker, 2007) from an inquiry into teacher education in Australia recommended a more collaborative approach to teacher education than existed at that time. Most particularly, in discussing practicum and partnerships, the need was identified for a stronger sense of shared responsibility between all stakeholders, that is, universities, schools and employing bodies, for preparing the next generation of teachers.

The research reported in this paper relates to a partnership based model being utilised to prepare primary school PSTs at Australian Catholic University (ACU) for their future work in schools. This partnership focused on building capacity for effective teaching of mathematics. The authors sought possibilities to innovate upon current teacher education practices that already existed at the university in relation to mathematics education. As stated by Kazemi et al. (2009),

> … the future viability of professional teacher preparation requires that we systematically pursue appropriate ways to develop, fine-tune and coach novice teachers' performance over a variety of settings. These activities must find their way into university coursework rather than be relegated to field placements. (p. 12)

Background

There is widespread concern for good teaching practice for mathematics learning (e.g., Australian Association of Mathematics Teachers, 2006; National Council of Teachers of Mathematics, 2000). Recent research has provided a range of insights into the practices of effective teachers. For example, Brown, Askew, Baker, Denvir, and Millett (1998) identified teaching that requires thought rather than practice, emphasis on establishing meanings and connections, collaborative problem solving, and autonomy for students to develop and discuss their own methods and ideas. Muir (2007) summarised characteristics of effective numeracy teachers as related to maintaining a focus on mathematical ideas, using a variety of teaching approaches to foster connections, encouraging purposeful discussion, and possessing knowledge and awareness of conceptual connections. The more detailed list of 25 characteristics of highly effective teachers of mathematics identified from case studies of six highly effective early years teachers (McDonough & Clarke, 2003), has commonalities with results from other studies but also offers insights into additional effective practices. A focus on such effective practices may provide an avenue for preservice teacher development, while at the same time further developing in beginning teachers an orientation to self-learning as called for by Sullivan (2002).

Goodlad (1991) defined a school-university partnership as a mutually beneficial inter-institutional relationship that is established through planned efforts. Goodlad purported that an essential aspect of school-university partnerships lies in drawing on the strengths of the parties involved in the partnership to advance the interests of the collaboration. Choice in participation is also an important aspect of establishing a school-university partnership (Stephens & Boldt, 2004).

All stakeholders involved in the school-university partnership have the opportunity to benefit from involvement through practices of sharing resources, expertise and facilities (Smedley, 2001; Smith & Lynch, 2002). Stronger school and university links, development of workplace capacity, and teacher and school renewal have been reported as benefits by those involved in successful school-university partnerships (Allen, Butler-Mader, & Smith, 2010). The sharing of knowledge and skills between the partnership sites (school classrooms and university campuses) is also possible, and this allows further opportunities to renew the sites during the partnership process (Stephens & Boldt, 2004).

Making the commitment to form a school-university partnership means that all parties involved in the collaboration also commit to learn together (Stephens & Boldt, 2004) through an on-going collaborative process of documentation, analysis and communication of successes and failures (Goodlad, 1991). However, there is no best way of organising school-university partnerships and debate about the most appropriate implementation approaches continues (Goodlad, 1991; Smedley, 2001).

In response to calls for re-thinking preservice teacher education, and the literature related to partnerships and effective teaching of mathematics, new possibilities for supporting PSTs for their future work as teachers of mathematics were pondered. Inspired by the work of Kazemi et al. (2009), the opportunity of developing school-university partnerships within the *Contemporary Teaching and Learning of*

Mathematics (CTLM) project[1] was identified by the authors of this paper. The CTLM project is a professional learning initiative conducted in partnership between the Catholic Education Office Melbourne (CEOM) and ACU. Professional learning, aimed at developing teacher pedagogical content knowledge of mathematics, took place through professional development sessions (including workshops, professional reading and discussions) and via in-school classroom visits by the ACU mathematics education lecturers who modelled mathematics lessons (Roche & Clarke, 2009).

In 2010, the CTLM project provided the opportunity to innovate upon mathematics education practices at ACU through the development of the *University Partnerships for Teaching and Learning Mathematics* (UPTLM) project. This partnership model is triadic in its nature involving CTLM schools, ACU PSTs (completing their final year of a Bachelor of Education) and ACU mathematics education lecturers.

The research question for the aspect of the study discussed in this paper was:

> What aspects of University Partnerships for Teaching and Learning Mathematics (UPTLM) did the preservice teachers perceive as most helpful in building capacity to be more effective teachers of mathematics?

Method

In 2010, the study involved 12 volunteer Bachelor of Education PSTs, undertaking a university project unit (taught by the authors). Within tutorials, these PSTs chose a pedagogical focus, selected from research findings on effective teachers of mathematics (McDonough & Clarke, 2003), that acted like a personal goal for further developing their mathematics teaching. Examples of pedagogical foci selected by the PSTs were

- hold back from telling children everything;
- structure purposeful tasks that enable different possibilities, strategies and products to emerge; and
- draw out key mathematical ideas during and/or towards the end of the lesson.

Following our partnership theme, during the first half of 2010, the PSTs visited CTLM schools where they observed ACU lecturers teach mathematics lessons. During the observations, the PSTs recorded evidence of their selected pedagogical focus in practice. Following these experiences, the PSTs engaged in focused lesson debriefings with CTLM teachers and ACU lecturers. Other UPTLM project practices included the planning and team-teaching of mathematics lessons with ACU lecturers in CTLM schools. Working with a fellow PST, they went on to "buddy-teach" a number of lessons in a CTLM classroom. The CTLM teachers in these classrooms volunteered their time and expertise to host the PSTs in their classrooms. The buddy-teaching experiences in CTLM schools provided further opportunity to give attention to the pedagogical focus and to offer and receive feedback within a collaborative and supportive relationship with each other and the CTLM classroom teacher. Tutorials at university also provided opportunities for members of the group to share, challenge, and support each other.

In November 2010, data regarding the PSTs' perceptions of UPTLM were gathered. Data were collected through individual written responses and a separate focus group semi-structured interview. In reporting data from the study, the authors draw on the

[1] The authors acknowledge the support of the Catholic Education Office Melbourne (CEOM) and that of Gerard Lewis and Paul Sedunary in particular in the funding of the CTLM project.

written responses and on results from a focus group ordering task. During this task, the PSTs were each invited to write what they perceived as a component of UPTLM on a card individually. These cards were then placed on a continuum from *most helpful* to *least helpful*. The PSTs involved in this ordering exercise were asked to share their insights with the group, identifying similarities and differences, and to develop consensus as a group as much as this was possible.

Results and discussion

The ordering task allowed access to some important insights into perceptions held by the PSTs about practices related to the UPTLM project. These practices are reported and discussed below, in order of their perceived helpfulness as expressed by the group overall. The two most helpful practices involved opportunities to work with ACU lecturers through mathematics lesson planning sessions and opportunities to debrief about lessons conducted in CTLM schools. The focused observation of lessons taught by ACU staff members was also believed to be a highly helpful practice.

The theme of partnerships featured as the next most helpful aspect of the UPTLM project. Aspects of the partnerships highlighted by the PSTs included the CTLM school communities, specifically opportunities to work with students who attended these schools. The PSTs also valued partnerships that developed between themselves and the ACU lecturers with whom they worked in primary mathematics classrooms. The PSTs saw the "buddy-teaching" experiences as another element of the partnerships forged through the project.

Team-teaching experiences with ACU lecturers were also reported to be a helpful practice. PSTs mentioned that the focused feedback on performance was valuable. Deemed as equally helpful by the PSTs were the UPTLM meetings that were conducted on campus during the year. It was articulated that opportunities for group reflection on UPTLM experiences were helpful in developing deeper understandings of effective teaching of mathematics. Feedback from parties involved in the partnerships (ACU lecturers, CTLM school teachers and the PSTs themselves) provided an *external voice* that supported critical reflection which was used to challenge current ideas and practices related to effective mathematics teaching (Muir & Beswick, 2007).

The final most helpful aspect of the UPTLM project, as perceived by the PSTs, was the pedagogical focus. This self-selected focus provided opportunities for the PSTs to reflect on current practices and it also provided focus for the lesson observations in CTLM schools. The role that the pedagogical focus played was also highlighted in the written responses by the PSTs.

Not surprisingly, time constraints were the least helpful aspect of the UPTLM project. The PSTs agreed that there was a greater need for more time and opportunities to participate in the UPTLM project, spending more time in the partnership schools. When asked to describe the opportunities in which they would engage if they had more time, the PSTs identified lesson observation and team-teaching experiences with ACU lecturers were deemed as valuable uses of time by the focus group of PSTs.

The following brief discussion provides further insights into how the helpful UPTLM aspects developed the PSTs' capacity for effective teaching of mathematics. As discussed earlier, the notion of partnerships was central to UPTLM. One PST expressed the value of partnership, not only in relation to her attitudes to mathematics teaching, but also for the contribution to the school in which she undertook her "buddy teaching":

> Being part of [UPTLM] made me a more confident and prepared preservice teacher for my placement. My teacher was impressed and appreciated the ideas I could bring to her class. (McDonough, Sexton, Miller, Mitchell, & Watson, 2010, p. 4)

Another school contacted the authors stating how impressed they were with the contribution by the two PSTs who were buddy teaching at their school. Indeed, they asked the PSTs to share their taught lessons with other teachers in the school as they felt this could be a valuable learning opportunity for staff. This suggests that UPTLM allowed the parties in partnership to draw on each others' strengths (Goodlad, 1991) and share expertise (Smedley, 2001; Smith & Lynch, 2002).

Other PSTs also expressed the value in terms of the influence on attitudes to teaching mathematics, for example,

> I have become a lot more confident in my teaching of Mathematics, not just in my content knowledge, but in my knowledge of what it takes to be an effective practitioner. I have also become a lot more enthusiastic about teaching maths. It is a great way to feel about a subject I will be teaching every day. (McDonough et al., 2010, p. 5)

The above quote also indicates the influence on content knowledge, a key focus of the CTLM project in which the teachers in the partnership schools were involved, and a focus of the preservice mathematics education units at ACU.

For these PSTs, having selected a specific pedagogical focus gave direction for a range of the UPTLM activities. This is expressed, for example, in the following:

> UPTLM has taught me the importance of focusing one aspect of your teaching. Having a pedagogical focus allowed me to really focus on one thing I needed to improve. I found this was more beneficial than trying to improve all areas of my teaching practice. … I can now effectively draw out the key mathematical understandings towards/at the end of a lesson. (McDonough et al., 2010, p. 5)

The PSTs also saw that by concentrating on one aspect of teaching, links with other effective teaching practices (e.g., McDonough & Clarke, 2003) could be seen. They also expressed the value of UPTLM experiences for an 8-week extended practicum that occurred in the second half of the 2010 academic year.

Conclusion

The UPTLM project has provided some insights into possible ways of establishing school-university partnerships. Throughout the project, there were opportunities for the participants in the partnerships to learn (Stephens & Boldt, 2004), including the authors who learned more about their work as teacher educators. The authors have come to understand more about ways of building PST capacities for more effective teaching of mathematics and exploring these practices in renewed, innovative, and collaborative ways (Hartsuyker, 2007; Kazemi et al., 2009; Stephens & Boldt, 2004). The authors also believe that they have challenged a model of teaching and teacher education that is referred to by Lampert and Graziani (2009) as "closing the classroom door" (p. 491), where individual learning is valued but the collective accumulation of knowledge is disregarded.

It is acknowledged that, in this project, the authors worked with a small number of PSTs and had the advantage of drawing on a professional learning project where lecturers were teaching in schools. Such opportunities may not be available presently to our colleagues in all tertiary institutions. However, the project has allowed a re-thinking of preservice teacher education and the value of partnerships in developing capacity for effective teaching of mathematics. Whilst respecting traditions, the authors see the possibility for this research to stimulate and contribute to dialogue that responds to calls for new practices in teacher education including partnership based models and approaches.

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LISTENING TO CHILDREN'S EXPLANATIONS OF FRACTION PAIR TASKS: WHEN MORE THAN AN ANSWER AND AN INITIAL EXPLANATION ARE NEEDED[1]

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Research has shown that children can offer the right answer but have mathematically incorrect reasoning (Clements & Ellerton, 2005). One-to-one task-based interviews enabled the researchers to engage in observational listening (Empson & Jacobs, 2008) and uncover the mathematical strategies used by Grade 6 students in fraction pair tasks. Some students' answers and initial explanations were similar, but different strategies were revealed by further questioning: the correct strategy of benchmarking or the misconception of gap thinking.

Introduction

Careful listening is essential for good teaching. If we want to know what to teach next, we need to know the mathematical thinking of individual students. Teachers cannot assume that a correct answer indicates misconception-free thinking. To teach within the framework of constructivism, we need

- specialised content knowledge
- observational listening skills, and
- classroom norms that value mathematical explanations

The analysis of specific strategies for the comparison of the relative size of two fractions, $^4/_5$ and $^4/_7$, illustrates the complexity of assessing children's fraction understanding. Two strategies, the gap thinking misconception and the mathematically correct strategy of benchmarking both gave the same answer ($^4/_5$ is larger) and had similar initial explanations. Responsive teachers know that the different strategies exist, have the listening skills to determine which is being used, and create classroom norms which value explanations from students that enable peers and the teacher to engage with their mathematical thinking.

Review of the literature

A co-ordinated fraction understanding encompasses several contexts. Kieren's model for understanding rational number knowledge identified *four* sub-constructs (measure, quotient, operator, and ratio) and *three* underlying concepts (partitioning, equivalence,

[1] The authors wish to thank Anne Roche and Doug Clarke for their help in developing tasks and record sheets, and double coding the data.

and unit-forming) (Kieren, 1988, 1992). These could be engaged with on *four* levels, ethnomathematic, intuitive, technical-symbolic, and axiomatic-deductive. We use the term the *four-three-four* model to distinguish it from Kieren's five-part model (1980). The concepts that concern us in this paper are the measure sub-construct, the concept of equivalence, and intuitive understandings.

Fraction tasks in the primary school considered part of the measure sub-construct include, number lines (Lamon, 1999), area and length diagrams (Kieren, 1992), and fraction pair comparisons (Ni, 2000). The relative size of fractions has been called *order* in order and equivalence studies. *As many as* equivalence, (multiplicative) partitioning, is used to generate equivalent fractions. For example, $^4/_8$ is equivalent to $^1/_2$. *As much a,s* equivalence (additive) unit-forming, is used to combine fractions (Kieren, 1992). For example, $^5/_8$ equals $^1/_2$ plus $^1/_8$. *Intuitive* approaches were planned mathematical activity, firmly located in a context developed from schooled or taught knowledge (Kieren, 1988).

Strategies for comparing the relative size of fractions include both correct strategies and misconceptions. Correct strategies include residual thinking, benchmarking and common denominators. Misconceptions, often inappropriate generalisations, include gap thinking, higher or larger numbers, and bigger denominator indicates bigger fraction thinking. Knowledge of these strategies forms part of a teacher's specialised content knowledge. This knowledge of mathematics and knowledge of students is necessary for pedagogical content knowledge (Hill, Ball, & Shilling, 2008).

The *residual thinking* strategy has been observed in the comparisons of fractions, such as $^5/_6$ and $^7/_8$ that are both one piece away from the whole (see, for example, Clarke & Roche, 2009; Cramer & Wyberg, 2009; Post, Behr, & Lesh, 1986). Students reason correctly that an eighth away from the whole is closer than one sixth away from the whole and so $^7/_8$ is the larger fraction.

Using half as a benchmark was a strategy that children could use when they combined the (additive) unit-forming aspect of equivalence, $^5/_8$ is as much as $^1/_2$ and another piece, and the (multiplicative) partitioning aspect of equivalence, $^4/_8$ is as many as $^1/_2$. Benchmarking had been reported in Australia (Clarke & Roche, 2009), and had been called the transitive or reference point strategy in the United States (Behr & Post, 1986; Post et al. 1986; Post & Cramer, 1987). For example, $^5/_8$ is larger than $^3/_7$ because $^3/_7$ is less than a half and $^5/_8$ is more than a half.

Gap thinking has been observed in Australia (Clarke & Roche, 2009; Gould, 2011; Mitchell & Horne, 2010; Pearn & Stephens, 2004) and was one of four whole number dominance strategies described by Post & Cramer, (1987) and observed in recent studies (Cramer & Wyberg, 2009). Children with this misconception looked at the numerical difference between the numerator and denominator and chose the fraction with the smallest gap as the largest fraction. For example, in a study of 323 Grade 6 students, 35.6% of the incorrect answers comparing $^3/_4$ and $^7/_9$ demonstrated gap thinking: $^3/_4$ was larger because it had a gap of 1 while $^7/_9$ had a gap of 2 (Clarke & Roche, 2009). Nearly 30% of Grade 6 students incorrectly said that $^5/_6$ and $^7/_8$ were equivalent because two fractions, both with a "gap" of one, were the same, instead of using a correct strategy such as residual thinking (Clarke & Roche, 2009). In a separate study, 50% of Grade 6 students used gap thinking on this same pair to conclude that the fractions were the same, and the misconception was shown to emerge at the same time as early equivalence understanding (Mitchell & Horne, 2010).

In these examples, $^3/_4$ and $^7/_9$, and $^5/_6$ and $^7/_8$, gap thinking gives the wrong answer with incorrect reasoning. However, in fraction pairs such as $^4/_5$ and $^4/_{7,}$ gap thinking gives the right answer for the wrong answer: $^4/_5$ is larger because the gap of 1 is less than the gap of 3 in $^4/_{7.}$ A matrix of answer and explanation types (Clements & Ellerton, 1995, 2005) has been elaborated as:

- correct answer, correct mathematical thinking;
- correct answer, incorrect reasoning;
- incorrect answer, mathematically correct not fully executed/partially correct reasoning; and
- incorrect answer, incorrect reasoning.

Directive listening by teachers focussed on whether a child's answer matched an expected response (Empson & Jacob, 2008). The term *directive listening* corresponds to the term *evaluative listening* used by Davis (1997). Teachers who used this type of listening in classroom contexts were *listening for* something, not *listening to* the students (Even, 2005) and this could result in teachers overestimating what students knew (Empson & Jacobs, 2008) by assigning understanding to correct answers with vague explanations (Even, 2005).

Observational listening (Empson & Jacobs, 2008), on the other hand, was a term used to describe teachers listening to students and trying to work out what the students were actually thinking. Davis had described this as *interpretive listening* (1997). Empson and Jacobs (2008) specified one-to-one task-based interviews as contexts for the use (and practise) of observational listening.

Responsive listening (Empson & Jacobs, 2008) by teachers encompassed trying to understand individual students' approaches and responding to them individually and instantaneously, whilst keeping 25 children engaged and included, in the group dynamic of a single lesson. Davis had termed this *hermeneutical listening* (1997).

Calculation explanations described the calculation steps of a strategy rather than communicated the purpose of the calculations (Cobb, Yackel, & Wood, 1992). For example, when adding three 19s, Grade 2 children used calculation explanations in their initial peer conversation "Nine and nine is ... 18 ... and nine more..." (p. 104). They assumed they were all using the same strategy (adding ones and then tens). However, they did not have *equivalent* strategies (the same), they had *parallel* (assumed the same when not) strategies because one was adding ones and tens, 27 plus 30, whereas the other was adding ones before adding three more ones (incorrectly treating tens as ones), 27 plus 3. The children did not explain what they were doing mathematically; they described the calculation steps that they were using to execute their mathematical thinking. In some classrooms, calculation explanations counted as an acceptable mathematical argument despite the fact that calculation explanations made it difficult for students to recognise whether they had equivalent strategies or parallel strategies (Cobb, 2011).

Methodology

One-to-one task-based interviews were conducted with 88 Grade 6 students, offering 65 tasks that assessed their understanding of length and area measurement, dynamic imagery, multiplication, and fraction understanding. Each student was interviewed for up to three hours over several sessions. Observational listening and non-directive prompts were used to elaborate further explanations. The students' responses were noted on record sheets during the interview, and as two thirds of the interviews were video-taped (and all audio-taped) transcripts enabled the classification of their answers and explanations. Pseudonyms have been used when quoting the students' explanations.

One question will be examined in detail in this paper. The Fraction Pair task assessed students' understanding of the relative size of fractions. The eight fraction pairs were the same as used by Clarke and Roche (2009): $^{3}/_{8}$ and $^{7}/_{8}$, $^{2}/_{4}$ and $^{4}/_{8}$, $^{1}/_{2}$ and $^{5}/_{8}$, $^{2}/_{4}$ and $^{4}/_{2}$, $^{4}/_{5}$ and $^{4}/_{7}$, $^{3}/_{7}$ and $^{5}/_{8}$, $^{5}/_{6}$ and $^{7}/_{8}$, $^{3}/_{4}$ and $^{7}/_{9}$. The children were shown a card with the two fractions (symbolic inscriptions) and were asked, please point to the larger fraction or tell me if they're the same. After they stated or pointed to their answer they were asked, and how did you work that out? Two of the fraction pairs are discussed in this paper, $^{4}/_{5}$ and $^{4}/_{7,}$ and $^{5}/_{6}$ and $^{7}/_{8}$.

Results

In the present study, Sarah provided an example of residual reasoning when comparing the fraction pair $^{5}/_{6}$ and $^{7}/_{8.}$ She chose $^{7}/_{8}$ as larger, "because if I imagine a pie cut into sixths and you do five of them. And I imagine a pie cut into eight and there's seven of them; that's a little more." When prompted, "How do you know?" she elaborated correctly, "Because eighths are smaller, and like seven of them would be closer to a whole than five sixths." In contrast, Meg used gap thinking to conclude incorrectly that "They're the same because five sixths has got one more to become a whole. And seven eighths it also has one more to become a whole." In the present study, gap thinking was used by 50% of the students for this fraction pair.

Table 1. Answers, initial and further explanations for the comparison of the fraction pair $^{4}/_{5}$ and $^{4}/_{7.}$

Strategy		
Gap thinking	Lara:	This one [points to $^{4}/_{5}$]
	I:	And how did you decide?
	Lara:	'Cause it's only one away from being a whole.
	I:	Mmm?
	Lara:	And this is three away from being a whole
Benchmarking	Chris:	[points to $^{4}/_{5}$]
	I:	How did you decide?
	Chris:	Well, five, ff; four fifths is almost a whole
	I:	Mmm?
	Chris:	And four sevenths is um, a bit higher than half
Benchmarking	Adam:	This one. [points to $^{4}/_{5}$]
	I:	And how did you decide?
	Adam:	Um four is closer to five.
	I:	Can you tell me a bit more about that?
	Adam:	Um. Four. The four and the seven, there's more less, like, um close to a half, but this one's like almost a whole.

The fraction pair $^{4}/_{5}$ and $^{4}/_{7}$ lent itself to the correct strategy of benchmarking because $^{4}/_{5}$ was close to one and $^{4}/_{7}$ was just over a half. However, it was difficult to hear the difference between benchmarking (a correct strategy) and gap thinking (a misconception) in the students' explanations (see Table 1).

Discussion

In the terms of Kieren's four-three-four model (1988), Sarah's explanation using residual thinking "Because if I imagine a pie cut into sixths and you do five of them", illustrated her engagement at an intuitive level. The difference between residual thinking (a correct strategy) and gap thinking (a misconception) was easiest to hear in the explanations of the comparison of the fraction pair $^5/_6$ and $^7/_8$ because residual thinking gave the correct answer with correct thinking and gap thinking gave an incorrect answer with mathematically incorrect reasoning. This was because the gap answer was distinctive: "They're the same".

In contrast, the correct answer, $^4/_5$ was given by Sarah, Chris, and Adam when comparing the fraction pair $^4/_5$ and $^4/_7$ and their initial explanations sounded similar:

- " 'cause it's only one away from being a whole."
- "Four fifths is almost a whole."
- "Four is closer to five."

However, in response to the non-directive prompting, "Mmm?", Lara elaborated, "And this is three away from being a whole" (see Table 1). Lara was using the gap thinking misconception, calculating the complement to one for each fraction, by working out the numerical difference between numerator and denominator, and choosing the fraction with the smaller gap. Lara had the right answer for the wrong reason.

In contrast, when prompted, "Mmm?" Chris added, "And four sevenths is um, a bit higher than a half." In Adam's case, after being prompted "Can you tell me a bit more about that?", he explained that, "The four and the seven, there's more less, like um close to a half, but this one's like almost a whole." These further explanations revealed that both Chris and Adam had been benchmarking and so had the correct answer with correct mathematical reasoning.

Implications

The similarity of the initial explanations with the correct answer for the responses by students who were benchmarking or were using gap thinking has implications for how teachers talk to students and how students explain their thinking to each other.

It has been observed that teachers using directive listening interpreted vague explanations as correct mathematical reasoning if the answer was also correct (Even, 2005). Prompting for further elaboration of the students explanations was needed to determine whether the students were correct (correct answer and mathematically correct strategy) or incorrect (correct answer and mathematically incorrect strategy). The relationship of observational listening had to be maintained, without cueing the student into a directive listening exchange. Teachers with high specialized content knowledge should be alert to this possible confusion and prompt for further elaboration of the strategy to specifically establish which strategy is being used by the student.

If a teacher were explaining the benchmarking strategy for the fraction pair $^4/_5$ and $^4/_7$ and said, "Four fifths is nearly a whole", Adam might hear his benchmarking strategy confirmed (four is closer to five) but Lara would also hear her gap thinking strategy confirmed (it's only one away from being a whole). Lara might not experience cognitive conflict between the teacher's strategy and her own. The difference between the two strategies (benchmarking and gap thinking) could not be distinguished by the researchers using the students' answers and initial explanations, so it would also be difficult for Lara to hear the distinction between the mathematically correct reasoning of the teacher and her own mathematically incorrect reasoning if only an initial explanation was offered.

It is possible that students participating in peer conversations could react in the same way: if the answer was the same as their own and the explanations were similar, they would assume their strategy was the same as the other student. For example, let us imagine that Lara, Adam, and Chris were working together to solve the fraction comparison task $^4/_5$ and $^4/_7$. The terminology of peer conversation would enable us to describe their initial explanations as calculational: "'Cause it's only one away from being a whole", "four fifths is almost a whole" and "four is closer to five." All three children describe a difference calculation and none explain why they are doing this. At this point they might imagine that they are all agreeing on the strategy (that they have equivalent strategies). Even if Lara added "And this is three away from being a whole", Adam might not realise that she was not benchmarking like he was, unless he knew to listen for gap thinking. Parallel interpretations have the same answer and the same initial calculational explanation, but are actually different strategies. Lara and Adam have parallel strategies. Adam and Chris who are both benchmarking have equivalent strategies.

Cobb, Yackel, and Wood's (1992) examples of calculational, parallel, and equivalent explanations were of addition by Grade 2 children. Excellent teaching by the classroom teacher in their study enabled students to increase their knowledge of addition strategies. Grade 6 teachers have access to a repertoire of descriptions of strategies and misconceptions to draw on when responding to student explanations. However, for students to recognise parallel explanations, they may need to acquire a similar sophisticated repertoire of possible strategies in order to make sense of other students' explanations.

Conclusion

A highly detailed knowledge of gap thinking is needed in teachers' pedagogical content knowledge. Observational listening by teachers may require interpretations not only of answers and initial explanations but also prompting for further explanations.

The students' responses when comparing the fraction pair $^4/_5$ and $^4/_7$ demonstrated that these initial answers were considered acceptable mathematical answers by the students in the interview context. If students are to learn through peer conversation then teachers must establish the classroom norm that calculational answers are only partly acceptable mathematical answers. Acceptable mathematical answers include

- an answer,
- an explanation describing the strategy, and
- a description of the calculational steps used to execute that strategy.

This means that students will also have to develop their own knowledge of strategies, such as gap thinking and benchmarking so that they recognise when they have equivalent explanations or parallel explanations.

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VICTORIAN INDIGNENOUS CHILDREN'S RESPONSES TO MATHEMATICS NAPLAN ITEMS

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It has often been reported that children of Australian Indigenous background do not perform as well as a group as the whole population. This paper addresses the question of whether Victorian Indigenous children have different patterns of responses from the general population. The analysis compares the responses on each item for Indigenous children with the responses for non-Indigenous children both directly and for those who achieved the same NAPLAN scores for the 2008 NAPLAN numeracy assessment for years 5, 7, and 9. The results indicate trends in the characteristics of items which successes or challenges for Indigenous children.

Introduction

It has often been reported that Indigenous Australian children do not perform as well as a group as their peers (Doig, 2001). Large-scale assessments such as the National Assessment Plan - Literacy And Numeracy (NAPLAN), while they may not provide the detailed understanding that smaller studies can, complement the findings of smaller studies by offering a wider perspective at a population level. This paper addresses the question of whether children of Indigenous background in Victoria, Australia, have different patterns of mathematical responses from the general population, which may have implications for teaching approaches.

The literature to date has been principally concerned with the socio-economic and environmental factors that contribute to the relatively low performance of Indigenous children. This analysis seeks to extend our understanding of the reasons underlying the lower performance by identifying differences in facility related to the topics and presentation of the mathematical NAPLAN items. This research shows that for items in the Space strand of the curriculum, Indigenous children are performing close to grade level. This paper seeks to provide a more detailed view of the NAPLAN results by examining the difference in facility of the items for the Indigenous children compared to the entire population. To ensure that the trends found are not due to differences in levels of mathematical understanding, the responses of Indigenous children and non-Indigenous children who achieved the same NAPLAN scores for the 2008 NAPLAN numeracy assessment are also compared. The data consistently show that Indigenous children show relatively strong performance on items in the Space strand, and have greater difficulty with items which are difficult for the general population.

Figure 1 below presents a visual representation of the relative performance of Indigenous and non-Indigenous children. The NAPLAN assessment score is calculated as the number of items correct that each child obtained. The box-plots, scaled in width to visually indicate relative population sizes, represent distribution the 2008 mathematical assessment scores for years 5, 7 and 9. The population size represented by each box is indicated at the bottom of each box-plot. Because the assessments are not directly comparable, either in the number of items or the relative difficulty, the score is represented as the number of standard deviations from a mean of zero.

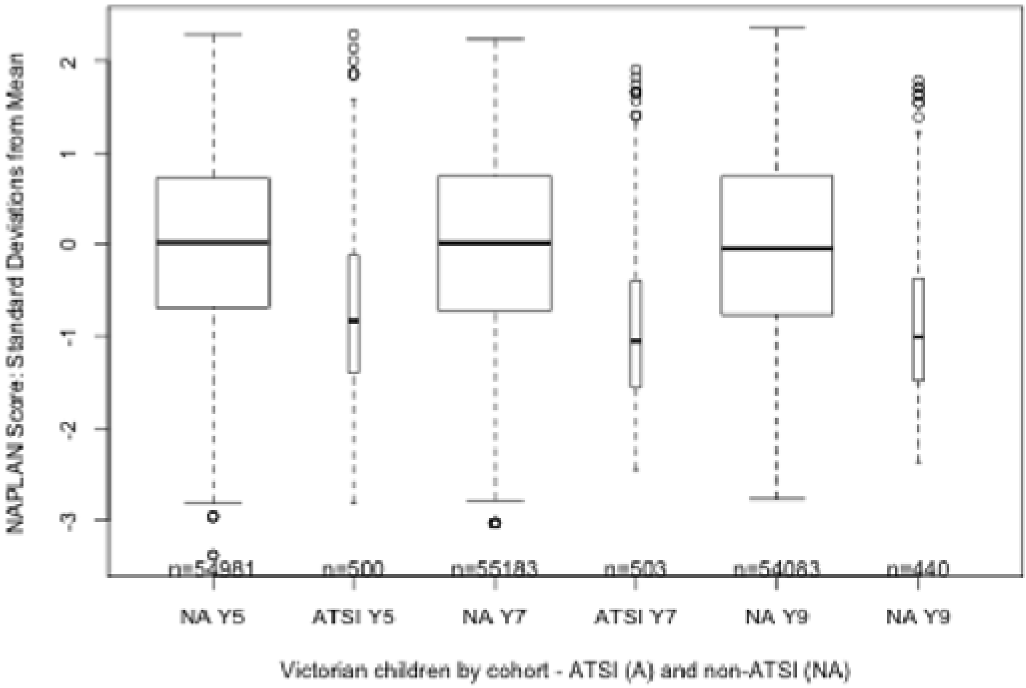

Figure 1: Distribution of 2008 NAPLAN scores.

Figure 1 shows the extent to which Indigenous children are performing less well as a group in comparison to the non-Indigenous population. For each year level represented, three quarters of Indigenous children scored in the same range as the lower half of the non-Indigenous population. The upper quartile included children who achieved very high scores, including one child who achieved a perfect score at the year 5 level. The second quartile of children are within the inter-quartile range of the main population, and the third and fourth quartiles are in the same range as the lower quartile of the main population, showing that although Indigenous children as a group perform less well than the general population, a significant proportion of the Indigenous children are performing within the main range of the general population, particularly in the earlier years.

Literature review

Hart (1980) demonstrates the use of results of assessments to obtain insights into children's understanding of mathematics. More recently, international large-scale assessments such as PISA and TIMMS have been used to measure and monitor academic outcomes, allowing researchers to gain insights from analysis of large scale data sets. One example is Thompson, de Bortoli, Nicholas, Hillman, and Buckley (2010) who used the 2009 PISA results to make various inferences about successes and challenges within mathematics education in Australia.

Some of the issues that affect the learning of Indigenous students that have been addressed include issues such as remoteness, attendance, and language (Jorgensen & Sullivan, 2010), mismatches between expectations and pedagogies (Cooper, Baturo, & Warren, 2005), and aspects such as learning style (Reeve, 2010).

Methodology

The data used in this analysis comprise Victorian children's responses to all multiple-choice items on the Australian National Assessment Plan—Literacy And Numeracy (NAPLAN) 2008 Numeracy assessment for years 5, 7, and 9[1]. Because the capability of large-scale assessments to inform the mathematical education community is not fully known, an exploratory analysis allows us to detect patterns in the data, thus providing a base of knowledge upon which to form and confirm hypothesis in further research (Tukey, 1980). Specifically, the exploration focuses on identifying patterns in items for which the Indigenous population responds in a different way from the non-Indigenous population. The analysis presents an initial overview, using boxplots showing relative distributions of performance, scatter plots to show trends in item facility, and Lowess curves to show the trends in differences in facility between the two groups for different curriculum strands where there are sufficient data points.

As a way of getting better insights into the differences between groups, the facilities of items for the Indigenous and non-Indigenous children who attained the same score on the NAPLAN assessment were compared. The Welch t-test is a variation of the non-parametric Student t-test which is appropriate when the variances of the populations differ, especially when the population size is unequal, as in this case the Student t-test is less reliable (Ruxton, 2006). Both of these conditions apply in this case, with the Indigenous population being 500, 503, and 440 for the year 5, 7, and 9 cohorts respectively. The variance within each population of the total NAPLAN score was calculated for the Indigenous and non-Indigenous population, and were found to be 42 and 49 for the Year 5 cohort, 109 and 147 for year 7, and 155 and 114 for the year 9 cohort. The Welch t-test was applied for each score level for which there were more than 20 Indigenous students. A total of 1288 tests were carried out, and the 150 tests for which a p-value of less than 0.05 was obtained were noted. Since approximately 70 of these could be expected purely by chance, only items for which the Welch test was positive for three or more score levels have been considered.

Analysis

The focus of the analysis is to identify trends in the characteristics of items that Indigenous children find relatively challenging or have relative success with. Figure 2 shows, for each item, the proportion of the 55,481 Year 5 children who answered the item correctly, or facility, on the 2008 NAPLAN numeracy assessment grouped

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[1] NAPLAN data are used and reproduced with permission of the Victorian Curriculum and Assessment Authority (VCAA). Analysis and findings using that data are not connected with or endorsed by the VCAA.

according to Indigenous background. The term facility is expressed here in terms of a proportion, rather than as a percentage, but is otherwise identical to the usage as introduced by Hart (1980).

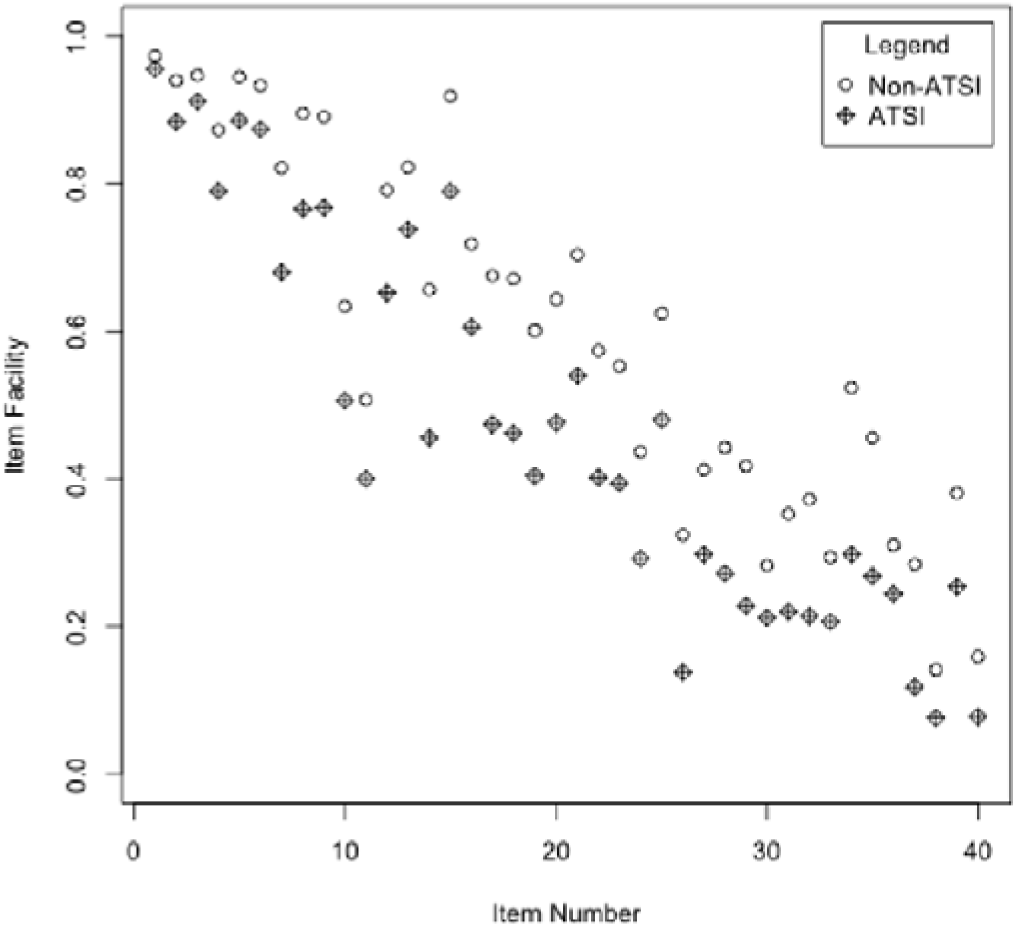

Figure 2. Proportion of Year 5 children who answered item correctly, grouped according to Indigenous background.

Figure 2 shows that the items are progressively more difficult to complete successfully throughout the assessment. The general trend appears to be the same for both groups, with children of both groups performing well on items at the beginning of the assessment, and finding the later items more difficult. The number of missed items is small—around 1%, for both Indigenous and non-Indigenous groups—even for items at the end of the assessment. The facility of each item is lower for Indigenous children than it is for non-Indigenous children, although some items have a smaller difference than others. These observations also hold for years 7 and 9 on the 2008 assessments.

Examining the magnitude of the difference in item facility provides more information. At year 5, the difference in facility between Indigenous and non-Indigenous children is greatest for the items of moderate facility. The items which are relatively easy, with a facility of greater than 0.9, are answered very well by Indigenous children, showing little difference between the two groups. Items of low facility only demonstrate that the items are very difficult for both groups, as the scope for differences in facility becomes smaller. There is no obvious differentiation between curriculum strands. Some items pose greater difficulty for Indigenous students relative to non-Indigenous children, and the analysis of these may be of interest in future research.

Figure 3 shows the difference in item facility between the Indigenous and non-Indigenous populations on the year 7 NAPLAN assessment for each item. The three curriculum strands Measurement, Number, and Space are depicted respectively as circles, crosses and triangles. A locally-weighted scatterplot smoothing, or Lowess curve, has been drawn for each curriculum strand.

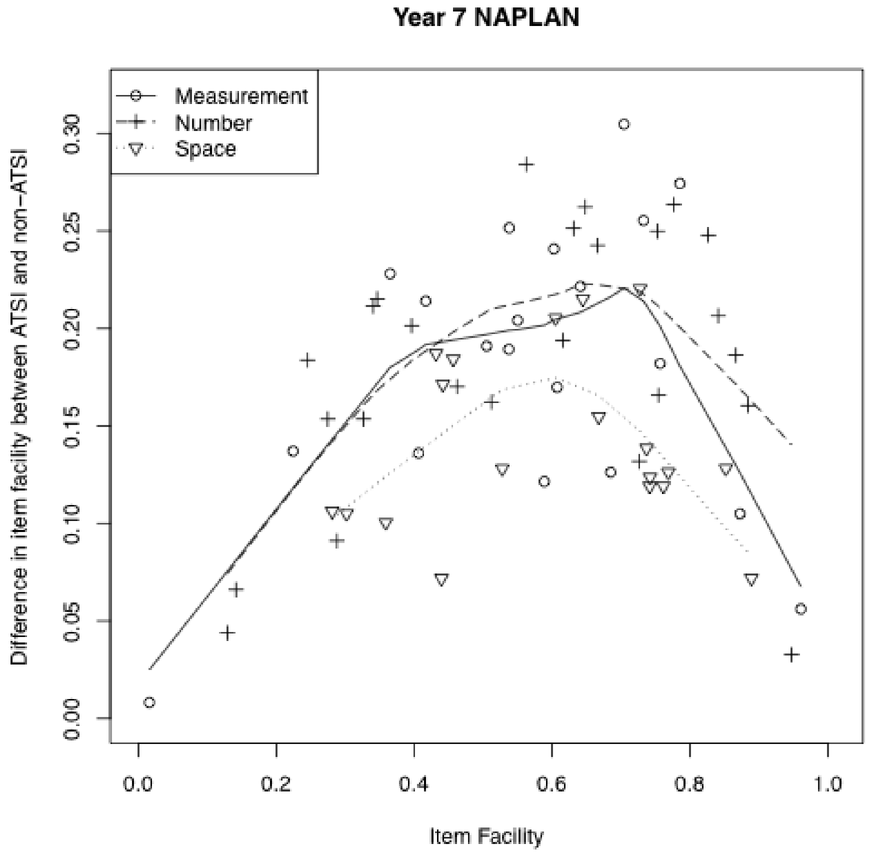

Figure 3. Difference in item facility between the Indigenous and non-Indigenous populations in the year 7 results.

Figure 3 shows that at Year 7, the differences of facilities between Indigenous and non-Indigenous children follow a similar pattern to Year 5. While Number and Measurement have similar Lowess curves, it is apparent that the Space strand of the curriculum has a smaller difference in facility between Indigenous and non-Indigenous groups. Lowess curves reflect all of the data, and the pointy peak of the Measurement curve indicates something unusual in the data. It draws our attention to the item which had the greatest difference in facility (0.3) between the two groups. Item 36 was answered correctly by 70% of non-Indigenous children, but only 40% of Indigenous children. This particular item assessed the ability to calculate the average of a number of items listed in a table. The most common error for both groups was to choose the sum of the numbers, rather than the average. In this case, the item could be solved by inspection, by eliminating all options other than the correct one using the knowledge that the average is in some way representative of a set of numbers. Possible sources of difficulty include the potential unfamiliarity of the item context (walking a dog for a certain amount of time); unfamiliarity with the term ―average"; or the number of steps involved in the task. Graphs such as this one highlight items that are particularly problematic to students, providing teachers with a focus for planning teaching strategies. This particular item would suggest that a problem-solving strategy that includes steps for inspecting an item before calculating, and afterwards checking if the result is reasonable, may be useful.

By year 9, some, but not all, of the items on the Space strand of the curriculum continue to be a strength of Indigenous children relative to other strands. The greatest difference is found in the Algebra strand, particularly in items of moderate difficulty. It is also apparent that some algebra items are more difficult for Indigenous children than others, independent of the facility of the item. It remains as a future research project to investigate those items to determine if there are identifiable factors that make these items relatively more difficult.

Analysis of items with achievement held constant

As a way of getting better insights into the differences between groups, the facilities of items for the Indigenous and non-Indigenous children who attained the same score on the NAPLAN assessment were compared. The items which were identified as having different facilities between the two groups on the Year 5 NAPLAN numeracy assessment are shown in Table 1. The skill is the description of the skill assessed by the item as given in the VCAA test answer booklet. The strand corresponds to one of the curriculum areas: Number (N); Space (S); Measurement (M); Chance and Data (D).

Table 1 shows that there are no items from the Measurement strand of the curriculum. The Space and Number strands of the curriculum contain some items, generally of high overall facility, that Indigenous children perform relatively well on, and items, generally of low overall facility that Indigenous children perform relatively less well on. These results indicate that the difficulty of the item has a greater impact on any difference between Indigenous and non-Indigenous children than the curriculum area of the item for children who are achieving the same NAPLAN score.

Table 1. Items on 2008 Year 5 NAPLAN Numeracy assessment for which Indigenous children and non-Indigenous children of the same NAPLAN score had different facilities.

Item	Strand	Multiple choice	Facility of item for Indigenous children	Overall facility	Skill
1	S	Yes	Higher	0.97	Identify symmetry in shapes
3	S	Yes	Higher	0.95	Compare the size of different angles
5	N	Yes	Higher	0.94	Complete number patterns based on simple criteria
6	S	Yes	Higher	0.93	Identify a 3D model given its individual components
13	N	No	Higher	0.82	Carry out simple money calculations
26	N	No	Lower	0.32	Recognise decimal numbers generated by dividing by 10
32	S	Yes	Lower	0.37	Identify and recognise properties of 2D shapes

38	N	No	Lower	0.14	Perform computations involving decimals
40	S	Yes	Lower	0.16	Recognise perspective in 2D representations of a 3D shape

The trend of Indigenous children doing well on the easier items and less well on the difficult items that was observed in the year 5 results is also evident in the year 7 data. Of the four items identified, only one was a multiple-choice item. Two items had an overall difference in facility that was less than chance and so the difference in facility can tell us little. Item 59 also had a low facility of 0.34. Item 3, with a facility of 0.83, was from the number strand, and asked children to select another way of writing 6^2 from the options 6 × 2; 6 × 6; 6 + 6 and 2 × 2 × 2 × 2 × 2 × 2. Of the 503 Indigenous children in this year level, 125 chose the first option, 6 × 2. The correct answer, 6 × 6, was chosen by 291 children; only 12 chose 6 + 6, and 67 chose the final option of 2 × 2 × 2 × 2 × 2 × 2. The remaining 8 children gave no legible response.

In the Year 9 data, the Algebra strand accounts for 2 of the 4 items identified as more difficult for Indigenous children. Overall, one of the most consistent findings is that Indigenous children score highly on high facility items, such as those where 90% or more of the population answer correctly. The implication of this finding is that children would benefit from increased exposure to more challenging mathematical material.

One of the assumptions made in measuring achievement by scores on multiple-choice tests is that a higher score reflects greater knowledge. As Sadler (1998) points out, albeit in the science domain rather than mathematics, this is not necessarily the case for difficult questions, where the performance dips from the expected performance level achieved by random guessing as the student gains an incomplete understanding of the topic being assessed, and is more likely to choose a distractor than the correct answer. This is an important issue, as for these very difficult items, the achievement of a higher score does not match with the goal of increased understanding of the topic unless the individual has achieved sufficient understanding to be able to answer correctly, making reliance on scores alone problematic, especially for low facility items. The items that fall outside of these patterns are also of interest, because these are the items that reveal opportunities to enhance teaching practices. For example, item 3 from the year 7 assessment, where children were asked to choose the option corresponding to 6^2, was answered correctly by 83% of children generally, but stood out as an item of difficulty for Indigenous children even when compared to children who scored at the same level. Since powers are an important component of algebra, this item is an early sign of the difficulties that Algebra poses for Indigenous children in year 9.

The implication drawn from this item, and the Year 7 item on calculating averages identified earlier, is that drawing the attention of children to the distinct use of language in mathematics, may be of benefit.

Conclusion

The exploratory analysis of children’s responses to the 2008 NAPLAN numeracy assessment for years 5, 7, and 9 described in this paper confirms and extends previous findings in the research literature. The analysis demonstrates that there is wide variation in the individual achievement, and that Indigenous children perform well on items of high facility, and less well on items of low facility, suggesting that the children may be more familiar with the simpler items. Indigenous children in year 7 have a relative advantage compared to their peers of similar achievement levels in the Space strand of the curriculum, but this advantage lessens for year 9, and that at the year 9 level, the Algebra strand is relatively difficult for Indigenous children, while the Number and Measurement strands are relatively difficult for all assessments. The implication for teaching is that a detailed analysis of results in large-scale assessments may provide insights that may be incorporated into teaching strategies.

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JOIN THE CLUB: ENGAGING PARENTS IN MATHEMATICS EDUCATION

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While extensive reference in the literature can be found in regard to professional learning sessions and workshops for teachers of mathematics, relatively little has been reported about engaging parents in learning about mathematics and mathematics education. The importance of home school partnerships is readily acknowledged, with parents being arguably the most influential factor in their children‘s educational success, yet many parents feel uninformed about current educational practices and how best to support their child‘s learning. This paper reports on an initiative undertaken with the parents from a District High School whereby they joined a ―Maths club" and attended information sessions designed to familiarise them with current mathematical practices and pedagogy. The results indicated that parents were appreciative of the opportunities provided to them and that they were supportive of contemporary mathematical practices.

Background and literature review

Projects designed to engage families in numeracy have provided opportunities for families and teachers to work together to enhance children‘s numeracy development (e.g., Morony, 2004; Ministry of Education, 2008). Studies have found that parental support for education in the home influences students‘ numeracy development (Anthony & Walshaw, 2007) and participation in mathematics-focused learning-at-home activities have been consistently associated with improved student performance.

A review of the literature has shown that there are a number of examples of home-school programs and initiatives designed to encourage numeracy partnerships. Goos (2004), for example, identified 606 numeracy programs Australia-wide whose purposes were to involve parents in school activities and/or inform them about syllabus changes, and to improve children‘s mathematics experiences and outcomes. Two issues which arose from her research included the need to forge parental and community involvement in mathematics education and change, and a recommendation to improve teachers‘, parents‘, and communities‘ understanding of the nature of numeracy and numeracy learning.

In terms of examples of accounts of particular projects and initiatives, Goos and Jolly (2004) describe a school‘s practice of offering ‗take home packs‘ of mathematics activities to parents who requested additional materials to use with their children. Reinfeld, Lountain and Mellowship (2008) describe an initiative whereby children took home a ‗Maths Monster‘ with whom they engaged in exploring and investigating mathematical challenges at home. In an earlier paper (Muir, 2009), the author described a project which involved parents engaging in mathematical activities with their children at home as part of an ongoing program. Muir (2009) found that while initially only 36% of parents indicated that they had a good understanding of how their child was taught mathematics, the project was effective in familiarising them with current mathematical practices and ways in which they could reinforce these practices at home.

Although Muir (2009) found that generally parents were willing to be involved in supporting their child‘s numeracy learning at home, others have found that parents can be hesitant in participating in their children‘s mathematical education (Anthony & Walshaw, 2007). This may be attributable to their own personal experiences with mathematics, feelings of anxiety and helplessness (Haylock, 2007) and lack of confidence in their ability to help their child (Bryan, Burstein & Bryan, 2001 as cited in Anthony & Walshaw, 2007; Civil, 2001). A lack of mathematical content knowledge can also limit the ways in which parents can be involved in their child‘s mathematical education (Peressini, 1998) and as Pritchard (2004) and Muir (2009) found, many parents feel uninformed about the mathematics curriculum and how it is enacted in their child‘s classroom.

Research has also shown that there is a tension between how mathematics is taught today compared with how it was learned by parents (e.g., Civil, 2006; Marshall & Swan, 2010; Peressini, 1998). According to Civil (2006), this perception is often reinforced through the superficial interpretations of reform mathematics education and adoption of practices, such as activities, group work and manipulatives, which do not necessarily focus on understanding students‘ thinking. Moreover, many parents tend to give higher value to their own forms of doing mathematics (Quintos, Bratton & Civil, 2005), which has implications for influencing the mathematical interactions they have with their children. In contrast, Quintos et al. (2005) found that children valued schools‘ form of knowledge more often over the parents‘ knowledge, hence demonstrating the potential tensions that may arise when engaging in mathematical tasks and assignments at home. Although generational differences are perhaps inevitable, Civil (2006) argues that parents were more concerned that they were not familiar with the homework tasks set, and therefore unsure about the best ways in which to help their children. It would seem sensible, therefore, to provide information and engage in dialogue with parents about what it means to learn mathematics today, how best to capitalise on the knowledge held by parents and how to involve them more actively in the mathematics education of their children. In order to inform and engage parents, the author of this paper initiated a project whereby parents were encouraged to join a ‗Maths Club‘ and attend information sessions designed to familiarise them with current mathematical practices and pedagogy. While the project evolved from the original ‗Numeracy at Home Project‘ (Muir, 2009), it was distinctly different in nature and involved participants that may or may not have been involved in the original project. The project helped to address recommendations raised by Goos and Jolly (2004) and Cai (2003) that further examination of parental roles is needed. Specifically, the three research questions were:

- What mathematical knowledge, skills and attitudes are held by a selected number of parents?

- How informed are a selected number of parents about current mathematical practices?
- What are the features of an initiative designed to inform parents about current mathematical practices?

Much of the extended research involving parental workshops has occurred in the United States, with Civil's research on linking home and school being particularly relevant to this paper. Programs such as *Math and Parent Partnerships in the Southwest* [MAPPS] aimed to assist parents to help children with their school mathematics work and to develop leadership capital among parents (Quintos, et al., 2005), while *Linking home and school: A bridge to the many faces of mathematics* [Project BRIDGE] focused on parents learning mathematics with understanding (Civil, 2001). Workshops encouraged the use of non-traditional approaches with a focus on investigation, often presenting a contrast with the participants' own schooling experiences. In Australia, there are limited examples of similar programs, with Marshall and Swan (2010) providing one example, involving a series of six workshops conducted with parents that focused on mathematical topics, such as place value and fractions. The workshops highlighted for them that the language of mathematics was a barrier for many parents and that parents were unsure about "the times tables" and confused about some aspects of fractions. The workshops proved to be successful in increasing parents' confidence about assisting their children with mathematics.

In summary then, most of the Australasian research involving parents has focused on home-school partnerships, with the general consensus being that such partnerships have the potential to contribute positively to students' educational outcomes. Other research (e.g., Civil, 2001) has involved parents in workshops that are aimed at increasing their own mathematical content knowledge and that of mathematics educational practices. The study discussed in this paper adds to the limited research in this area through providing an account of a parental involvement initiative and the feedback received from parents as a result of this.

Methodology

The participants were parents from a local district high school who received an open invitation to join the "Maths Club". Three workshops were offered over five months and the number of participants varied from six to eighteen. Each sixty-minute workshop had a different topic and some parents attended all sessions, while others attended one or two. The ages of their children varied from pre-school to middle school. At the beginning of each workshop, parents were asked to complete an 'Anticipation Guide' (Tierney & Readance, 2005), which varied in nature, but usually included levels of agreement responses to a number of statements and completion of some mathematics problems. A summary of the workshop topics, dates, participant numbers, and anticipation guide overview is presented in Table 1.

Because the attendance varied at each session and not all parents attended every session, it seemed sensible to evaluate and seek feedback on each individual session, rather than attempt to evaluate the effect of being a member of the maths club in general. Qualitative data analysis commenced during the data collection process and a frequency count was conducted for the survey items and levels of agreement statements.

Table 1. Overview of the workshops.

Date	No. of participants	Topic	Description of workshop	Anticipation Guide overview
3/5/10	18	Helping your child with numeracy at home	Powerpoint presentation involving responses to statements (e.g., there is one right way to do a maths problem; calculators make you lazy) and interactive activities	Circle level of agreement to 8 different statements (e.g., Maths is about learning the correct procedures to solve problems) 3 open-ended questions (e.g., Were any of your ideas challenged as a result of this session?)
23/8/10	6	Algorithms: What are they? How are they taught?	PowerPoint presentation, including alternative algorithms and traditional algorithms; demonstration; reading article (Clarke, 2005)	Questionnaire (21 Likert scale items; e.g., I am confident with my own mathematics ability); 4 open-ended questions (e.g., Why did you decide to join the maths club); 6 maths problems (e.g., 300 – 124)
11/10/10	7	Tables and mental computation	Hands-on activities and games; reference to Mental Computation: A strategies approach and *Turn the Tables* (De Nardi, 2004)	10 multiplication facts; confidence rating scale; 3 open-ended questions (e.g., Are you familiar with how children are taught multiplication today?)

Results and discussion

Parental mathematical knowledge and skills

Data obtained from the anticipation guides administered in the second and third workshops were used to determine some of the mathematical knowledge held by this group of parents. At the beginning of the second workshop, for example, the parents were asked to solve the following four problems and to provide a preferred method, an alternative method, and a child‘s method (if appropriate or different):

1. Susan has $5.80 and John has $6.35. How much more money does John have then Susan?
2. 300 – 124
3. $\frac{1}{2} + \frac{2}{3}$
4. 2.06 + 1.3 + 0.38

In addition, Question 5 required them to order three different fractions and Question 6 asked them to circle the bigger decimal number.

All six participants provided the correct response for Questions 1 and 2, and all provided a preferred method of solution. Two participants used a traditional algorithm for each of the problems, while the other participants either solved the questions mentally or informally (e.g., —I added 20 cents to get $6.00, then added 35c). Two child solutions were provided which were —counting on fingers" and —the same as myself" [round up to $6.00 and then add 35 cents].

Question 3 posed problems for all but one participant who achieved the correct answer by using the following method: $\frac{3}{6}+\frac{4}{6}=\frac{7}{6}=1\frac{1}{6}$. Incorrect answers included $1\frac{1}{3}$ (provided by two participants) and $\frac{3}{5}$ (also provided by two participants). Not surprisingly, these results indicate that parents may hold similar misconceptions about fractions as do students and pre-service teachers (e.g., Ball, 1990; Siemon, Virgona & Corneille, 2001). A preferred solution, which resulted in $\frac{3}{5}$, was achieved by —adding the bottom figure, then the top". Interestingly, there were no attempts to show how a child may attempt the question, with one response being, —not sure at this age". This could be attributable to the age of the participant‘s child, or unfamiliarity with observing or discussing fraction situations with their child. Five participants could correctly order $\frac{1}{10}, \frac{1}{4}$ and $\frac{1}{3}$, with one participant identifying $\frac{1}{3}$ as smaller than $\frac{1}{4}$. In contrast, only one participant recorded an incorrect response to question 4 (3.17) and all participants correctly identified 2.46 as being larger than 2.3489.

The results from this set of questions indicate that this admittedly small selection of parents could add and subtract 3 digit numbers, calculate with and order decimals, but had less success with adding simple fractions. There is scant evidence in the literature related to parental mathematical knowledge, but extensive research conducted with teachers, pre-service teachers and students (e.g., Ball, 1990; Siemon, et al., 2001) indicate that similar problems with fractions exist elsewhere and further exploration into parental knowledge of fractions and other mathematical areas needs to be undertaken. It would also be interesting to further explore the reasons for the general lack of response to recording children‘s solution methods. This may indicate that the participants did not engage in doing these types of questions with their children and could not therefore record what their likely response would be, or they simply may not have had the opportunity to do this as they may have been parents of younger children.

At the beginning of the third workshop on tables and mental computation, the parents were asked to complete 10 multiplication questions (e.g., 6 × 7). They were timed and then asked to circle their level of confidence in completing the task. All but one participant (n=7) correctly answered all 10 questions, with the one error being an answer of 63 for 7 x 7. This participant indicated that they were ‗not confident‘ with completing the task, while the other participants all circled ‗quite confident‘. The results indicated that this selection of parents at least had the knowledge and skills to be reasonably confident with recalling multiplication tables.

Table 2 provides a summary of the data that was obtained from the Likert scale items (ranging from Strongly Disagree to Strongly Agree) that particularly relate to parents‘ mathematical knowledge, skills, and attitudes. Percentages have been used for ease of comparison.

The table shows consistency with the results obtained from the number questions, in that most participants indicated that they preferred to do most calculations mentally and that they knew their multiplication tables.

Table 2. Summary of parents' perceptions of mathematics knowledge, skills, and attitudes.

Statement	SD	D	N	A	SA
I prefer to do most calculations mentally	0	0	0	83	17
I understand what algorithms are	50	33	17	0	0
I know my multiplication tables	0	0	0	67	33
I get confused with different words and terms used in mathematics	0	0	0	83	17
I am confident with my own mathematics ability	0	17	0	66	17
There is a ‗correct' way to do any maths problem	0	33	33	33	0
I think rote learning is a good way to learn your tables	0	33	17	17	33

Not surprisingly, many participants agreed that rote learning is effective and all participants indicated that this was how they had learnt their multiplication tables. As Marshall and Swan (2010) found, one of the common themes that emerged from the workshops, and a topic that parents wanted further information about, was the use of mathematics terminology. There was 100% agreement with the statement that this was a source of confusion for parents. (A maths glossary of terms was provided to parents in a subsequent session.)

There was a mixed response to the open-ended question ‗How did you feel about mathematics when you were at school?' Although some responded positively (e.g., ―Loved it! Found it easy), others indicated that they ―didn't feel comfortable with it" and ―I didn't enjoy it because I wasn't very good at it, with the impression there was only one correct way to answer a problem".

Parental understandings about current mathematical practices

In order to answer the second research question, items from the anticipation guides distributed in the first two sessions were analysed, along with parents' responses to open-ended questions from all sessions. In the first session, parents were asked to circle their level of agreement (agree, disagree, unsure) with eight different statements. Following the session, they were then asked to review their responses and identify whether or not any of their ideas were challenged as a result of the session. The results showed that eight of the 18 participants indicated that they no longer agreed that ―Chanting is an effective way to learn tables", and while initially five people agreed that ―There is a correct way to do a maths problem", all of these changed their opinion at the end of the session. While no claims can be made as to the sustainability of these results, it is encouraging that the sessions did result in positive changes in some of the parents' beliefs. The following comments are illustrative of the changes in responses received:

> Most maths calculations can be done a number of ways
>
> I had the impression that maths was more recall than process. I have noted processes that I can now discuss with children.

The parent questionnaire referred to in Table 2 also contained items that were particularly related to parents' knowledge of current mathematical practices and their confidence in helping their children with mathematics at home. Table 3 shows the items and levels of agreement received for these aspects.

Table 3. Parents' understandings of current mathematical practices.

Statement	SD	D	N	A	SA
Mathematics teaching today is different to when I went to school	0	0	33	33	33
I am confident with helping my child with mathematics homework	0	0	0	50	50
I am confident with helping my child with the following specific maths topics:					
Addition and subtraction	0	0	0	50	50
Multiplication and division	0	0	0	67	33
Place value	0	33	0	50	17
Fractions	17	33	0	50	0
Multiplication tables	0	0	0	83	17
Decimals	0	17	0	66	17
Algebra	17	66	0	0	17

When interpreting the responses in relation to confidence with helping with specific maths topics, it must be remembered that many of the parents had young children and may have subsequently interpreted their confidence with younger grade levels, rather than the topic itself. Similarly it is difficult to conclude whether levels of agreement or otherwise relate to parents‘ own personal knowledge and confidence, or whether they are referring to their ability to ‗help‘ their child.

Qualitative comments revealed that the parents were not familiar with current mathematical practices and were concerned that they ―would teach them the wrong way”. Although this issue was challenged at the workshops, parents still indicated that ―I am not familiar with current practices” and ―I‘m not sure how it is taught”. This is consistent with Muir‘s (2009) findings whose survey revealed that only 36% of parents agreed that they had a good understanding of how their child was taught mathematics.

Evaluation of workshops

The results indicated that the parents in this study were willing to participate in their child‘s mathematical education and appreciated the opportunity to become more informed about current mathematical practices. Furthermore, as Civil (2001) found, parents were also appreciative of the opportunity to engage in discussions about mathematics teaching, enjoyed doing mathematics, and were keen to improve their own mathematical content knowledge, while gaining a better understanding of reform mathematics.

The workshops were designed to be informal, with some information sharing, opportunities to interact with the presenter and each other, participation in hands-on activities and provision of resources. In the second workshop parents were asked to give their reasons for joining the maths club and what they hoped to get out of it. Interestingly all reasons given referred to the benefits they saw for their children, rather than themselves. ―To get a better understanding of teaching methods to help kids at home” was typical of the comments received. Parents also identified that they would like future workshops to include fractions and ideas for extending children who are confident and helping those who are not.

As Table 1 shows, the numbers attending the workshops varied, with some parents attending all sessions and others attending one or two. The high numbers at the first workshop were attributed to it being conducted at the same time as another initiative, which involved the taking home of numeracy packs in the early grades and which was very actively promoted with the parents. Informal feedback received from parents indicated that the sessions were valuable, with one parent expressing, ―That was fantastic", and there was overwhelming support for the sessions to continue in 2011. The success of the workshops could also be attributed to the support of the school‘s teachers and senior staff, who prepared fliers, advertised in the newsletter, and personally approached parents to attend.

Conclusions and implications

There is still more to be done in terms of finding out the knowledge and skills held by parents and what their perceptions of mathematics are. Many parents in this study indicated that they did not have a good understanding of how their children were taught mathematics, which can be a source of tension when trying to help children with mathematics at home. More research needs to be undertaken into the reasons for these tensions – it may be because mathematics is taught in a different way to years ago, or even that parents see teachers as having the responsibility for mathematics education (Civil, Diez-Palomar, Menendez-Gomez & Acosta-Iriqui, 2008). Nevertheless, parental workshops such as the one described in this paper, and programs such as *Maths for Parents* (Civil, 2001) may help to address these concerns.

This study has also added to the literature in relation to highlighting the mathematical knowledge held by parents—such as indicating that some of the misconceptions held by pre-service teachers, teachers, and students are also held by parents. It is recommended that parental workshops should therefore address mathematical content, along with familiarising them with current mathematical practices. As Marshall and Swan (2010) found, these parents also indicated that there were mathematical topics that they would like further information about, including fractions and place value. In addition, Civil (1999) recommends that a two-way dialogue needs to be established, whereby parents are seen as intellectual resources and provision made for their beliefs, ideas and concerns to be heard.

The next phase of the study will examine ways in which to empower parents to contribute more to their child‘s mathematical education, particularly in the higher grades, and to explore different models or approaches to help parents understand the mathematics teaching that occurs in today‘s classrooms. It is hoped that the documentation of such programs will assist teachers and educators with recognising the importance of parental influences and the difference they can make to their child‘s education.

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TEACHING MATHEMATICS IN THE PAPUA NEW GUINEA HIGHLANDS: A COMPLEX MULTILINGUAL CONTEXT

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The classrooms of Papua New Guinea are multilingual. For many years only the official language of education, English, was permitted for teaching. In the mid 1990s the curriculum changed to declare that multiple languages would be used in teaching in the first three years of schooling. In the next year English is introduced, and gradually over the next few years becomes the dominant language of teaching. This paper examines how eight teachers in the crucial transition year 3 use their multiple languages to teach mathematics, although they seem to use their other available languages to privilege English learning.

Introduction

Papua New Guinea (PNG) has the most languages per head of population in the world. With a population of about six and a half million people there are some 820 living distinct languages. Situated on the eastern half of the island of New Guinea, the people of tropical PNG live in many coastal villages through to villages in the deep valleys in the highlands that make travel from one valley to the next difficult at the best of times. Clearly the many languages spoken have always impacted on the education system. The first schools were founded by Christian missionaries in the late 1800s in coastal areas.

The highlands were so inaccessible that westerners thought they were largely uninhabited until they were reached by Australian „explorers" in the early 1950s. In fact the majority of the population has always lived in the highlands. It was only then that the colonial Australian government began the extension of the school system into the high valleys. The missionaries had favoured the use of indigenous languages in schooling, with the better students who reached the later years of primary school being taught English.

With the coming of a whole country colonial policy in the late 1940s, an English only policy was imposed for teaching in all schools. The first author can remember sitting during his early years of school wondering what was going on, since he as a little boy never experienced the language of English until he went to school at age 7. The second author has photographs from the early 1980s of „classroom rules" and „school rules" insisting that non English languages should not be used in class or the playground, with various reprimands detailed if students were caught disobeying these rules.

The school curriculum mirrored one from the colonial power Australia. With independence gained in 1975, gradually elements of the curriculum drew more and more on PNG cultures, although as in many ex-colonies, the impact of the western curriculum is still very obvious. Teaching too is still heavily influenced by western ideas with Australia regularly providing „aid money‟ to „upgrade‟ the quality of teachers and their preparation (Clarkson, Hamadi, Kaleva, Owens, & Toomey, 2004). There is a growing voice however that such aid money is perpetuating the global education hegemony, and it is time for PNG to develop its own style of teaching (Nongkas, 2007), which indeed is emerging (Pickford, 2008). But one far ranging decision made in the late 1990s that saw a definite break with much western education practice was for the early years of schooling in PNG schools to become multi lingual.

The PNG mathematics curriculum has also been impacted by the various general trends in the school curriculum. Some use has been made of indigenous mathematics (Lean, 1994; Muke, 2001), although there is much scope for more of this to take place. It has been recognised for many years that the mathematics performance of PNG students in part relies on their language abilities (Clarkson, 1983), but more their performance on a variety of mathematics tests, and indeed on system examinations covering language and general studies as well, is in part dependent on their competence in their various languages (Clarkson, 1992; Clarkson & Clarkson, 1993). Such results are mirrored by other studies elsewhere in the world (Barwell, 2008). However these early research projects in PNG only studied urban students, and only looked at the interplay of two of the multi lingual students‟ languages; Pidgin (the common lingua franca in the northern parts of PNG) and English (the language of schooling and the dominant language of commerce). However, most school students attend rural schools, and know three or four languages, but rarely English, when commencing school. The early studies also did not analyse the teaching of mathematics but concentrated on students‟ learning and understanding of mathematics.

The present study

The study describe in this paper focuses on the teaching of year 3 mathematics in four PNG rural primary schools. The year level is important. In the new curriculum the first three years of schooling is undertaken in Elementary Schools (Prep, years 1 and 2). In these schools the curriculum indicates that local languages should be used for teaching, although some schools in urban areas do opt to use English. After year 2, students move to primary schools which span years 3 to 8. Year 3 is marked as the „bridging year‟ in teaching. During this year it is anticipated that the language of teaching will be a mixture of the languages used in the Elementary school, with a lingua franca if not already used in year 2, and the gradually introduction of English. It is expected that by year 5 all teaching will be in English, although the curriculum documents suggest use of other languages if the teacher gauges that would help the learning of students. This situation applies to all curriculum areas including mathematics.

Potentially, there are a number of possibilities for teachers teaching in a multilingual context to pursue. They could just decide on the simplest approach to stay with the dominant teaching language. On the other hand they may decide to use other languages available, but only switching from the dominant teaching language when students are having difficulties in understanding mathematical concepts, or an indigenous language may be called for when analysing a problem drawn from the local culture. The context becomes more complex if the teacher takes seriously the progression from using the everyday language of the students through to formal mathematical language, Some of this complexity has been portrayed diagrammatically elsewhere (Clarkson, 2009). For this study the work by Setati and Adler (2001) was important. They mapped out possibilities for teachers‟ use of language in multi lingual mathematics classroom in South African classrooms on a number of dimensions. One dimension was the possibilities available as teachers and students moved from the informal to the needed formal mathematical language. Another was moving between the various vernaculars spoken by teachers and students. A third was moving back and forth between managerial and conceptual teaching discourses.

The authors were aware that many teachers in PNG also believe that the learning of mathematics has little to do with student language competencies. Moreover we were cognisant of the fact that teacher college educators did not believe that the notions of the „bridging year‟ had implications for how mathematics was taught in year 3, since mathematics was a language free zone (Clarkson et al., 2004). Nevertheless anecdotal evidence was available to both authors from their own observations of PNG classrooms that occasionally good teachers of mathematics, even before the change to the curriculum, would switch languages when teaching mathematics, if they felt the need to do so. However why they did so to our knowledge has never been explored. Thus the focus for this study became:

1. Did the teachers use a variety of languages when teaching mathematics and if so was there a consistent pattern to this usage for individual teachers and/or topic?
2. If teachers did use multiple languages in their teaching, why did they?

Methodology

This study was conducted in four primary schools in the rural Wahgi Valley of the western Highlands of PNG. All schools are some days‟ travel from the main town of the province Mt Hagen. The eight year three teachers were all fluent English and Pidgin speakers, and all knew the local vernacular Wahgi, and could speak other languages as well. The year 3 students were for their age fluent in Wahgi, Pidgin, knew some English, and often knew some other language(s) as well. The schools by western standards had few resources, but by PNG standards had normal resources to draw on. They certainly had dedicated teachers.

The first author observed a number of classes taught by the teachers. Although it had been planned to observe three classes for each teacher, each separated by a six month interval, because of logistic difficulties this did not occur (Valero & Vithal, 1998). As it turned out three teachers were observed for three lessons, two for two lessons, and the remaining three teachers for a single lesson giving 16 observed lessons in total. Each lesson was video and audio recorded. Teachers were interviewed briefly before each lesson, and a post lesson interview of some 60 minutes was conducted on the day of the lesson. All recordings of the lessons were transcribed as were the interviews. During each lesson the first author also completed field notes, concentrating particularly on the language of the teacher, the context in which that language occurred, the segment of the lesson, and content of the teaching. The video recording of the lesson was available during the interview, and if it helped the discussion, segments were often replayed at either the author"s or teacher"s instigation.

The transcription data from the lessons were analysed by sentence for the types of languages used during the lesson, and at what points the teacher switched between languages. The author made a judgement of language use on the basis of semantics and syntax of the sentence. For the vast majority of sentences this was clear cut. The instances of using an isolated borrowed word (often a formal mathematical term in English) were noted, but did not impact on the decision of language categorisation. The interviews were analysed to find the perception of the teachers as to why they used multiple languages in their teaching, and why they switched between languages when they did (Gee, 1999).

Results

Research question 1

In the 16 observed lessons, instances of the use of Wahgi, Pidgin and English were all noted except for two lessons when Wahgi was not used and one lesson when English was not used. As shown elsewhere (Muke & Clarkson, in press) about half the teachers" language use was in Pidgin with the remainder divided about equally between Wahgi and English. The same pattern of language use was not consistent from lesson to lesson for teachers who gave multiple lessons. Nor did the mathematical topic of the lesson seem to be the determinant of language usage.

It has been noted above that sentences were the unit used to estimate the frequency of „language use". However many formal mathematical terms/phrases expressed in English were borrowed even though the overall sentence was in one of the local languages. Table 1 shows the topics taught by the teachers and the specific terms borrowed into Wahgi and Pidgin. One feature of this listing is the variety of terms borrowed. The higher frequency of terms borrowed into Pidgin is probably just a function of the greater use made of Pidgin by teachers.

Research question 2

One way to explore why teachers switched languages in their teaching is to look at the types of language switches Setati and Adler (2001) referred to as code-borrowing and code-mixing. Both were observed in this study. Code-borrowing refers to a switch that involves borrowing either a term or a phrase from a different language and using it in a sentence constructed in another language. Similarly code-mixing refers to a sentence made up of two languages, where one language is used to start the sentence and the other completes the sentence.

First, code-borrowing that involved a single term in another language and used within a sentence constructed in another language will be considered, followed by that of borrowing a phrase. In this study, most terms that were borrowed by teachers were from a mathematical register, and the overwhelming majority of these terms were from the mathematical English register. As Skiba (1997) noted, one of the skills of a bilingual or multilingual speaker is to use such borrowed terms within the grammatical rules of the sentence, which is in the other language. The two main parts of most sentences are a noun phrase and a verb phrase (Skiba, 1997). Teachers observed in this study always used terms from the formal mathematical English register as a noun. This meant that the verb phrases, the rest of the sentences, were commonly in one of the local languages. To illustrate this, an excerpt from the transcription of one of the three lessons given by Mr. W will be examined.

Table 1. English formal terms borrowed when using Wahgi or Pidgin.

Teacher / Lesson no.	Lesson topics	English mathematical terms borrowed into:	
		Wahgi	Pidgin
Mr M / L1 Mr M / L2 Mrs K / L1 Mrs K / L3 Mr J / L1 Mr J / L2 Mrs T / L1	Number operations	*a* groups of *b*, carry, division, group, multiplication, multiply, names of place columns, number names, multiples of *x*, number, plus, subtract, times table, times	*a* groups of *b*, $a \times b$ equals, addition, all together, carry, count, divide, divided, division, equals, groups of, multiples, multiplication, multiply, names of place columns, number names, number, place value, plus, put down, subtract, takeaway, times table, times, zero
Mr M / L3 Mr W / L2 Mr A / L1	Fraction	half, quarters, one whole, quarter	whole, half, quarter, number names, fraction, one whole, parts, one fourth, two thirds, three fourths, one sixth, four sixths, fractions, square
Mrs K / L2 Mr D / L1 Mr D / L2 Mr K / L1	Measurement	guess, measurement, meter, perimeter, meters, number names, weight, length, units, grams, kilograms, tonnes, true, false, units of measuring, weight	measurement, millimetres, centimetres, meters, kilometres, 10mm = 1cm, 100cm = 1m, 1000m = 1km, meter ruler, number names, guess, weight, grams, kilograms, perimeter, shapes, metres, number, milligrams
Mr W / L1	Number		number chart, numbers, words, objects, number names
Mr W / L3	Shape		shapes, kite, corner, rectangle, measurement, triangle, square, oblong, angle, rhombus, trapezium, diamond, pentagon, number names

An examination of the transcriptions of Mr W‟s overall language combinations in these three lessons showed that he responded to the language need of each lesson without using a particular language combination. The topic for this lesson was fractions, and Mr W is asking students what a fraction is:

Mr W: Lesson 2, Paragraph 17 & 18 (original in Pidgin & English)
17. Mr W: Okay, what is a fraction? … Fraction, em wanem samting? …Meaning bilong em olsem … a small part of a thing. A small part of a …
18. Children: Thing

English Translation
17. Mr W: Okay, what is a fraction? … Fraction, what is it? The meaning belongs to it/him/her is… a small part of a thing. A small part of a …
18. Children: Thing

The language combination Mr W used in lesson 2 was 78% of Pidgin and hence the leading language for this lesson, with 16% of English as the first supportive language, and 2% of Wahgi as the second supportive language. The two languages used in paragraph 17 by Mr. W were Pidgin and English. The first sentence is a question in

English, asking for the meaning of the term „fraction‟. Although in this lesson English was used far less often than Pidgin, the teacher felt the need to ask the question in English at the beginning of this paragraph. Later analysis showed that Mr W had a desire to model English, particularly so that students could listen to questions in English and hence be more prepared for test situations where items were always written in English. The second sentence of the excerpt is in Pidgin and repeats the question first asked in English. What is of immediate interest here is the code-borrowing occurring in the second sentence with the word „fraction‟. The sentence is constructed in Pidgin in such a manner that enables the borrowing of mathematical term in English, but uses correct grammar for Pidgin. The question asked in Pidgin is; "Fraction*, em wanem samting*?" and when translated word for word; "fraction; it (em) what (wanem) thing (samting)‟, which is understood as; fraction, what is it? The word „*em*‟ in Pidgin is a pronoun and it is used as „it‟ to refer to „fraction‟ as a thing. It is common in Pidgin to use the pronoun immediately after the name of a thing is used. Such an expression in this type of sentence shows that in this case the term fraction was used as a noun phrase, and the rest of the sentence in Pidgin formed the verb phrase. This means that the local language, in this case Pidgin, was used as a verb phrase, promoting the noun. The implication follows that for this sentence the local language took up a supportive role to the promotion of English.

The third sentence in the above excerpt involved code-mixing. Code-mixing is where a single sentence is completed by two different languages. In this case, the first part of the sentence is in Pidgin and the second part is in English. As translated, Pidgin was used to introduce the formal meaning of fraction. In Pidgin, the teacher said; "*meaning bilong em olsem"* which means the definition belongs to it, where the word *em* used as „it‟ in Pidgin referring to the term fraction, and then switched to English to actually say the formal meaning in English; „a small part of a thing‟. The way Pidgin, the local language, is used here is that it is used as a pointer; directing students to be aware of the coming of an important thing. In this case it is not only the formal definition that is pointed to, but coincidentally this definition is expressed in English, and this language switch becomes part of the important designation of which students are to take note. Hence the students‟ fluently spoken local language is used to help students be aware of the formal mathematical concepts expressed in English. Therefore, the local language is given only the supportive role in this not unusual switching incident.

Another example will elaborate this issue further. One lesson given by Mr K was observed in this study. In this lesson Mr K used nearly the same amount of Wahgi (46%) and English (43%) with only very few sentences in Pidgin (5%). The following excerpt from the transcription of the lesson shows how Mr K borrowed a phrase in one language, a formal mathematical expression in English, but used this in a grammatically correct sentence constructed in Wahgi. The topic of this lesson was measurement and Mr K is singling to the class a new direction that the lesson will take:

Mr K: Lesson 1, Paragraph 13 (original in Wahgi & English)
13. Mr K: kinim ya units of measurement, ah units of measuring weight kanamin eh. Kanamin eh, mi mene units kembis woi kan wo ep mine units okma kanamin eh.

English Translation
13. Mr K: We will now look at „units of measuring weight‟. We will look at the smallest unit to the biggest units.

The sentence is constructed in Wahgi and the borrowed mathematical English terms and phrase are borrowed and inserted within the sentence. The phrase borrowed here is „units of measuring weight‟ and the word borrowed is „unit‟. Both are examples of formal mathematical language, but both are expressed in English. The way this was expressed in Wahgi was, „we will look at *units of measuring weight*, from smallest to the biggest *units*‟. Both the borrowed formal mathematical English term and phrase acted as nouns in each sentence. The rest of the sentences were in Wahgi, and formed verb phrases. In the first sentence, Wahgi is used to say that they (the class as a whole) were going to do the „looking at‟ or studying. As the teacher continued he said that the looking at or studying was going to involve the „units of measurement‟. This indicated that the teacher used the fluently spoken local language to inform the students what they would be doing, which is obviously forming the verb phrase, to the unit of measuring weight, the noun phrase. In the second sentence, the teacher becomes more specific regarding what they will be looking at or studying in the lesson. In Wahgi the teacher explains that they will be looking at or studying the smallest to the biggest, and this will involve the units, the core term which forms the noun phrase but again expressed in English.

These two examples show common constructions of teachers observed in the study. Often teachers used the local languages to construct grammatically correct sentences, but inserted borrowed formal mathematical terms using English. In doing so the mathematical terms and the language in which they were expressed, English, became the focus of the discourse, with the local languages playing supportive roles only.

Summary

After more than a century since schooling was introduced to PNG the indigenous cultures are starting to impact on teaching. Although the results noted in this paper are from only a limited number of teachers, their purposeful use of the variety of languages available to them and their students we suspect is mirrored in many classrooms throughout PNG. We note that although research for some years has suggested that multilingual students gain cognitive advantage if they are encouraged to use all their languages, this was not a factor for the reasons given by teachers for their exploiting of the multiple language environment. A key finding which would be well worth exploring with many more teachers is the way Wahgi and Pidgin were used, not to explore the nuances of the languages, but to learn the dominant language of English. It would be interesting to know whether this is an indication of the hegemonic impact of globalisation, even in the remote villagers of the western highlands of PNG.

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AN EVALUATION OF THE PATTERN AND STRUCTURE MATHEMATICS AWARENESS PROGRAM IN THE EARLY SCHOOL YEARS

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This paper reports a 2-year longitudinal study on the effectiveness of the Pattern and Structure Mathematical Awareness Program (PASMAP) on students' mathematical development. The study involved 316 Kindergarten students in 17 classes from four schools in Sydney and Brisbane. The development of the PASA assessment interview and scale are presented. The intervention program provided explicit instruction in mathematical pattern and structure that enhanced the development of students' spatial structuring, multiplicative reasoning, and emergent generalisations. This paper presents the initial findings of the impact of the PASMAP and illustrates students' structural development.

Mathematics learning that focuses on pattern and structure can not only lead to improved generalised thinking but can create opportunities for developing mathematical reasoning commensurate with the abilities of young learners. Pattern has been described as any predictable regularity involving number, space or measure; and structure, as the way in which various elements are organised and related (Mulligan & Mitchelmore, 2009). Over the past decade a suite of studies with four- to nine-year olds has examined how children develop an Awareness of Mathematical Pattern and Structure (AMPS), found to be common across mathematical concepts (Mulligan, 2011; Mulligan, English, Mitchelmore, & Robertson, 2010). An assessment interview, the Pattern and Structure Assessment (PASA) and a Pattern and Structure Mathematics Awareness Program (PASMAP) focuses on the development of structural relationships between concepts. Tracking, describing and classifying children's models, representations, and explanations of their mathematical ideas—and analysing the *structural* features of this development—are fundamentally important.

Our goal is a reliable, coherent model for assessing and describing structural development with aligned learning and pedagogical frameworks. In this paper we focus on the development of the Pattern and Structure Assessment (PASA) interview and a

Rasch modelled scale for measuring student growth over time. An exemplar of the qualitative analysis of structural development is provided.

Background

Previous studies have examined independently, counting, grouping, unitising, partitioning, estimating, and notating as essential elements of numerical structure (Thomas, Mulligan & Goldin, 2002); multiplicative concepts (Mulligan & Mitchelmore, 1997); combinatorial thinking (English, 1993); and spatial structuring in geometric figures and arrays (Battista, 1999). Recent studies of young children's mathematical reasoning have provided complementary evidence of the importance of early patterning skills, analogical reasoning and the development of structural thinking (Blanton & Kaput, 2005; English, 2004; Papic, Mulligan, & Mitchelmore, 2011).

There is also increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships such as equivalence and functional thinking in early childhood (Warren & Cooper, 2008). Recent initiatives in early childhood mathematics education, for example 'Building Blocks' (Clements & Sarama, 2007; Clements & Sarama, 2009), 'Big Maths for Little Kids' (Greenes, Ginsburg, & Balfanz, 2004), and 'Curious Minds' (van Nes & de Lange, 2007) provide research frameworks to promote 'big ideas' in early mathematics education. Papic's assessment of preschoolers using an Early Mathematical Patterning Assessment (EMPA) show that children are capable of abstracting complex patterns before they start formal schooling (Papic et al., 2011). Thus in designing PASMAP and an accompanying assessment, we focussed on the relationships between children's patterning skills, structural relationships and the big ideas in mathematics.

Method

A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from diverse socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. At the follow-up assessment in September 2010, 303 students were retained. Two different mathematics programs were implemented: in each school, two Kindergarten teachers implemented the PASMAP and two implemented their standard program. The PASMAP framework was embedded within but almost entirely replaced the regular Kindergarten mathematics curriculum. The program focused on unitising and multiplicative structure, simple and complex repetitions, growing patterns and functions, spatial structuring, the spatial properties of congruence and similarity and transformation, the structure of measurement units and data representation. Emphasis was also laid on the development of visual memory and simple generalisation (for details see Mulligan et al., 2010). A researcher/teacher visited each teacher on a weekly basis and equivalent professional development for both pairs of teachers was provided. Incremental features of PASMAP were introduced by the research team gradually, at approximately the same pace and with equivalent mentoring for each teacher, over three school terms (May-December 2009). Implementation time varied considerably between classes and schools, ranging from one 40-minute lesson per week to more than 5 one-hour lessons per week.

Students were pre- and post- tested with I Can Do Maths (ICDM) (Doig & de Lemos, 2000) in February and December 2009 and September 2010; from pre-test data two 'focus' groups of five students in each class were selected from the upper and lower quartiles, respectively. These 190 students were interviewed by the research team using a new version of a 20-item Pattern and Structure Assessment (PASA1) in February 2009, a revised 19-item PASA2 in December 2009 (n=184), and the PASA2 and "extension" PASA in September 2010 (n=170).

Focus children (n=190) were monitored closely by the teacher and the research assistant collecting detailed observation notes, digital recordings of their mathematics learning and work samples, and other classroom-based and school-based assessment data. These data formed the basis of digital profiles for each student.

In summary, the qualitative analysis of the focus students' learning is complemented by the quantitative analysis of the ICDM and the PASA data presented here as a scale (see Figure 1). Further analysis of students' level of structural development at the three assessment points on selected PASA items supports the quantitative analysis. (For methods see Mulligan, 2011). For example, 190 students drawn representations for selected items were systematically coded for one of five levels of structural development. This enabled the description of developmental features (see Mulligan & Mitchelmore, 2009). Other evaluation data includes the implementation of PASMAP and teachers' views of the impact of the program on student learning and their own professional learning.

The development of the PASA assessment items

The assessment interview sought to complement interview-based numeracy assessment instruments such as the SENA (NSW DET, 2002) by extending counting and arithmetic strategies (addition and subtraction) to multiplicative reasoning. Our earlier studies highlighting the relationship between multiple counting and patterning, the development of composite units and unitising, base ten structure, partitioning and multiplicative reasoning influenced the design of the items [Items 4, 5, 6, 9, 10, 11, 12, (Mulligan & Mitchelmore, 1997; Thomas, Mulligan & Goldin, 2002)]. This included the work of English on combinatorial thinking and problem solving (English, 1993). Particular attention was paid to representations of 2-dimensional and 3-dimensional arrays (Items 7, 8, 18) and understanding the relationship between unit size and number of units (Outhred & Mitchelmore, 2000). The patterning tasks (Items 1, 2, 15) were based on our earlier studies with Kindergarten and Year 1 students and Papic's studies with preschoolers. These were extended to include an item integrating multiple counting and emergent functional thinking (Blanton & Kaput, 2005; Warren & Cooper, 2008). The ability to subitize was considered fundamental in developing visual memory and pattern recognition (Bobis, 1996; Hunting, 2003; Wright, 2003). The subitizing tasks extended those in the Schedule for Early Number Assessment 1 (SENA 1) (NSW DET, 2002) as it was considered important to compare responses on this item with those elicited on other patterning items. The inclusion of items on analogical reasoning (Item 13) and transformation (Item 14) was inspired by the work of English (2004), based on the notion that there were strong links between analogical reasoning and spatial patterning. As well Item 14 served to inform our assessment of students' transformational and sequencing skills. Further, several items required students to draw and explain representations such as the structuring features evident on a clockface. We included this item and another on drawing a ruler (in the extension PASA) based on our previous analyses of structural development (Mulligan & Mitchelmore, 2009). Additional items such as composite units in 2- dimensional shapes, the structure of ten frames, hundreds charts and counting patterns, the pattern of squares, equivalence and commutativity, and unitising length were formulated as an extension PASA.

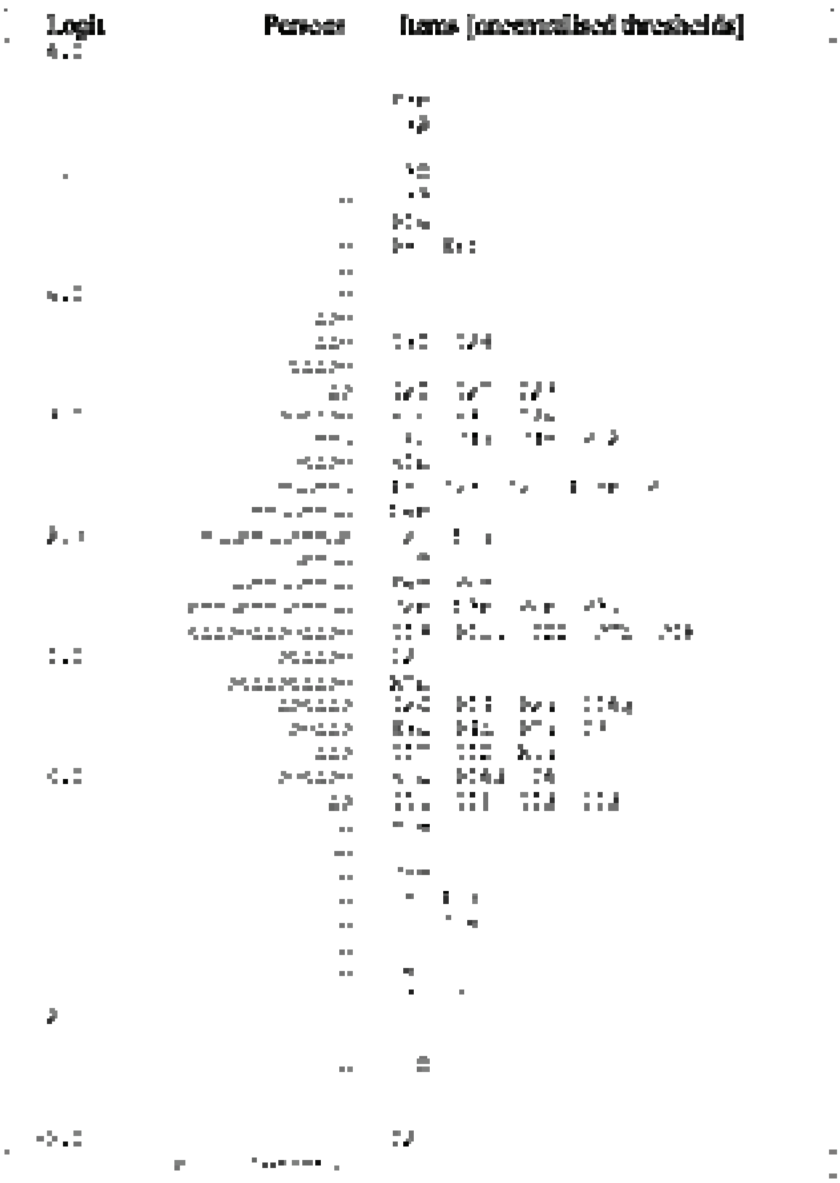
The development of a PASA scale

Although the project focused on descriptive analyses of students' structural development, we complemented these by producing measures of students' ability that could define and assess growth (growth is defined as the difference between a student's performances at two points of time). The PASA data was analysed to construct a unidimensional scale that incorporated graded items along a continuum, for students aged 4.5 to 7.5 years. In order to establish the integrity of these items within a single construct, 'Pattern and Structure', it was advantageous to conceptualise these items on a linear scale. The main advantage of using Rasch analysis for constructing the PASA scale was that it could be used to link different versions of the PASA containing different subsets of items (see Looveer & Mulligan, 2009). As well students' performance on the ICDM, also using a Rasch scale, could be later integrated into the one scale to give a broader view of mathematical growth across the three assessment points. In order to measure this growth, ability estimates of students' location on the continuum could be determined and changes in students' ability locations could provide measures of growth. Rasch Unidimensional Measurement Models (RUMM) computer software (Andrich, Lyne, Sheridan & Luo, 2001) was used to generate scale scores for PASA items and student measures for the construction of the PASA scale. Item analysis was used to discard items not functioning well in PASA1 to reformulate PASA2 and the extension PASA. Following this, a separate Item map produced for the ICDM scores was integrated into the PASA scale (each scale can be viewed separately).

Results

Figure 1 shows an integrated ICDM and PASA scale. The distribution of ICDM (code I), PASA2 (code P) and extension PASA (code E) items and students. The right-hand side of Figure 1 shows 73 item locations; Item P3b (PASA2) is the hardest item and Item I2 (ICDM 2) the least difficult. (The PASA 3b item was difficult because the students were required to visualise and calculate the number of items in 5x5 array from memory; the ICDM item 2 required students to indicate the longer of two lines.) On the left-hand side of Figure 1, each o represents 2 students. The scale extends from -3 logits to 6 logits, representing students' ability measured in March of Kindergarten to September of Year 1. The item map indicates that the items and the students were reasonably well matched; the PASA and extension PASA together performed reliably with several items challenging students beyond 2.0 logits. In comparison, the ICDM items at the lower end of the scale did not sufficiently challenge the majority of students, although some more difficult ICDM items fill a gap in the scale between 3.0 and 4.0 logits. Taken separately, the extension PASA also performed reliably.

Figure 1. Item map of integrated PASA and ICDM scale.



The scale's order of item difficulty on PASA items provided a measure of pattern and structure that reflected the students' overall level of AMPS. For example, items that challenged the most able students (Items P14 and P8) clearly assessed their visualisation and spatial structuring; and discrimination between simple repetition and recognition of a transformational (rotational) pattern respectively. (Item 14: Provide a net of an open box (2cm x 2cm x 2cm) and one multilink cube. *Imagine this shape folded up to make a box. How many cubes like this would fill the box without any spaces left?* Item P8: Show three arrows in a sequence (pointing upwards, sidewards, and downwards). *Show the way you think the arrow will go next? And which way after that? Tell me why you think that?*). Thus a conceptual analysis of the item and its position on the scale reflected the complexity of the task in terms of pattern and structure as well as the reasoning required to complete it successfully. What we aimed to achieve with the scale was an indicator of AMPS aligned with student ability.

We now present some general findings from the quantitative analysis to date. The ICDM scores were analysed as a standard measure of mathematical competence for all students at each assessment point. We did not expect ICDM to provide evidence of students' development of pattern and structure. Rather the ICDM served as a measure to validate the sample with Australian norms for Grade 1 (Doig & de Lemos, 2006) and to assign students initially to 'high-achieving' or 'low-achieving' ability focus groups. Figure 2 indicates that the sample's mean ICDM scores were slightly above the ICDM norms. There were no significant differences found on ICDM scores between PASMAP and regular students at any of the three assessment points but there were significant differences found between states.

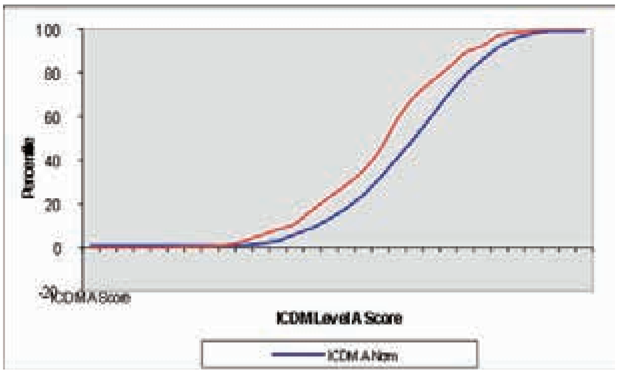


Figure 2. ICDM norms compared with the sample (n=316).

Both groups of students made substantial gains on the ICDM and the PASA1 across the three assessments with PASMAP students' overall mean scores consistently higher than the regular group (see Table 1). However, these were not significant (p=0.105). An analysis of variance revealed significant differences between states (p=0.035) and between schools (p=0.040) with NSW students outperforming Queensland students at each assessment point.

		PASA1	*n*	2009 PASA2	*n*	2010 PASA2	E-PASA	*n*
NSW	PASMAP	9.40	*40*	15.05	*38*	14.14	9.09	*35*
	Regular	9.74	*50*	14.66	*50*	12.19	7.79	*42*
QLD	PASMAP	11.25	*51*	14.42	*50*	12.00	7.45	*47*
	Regular	10.94	*49*	15.67	*46*	10.80	7.07	*46*

Table 1. Mean scores for all PASA assessments.

Discussion

Clearly these data showed consistently that NSW students were more advanced in their general mathematical competencies than the Queensland students. Queensland students had not necessarily experienced a preparatory curriculum and 2009 was the first year of a formal mathematics curriculum for 5 year olds. Nevertheless PASMAP students in Qld demonstrated growth in structural development in similar ways to the NSW students once they participated in the PASMAP program. Although we found consistently higher mean scores for PASAMP students, we expected that this finding may not necessarily prove to be statistically significant. We interpreted these findings in light of one confounding factor; the amount of time that individual PASMAP teachers devoted to the program implementation which had differential effects on students' learning outcomes. The time devoted to PASMAP ranged between one 40-minute lesson to more than 5 x 1 hour lessons per week. Some PASMAP teachers completed only half of the program components while others completed almost the entire program and revisited concepts regularly. Qualitative analysis of the NSW students' profiles and the classroom observation data showed stark differences in the way that the PASMAP students developed their knowledge and reasoning skills. Because the program focused intently on developing structural relationships, only the PASMAP students made direct connections between mathematical ideas and processes, and formed emergent generalisations. For example students began to link simple multiple counting to more complex multiples, arrays and multiplicative structures through their experience of the notion of unit of repeat in patterning, partitioning, in spatial tasks and in measurement contexts. Able students used particular features of pattern and structure to build new and more complex ideas. Regular students could also solve tasks requiring multiplicative thinking but these were considered as separate mathematical ideas, i.e., these students could not explain what was similar or different, what was the connection between ideas, or form simple generalisations.

Categorising responses for stages of structural development

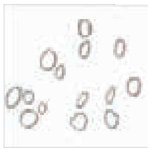
The analysis of PASA assessment interviews indicated marked differences between groups in students' levels of structural development (AMPS) at the second and third assessments. Students participating in the PASMAP program showed higher levels of AMPS than the regular group, made connections between mathematical ideas and processes, and formed emergent generalisations. Students' drawn responses and their explanations, at the three assessment points, were categorised using the levels of analysis from previous studies (Mulligan & Mitchelmore, 2009) as follows:

- *Pre-structural*: representations lack evidence of numerical or spatial structure
- *Emergent (inventive-semiotic):* representations show some relevant elements but numerical or spatial structure is not represented
- *Partial structural:* representations show most relevant aspects but are incomplete
- *Structural:* representations correctly integrate numerical and spatial structural features

An independent coder categorised each response for level of structural development with reference to each interview script (reliability of 0.91). We present an exemplar of the analysis of structural development drawn from 600 responses for Item 15 including approximately 10% as "second attempts". The coding was consistent with that used in previous studies but allowed for comparison of more challenging items.

Figures 3 to 8 show typical examples of developmental features of students' AMPS in response to Item 15. Figure 3 presents a circular border of dots and a random formation in the centre. There is some perception of outer and inner dots but it is largely idiosyncratic and depicts a 'crowded' image. Figures 4 and 5 show some awareness of triangular formation but there is little structural extension of the pattern. Figures 6, 7, and 8 represent the correct formation but vary in structural complexity. Note that Figure

7 depicts a simple repetition rather than a growing pattern. In Figure 8, the student explains the growing pattern numerically and as a simple generalisation.

					
Figure 3. *Pre-structural*	*Figure 4.* *Emergent*	*Figure 5.* *Emergent*	*Figure 6.* *Partial*	*Figure 7.* *Structural features*	*Figure 8.* *Advanced*

Conclusions and Implications

The study produced a valid and reliable interview-based measure and scale of mathematical pattern and structure that revealed new insights into students' mathematical capabilities at school entry. Clearly young students were able to solve a broad range of novel mathematical tasks, including repetitions and growing patterns, and multiplicative problems, not usually asked of students of this age. Generally all students were able to construct and use counting and arithmetic strategies up to 20 and beyond. About 25% of PASMAP students recognised complex number patterns effectively on a hundreds chart in Kindergarten. The ICDM measures could be integrated with the new PASA scale to provide a comparative measure, although it assessed numeracy in traditional ways and did little to complement the PASA data.

PASMAP explicitly focused on the promotion of students' awareness of pattern and structure (AMPS): the analysis of students' learning showed that it had achieved its aims. Particular gains were noted in the related areas of patterning, multiplicative thinking (skip counting and quotition), and rectangular structure (regular covering of circles and rectangles). As expected, a focus on pattern, structure, representation, and emergent generalisation advantaged the PASMAP students. However, students in the regular program were also able to elicit structural responses but had not been given opportunities to describe or explain their emergent generalised thinking that may have been developing. Thus, it was not possible to determine whether more advanced examples of structural development could be directly attributed to the program or innate developmental advances of more able students. One of the most promising findings was that the focus students categorised as low ability were able to develop structural responses over a relatively short period of time. Further analysis of the impact of PASMAP on structural development must consider individual teacher effect and school-based approaches to evaluate the program's scope and depth of achievement.

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REVIEWING THE EFFECTIVENESS OF MATHEMATICAL TASKS IN ENCOURAGING COLLABORATIVE TALK WITH YOUNG CHILDREN

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This paper presents results from the *Resourcing Talking in Maths* project. The project aimed to review the resourcing, management and orchestration of collaborative mathematical tasks with young children (6 years old) and to examine the tensions involved in developing tasks that are accessible but also provide a challenge. It was found that the level of explicitness of the attended focus of the task needed to be balanced and that this balance was informed by the precision of the teacher's explanations and the definition of the mathematical relations as presented in the use of resources.

Introduction

The *Resourcing Talking in Maths* project, funded by the National Centre for Excellence in Teaching Mathematics (NCETM) in England, built on previous work that has shown the effectiveness of pupil-pupil talk on attainment in mathematics (Mercer & Sams, 2006; Murphy, 2011). It also acknowledges the difficulty teachers face in presenting tasks that encourage engagement and talk with younger lower-attaining children. Although rich problem solving tasks can overcome barriers to mathematical learning (Sullivan, 2003; Lubienski, 2000), it is seen that the 'richness' of a task depends on the teacher's management and orchestration, and that there is little guidance for teachers on this.

The project was based on a collaborative classroom teaching experiment involving two primary school teachers over one term. Classroom experiment is seen as one type of setting for a design experiment methodology (Cobb, Confrey, diSessa, Lehrer, R & Schauble, 2003) in which the researcher collaborates with the teacher to investigate instructional design. In this way pedagogical design is used to inform theory within a specific domain. Results are presented from two groups of three children (6 years old), where each group is engaged in three mathematical tasks. These six tasks and are used to develop a framework to support the development of effective collaborative tasks. Although based within a specific domain it is hoped that the framework is useful in supporting teachers in developing collaborative tasks more generally.

Background to research aims

The research is based on the assumption that children's engagement in collaborative mathematical tasks will enable children to participate actively in learning arithmetic. From a social constructivist perspective transmission of knowledge is seen to happen in the context of solving a problem where solutions are proposed and responded to (Wells, 1999). Barnes (1976) had proposed that encouraging children to talk in an exploratory way allowed them to use language as a way of thinking aloud. Exploratory talk has been further typified as "a way of using language effectively for joint, explicit, collaborative reasoning" (Mercer, Wegerif and Dawes, 1999, p. 97). The development of such talk would seem to support children's collaborative exploration of ideas and discussions within mathematics. As children engage in pupil-pupil talk they test out their understanding and applications of procedures in key mathematical ideas.

Encouraging young children to work in this way requires a different pedagogy and for teachers this may mean learning new skills. The development of exploratory talk through explicit teaching strategies has been seen to be effective in supporting children's use of talk as well as helping teachers to change their practices. The teachers involved in this study had participated previously in research on the introduction of explicit talk strategies and the children were familiar with this approach to mathematical tasks. A further element to consider is the task that the children are engaged in. Blatchford, Kutnick, Baines and Galton (2003) have suggested that in developing strategies for effective group work the learning task is a critical factor. If tasks are simplified they do not necessarily lead to success (Houssart, 2002). The difficulty would seem to be in developing rich, problem solving tasks that are accessible but also provided a challenge.

A key aim of the project was to examine the effective management of learning tasks and, in particular, to examine the balance between the precision of the explanations given by the teacher and the definition of the mathematics represented in the use of the resources. Developing the tasks within a classroom-based experiment required the teachers to be reflective and innovative and it was anticipated that the teachers' involvement would support professional development.

The study

The study involved a series of three workshops interspersed with the trial of group tasks in the teachers' classrooms. Each of the group tasks were videoed and observed in the workshops by the teachers in collaboration with the researcher. The workshops were used to analyse the way the children negotiated ideas and the way they engaged with the mathematical relations intended in the task. This analysis was used to identify the next step in instructional design. Final analysis was carried out using the video data from the group tasks to inform a theoretical framework.

Based on Nunes, Bryant and Watson's (2009) studies on key understandings in learning mathematics, the tasks aimed to help children connect their understanding of quantity with their knowledge of counting. It was decided to look at comparison as a mode of enquiry in order to make distinctions and to sort representations with regard to equivalence as a mathematical relation. The tasks were based on a type of rich task identified by Swan (2006) as comparing representations, in this case to make connections between quantities and numbers. The teachers developed tasks that involved sorting and matching representations that included both quantities and numbers. The learning intention was that children worked with the mathematical relations rather than perceptual differences and similarities in order to find equivalent quantities (Nunes et al., 2009). Although the key purpose of each task was the same, the teachers took different approaches in resourcing, managing and orchestrating the tasks.

In this paper, six of the tasks are presented (3 tasks across 2 groups of 3 children). The tasks were developed by the two teachers, Teacher 1 and Teacher 2, and carried out with a group of three children in each of their classes. In the first task the teachers used cards with pictorial representations such as stars, cars, blocks and number lines as well as numerals (figures 1 and 2), and the children were asked to sort the quantities in relation to operations on numbers. In the second task the purpose was to find matching pairs of quantities or quantities and numbers. Teacher 1 used the same resource as task 1. Teacher 2 used cards with different calculations showing commutative pairs (figure 3). In the third task, both teachers used money to represent quantities and children were asked to find representations of equivalent amounts (figures 4 and 5).

Results

Task 1. Sorting representations

Teacher 1 gave the children a set of cards that provided a wide range of representations, each totalling 15 (figure 1). The teacher gave no initial explanation other than to sort the cards. The children negotiated ideas but sorted by perceptual similarities such as shape and colour. The teacher gave prompts such as "Why do you think some are split into two colours?" but the children did not notice the mathematical relations. Teacher 2 limited the range of representations (figure 2) on the cards that she gave the children. She did not include the number line and used fewer representations. She also gave the children cards in stages as sets according to the representations. The teacher provided a grid for organising the resources. The teacher modelled how she would sort the cards and asked questions such as, "Why have I put these two together?" The children focused on the mathematical relations but there was limited negotiation of ideas.

Figure 1. Task 1and task 2, Teacher 1: Sorting representations and matching pairs.

Figure 2. Task 1, Teacher 2: Sorting representations.

Task 2. Matching pairs

Teacher 1 used the resources from Task 1 but with representations totalling 12. Again she did not model what to do and gave no initial explanation other than to sort the cards. As the children started to sort by perceptual similarities the teacher prompted the children to find pairs, "Are there any you could match together as a pair?" The children negotiated ideas and with further prompts in finding matching pairs they sorted according to mathematical relations. Teacher 2 provided three calculations; 4+5, 5+4, 5+3, and asked for the 'odd one out'. There was limited negotiation in this initial task. Then the teacher provided a wider range of calculations to find other 'odd one out' calculations (figure 3). The children engaged in negotiation in this subsequent task and the teacher questioned the children after they had completed the task, "6+6; would that have a partner?"

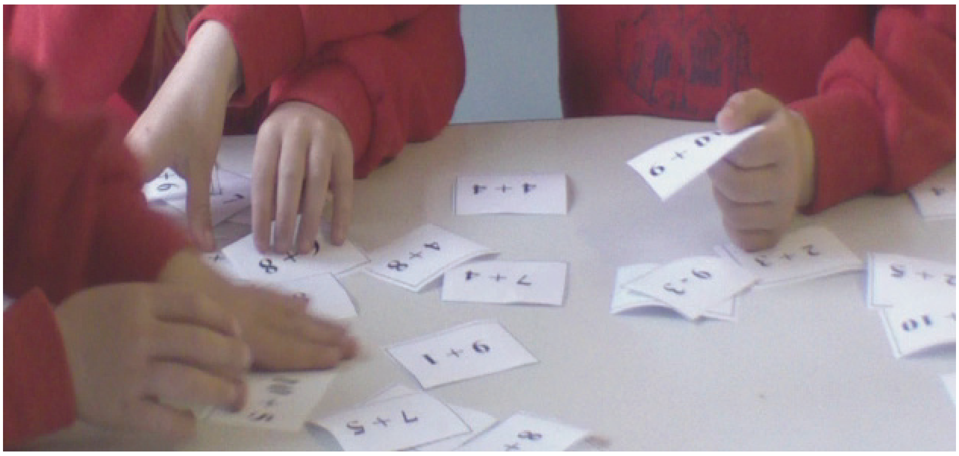

Figure 3. Task 2, Teacher 2: Matching pairs.

Task 3. Money representations

Teacher 1 modelled to the children how she would find equivalent solutions for making 10p. The children were then asked if she had found all the ways. The children were given blank cards to record further solutions (figure 4). As the children investigated other equivalent solutions the teacher prompted the children in working systematically, "Where do you think that would go in your order?" and in recognising equivalence, "Is this one different, why is it different?" The children noticed the mathematical relations and negotiated ideas. Teacher 2 gave the children a set of cards with different amounts of money to put into pairs and a grid to organise the pairs (figure 5). The teacher allowed the children to decide how they would use the grid. The children noticed the mathematical relations and negotiated ideas. At the end of the task the teacher supported the children's organisation of the pairs on the grid.

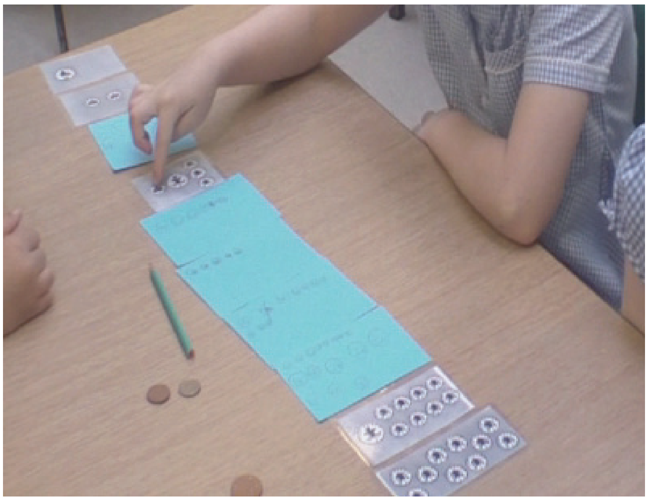

Figure 4. Task 3, Teacher 1: Money representations.

Figure 5. Task 3, Teacher 2: Money representations.

Analysis and discussion

Sfard and Kieran (2001) identified different components that children focus on as they work together on mathematical tasks; the attended focus, the intended focus and the pronounced focus. The attended focus relates to an individual pupil's focus as they attend to the process of a task. The intended focus is mainly private and relates to the experiences evoked by the other focal components. The pronounced focus is the publicly agreed focus. The relationships between these different focal components are seen to have an effect on collaboration.

The attended focus mediates between the public pronounced focus and the private intended focus. It is how the children attend to the process of the task and share their own private intended foci. In other words, how the private intended focus becomes pronounced or public. This mediation is influenced by the explicitness of the attended focus. Various pedagogic strategies, such as scaffolding the tasks, questioning and prompting, and repetition of similar formats can make the attended focus more explicit. Also, the resources used can more or less define the mathematical relations and hence make the attended focus more or less explicit. The pedagogic strategies, along with the use of resources, can help the children attend to the process of the task, to share private intended foci and make them public. In this way the level of explicitness is related to the teacher's presentation and orchestration of the task, and how her intended focus is made sufficiently public. If the teacher's presentation is prescribed, her intended focus is made public in a precise way.

In relation to these focal components the tasks are analysed according to two factors:

1. Level of precision provided by the teacher in their explanations and organisation. This is determined by how the teacher prescribes the task.
2. Level of definition of the mathematical relations presented in the resources used in the task.

Task 1. Sorting representations

Teacher 1 did not provide any precise explanations nor did she define the use of the resources, so the attended focus was not explicit. The children may have negotiated ideas but they did not notice the mathematical relations as expected. On the other hand Teacher 2's explanations were precise and she defined the use of the resources. In this case the attended focus was very explicit. The teacher's intended focus was made public in a precise way. The children did notice the mathematical relations as expected but they did not need to negotiate ideas and there was little collaboration.

Task 2. Matching Pairs

Teacher 1 repeated the format of the previous task and in this way the use of resources became more defined. The prompts and questions focused the children on the process of finding pairs and the teacher's intervention was more precise. In this way the children were able to attend to the mathematical relations as expected but there was still a need for the children to negotiate ideas. In Teacher 2's initial task the use of resources was clearly defined, the presentation was prescribed and the attending focus was very explicit. The children did not need to negotiate ideas. However this initial task helped to define the use of the resources in the wider subsequent task and the attended focus was sufficiently explicit for the children to focus on the mathematical relations as expected and also to negotiate ideas.

Task 3. Money representations

In this task, Teacher 1 provided an initial stimulus that made her intended focus precise and defined the use of resources. This then informed the subsequent task as the children found all the solutions. The attended focus was sufficiently explicit. The children negotiated ideas and focused on the mathematical relations as expected. Teacher 2 gave no precise explanations but the resources were defined through the repeated format of finding pairs. The attended focus was sufficiently explicit to enable the children to negotiate ideas and focus on the mathematical relations as intended.

Level of definition (provided by use of resources) ↕	Level of Precision (provided by the teacher in explanations and orchestration): High	Level of Precision (provided by the teacher in explanations and orchestration): Low
High	Attended focus is too explicit. Task completed individually with no need to negotiate	Attended focus is sufficiently explicit. Children negotiate ideas and interpret the intended focus of the task as expected
Low	Attended focus is sufficiently explicit. Children negotiate ideas and interpret the intended focus of the task as expected	Attended focus is not explicit. Children negotiate ideas but do not interpret the intended focus of the task as expected

Figure 6: Balancing the level of explicitness in the attended focus of a task.

It would seem that the level of explicitness requires a critical balance in order to enable children to engage collaboratively in a task and also to focus on the mathematical relations as expected. If the mathematical relations are very well defined in the resources and the teacher's explanations are prescriptive, the intended focus of the teacher is made public in a precise way and the attended focus is very explicit. There is little need to engage in discussion or share ideas. If the resources are ill defined, and the teacher's explanations are not precise, the attended focus is not sufficiently explicit. The children may not be able to negotiate ideas, or they may negotiate ideas but interpret the teacher's intended focus in an unexpected way. Tasks that had a balance between precision and definition seemed to encourage talk and collaboration. This is summarised in Figure 6.

Conclusion

A key aim of the research was to investigate the development of mathematical tasks that encouraged talk and collaboration with young children. It was hoped that such tasks could be used to help children see the relations between number and quantity as a key understanding in arithmetic. The tension was seen to be in developing tasks that were accessible but that also provided a challenge.

The learning task was seen to be a crucial factor in enabling collaboration and talk to happen. From the analysis of the six tasks it would seem that the effectiveness of a task is determined by the explicitness of the attended focus, and that this explicitness is, in turn, determined by the precision of the teacher's intended focus through the level of prescription and the definition of the mathematical relations in the use of resources (Table 1). If the attended focus is sufficiently explicit the children are able to negotiate ideas and focus on the intended mathematical relations. This enabled the children to see the relationships between number and quantity as expected. Using this theoretical framework it would seem that the tasks that were accessible but that also provided a challenge were those tasks that had a balance between the precision of the teacher's explanations and the definition of the intended mathematical relations as presented in the resources.

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THE USE OF PROBLEM CATEGORISATION IN THE LEARNING OF RATIO

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Mathematics pupils in Singapore are not performing to expectation. Pupils fail to apply learnt concepts, and new concepts are learnt in isolation instead of through a ‗build-up' based upon known older ones. This ongoing study investigates relating students' prior knowledge of the topic Ratio to new concepts. *Case Based Reasoning* and *Cognitively Guided Instruction* are used in this research. Their frameworks are combined, creating ‗categorisation' where items are grouped, based on the concepts.

Introduction

Ratio is taught in Primary 5 and 6 in Singapore. The Curriculum Planning and Development Division (CPDD, 2007), has spelt out justifiable expectations, but these and assessment do not meet. Many pupils are unable to apply what they have been taught to solve new problems, while others are unable to reason logically or use information correctly, possibly because of the lack of effective problem solving strategies. The study reported in this paper was designed to investigate the use of categorisation and its effectiveness in solving ratio problems. Tied to categorisation is the use of solving strategies that are based on the concepts studied. In order to provide a focus for these objectives, the following research questions were formulated.

1. Does categorisation of problems result in meaningful differentiation of *student thinking* about ratio?
2. Does categorisation of problems result in meaningful differentiation of *student performance* about ratio?
3. What kinds of informal strategies do children use to solve ratio problems before and during instruction?
4. What instructional implications (teachers' and children's) can be drawn from children's pre-instructional knowledge in relation to problem categorisation?

Literature review

The Singapore Mathematics Framework (Ministry of Education, 2006) considers mathematical problem solving to be central to mathematics learning. Students are to attain and apply the mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems; however evidence has shown that problem solving is not well developed in our pupils. Kaur and Yap (1999) reported that when students were given concepts in unfamiliar situations, many did not perform as well as expected. There is a need to address the gap between students‘ initial ability to understand concepts and the new concepts they are to learn. To achieve this goal, the principles of *Cognitively Guided Instruction* (CGI) (Carpenter, Fennema, Franke, Levi & Empson, 1999) and *Case Based Reasoning* (CBR) (Kolodner, 1997) were adopted for the study. CGI aims for teachers to ―work back from errors to find out what valid conceptions students have so that instruction can help students build on their existing knowledge" (Carpenter, Fennema & Franke, 1996, p.14). In CGI, the emphasis shifts from teachers finding ways to teach mathematical knowledge to students constructing their own knowledge based on their intuitive problem solving strategies. Supporting CGI is the idea of CBR, based on previously acquired experience.

Kolodner and Guzdial (2000) explained that the process of carrying out CBR includes: using case libraries (in this study, a collection of similar word problems) as a resource; indexing problems (identifying and classifying questions that are similar, or questions with a common concept); retrieval processes (recalling previously done questions to help solve current ones) and partial matching processes (matching similar questions to existing ones). One major issue with CBR involves the process of indexing problems. This means identifying old situations that are relevant to a new one. Suitable cases can be recalled if they are indexed well. Good indexes and the ability to apply knowledge or skills from one situation to another different situation are critical in CBR.

CGI (use of categorisation) is tied together with the CBR (instructional process flow) framework to bring about optimum learning. In the CGI framework, there are three main components: problem types, pupils‘ informal knowledge of strategies, and pedagogy concerns. This study focuses on only one component of CGI, namely problem types, as the ability to solve word problems depends so much on pupils‘ ability to recognize the differences among the problem types (Carpenter et al., 1999).

CGI therefore involves examining the various structures of problems. In ratio, problems are placed in various categories based on the distinctive feature each structure offers. Each category influences the strategies that pupils use to solve the problem. Hence, these categories are tagged not only based on their structure of questions but also on the concepts used to solve problems. There are seven categories in all. In order of increasing difficulty they are:

- (Category 1): *Ratio with Values assigned*. Example: The sides of a triangle are in the ratio of 2:3:4. The longest side is 68 cm. Find the perimeter of the triangle.
- (Category 2): *Ratio with one quantity remaining the same*: Example: The ratio of the number of Lynette‘s stamps to Joel‘s was 5:4 at first. After Joel collected another 58 stamps, the ratio became 15:14. How many stamps did Lynette have?
- (Category 3): *Ratio with a constant difference*: The ratio of Jessie‘s age to her father‘s age is 3:7. 12 years later, the ratio will be 3:5. How old is Jessie now?
- (Category 4): *Ratio with a fixed total.* Example: Ron and Kat shared some mangoes in the ratio 3:8. When Kat gave 48 mangoes to Ron, the new ratio of Ron‘s mangoes to Kat‘s was 9:2. How many mangoes did Roland have at first?
- (Category 5): *Fractional Parts of a ratio.* Example: Mark and Sean shared some marbles in the ratio 5:4. After Mark gave half of his marbles to Sean, Sean had 96 more marbles than Mark. How many marbles did they have altogether?

- (Category 6): *Ratio with changing quantities.* Example: The ratio of the number of men‘s watches to the number of ladies watches in a showcase was 4:1. After putting another 48 men‘s watches and 36 ladies watches into the showcase, the ratio became 8:4. How many watches were there at first?
- (Category 7): *Ratio of a ratio.* Example: At a party, the ratio of the number of boys to the number of girls is 3:2. If each boy and each girl is given stickers in the ratio 2:3, a total of 1992 stickers are needed. How many boys and girls are there?

Recognizing the differences among the problems alone is insufficient – pupils must be able to apply the correct strategy to solve the problem. Applying the correct strategy comes about from being able to identify the concept within the problem. To apply the correct strategy, pupils must first overcome conceptual difficulty. Lo and Watanabe (1997) claimed that technical difficulties usually are not the main obstacle in curtailing students‘ solving process. Conceptual difficulty apparently is much greater and more complicated. Categorisation can solve this problem of conceptual difficulty, as it trains pupils to identify concepts involved in a particular question. Lamon (1993) believed that there is a need to move beyond identifying the litany of tasks variables that affect problem difficulty, toward the identification of components that offer more explanatory power for children‘s performance. That is, there is a need to do more than just look into pupils‘ cognition. Combining frameworks of CGI and CBR could eliminate a litany of tasks variables.

Research method

Sample: Thirty-two Primary 6 pupils from a primary school in Singapore were selected to form the non-random purposive sampling group. A teacher also participated. The school was co-educational, with the ratio of boys to girls being approximately 1:1. Most of the students‘ mathematics ability met the nation‘s national average. None of the pupils had been exposed to problem categorisation. In Primary 5, these pupils used various heuristics such as model drawing, listing, and guess and check to solve ratio problems—part of the heuristics package used by the school. Categorisation was a newly developed process and had not been tried with these pupils before.

Research Design: Mixed methods were adopted where quantitative and qualitative data were simultaneously collected and merged. Creswell (2008) believes that the strengths of one data form offset the weaknesses of the other form. Both forms of data were collected at the same time and the results were used to validate each other.

The research was carried out in three phases:

Phase 1 (Categories 1, 2 and 3): A pre test was administered before the start of phase 1. This phase dealt with pupils‘ first interaction with ratio and only basic ratio categories were covered. New concepts were built on the old ones using the CBR lesson flow framework. In this phase, the effectiveness of categorisation for the basic ratio problems was being investigated. Strategies used were being considered in order to determine if pupils were able to move away from model drawing to using ratio concepts. Difficulties and misconceptions pupils faced were also examined. To do this, the use of voice recording with MP3 players was adopted. Pupils recorded their thought processes into the device and the recording was played for the whole class to listen to. This is where students‘ peers, the teacher, and researchers heard their thought processes while solving ratio problems. Discussion was opened for all to comment constructively on what they heard. Misconceptions that arose at that point of time were rectified, so pupils were made aware of the correct concepts and approaches to solving ratio problems.

Phase 2 (Categories 4 and 5): This intermediate phase was one where pupils had become familiar with the lesson structure. The teacher used questioning more in this phase in order to probe thinking at a deep level and to delve beneath the surface of ideas.

Phase 3 (Categories 6 and 7): Strategies applied (from simple model drawings/guess & check/listing→ Ratio concepts→ algebra) were expected to be more sophisticated as students moved away from their initial (informal) strategy. A post test (7 questions) was administered in this phase. MP3 recording, written formative tests (4 questions each), interviews (with the teacher and 4 – 5 pupils) and journal writing were carried out in all three phases.

Data collection

(i) Classroom observations: Classroom observations were conducted, where the researcher/observer blended into the setting, ‗becoming a more or less ‗natural' part of the scene' (Bogdan & Biklen, 2003). Pre and post lesson discussions were carried out to probe the teacher's personal opinions and pupils' conceptual understanding of each lesson.

(ii) Student interviews: Interviews generally allow for open-ended responses and are ‗flexible enough for the observer to note and collect data on unexpected dimensions of the topic' (Bogdan & Biklen, 2003). Pupils were interviewed to bring out their knowledge of ratio. All interviews were tape recorded and transcribed.

(iii) Journal Writing: This process, as mentioned by Yazilah & Fan (2002), is a good avenue for pupils to provide feedback on mathematics teaching and learning (Fan, 2006). In this form of assessment, questions were asked in written form to determine factors such as students' feelings, difficulties, discoveries, and thoughts.

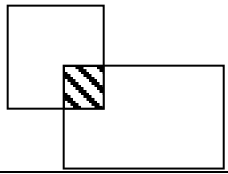
(iv) Performance Test: A written test measured pupils' explicit understanding and performance of ratio concepts through problem categorisation. Making them solve problems in a pen and paper test served as reaffirmation of their understanding. This way, there was a basis for determining whether an individual's ability had changed (Malone, Douglas, Kissane & Mortlock, 2007) and whether problem categorisation was effective in developing ratio concepts. Two forms of written tests were given: formative and summative (in the form of a post test). A scoring scale on marking and measuring problem solving (Malone et al., 2007) was used.

Findings

Changes in student thinking

Voice recording through MP3 player: When transcripts of pupils' thought processes during pre and post test were compared, it was found that there was a noticeable change in student thinking. In the post test, pupils were able to reason logically and correctly, hence they were able to categorise questions correctly. By doing so, pupils get to the concepts tied to the category identified. One example follows (Figure 1).

The figure is made up of a rectangle and a square. The ratio of the area of the square to the area of the rectangle is 1 : 3. A shaded area of 20cm² is being cut out. The ratio of the area of the unshaded square to that of the unshaded rectangle is now 2 : 7. What is the length of each side of the square?



Pre-Test

(7 – 3) parts → 20cm²

1 part → 20cm² ’ 4

= 5cm²

7 parts → 5cm² × 7

= 35cm²

35cm² + 20cm² = 55cm²

55cm² ÷ 2 = <u>27.5cm</u>

Post-Test

[Constant difference because equal parts are taken away]

Before	After
S : R	S : R
1 : 3	2 : 7
5 : 15	4 : 14

(5 – 4) parts → 20cm²

1 part → 20cm²

4 parts → 20cm² × 4 = 80cm²

80cm² + 20cm² = 100cm²

100cm² ÷ 2 = <u>50cm</u>

Note: Pupil had a misconception in area and this misconception was not corrected because the focus of this study was on ratio.

Figure 1: Pre- and post-test solutions to question 3

Pupil C12 : Transcript for Pre-Test question 3

> 7 – 3 because they are the two parts for rectangle. One is before cutting and the other after. So I minus them to get the shaded area that has been cut. Then the answer is equal to the shaded area which is 20cm². I find 1 part first and then multiply it by 7 parts to get the unshaded area. Answer is obtained by adding the cut part to the unshaded part. This gives me the area. Then I divide by 2 to get each side.

Pupil C12 : Transcript for Post-Test question 3

> The same parts are taken away. So, this is constant difference. I must make the difference between the before and after ratio to be the same. Then I can see that the old part for rectangle is 15 and the new part is 14. So the difference is 1. The old part for square is 5 and the new part is 4. The difference is also 1. Now, the difference is the same. So, 1 part is 20cm². Then I find the unshaded part of square, 4 parts and multiply it by 20 cm². This gives me 80 cm². I must add 20cm² to 80 cm² to give the area of a whole square. To find the length, I divide the area by 2.

Based on the thought processes revealed, all students (32/32) managed to reason correctly, placing problems in the correct categories 1 to 3 and obtaining correct solutions. The reasons used to identify the three categories were also correct.

In category 4, 93.75% of the pupils (30/32) managed to identify the category for the given problem. One pupil said that he knew the concept after reading the question, but found the headings to the categories difficult to recall.

In category 5, pupils‘ reasoning was very good, with almost everyone doing well in this category. In the MP3 recording every pupil mentioned the use of _lowest common multiple‘. One pupil (C6) actually made a comment that category 5 (fractional parts of ratio) is usually tied to another category. The following describes his experience.

Mark and Sean shared some marbles in the ratio 5:4. After Mark gave half of his marbles to Sean, Sean had 96 more marbles than Mark. How many marbles did they have altogether?

Pupil C6 : Transcript for Pre-Test question 5

> I get half of 5 and add it to 4 to get 6½. This leaves Mark with…2½. Then I minus 6½ with 2½ to get 4 parts. This 4 parts is the ―more" parts which is the same as 96. Then I find 1 part and multiply it by 9.

Pre Test

Before	**After**	
M : S	M : S	4 parts → 96
		1 part → 96 ’ 4 = 24
5 : 4	$2\frac{1}{2}$: $6\frac{1}{2}$	(5 + 4) parts → 24 × 9
		= 216

Post Test

Before	**After**	
M : S	M : S	(13 – 5) parts → 96
5 : 4	? : ?	1 part → 96 ’ 8 = 12
10 : 8	5 : 13	(5 + 13) parts → 12 × 18
		= 216

Figure 2: Pre- and post-test solutions to question 5

Pupil C6 : Transcript for Post-Test question 5

> I must find a common multiple of 2 and 5 first. It is 10. Then I must multiply 2 to 5 to get 10 parts. I must also multiply 4 by 2 to get 8 parts. So, the new ratio is 10:8. Now, half of 10 is 5 parts. This 5 parts must be given to Sean. Now, he has (5 + 8) parts ….

The pupil worked quietly and started to complete the solution. Later during an interview, both solutions (pre and post tests) were put to him (Figure 2) and he was asked which of the two he preferred. He quickly pointed to the post test and said that he did not like to work with fractions. He added that sometimes, when he had to divide the value (points to 96) by a fraction, he usually ‗messes up‘ his answer by getting it wrong. With this new approach of categorisation, he felt that he did not have to work with fractions at all. Also, he commented that this problem can also be a Cat 4 (total before = total after) problem. He noticed that category 5, fractional parts of ratio, usually comes accompanied by another category.

In category 6, 75% (24/32) of the pupils were able to identify the correct category and 62.5% (20/32) managed to obtain the correct solution. One particular reason for this was because the category involves the use of algebra. Those who managed to get this completely correct used basic algebra to solve this. The rest did it by algebra too, but were stuck halfway through the working. When MP3 recordings of 12 pupils who failed to get this question correct were played, it was discovered that all of them could identify the correct category and the concept, but were stuck when it came to the technical part of algebra where they could not manage when they transposed to the other side of the equation. They worked through the equation using their understanding of equivalence and constructed their knowledge based on their intuitive problem solving strategies.

In the last category, pupils were able to identify this category as the only one with two given ratios that do not refer to the same thing (Figure 3, example 1). In the post test, pupils were able to get through this problem easily. All of them managed to get this correct as they were quick to notice two given ratios that represent different items; pupils and stickers. However, in the formative test that was given at the end of the third phase, pupils were not able to solve a particular question (Figure 3, example 2).

In the MP3 recording, most of them could not find the other ratio; G : B : T = 10 : 5 : 100. It did not cross their minds that the amount of money donated can be written as a ratio. Because of this, the class was not able to obtain a solution. More scaffolding was done to help address the gap between students‘ initial ability to understand concepts and the new concepts they had to learn in category 7. From then on, pupils were more aware (as found in the transcripts) of the ‗other ratio‘ in category 7.

Example 1	Example 2
At a party, the ratio of the number of boys to the number of girls is 3 :2. If each boy and each girl is given stickers in the ratio 2 : 3, a total of 1992 stickers are needed. How many boys and girls are there?	The ratio of the number of girls to the number of boys to the number of teachers in a school is 5 : 6 : 1. Each girl donates $10, each boy donates $5 and each teacher donates $100 in a fund-raising event. If a total of $27000 is donated, how many pupils are there in the school?

Figure 3: Examples in Category 7

Interviews: Five pupils who were interviewed said that learning where new concepts were built on old ones made learning ratio easy. These pupils were able to link new knowledge to the old. This is important as proportional reasoning becomes more complex and detailed as pupils go deeper into the categories of varying content. To overcome this, each category must scaffold the next. This way, pupils are able to see that all ratio problems are connected, and that concepts build from prior knowledge instead of new ideas, thus encouraging a transfer of knowledge between categories.

It was noticeable that pupils began to apply analogical reasoning that focused on reasoning based on previous experience in category 3. They were able to explain, correct and engage the teacher during lessons and had cultivated the habit of reading a question seriously as they realised that each question contained clues to the answer. They were now more conscious than before of the importance of reading to understand. When asked if they faced any difficulties, two felt that the headings of each category were difficult for them to recall. Another two felt that category 5 was very difficult, but were quick to say that if they could overcome that, they would be able to solve more ratio questions. All of them agreed that categorisation should be practiced in other topics, especially in fractions.

Journal writing: In this qualitative aspect of the study the three items discussed were: (i) student thinking, then and now, (ii) preference; categorisation versus current heuristics, (iii) confidence in solving ratio questions.

(i) Student thinking, then and now: Pupils were asked if there were changes in the way they thought when solving ratio questions before and after categorisation. Everyone agreed that there was a change after the categorisation intervention. Unlike before, when they attempted ratio questions after the intervention, they first read before deciding on the category. They realised that they were indirectly ‗forced‘ to read the question. Also, the availability of a strategy (that comes from the concept) was a plus for them. This way, time spent deciding on a strategy was saved and put to better use. Six pupils (18.75%) commented that before, the given information meant very little to them and they did not know what to do with it. The reason for this was that the pupils did not understand the question, hence the information was not fully and correctly utilised. Now, with categorisation, pupils were starting to read the question for understanding first. Twenty-eight pupils (87.5%) agreed that categorisation also helped them give structure to their thinking and made the solving process easier.

(ii) Preference – categorisation versus current heuristics: In the journal entries collected, everyone agreed that categorisation helped them solve ratio questions better than the usual heuristics. The features each category contained [e.g. receiving the same amount → constant difference (cat 3), A shares with B → total before = total after (cat 4)] made it easy for them to identify questions and match them to the respective categories. Each category had a concept that led to a solution. The pupils felt that knowing *what* strategies to use and *how* to use them gave them the confidence they needed to solve the questions. This addressed the issue of *how* learning was facilitated.

(iii) Confidence: It was found that all pupils were more confident in solving ratio questions. With categorisation, pupils did not have to worry about finding the right strategy as this was tied to the categories. Pupils became more confident as they had to only focus on getting to the correct category. Pupils were able to communicate their solutions clearly and logically. It was noticeable that the high performers' explanations were very short and to the point, mentioning only the relevant points, whereas the average performers were very detailed and systematic in their explanation. The low ability students struggled to explain their steps and they took longer to record their thoughts.

Changes in student performance

On the whole, pupils' performance in the post test improved, especially in categories 6 and 7 (Table 1).

Table 1: Student performance on pre and post-test

	Pre-test		Post-test	
Categories	**Number of pupils**	**%**	**Number of pupils**	**%**
Category 1	32	100	32	100
Category 2	25	78	32	100
Category 3	19	59	32	100
Category 4	13	41	30	94
Category 5	15	47	30	94
Category 6	0	0	20	63
Category 7	0	0	32	100

Strategies used

Most pupils started ratio using model drawing; something they have been taught to do since primary 1. Therefore, it was not surprising that most pupils used model drawing in the pre test. Those who were neither good nor confident in model drawing used guess and check and listing as alternatives. These pupils usually fall in the group of low performers in mathematics. As opposed to the post test none of the 32 pupils solved any of the pre-test questions using ratio. Surprisingly, no model drawings were used in the post-test. Listing, and guess and check were heavily used by the low performers in categories 4, 6 and 7. These pupils managed to apply the ratio concepts learnt to solve categories 1, 2, 3 and 5. It was noticeable that these pupils reverted to their old strategies when they were faced with either an unfamiliar or difficult question. A big shift in strategy was seen in the average performers. These pupils were able to move away from direct modelling to writing it out. They found category 6 very challenging as they struggled with the use of algebra. The use of negative numbers and the transformation of the operations were the reasons why they were unable to solve category 6 questions, but the concept in that category was fully understood as shown in the post test.

Implications

Pupils gave positive feedback on learning ratio through categorisation. The results of the post-test confirmed the value of this approach. Through the journal entries, it was clear that pupils hoped to have more learning conducted this way.

From a pedagogical point of view, the teacher participant felt that it was easier to teach ratio this way and thought that it would be good to extend this approach to other topics in mathematics. One particular aspect she liked was the construction of new knowledge from old. She felt that this form of learning was effective and it formed a strong foundation in pupils‘ learning of ratio. She also noticed that her pupils were beginning to read the questions, something she had been asking the pupils to do with little success. In addition, she found the environment in her class more active as, unlike before, pupils were participating in the discussions. She also noticed a positive change in pupils‘ reasoning skills.

Resources used were carefully selected and studied before the categories were decided on. Obtaining questions was easy, but identifying the concepts in the questions was not. Concepts identified had to be vetted to ensure that they could be applied to all questions in that category. In short, considerable effort went into planning the resources in order to achieve the desired outcome.

Pupils do not have to worry about coping with new information regardless of their readiness when using the approach described in this paper. CBR relies heavily on prior knowledge and appears to work well with CGI problem categorisation and shows promise for assisting students towards a better understanding of learning the mathematics of ratio.

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NATIONAL TESTING OF PROBABILITY IN YEARS 3, 5, 7, & 9 IN AUSTRALIA: A CRITICAL ANALYSIS

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This paper provides a critical analysis of the probability questions in the 2009 & 2010 NAPLAN numeracy tests for Years 3, 5, 7, and 9 from a number of perspectives. The analysis revealed that probability is under-represented in recent NAPLAN tests, with only one probability item included in each year level in the 2010 test. The items reveal a limited focus regarding type of task and response method. There is poor coverage of the National Statements on Learning in Mathematics, and weak alignment with the constructs in the *Probabilistic Reasoning Framework* (Jones et al, 1997).

Introduction

In Australia, schools now have to provide evidence to the community that their students perform to 'acceptable standards'. In particular, in 1997 the Australian Federal Government adopted compulsory national testing of literacy and numeracy to (a) identify students at risk, (b) conduct intervention programs, (c) assess all students against national benchmarks, and (d) introduce a national numeracy reporting system (Department of Education, Training, & Youth Affairs, 2000). Since 2008, the writing of the numeracy tests has been done nationally, and the test items have been based on the *National Statements of Learning for Mathematics* (Curriculum Corporation, 2006). The tests are currently conducted in all schools in May, yearly.

The introduction of state and national testing has not been supported widely in the educational community in Australia. Nisbet & Grimbeek (2004) found that many Queensland teachers had very negative attitudes to state tests. They did not believe the tests were valid, and hence the results of the tests did not influence their teaching practices. Nor did the teachers use the test results to any great extent to inform their planning, apart from some identifying gaps in their schools' programs. National testing has limited validity considering (i) the accepted definitions and purposes of assessment and (ii) the nature of large-scale tests. First, assessment can be defined as the comprehensive accounting of a student's knowledge and a means to achieve educational goals, and not the end of an educational experience (Webb, 1993). The purposes of assessment according to Clarke, Clarke, and Lovitt (1990) are to: (i) improve instruction by identifying sources of error; (ii) improve instruction by identifying instructional strategies that are successful; (iii) inform the learner of strengths and weaknesses;

(iv) inform subsequent teachers of students' abilities; and (v) inform parents of their child's progress. The National Assessment Program—Literacy and Numeracy [NAPLAN] can only provide a 'snapshot' of students' achievements and satisfy these purposes to a limited extent.

Second, large-scale numeracy testing brings with it a number of logistical limitations. For instance, the methods of students' responses adopted in the NAPLAN tests are limited to just two types: (i) multiple choice ("Colour in the bubble"); and (ii) single numerical answers ("Write the number in the box"); that can be marked by computer scanning. These limit the questions and tasks that can be included in the tests. It is commonly accepted that through assessment, teachers communicate to students which activities and learning outcomes are valued, so that assessment should be comprehensive and give recognition to all valued learning experiences (Clarke, Clarke, & Lovitt, 1990). Thus national testing brings limitations including a bias towards mechanical processes, and away from problem solving and creativity.

It is known that items in a related area, statistics, in traditional pencil-and-paper mathematics tests have shortcomings; in that they test skills in isolation from the problem context, they do not test whether or not students understand how statistical measures are interpreted, and they fail to assess students' ability to communicate using statistical language (Garfield, 1993). More recent research by Nisbet revealed that the statistics items in NAPLAN were limited in type of tasks, and related to just one aspect of the requirements stated in the *National Statement of Learning: Mathematics* (Curriculum Corporation, 2006), namely, analysing data, and ignored data collection, representation and interpretation. Similarly, most of the statistics items (94%) aligned with just one construct of the *Statistical Thinking Framework* (Jones et al, 2001), and disregarded the other three constructs.

National testing programs have tended to result in classroom assessment moving away from authentic formative practices and towards techniques closely aligned to the national test format (Stiggins, 1999). Teachers feel compelled to spend time preparing their students to master the skills included on the tests. It is against this background that this analysis has been conducted.

NAPLAN numeracy tests

The Australian national numeracy tests are based on a broad definition of numeracy, viz. "Numeracy is the effective use of mathematics to meet the general demands of life at school, at home, in paid work, and for participation in community and civic life" (MCEETYA Benchmarking Task Force, 1997, p. 4). Hence the tests cover the strands of mathematics which most people would meet in daily life; i.e., number, measurement, geometry, chance, and data. The Year 3 test usually contains 35 items, and the Year 5 test 40 items. There are two tests at the Year 7 level, one done without a calculator (32 items), and another where a calculator is allowed (also 32 items). Similarly, there are two tests at the Year 9 level, one with and one without calculators (each 31 items in 2009, and 32 items in 2010). The methods of response to the test items vary across items; some items are answered by multiple choice, in which students have to colour in with pencil one of the small 'bubbles' placed under four alternatives, and some items are open response, in which students write their numerical answers in a small box

(30mm x 12mm) provided on the page. Each paper has a set of three practice items to assist students become familiar with how to answer the questions.

Another feature of the tests is the use of ‘link items’, i.e., items that are common to two year levels. These are inserted to compare performance across grades levels. Seven of the 32 items in the Year 7 non-calculator test are linked to the Year 5 test, and another seven items in the Year 7 non-calculator test are linked to the Year 9 calculator test. Typically, the percentage of students who answer a link item correctly in Year 7 is greater than the percentage of students who answer it correctly in Year 5.

Analysis of probability items in 2009 and 2010 tests

This paper provides an analysis of probability items with respect to the following issues: number of items included, definition of numeracy and use of real-world contexts; type of stimulus material; type of task; alignment with the *National Statements of Learning* [SOLs] for Mathematics for Years 3, 5, 7, and 9 (Curriculum Corporation, 2006); and alignment with the *Probabilistic Reasoning Framework* (Jones et al, 1997). Joliffe (2005) argues that it is useful to compare assessment tasks with curriculum goals and teaching and learning frameworks “to check what dimensions are being assessed” (p. 328).

In comparison with other topics in the mathematics syllabus, probability is under-represented in the NAPLAN tests, with respect to number of items included. In 2010, there was only one Chance item in each year-level test (approximately 3% of the items). In 2009, there was only one Chance item in each of the Year 3 and Year 5 tests (also 3%), two items in the Year 7 test (6%), and four items in Year 9 (6.5%).

The definition of numeracy underlying the Year 3, 5, 7, & 9 numeracy tests was adopted by the Australian Ministerial Council on Education when national benchmarks were set in 1997, and reads as follows: ‘Numeracy is the effective use of mathematics to meet the general demands of life at school, at home, in paid work, and for participation in community and civic life’ (MCEETYA Benchmarking Task Force, 1997, p. 4). All chance questions in the 2009 and 2010 tests referred to various types of hands-on materials, namely, spinners, dice, coins, marbles, jelly beans, and polyhedral blocks. However, such contexts may not be authentic for some students, as the authenticity of the contexts depends on the students’ varied personal backgrounds and experiences, and even the extent to which their teachers had included activities utilising such materials in their classroom mathematics programs. The stimulus material for each item was a statement about particular concrete materials, mainly spinners and jelly beans. Other materials were dice, coins, marbles and blocks. Six out of 12 of the items (50%) included a drawing of the materials (e.g., a spinner), and three out of the 12 items (25%) included a table of results from a Chance experiment.

The items required the students to do one of the following tasks: (i) identify an outcome which is impossible to occur; (ii) identify an outcome which is most likely to occur; (iii) identify the most likely set of results; (iv) identify a situation or concrete material that optimises a particular outcome; (v) calculate the probability of an event occurring based on a diagram of particular concrete materials (e.g. spinner); and (vi) calculate the probability of an event occurring based on a given table of results. Only one item (out of the 12 items in total) required the students to write an answer after performing a calculation. The other 11 questions were multiple-choice items, which allowed students to guess rather than think about the item. Research has shown that it is possible to get multiple-choice items correct without knowing much or doing any real thinking (Northwest Regional Educational Laboratory, 1988). A student may pick the correct response by either knowing or calculating the correct answer, making an informed guess, or just guessing wildly (Parkes, 2010).

The Year 3 Statement of Learning relating to probability (Curriculum Corporation, 2006) is as follows: '[Students] make simple statements, including predictions about likelihood, what is possible and what is not.' (p. 6). The test item given in 2009 asked the students to identify which colour marble was impossible to pick from a box of coloured marbles (with information given about numbers of each colour), so it aligns with the SOL. The item given in 2010 asked the students to identify which spinner from a group of four spinners (illustrated) was most likely to stop on 'white', so it aligns with the SOL also. Hence it is concluded that the Year 3 test items addressed the Year 3 SOL, albeit it in a limited way, with only one item included each year.

The Year 5 Statement of Learning (Curriculum Corporation, 2006, p. 10) refers to students:

- identifying and describing all possible outcomes for familiar chance events;
- making judgments about their likelihood;
- predicting whether some outcomes are more likely than others;
- using suitable language including most unlikely, never, probably;
- collecting data from experiments to justify or adjust these predictions; and
- distinguishing situations that involve equally-likely events from those that do not.

The probability item included in the Year 5 test in 2009 asked the students to identify "Which jar gives Annie the best chance of taking a black jelly bean?" so it aligns with the third dot point. The item in 2010 asked "On which number on a spinner was the arrow most likely to stop?", so it aligns with the third dot point also. It is concluded that the Year 5 test items assessed just one out of the six aspects of chance referred to in the Year 5 SOL. This seems unsatisfactory, but is not surprising considering that the tests contained only one item on probability each year.

The Year 7 Statement of Learning refers to students doing the following:

- comprehending that many events have different likelihoods of occurrence;
- making and interpreting empirical estimates of probabilities;
- comparing experimental data for simple chance events with theoretical probability obtained from proportions expressed as percentages, fractions or decimals; and
- distinguishing events that are equally likely from those that are not. (p. 7)

There were two probability items in the Year 7 test in 2009. The first asked the students "What is the chance that a jelly bean (drawn from a tin) is red?" so it aligns with the second dot point. The second item showed a spinner with unequal sectors and asked "Which table is most likely to show the results of spinning it 100 times?" It aligns with the first and fourth dot points. The only chance item in the 2010 test asked the students to identify which "On which number on the spinner is the arrow most likely to stop?" so it also aligns with the first and fourth dot point of the Year 7 SOL. Hence it is concluded that the Year 7 test items assessed only two out of four aspects (dot points) of the Year 7 SOL.

The Year 9 Statement of Learning refers to students doing the following:

- using a variety of sources, including surveys, data-bases, experiments and simulations to estimate probabilities associated with events;
- assigning or making estimates of probabilities based on personal experiences;
- specifying sample (event) spaces for single and straightforward compound events using a variety of suitable representations;
- determining corresponding probabilities by counting, measure and symmetry; and
- being familiar with the notion of equally likely events; and
- being familiar with the use of random event generators, including technology. (p. 11)

There were four probability items included in the Year 9 test in 2009. The first item asked the students "What the probability is of rolling a 3 on a standard six-sided die?" This aligns with the second dot point. The second item asked "What is the chance that a jelly bean (drawn from a tin) is red?", so it aligns with the fourth dot point. The third item showed a spinner with unequal sectors and asked students to indentify "Which table is most likely to show the results of spinning it 100 times?" It aligns with the fourth dot point of the Year 9 SOL. Item 4 said "Calculate the probability of getting 2 tails and 1 head in any order when a coin is tossed 3 times." It aligns with the third and fourth dot points. It is concluded that the Year 9 test items in 2009 assessed four out of six aspects of the Year 9 SOL. The only Chance item in the 2010 test required the students to calculate the probability of rolling a 2 on a non-regular hexahedron block, given the frequency results for it being thrown 1000 times, so it also aligns with the first and fifth dot point of the Year 9 SOL. Hence it is concluded that the Year 9 test item in 2010 assessed two out of six aspects of the Year 9 SOL. A summary of the aspects of the Year 3, 5, 7, & 9 SOLs covered by NAPLAN is shown in Table 1.

Table 1: Aspects of National Statements of Learning covered in NAPLAN tests by Year level.

Year of test	Year level			
	Year 3	Year 5	Year 7	Year 9
2009	1 out of 1 (100%)	1 out of 6 (16.7%)	2 out of 4 (50%)	4 out of 6 (66.7%)
2010	1 out of 1 (100%)	1 out of 6 (16.7%)	2 out of 4 (50%)	2 out of 6 (33.3%)

Analysis in relation to the *Probabilistic Reasoning Framework*

The *Probabilistic Reasoning Framework* (Jones, Langrall, Thornton, & Mogill, 1997) was developed after research and validation with elementary school pupils. Its purpose is to describe students' probabilistic reasoning skills. It consists of six constructs that are described at four levels (Level 1, Subjective; L2, Transitional' L3, Informal quantitative; & L4, Numerical). The construct *Sample Space* refers to students being able to list the outcomes of one- and two-stage experiments. *Experimental probability* involves recognising the role of the number of trials, and calculating probability values from data. *Theoretical probability* involves being able to predict least likely and most likely events in one- and two-stage experiments, and assign numerical values to probability situations. *Probability comparisons* involve reasoning to distinguish 'fair' and 'unfair' probability situations. *Conditional probability* relates to how probability values differ in replacement and non-replacement situations. Finally, the construct *Independence* refers to students being able to distinguish between independent and non-independent events, and understanding the independence of consecutive trials.

In the Year 3 tests, only two out of the six constructs were assessed (Sample Space in 2009 and Probability comparisons in 2010). In the Year 5 tests, only one construct was assessed (Probability comparisons in both years). In Year 7, two constructs were assessed in 2009 (Experimental & Theoretical probability), and only one construct (Probability comparisons) was assessed in 2010. See Table 2.

Table 2: Probability constructs tested and levels of thinking required according to year level.

	Test items and levels of thinking required				
Construct	Year 3	Year 5	Year 7	Year 9	Totals
Sample Space	2009, Q15: L2				1
Experimental Probability			2009, Q17: L4	2010, Q16: L4	2
Theoretical Probability			2009, Q12: L4	2009, Q2: L4 2009, Q 10: L4 2009, Q28: L4	4
Probability Comparisons	2010, Q7: L2	2009, Q29: L2 2010, Q 7: L2	2010, Q 3: L2	2009, Q 15: L4	5
Conditional Probability					0
Independence					0
Total items	2	2	3	5	12

Overall the majority of the test items (9 out of 12 or 75%) were limited to two of the six constructs (Theoretical probability and Probability comparisons). Two of the 12 items related to Experimental Probability, and one item related to Sample Space. Two constructs (Conditional Probability & Independence) were not tested at all over the two years.

Despite the inadequate coverage, the levels of thinking expected of students followed a reasonable progression from Year 3 to Year 9.

Conclusion

The probability questions in the 2009 and 2010 tests relate to the MCEETYA (1997) definition of numeracy to a limited extent only, because of the variable levels of authenticity of the item contexts. There is also the potential for language or cultural bias, as has been noted in research concerning multiple-choice questions (Parkes, 2010). Further, the probability items are limited in the type of task required, with most being multiple-choice items in which students select a spinner, choose an outcome, or pick a probability value. The multiple-choice items are fundamentally recognition tasks, where students identify the correct response (Parkes, 2010). The test results do not indicate whether the students knew the answer, or made an informed or blind guess (Gronlund & Linn, 1990).

This analysis reveals that the *National Statements of Learning* (Curriculum Corporation, 2006) relating to probability are poorly served by recent NAPLAN tests.

Probability deserves to have more than one item included in the tests each year, and the items should cover more aspects of probability in the statements than those covered in the 2009 and 2010 tests. The current situation promotes a limited view of probability that may be noted by teachers and students and lead to a down grading of the topic in class.

The probability items in the 2009 and 2010 tests relate to a minority of the constructs defined by the *Probabilistic Reasoning Framework* (Jones et al, 1997). With no items on sample space, independence, and conditional probability the tests imply that these constructs are not important, and teachers may omit reference to them in their teaching. As noted by Sadler (1998), when narrow tests define learning, instruction often gets reduced to “drill and kill”— practice on questions that look like the test. If schools place emphasis on the content and style of items contained in NAPLAN tests, and teachers teach to the test (Stiggins, 1999), there will be an unfortunate narrowing of the enacted curriculum and students’ knowledge and understanding of probability will suffer.

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A POPPERIAN CONSILIENCE: MODELLING MATHEMATICAL KNOWLEDGE AND UNDERSTANDING

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Goldin (2003) and McDonald, Yanchar, and Osguthorpe (2005) have called for mathematics learning theory that reconciles the chasm between ideologies, and which may advance mathematics teaching and learning practice. This paper discusses the theoretical underpinnings of a recently completed PhD study that draws upon Popper's (1978) three-world model of knowledge as a lens through which to reconsider a variety of learning theories, including Piaget's reflective abstraction. Based upon this consideration of theories, an alternative theoretical framework and complementary operational model was synthesised, the viability of which was demonstrated by its use to analyse the domain of early-number counting, addition and subtraction.

Introduction

An *alternative theoretical framework* has been proposed (Nutchey, 2011) that explicitly differentiates, and is thus able to describe, the knowledge shared in the learning community and each learner's idiosyncratic understanding. This proposition is an attempt to address the perceived challenges of reconciling student-centred, constructivist learning and the state-able, objective structure of mathematics shared by a community of mathematicians. In this paper, literature substantiating this need is first identified, and then key theoretical constructs that inform the proposed alternative theoretical framework are summarised. The proposed theory is complemented by an *operational model,* of which a significant component is a graphical language for describing the organisation of a domain of mathematical knowledge shared by a community. This language is introduced, and then its viability is illustrated by applying it to the analysis and description of one perspective of early-number counting. A broader discussion of the viability and significance of the alternative theoretical framework, operational model and graphical language is then provided.

Background

Various theoretical bases are promoted for the teaching and learning of mathematics. Objectivist theories are often criticised for not taking into consideration the learner's prior experience (Lesh, 1985). Despite these criticisms, objectivist-based practice remains prevalent in many of today's classrooms (Falk & Millar, 2001) and computer-mediated learning environments (McDonald et al., 2005). Constructivist-based reformists argue for the development of experientially-based richly connected schemas of understanding. However, such practice is often stymied by the difficulties encountered when attempting to translate the constructivist theory into classroom practice (Baroody & Dowker, 2003; Scardamalia & Bereiter, 2006; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 2004). Some of these difficulties have been attributed to the lack of focus upon the highly structured nature of mathematical domain knowledge (Kirschner, Sweller, & Clark, 2006; Mayer, 2004). The often occurring polarisation of these two theoretical viewpoints has been criticised (Goldin, 2003; McDonald et al., 2005) as having a deleterious effect on mathematical education practice. Instead, such critics argue that each viewpoint has its associated strengths, and that these should be drawn upon to develop effective educational practice which recognises both learner idiosyncrasy and the state-able and thus objective nature of mathematical domain knowledge. Through the development of such a consilience (Goldin, 2003) of learning theory, contemporary mathematics education practice may be advanced.

Popper's (1978) three-world model of knowledge and understanding has been adopted as a lens through which to re-consider the objectivist and constructivist theories. Popper's three-world model permits the explicit differentiation of World 3 knowledge shared in a community from the unique, experience-based World 2 understanding of that knowledge held by each community member. In this model, World 2 understanding mediates between the World 3 knowledge of the community and the individual's physical actions of World 1. That is, if the organisation of mathematical ideas that define the shared World 3 knowledge of some mathematical domain can be described, then the individual learner's idiosyncratic and experientially developed and demonstrated World 2 understanding can then be mapped against this description of knowledge. This differentiation and modelling of both World 3 knowledge and World 2 understanding is at the core of the proposed alternative theoretical framework.

Piaget (1977/2001) proposed reflective abstraction as a process of accommodation by adaptation that is sufficiently powerful to describe a learner's entire conceptual development in mathematics. Five specific processes, or transformations, of reflective abstraction are noted in Piaget's work (Dubinsky, 1991): interiorisation, coordination, encapsulation, generalisation and reversal. Interiorisation involves the internalisation and then re-presentation of some phenomena in a de-contextualised, more abstract way. Coordination involves the composition of two or more existing processes to form a more complex process. In a related way, encapsulation involves the bringing together of what were previously independent parts into a manipulable whole. This whole may represent the abstraction of a commonality between a set of concepts or the abstraction of a detailed process into a single object. Generalisation is the broadening of understanding by the application of existing processes and structures to a wider collection of problem phenomena. Finally, reversal involves the consideration of the differences between concepts and the subsequent abstraction of inverse or 'undoing' relationships.

Reflective abstraction is typically associated with the cognitive processes by which individuals construct idiosyncratic understanding. However, when Popper's three-world model is adopted and World 3 knowledge is defined as the expression and thus sharing of idiosyncratic mental thought and cognitive process (i.e., World 2 understanding), then the description of shared World 3 knowledge will reflect these transformations of reflective abstraction. Thus, reflective abstraction forms the basis of the proposed graphical language for describing mathematical World 3 knowledge.

The graphical language

The graphical language for describing World 3 knowledge is a major element of the proposed operational model that embodies the alternative theoretical framework. The graphical language is used to create *genetic decompositions*, a term borrowed from the work of Dubinsky (1991). A genetic decomposition is a network-like structure of nodes and links, and is in keeping with the notion of schema discussed in constructivist literature. Each node in a genetic decomposition is referred to as a *knowledge object*, and these knowledge objects are linked by one or more *knowledge associations*. Each knowledge association in the genetic decomposition describes some reflective abstraction-based relationship between the knowledge objects involved. In the following sub-sections, these constructs of the graphical language are discussed in greater detail.

Knowledge objects

Knowledge objects form the nodes in the network-like genetic decomposition, and of these there are three different types. At the core of learning in mathematics is the solution of problems, and so one type of knowledge object is the *problem object*. To form solutions to such problems, conceptual knowledge (i.e., principles, facts) may be drawn upon as well as procedural knowledge (i.e., skills and processes). Considering Baroody's (2003) suggestion that to foster adaptive expertise conceptual and procedural knowledge should be integrated together, the second type of knowledge object is the *concept object*. Central to mathematical activity is the use of language to express the problems and concepts of the domain. To this end, the third type of knowledge object is the *representation object*, which is used to identify the different signs and symbols of the domain. The three knowledge object types are denoted in the graphical language using three different icons, as shown in Figure 1.

Figure 1. Knowledge objects.

Knowledge associations

To organise the knowledge objects in a genetic decomposition, six different knowledge associations have been derived from Piaget's reflective abstraction: *inheritance*, *aggregation*, *solution*, *inversion*, *formalisation* and *expression*. These associations are discussed in the remainder of this section: the syntax of each in the graphical language is shown in Figure 2, and then each of the associations and their derivation from Piaget's reflective abstraction are summarised.

Inheritance – the *Parent* concept (or problem or representation) is a super-class, of which *Child 1* and *Child 2* are more specific sub-types.

Aggregation – the *Aggregate* concept (or problem or representation) is composed of *Component 1* and *Component 2*.

Solution – the *Problem* can be solved using the coordination of *Concept 1* and *Concept 2*.

Inversion – the *Normal* and *Complement* concepts (or problems) have differences.

Formalisation – the *Formal* representation is a de-contextualised representation compared to the *Informal* representation.

Expression – the *Concept* (or problem) is expressed using the *Representation*.

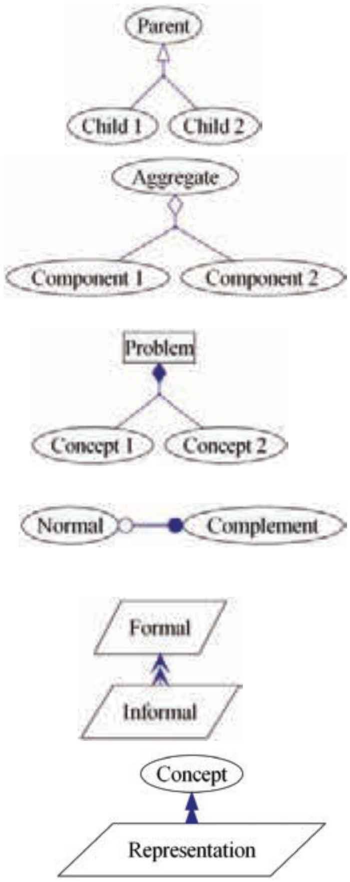

Figure 2. Knowledge associations.

Derived from Piaget's encapsulation, the inheritance association describes either problem, concept or representation objects that share a super-ordinate relationship: The *child* objects are sub-types of the more abstract *parent* object. Inheritance is denoted by an open triangle attached to the parent, from which two or more lines connect the parent to each child.

The aggregation association is derived from Piaget's coordination, and is used to describe the coordination of several *component* parts to form a more complex *aggregate* whole. Aggregation may be applied to either problem, concept or representation objects. The association is denoted by an open diamond attached to the aggregate, from which one or more lines connect the aggregate to each component.

The solution association defines the relationship between a problem and the concepts used to solve it. This association is derived from Piaget's coordination and encapsulation (i.e., the use of two or more concepts in a coordinated manner to solve a problem) as well as generalisation (i.e., since a problem may be solved using several different co-ordinations of concepts, or a set of coordinated concepts may be used to solve a range of different problems). The solution association is denoted using a closed diamond attached to the problem, from which one or more lines connect the problem to each concept.

The inversion association is derived from Piaget's reversal, and is used to describe two knowledge objects (either problems or concepts) that are in some way complementary to each other. The two knowledge objects are referred to as the *normal* and *complement* objects. The association is denoted by a line connecting the two objects that is terminated by open and closed circles.

The formalisation association describes the increasingly abstract signs and symbols used in mathematics. Derived from Piaget's interiorisation, formalisation captures the relative degree of de-contextualisation between two representation objects, which are referred to as the *informal* object and the more de-contextualised *formal* object. The formalisation association is denoted by a line connecting the informal and formal representation objects which is terminated by open arrow-heads.

The expression association describes the various ways by which a problem or concept may be represented, that is, how the signs and symbols of the domain may be used. This association is also derived from Piaget's interiorisation, since when considered in combination with formalisation, the expression association suggests opportunities for interiorisation to occur. The expression association is denoted by a line connecting a representation object to a problem or concept object which is terminated by closed arrowheads.

Application

To demonstrate the viability of the proposed alternative theoretical framework, operational model and graphical language, literature regarding the domain of early-number counting, addition and subtraction was analysed and described (Nutchey, 2011). In this section, a summary of the analysis and description of Gelman and Gallistel's (1978) work on children's counting is provided to demonstrate the use of the proposed graphical language.

Gelman and Gallistel (1978) theorised that a child's ability to count is based on the coordination of five principles: the one-one principle, the stable-order principle, the cardinal principle, the abstraction principle, and the order-irrelevance principle. A genetic decomposition summarising this organisation of counting principles is presented in Figure 3 (next page), which is then explained.

Gelman and Gallistel discussed a stage-like development of a child's counting ability; this has been described by the use of solution associations to describe the use of increasingly complex *pre-counting*, *simple counting* and *counting concepts to solve the problem of single collection counting*.

Pre-counting is described by the aggregation of the *one-one principle* and the *stable-order principle*. The *stable-order principle* aggregates the concept of the *numeron sequence*, itself aggregating the notion of *no tag repetition*. An inheritance association describes two more specific types of *numeron sequence:* the *conventional sequence* and the un-conventional sequence. The difference between these two numeron sequences is highlighted by an inversion association. The *conventional sequence* is described by the aggregation of *conventional numerons*, whereas the un-*conventional sequence* is described by the aggregation of *numerons*, and thus the more specific *conventional numerons* or *un-conventional numerons*. The *numeron sequence* is also a component of *synchronous tagging*, which is in turn aggregated along with the skill of *set partitioning to define the one-one principle*.

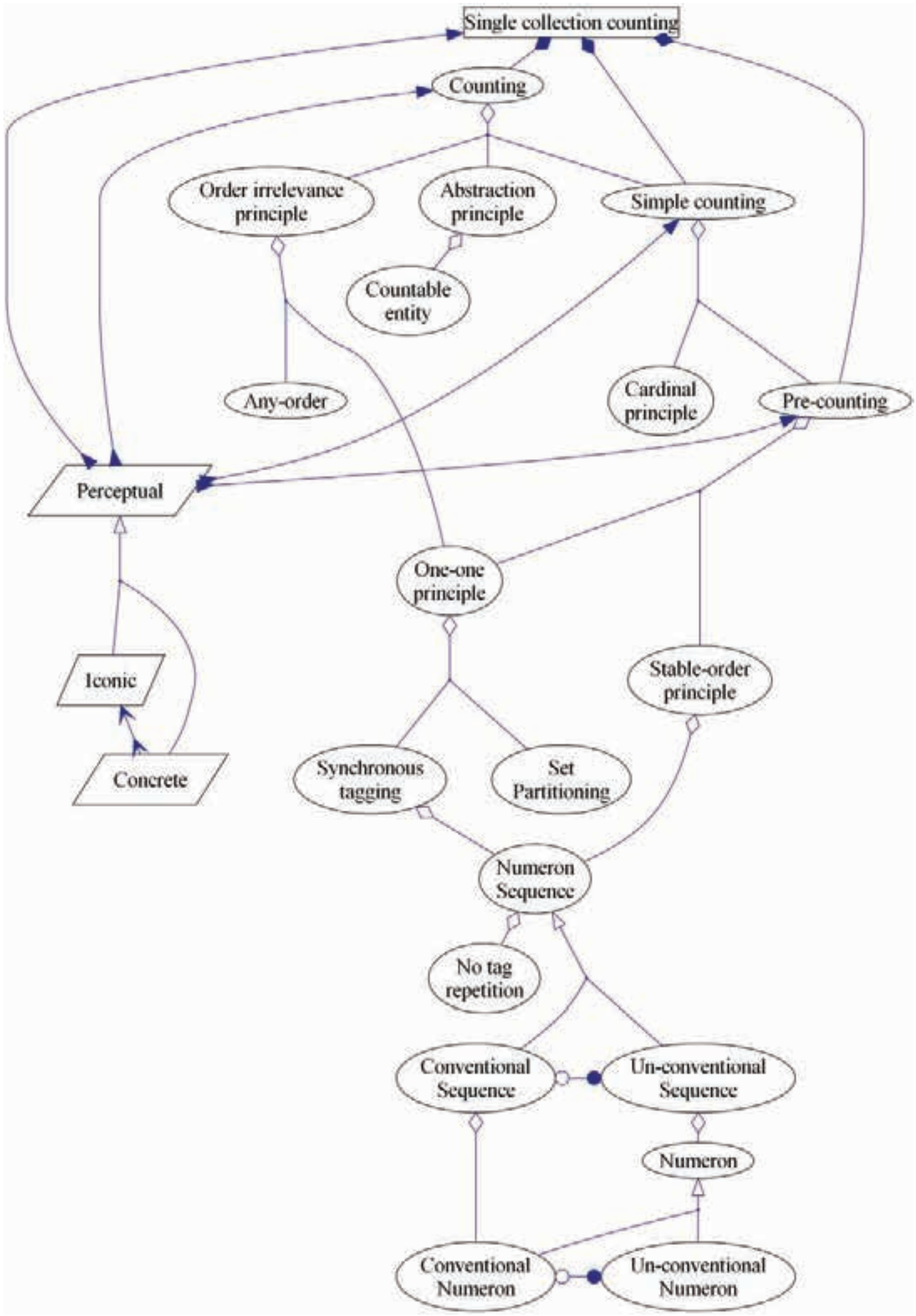

Figure 3. A genetic decomposition of Gelman and Gallistel's counting principles

The more complex concept of *simple counting* is described by the aggregation of *pre-counting* and the *cardinal principle.* The *order irrelevance* principle aggregates the concepts of the *one-one principle* and the notion of *any-order.* The *abstraction principle* aggregates the notion of *countable entity.* Together, the *order irrelevance principle and* *abstraction principle* are aggregated along with *simple counting* to describe the most complex *counting* concept.

Gelman and Gallistel's work focussed on children's counting of perceptual objects, as described by the expression association that indicates the problem of *single collection counting* may be expressed using *perceptual objects.* The perceptual objects have two more specific forms, *concrete* and *iconic*, as described by the inheritance association. Similarly, *perceptual objects* may be used to express *pre-counting, simple counting* and *counting.*

Discussion

Using the proposed alternative theoretical framework, operational model and graphical language, other early-number literature has also been analysed and described (Nutchey, 2011), including early-number word problem classification (Carpenter & Moser, 1983; Fuson, 1992), the development of number-word and number-sequence meaning (Fuson, 1992; Olive, 2001; Steffe & Cobb, 1988) and the strategies used to solve early-number word problems (Carpenter & Moser, 1983; Fuson, 1992). The resultant genetic decompositions were then synthesised together to form a composite description of early-number; a process that revealed similarities, differences and sometimes discrepancies in the literature. The resulting complex genetic decomposition, presented in Nutchey (2011), includes 56 problem objects, 49 concept objects, three representation objects, and over 200 associations to organise these objects. This activity of analysis and description has demonstrated the viability of the proposed graphical language as a tool with which to characterise World 3 mathematical knowledge. In the future, further analysis and description activity should extend the composite description to include the various representations commonly used in early-number, in particular those that scaffold the development of counting, addition and subtraction strategies.

The composite description of early-number may provide a basis for the analysis and description of an individual learner's World 2 understanding. A mechanism has been proposed (Nutchey, 2011) which suggests that World 2 understanding can be described (and thus analysed) in terms of a chronological sequence of *images* – collections of problem, concept and representation objects that each describe an activity in the learner's conceptual development. Guided by the notion of reflective abstraction, the analysis of an image sequence may lead to the assessment of a learner's understanding and the identification of future activities that may enhance their understanding.

The alternative theoretical framework, operational model and graphical language may potentially advance mathematics teaching and learning practice in several ways. The modelling technique may form the basis of computer-mediated learning environments that are responsive to learner's mathematical conceptual development. The graphical language, when used to express the highly connected nature of mathematics, may support a teacher's development of learning activities that scaffold students' constructive exploration of this organisation of mathematical ideas. This potential for theory to impact practice will be the topic of future research and development activity.

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WHAT ASPECTS OF QUALITY DO STUDENTS FOCUS ON WHEN EVALUATING ORAL AND WRITTEN MATHEMATICAL PRESENTATIONS?

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University students' evaluations of mathematical presentations are examined in this paper, which reports on part of a pilot study about different types of presentations, regarding different topics, formats (oral or written), and discourses (process- or object-oriented). In this paper focus is on different formats; oral lectures and written texts. Students' written comments about what is good or bad about given presentations are analysed in order to examine what students focus on when evaluating the quality of presentations. In addition, evaluations given about written and oral presentations are compared in order to examine if/how format affects students' evaluations regarding quality.

Introduction

The topic in the present paper relates to questions asked by other researchers regarding student perspectives on lectures in mathematics courses at university level. In particular, results have revealed why students attend lectures, for example for enjoyment, finding out what is central and important in the course/unit, and getting notes for later study (Hubbard, 2007) or due to interest, enjoyment, and good lecturers (Hunter & Tetley, 1999). What students see as a good lecture has also been examined, which for example includes aspects of the lecturer, such as pedagogical awareness (clarity, good language, structure, suitable pace etc.) and charisma (inspiration, engagement, humour etc.), and the possibility to get good lecture notes (Bergsten, 2011). These prior studies do not focus on students' comments about specific lectures but on their more general conceptions about attending lectures and the quality of lectures. The data described in the present paper consist of students' evaluations of specific presentations given to them. The results from analysing these data are in this paper also compared to results from previous studies mentioned above.

In addition, focus of the present paper is not only on oral lectures but on mathematical presentations more generally, by including a comparison between oral and written presentations. The interest of comparing these different types of presentations is related to the question why students attend lectures, in particular in relation to reading a text that presents the same topic. In my previous studies about students' reading of mathematical texts, a common comment from students was that it is essentially impossible to learn something from reading a text (Österholm, 2006), that is, they were asking for something more or different. At the same time, a common purpose of attending lectures seems to be to get the notes from the lecture (Hubbard, 2007; Hunter & Tetley, 1999), that is, students want a written text. This conclusion is in line with the results from another study where students preferred written lecture summaries when they had access to both audio recordings of lectures and text-based lecture summaries (Grabe & Christopherson, 2008). Since students seem to prefer the written text produced from a lecture compared to a written text from a textbook, there seems to be something about an oral presentation that is favoured among students, besides such things that can only be given orally, such as the opportunity to ask questions and have a dialogue. In general, there can be differences between oral and written presentations regarding aspects of grammar, for example that "the textbook uses more of the standard conventions of mathematical writing" (Wood & Smith, 2004, p. 11). This type of difference could perhaps make students prefer notes from an oral presentation over a "purely" written presentation. Students participating in the study by Bergsten (2011) say they prefer intuitive descriptions over formal definitions and proofs in a lecture, which could be seen as a confirmation that students want something different than what is usually given in textbooks, in particular regarding aspects of grammar and the type of language used. However, could there also be something else that is relevant from a student perspective when comparing different formats of presentations? This question is examined in the present paper, by reducing the affect of type of language usually used for different formats, and focusing on the effect of format per se.

Purpose

As outlined in the introduction, the study described in this paper focuses on students' perspectives on the quality of mathematical presentations of different formats, regarding oral and written presentations. The main interest in the present paper is the following two questions:

- What do students focus on when giving qualitative evaluations of presentations?
- What differences exist between different formats regarding students' qualitative evaluations?

In this paper, focus is on students' evaluations of specific presentations that they are exposed to, where the aim of the first question is to create a structured description of different quality aspects mentioned by students. The created description is also compared with results from other similar studies that have examined students' more general conceptions of lectures, in order to investigate if such different types of studies yield different types of results.

The main purpose of comparing different formats of presentations is to examine a possible more "pure" effect of format than described in other studies. In particular, the aim is to avoid comparing different types of descriptions, such as intuitive or formal, and instead primarily examine an effect of format per se.

Method

A total of 22 Swedish university students voluntarily participated in this study, all enrolled in a calculus course. The procedure consisted of mostly individual tasks, but some parts they did together with another student, according to the following six steps:

1. Listen to a pre-recorded oral presentation (16–17 minutes long) or read a written presentation (approximately four pages long).

2. Give written comments about the presentation, including multiple-choice and open-ended questions about evaluating the presentation.
3. One student gives an oral description of the content of the presentation to another student (who had not the same topic in her/his presentation).

4-6. Repeating steps 1–3 with new presentations. For each student, the new presentation covered the same topic as the first presentation but had different format and/or different discourse.

This data collecting procedure was used in order to fulfil several different purposes, for example to examine students' discourse when describing the presentations and to examine students' preferences for different types of presentations through quantitative analysis of answers to multiple-choice questions. However, for the purpose of the present paper, focus is on students' answers to six open-ended questions given as evaluations of the presentations. For each presentation, students answered two questions as parts of both step 2 and step 5:

- What made you think this presentation was good or bad?
- What made you think this presentation was clear or unclear?

A reason for asking specifically about the aspect of clarity is to try to trigger some more specific comments than only saying that the presentation was clear or unclear, which otherwise was believed to become a common answer. After both presentations, as the last part of step 5, the students also answered two questions about direct comparisons between the presentations:

- What made you think one presentation was better than the other? OR What made you think that they were equally good (or equally bad)?
- What did you think was the most important difference between the two presentations and why?

In this study, mathematical presentations of different kinds are used, with variations regarding topic (sequences or Maclaurin polynomials), discourse (process- or object-oriented), and format (oral or written). The students were randomly assigned to receive certain types of presentations with respect to these variations.

The content of the presentations were scheduled to be covered within a few weeks in the students' calculus course, therefore judged to be suitable for use in this study, as presenting something new but relevant for the students.

For the analysis in this paper, focus is not on different discourses (different ways of describing), but nonetheless I have included variation of discourse in the data collection. This variation is included partly because it is focused on in other analyses not discussed in the current paper, but for the purpose of the paper this variation is included primarily because there is empirical evidence that different discourses are more or less commonly used for different formats. In particular, aspects of nominalisations are more common in written than in oral format (Einarsson, 1978). Thus, if I would choose a certain type of discourse with respect to aspects of nominalisations, this could be seen as more suitable for a certain format and resulting in analysing an effect of discourse and not format. Therefore, I choose to use a variation of discourse for the presentations used in the data collection, so that both oral and written presentations are given in different versions regarding discourse.

The oral and written presentations are similar word by word (when the same discourse is used), in order to pinpoint a potential specific effect of format. The wordings of all presentations were created by first recording and transcribing a spontaneous oral presentation of each topic. Two written versions of each topic were then created; one process-oriented version and one object-oriented version. Finally, oral presentations were recorded using these written versions as manuscripts.

The differences between the types of discourses are here shortly characterised through an imaginary change from process- to object-oriented discourse. In this change things are described as objects and properties of such objects instead of processes, activities or events of some kind, in particular through the following types of changes into more object-oriented discourse:

- *Nominalisations*: Using nouns or adjectives instead of verbs, e.g., “the usage of symbols is the same” instead of “we use the same symbols”.
- *Structuralisations*: Using structural types of verbs instead of process types of verbs, e.g., “it is the same” instead of “we do the same”.
- *Change of voice*: Using passive verbs instead of active verbs, e.g., “this can be calculated” instead of “we can calculate this”.

Data analysis

There are two main parts in the data analysis: A bottom-up type of analysis regarding what types of aspects are commented on in students’ evaluations, followed by a comparative analysis of the types of comments given about different formats.

In the first main part of analysis, all answers from all six open-ended questions are used. All comments are divided into smaller units of analysis so that only one aspect is included in each unit. For example, the comment “I understand more if I can see things and not only hear things” is divided into three parts; “I understand”, “see things” and “hear things”. Thereafter, a recursive procedure is performed: A structure of a few categories is created that seems to describe some central aspects in the data. All data are then categorised using the created categories, during which the (description of) categories also can be altered to better fit the data. Within each category, this process is repeated to create a more fine-grained structure of categories.

In the second main part of analysis, the categories created in the first part are used as a tool for comparing different formats. The comparison is done firstly by a quantitative analysis, using a chi-square test with significance level 0.05, of the proportions of students who commented on each aspect. In the statistical analysis, three partly overlapping student groups are compared; students who commented on oral presentations, students who commented on written presentations and students who compared presentations with different formats. Those aspects showing significant quantitative differences are then compared in a qualitative manner regarding what is commented on with respect to the aspect in question. Thus, the results consist of what aspects are primarily focused on for different formats and when directly comparing formats, and also what is commented on regarding these aspects. In this part of the analysis, data from all questions and all students are not used. When a specific format is analysed, answers to questions about the second presentation are not used when this presentation had a different format than the first presentation. This limitation is used in order to analyse answers that focus on one format and avoid getting comments about comparisons between formats triggered by the exposure of different formats. When explicit comparisons between formats are analysed, answers to questions about explicit comparisons are used only from students that have been exposed to different formats.

Results

The structure in this section corresponds to the two main research questions: First, the different aspects focused on by students are described through a created structure of categories, which is also compared to similar structures from other studies. Second, the comparisons between formats are discussed.

Aspects of quality

In the process of analysis, three main categories were first noted, which distinguish between some aspects of ontology; whether comments focus on the student, the lecturer or the presentation. Within these categories, sub-categories were created as described in Table 1.

Table 1. Structure of categories describing aspects of quality mentioned by students.

Ontology	Category	Description/exemplification
Student input	Prior knowledge	Relating to the content of the presentation, e.g., that the student did not know what a certain word meant or had studied similar things before.
	Preference	Expressed opinion that one thing is better than another, either in general or for the student personally, e.g., that it is more difficult or easier to listen to something than to read about it.
	State of mind	For example tired.
Student activity	Reading	For example the possibility to read several times.
	Listening	For example not have to write and listen at the same time.
	Writing	For example the difficulty to take notes at high speeds.
Student output	Cognition	Some aspect of understanding or learning, e.g., that something was difficult to follow or that the student could grasp what was presented.
Lecturer	White board	If/how the white board is used, e.g., that more should be written on the white board.
	Talking	For example regarding the tone of voice.
	Pace	For example that everything was too quick or that the lecturer behaved calmly.
Presentation	Thoroughness	That some certain things do exist or do not exist, e.g., the use of examples or that some part could have been explained more.
	Structure	Relations between different parts of the presentation, e.g., good mixture between descriptions and examples.
	Language	Choice of wording, e.g., that a certain phrase sounded strange or a certain word was used too much.
	Descriptions	Type of residual category for presentation, regarding good/bad aspects of the way of describing/explaining something, but characterised only by singular words, e.g., methodical, (un)clear, fuzzy or messy.

Complete statements from students are often not located in only one category, as some examples above also show, but it is common that students connect different aspects, also in a logical manner (i.e. that some aspect is a cause of another). However, due to space restrictions, it is not possible to analyse these types of comments in the present paper.

Comparisons with structures from other studies

An affective dimension (regarding enjoyment and interest) is included in the study of Hunter and Tetley (1999) as a high-ranked reason for students to attend lectures. An affective aspects is also included in the studies by Hubbard (2007), but this time not ranked very high as a reason to attend lectures. Students in Bergsten's (2011) study also mention affective aspects, regarding inspiration and humour, as important parts in a good lecture.

In my own data, no clear affective aspects exist in the students' evaluations, but could fit as a category under student output (e.g., enjoyment) or as a category under student input (e.g., interest). The lectures used in this study did not use humour in any way and was probably not very inspiring, in particular since they were recorded using a manuscript. Perhaps the students did not want to be negative in their comments about the lecturer or perhaps this result shows a difference between describing a good lecture more generally and evaluating a specific lecture. In particular, it could be that affective aspects are not always important, but that this dimension adds something when it do exist; that it is not directly negative if it does not exist, but it is positive when it does exist.

To get the notes from a lecture is stated as important by students in several studies (Bergsten, 2011; Hubbard, 2007; Hunter & Tetley, 1999). Comments about notes are also part of my data, where some students mention the importance of having the time to take notes during the lecture, and that the pace needs to be adjusted for this. Some students also mention that the lecturer should write much/more on the white board, and not just talk, in order to be able to take notes on what is being said.

Other common types of answers from students in Bergsten's (2011) study are also present in my data, primarily regarding what I have labelled as the main category of presentation and what Bergsten describes as aspects of pedagogical awareness of the lecturer. For example, students in Bergsten's study mention good explanations to support understanding; aspects of clarity, including not skipping details and structure; pace; coherence; and a good mixture between theory and method.

Comparisons between formats

The data in Table 2 describe how frequent different types of comments are in three different situations: When students evaluate only one format, either written or oral, and when students compare presentations of different formats. Statistical analyses of these data using a chi-square test with significance level 0.05 show that:

- Aspects more frequent for written format than for oral format are structure and descriptions.
- Aspects more frequent for oral format than for written format are all specific to oral format; white board, talking and pace.
- Aspects more frequent for comparisons than for the other two situations are preference, reading and listening.

Regarding structure and written format, students comment on the connection between examples/calculations and parts describing/explaining something, and they also note that different parts are building on each other. For the oral format, the describing parts of the presentation were given only orally and calculations were written on the white board (and read out loud), which could be a reason for this aspect not being commented on in the same way for the different formats. It could be that for oral format relationships between oral and written parts of the presentation are more in focus than the content of the presentation, which is more in focus for the written format.

Table 2. Proportion of students (in percent) that have commented on a certain aspect.

Situation	Prior knowledge	Preference	State of mind	Reading	Listening	Writing	Cognition	White board	Talking	Pace	Thoroughness	Structure	Language	Descriptions
Written (N=11)	27	0	0	9	0	0	64	0	0	0	82	36	27	91
Oral (N=11)	18	9	18	18	18	27	91	91	36	64	82	0	0	55
Compare (N=18)	17	50	6	78	56	28	89	11	0	39	17	0	0	39

Regarding descriptions, the comments given for the different formats are very similar, where something usually is described as being (un)clear, messy or well/badly described/explained. Thus, this difference is not of a qualitative kind but can be interpreted as a shift in focus, away from the content of the descriptions in the presentation for oral format compared to written format. The same tendency exists for the category of language, although the quantitative difference is not statistically significant.

Regarding white board, talking and pace, these aspects can be connected to the need of taking notes. Several students make the explicit connection to taking notes, while others only state that more should be written on the white board and the pace should be reduced (or is good), which can be assumed to relate to taking notes. However, such comments can also relate to the wish to have the time to understand, which some students also mention, but such comments are not as common as those relating to taking notes.

Regarding preference, some students express only an opinion (e.g., that it is better with an oral format) while others connect this opinion logically to another aspect (e.g., that some format is better because learning is better with this format or that written presentations are better since you then can read several times). In addition, some students describe the preference as highly personal and that different persons can have different preferences regarding which format is best for them.

Regarding reading and listening, for oral format some students express a wish that more should be written on the white board so that you can both read and listen. For written format, the possibility to read the same thing several times is mentioned. For comparisons, the same things are mentioned as for the different formats, but then usually expressed as a preference.

Conclusions

When comparing students' comments about specific presentations in my study with results from other studies, where focus has been on students' general conceptions about lectures, a great similarity can be noted regarding aspects of quality included in the data. However, one aspect that could need further attention is an affective dimension, including what Bergsten (2011) describes as teacher immediacy, which is commented on by students in several other studies but not at all in my study. Is this because students do not focus on this dimension when evaluating specific lectures or is this aspect only noted when a lecture do include some affective aspect? Further studies are needed to decide on this issue, regarding how important affective aspects really are for students.

When I used presentations of different formats but with the same type of descriptions and type of language/discourse, the results show that depending on which format is evaluated, there is a shift in focus among the students: There is a tendency for students to reduce the attention on aspects of content and instead direct the attention on the lecturer when evaluating oral presentations compared to written presentations.

This shift in focus seems at least partially to be about an effect of interactions between oral and written parts of oral presentations. Several students comment on such an interaction, often in connection with taking notes, for example that the lecturer should write more on the white board and not only talk, in order to have time to take notes. Pace is also often mentioned by students in relation to this; that it should be suitable in order to have time to take notes. This focus on taking notes in lectures is similar to what has been observed in several other studies (Bergsten, 2011; Hubbard, 2007; Hunter & Tetley, 1999). Other researchers have characterized this focus on note-taking as a pedagogical problem, and have developed alternate ways of structuring lectures in order for students not only to focus on taking notes or to be totally passive if complete notes are given in advance. Tonkes, Isaac and Scharaschkin (2009, p. 496) utilise what they call partially populated lecture notes, which is "pre-printed paper notes with carefully selected sections left empty for students to write during lectures". In the authors' evaluations, students have been positive about this alternate method.

When students in my study directly compared different formats, they focused more on themselves compared to when they commented on only one format. This focus is expressed through preferences regarding reading and listening, where students state that they prefer one format over the other, usually either expressed without any further explanation or elaboration, or that one format is better for comprehension or learning. Any other aspect or property of the presentations is seldom referred to, making this aspect primarily about a personal preference regarding format per se. This result could possibly be related to aspects of learning style, which can include many dimensions but where auditory and visual are sometimes included as different styles (e.g., see Honigsfeld & Dunn, 2003), which perhaps could be relevant to the preference for oral or written presentations observed in the present study.

In summary, the differences that have been noted when students evaluate different formats of presentations are that more focus is on the content of the presentation for written formats, more focus is on the lecturer for oral formats, and more focus is on the students' preferences and activities when directly comparing different formats.

The results in the present paper are primarily descriptive regarding observed differences, making it interesting as a next step to try to further explain the observed differences. One aspect of explanation could be regarding what is primary in the process of making evaluations about mathematical presentations among students, which can relate to the ontological aspects in the main categories in the present paper, for example:

- That different specific criteria, regarding properties of presentations, perhaps are used for different formats, i.e. that there is a direct focus on different aspects.
- That the feeling of understanding or not perhaps is primary, which can be different for different formats (perhaps due to a specific learning style), and based on this feeling, other properties of a presentation could possibly be related to.
- That a general conception about one's own preference for a certain format perhaps is primary, which is independent of the specific situation, thus always causing a certain format being evaluated as better than the other, and based on this general evaluation. Again, other properties of a presentation could possibly be relevant.

These issues can be related to more general aspects and theories of metacognition, which can include both a more static component of metacognitive knowledge and a more dynamic component of metacognitive control processes (Schraw & Moshman, 1995). Different types of criteria and conceptions about one's own preference can then be described as parts of metacognitive knowledge while the feeling of (i.e. evaluation of one's own) comprehension is a part of metacognitive control processes. A question for future studies can then be if/how students primarily rely on different parts of metacognition, in general or specifically when evaluating mathematical presentations.

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PROMOTING POWERFUL POSITIVE AFFECT: USING STAGES OF CONCERN AND ACTIVITY THEORY TO UNDERSTAND TEACHERS' PRACTICE IN MATHEMATICS

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This paper describes how one teacher‘s attempts to promote powerful positive affect in her mathematics classroom gave rise to concerns and tensions related to her practice. The paper shows how using a combination of Activity Theory and the Stages of Concern provides a helpful lens for researchers to understand the challenges of change and professional development. It is argued that the identification and resolution of these tensions is crucial to understand and facilitate the efforts of sustainable pedagogical change.

Introduction

Researchers such as Goldin (2000) and Epstein et al. (2007) have identified a need for teaching strategies that incorporate the affective domain into the mathematics classroom to promote powerful positive affect (PPA), or ―patterns of affect and behaviour that foster children‘s intimate engagement, interest, concentration, persistence and mathematical success" (Alston, Goldin, Jones, McCulloch, Rossman, & Schmeelk, 2007, p. 327). The use of PPA is proposed in response to suggestions that essential affective elements are often considered to be an incidental ‗add-on‘ to mathematics learning (Goldin, 2007).

Affective elements are commonly defined as encompassing feelings, emotions, attitudes, beliefs, and values attached to a subject or object (Leder & Forgasz, 2006). Challenges arise as teachers attempt to implement new teaching strategies and tools that reveal the affective dimensions of students‘ thinking, perhaps because teachers perceive these to be additions to their existing practice rather than integral aspects of student learning. Currently, tools include student surveys (Fennema & Sherman, 1976) and journaling activities (Jurdak & Zein, 1998; Scott, 2007), but there is a need for professional development (PD) and strategies that involve teachers‘ own perspectives and experimentation with tools (Flack & Osler, 1999; Smiles & Short, 2006).

This paper argues that developing teaching strategies that promote PPA requires a research approach that is sensitive to the affective dimensions of teachers‘ perspectives, professional development, and learning. According to Hall and Hord (2006), teachers approach PD and classroom change with many thoughts, feelings, and concerns due to the affective dimensions of change. Change implementers do not simply ‗do‘ the change but are constantly thinking about how the process is unfolding. This research explored whether it is helpful to refer to stages along a developmental continuum and sought to identify the characteristics of each stage as teachers attempted to modify their practice. The identification of teacher‘s individual concerns and the resolution of tensions related to the promotion of PPA in their mathematics classroom were fundamental.

This study is part of a doctoral research project that has adopted a critical ethnographic case study approach to examine how teachers promote PPA over a six-month period. While descriptive, ethnography essentially means ―learning from people" (Spradley, 1980, p. 3), and was chosen for two reasons: (a) teachers are the key to teaching and learning processes in mathematics; and (b) their perspectives and knowledge are often neglected in education research. In this study, the participants were empowered through valuing their voices in the analysis and presentation of the results (Kincheloe & McLaren, 2005). The researcher explored and acknowledged the ―self-other interaction" (Foley & Valenzuela, 2005, p. 218) and was self-aware of her role. Her positioning in the research was upfront and acknowledged (Kincheloe & McLaren, 2005).

This paper highlights the tensions and concerns of one participant as she attempted to promote PPA. The participant worked collaboratively with the researcher over a six month period to contribute to the theorising about her work by focusing on the interaction between her thoughts, affect, and actions and the factors that facilitate and constrain pedagogical change (Mahn & John-Steiner, 1998).

The theoretical framework: Activity Theory and Stages of Concern

This paper specifically reports on the use of Activity Theory (AT) and the Stages of Concern (SoC) to investigate one case study. The essence of these theories is presented below.

Activity Theory. Activity Theory provides a versatile tool to inquire into aspects of mathematics education, and its value is well documented (Daniels, 2001; Fai-Ho, 2006; Hardman, 2006). The main unit of analysis in AT is the activity system (Engeström, 1999). A model of the Third Generation Activity System, which is intended to develop conceptual tools to understand dialogues, multiple perspectives, and networks of interacting activity systems, is represented in Figure 1.

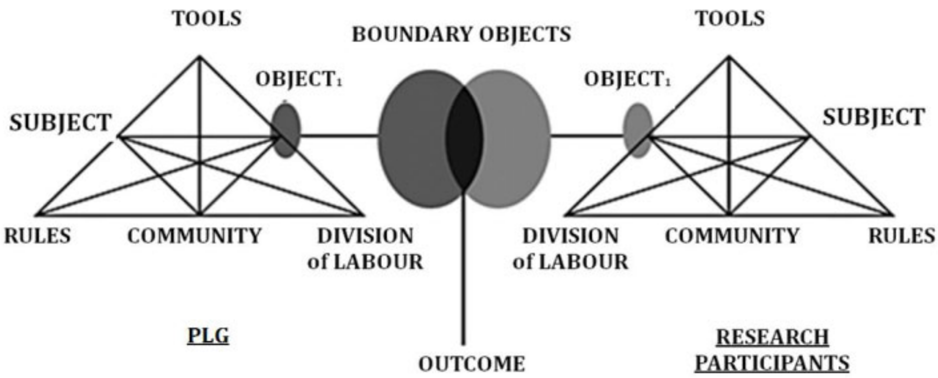

Figure 1. Third generation Activity Theory (Engeström, 1999).

The Subject node refers to the individual or group whose point of view is taken in the analysis of the activity. The identity and activity of the Subject is directed towards the

Object node or goal and is transformed into Outcomes with the help of physical and symbolic external and internal Tools that mediate the Object into an Outcome (Engeström, 1993). Thus, the Object embodies the meaning, motive, and purpose of the system. The base of the triangle represents the contextual characteristics of the activity system. The Community node refers to the participants who share the same general Object with the Subject. The Division of Labour node refers to how tasks are divided between community members. Rules are explicit or implicit regulations, norms and conventions that constrain actions and interactions within the activity system (Centre for Activity Theory and Developmental Work Research, 2003). Boundary Objects operate at the interface of different activity systems. For example, if teachers engage in discussion, debate and reflection whilst participating in professional learning groups, then their learning may be expanded beyond what is possible within their own classroom activity systems (Russell, 2002).

Stages of Concern. The Stages of Concern is a helpful construct to monitor, describe, and quantify the emotional part of change that is often neglected, with resulting arousal of unnecessary resistance to an innovation (Hall, 2010). The SoC describe a predictable pattern of developmental stages that teachers move through as they become increasingly sophisticated and skilled in using new innovations. The seven stages are: (0) Awareness, (1) Informational, (2) Personal, (3) Management, (4) Consequence, (5) Collaboration, and (6) Refocusing (Hall & Hord, 2006). The first stage typifies little concern or involvement in an innovation. The second and third stages involve ―self” concerns that focus on teachers‘ personal feelings of uncertainty and a need to find out more about the innovation such as its general characteristics, effects, and demands. The third stage is ―task” oriented, where attention is focused on the processes and tasks of using the innovation and issues related to efficiency, organisation, management, and time. The last three stages are ―impact” related concerns that deal with teachers‘ external concerns about how the innovation may affect students, colleagues, and future work. At the final stage, individuals have definite ideas about major changes or powerful alternatives to the existing form of the innovation (Hall, 2010; Hall & Hord, 2006). It is noteworthy that progress through the stages is not guaranteed and is not necessarily in one direction.

Methodology

Data were collected using two individual interviews at the commencement and one at the conclusion of the study, three to five classroom observations interspersed throughout the six months, reflective journals, and eight group interviews as part of a Professional Learning Group (PLG)—a process of collective and collaborative learning with people ―who share a concern or passion for something they do and learn how to do it better as they interact regularly” (Wenger, 2004). The PLG was a safe and supportive environment, meeting approximately every three weeks. It provided participants with opportunities to engage with current literature and to develop and reflect on tools to promote PPA.

Data analysis was a cyclic process involving both AT (Engeström, 1987) and SoC (Hall & Hord, 2006). AT focused on the identification of attempts to change behaviour in the activity system, whilst the SoC focused on identifying the affective aspects of change. All data were transcribed and coded into themes based on the nodes from AT using NVIVO7, a computer software program. The data were first analysed using AT and then again using the SoC. There was movement back and forth between the two theoretical lenses.

The use of AT and SoC complimented each other in two main ways. Firstly, the SoC was useful for refining the analysis of tensions identified using AT in the first round of analysis, by providing further detail about the affective aspect of change related to how participants felt about the implementation of a particular PPA innovation. Secondly, data analysis using the SoC provided more description of the tensions and in many cases highlighted tensions, which had not been revealed by using AT alone. This prompted further analysis using AT to analyse the transcripts and individual nodes related to the new tension to confirm if this tension was associated with attempts to change behaviour and teaching practice.

The SoC lens is used first in the following discussion of the case, to understand and describe the participants‘ concerns and affective aspects of change. Then AT is used to refine and theorise attempts to change behaviour and teaching practice in the activity system to promote PPA.

Theoretical analysis and discussion

―Leonie” has been teaching for 17 years in various school settings, and at the time of the study was a contract teacher at ―Hillsview Primary School” located in Adelaide, South Australia. Leonie taught a Year 3 class in tandem with her colleague, Violet. In her first interview, Leonie portrayed strong images of not being ―good at maths” and described that her learning of mathematics ―didn‘t come easy”. These negative perceptions and experiences resulted in a lack of confidence, which appeared to contribute to her many concerns as well as tensions within her mathematics teaching practice. Leonie identified the use of concrete materials, reflection and mathematics story books as useful innovative tools for promoting PPA in her mathematics classroom. Leonie appeared enthusiastic about experimenting with different tools to promote PPA. However, she also raised concerns about making changes to her practice: lack of time to accommodate new tools into the school day and a need to maintain classroom order.

Leonie’s First Dominant Concern: Lack of time for implementing new tools being shared at PD sessions. Leonie explained this concern at the beginning of the study during her second interview:

> How do we bring it [new tool] in? We‘ve got to let something go and it‘s making those choices about what we let go in order to fit this into our day because everything we‘re doing is really important … but this new bit of information is important, really important as well, so how do you make those choices within the time constraints?

This suggested that Leonie is struggling to prioritise the teaching strategies and tools she needs to use, as ‗everything is important‘. At this stage, Leonie is attempting to promote PPA by using new tools in addition to her existing practice. This is causing a tension related to a perceived lack of time to ‗fit‘ everything in. This aligns with Dennis and O‘Hair (2010) who suggest perceived time constraints are a significant concern that influences the use of teaching strategies. However, the results of the study presented in this paper suggest that a PLG can support individual teachers in overcoming concerns about time because the PLG responded to Leonie‘s concerns about time constraints. The

PLG discussions revealed this concern was not just individual, but collective. Connections across the curriculum were suggested by participants as a potential solution. In particular, the use of mathematics story books and journaling were considered to have strong connections to literacy activities, thus relieving timetabling tensions with a lack of time to fit ‗everything' into the school day. The PLG supported Leonie in meeting her individual PD needs according to her concerns.

Leonie's Second Dominant Concern - Maintaining a high level of order within the classroom. During a PLG meeting there was a discussion about the use of dialogue to enhance student learning and reflection in mathematics and the problems Leonie had encountered when implementing this tool. Leonie stated:

> Talking is crucial, but how often do we let them talk about maths and their learning? We often say as teachers that the classroom is too noisy. People are coming into the school and we want to keep the students settled – we don't want them talking. We have that constant struggle with our identity as teachers, you know, we want them to talk but we want them to be quiet, we want them to be hands-on and explore things but we want them to stay in their desks and not to touch or annoy other people. There's always that contradiction in teaching … it's not always easy as a teacher … especially when we have people walking into the room – it looks like total chaos (PLG2).

Leonie seemed concerned about how this tool would affect her personally and her reputation as a teacher. Her use of the pronoun ―we" suggests that she believed that others shared her concern. She did not want her classroom to look chaotic even if it was for the benefit of her students. In fact, Leonie's concerns about a perceived need to maintain order remained evident at the conclusion of the study but it appeared that she felt that a lack of order was acceptable at times. Leonie explained in her final interview:

> There's a fair bit of pressure from parents and staff … especially in maths – that it is done in a particular way. … Now having been through this process, it's ok to have lots of noise. … You feel that pressure to have kids quiet because you might walk along the corridors and all the other classes are quiet. I think: Why is my class so noisy? There're other classes that are probably noisy but I just don't walk past them when they are.

As discussed in the PLG, the importance of social interactions and talking are emphasised in policy documents and mathematics education research: for example, the *Learner Wellbeing Framework for Birth to Year 12* (Department of Education and Children's Services, 2007) and the *Aspects of Working as a Mathematician* model (Grootenboer & Jorgensen, 2009). By the end of the study, Leonie was aware of the value of social interactions and dialogue and this influenced her valuing of these aspects in her classroom, even if this meant that her classroom appeared chaotic at times. The quotation above suggests that Leonie was approaching a resolution to the tension related to the need to maintain order in her classroom by coming to terms with her personal concerns about the expectations of others.

Applying the lens of the SoC (Hall & Hord, 2006) to the case study of Leonie, it appeared that Leonie's concerns were characteristic of Stage 3 Management and Stage 2 Personal concerns. Specifically, Leonie's first dominant concern related to a perceived lack of time to accommodate new tools or strategies to promote PPA into the school day was characteristic of Sage 3 Management concerns from the ―task" area, as Leonie focused on implementing a range of tools, teaching strategies and issues related to efficiency, organisation, management, and time to ‗fit' them into the school day.

Leonie‘s second dominant concern suggests a Stage 2 Personal concern that involves personal feelings of uncertainty and a need to find out more about the innovation: the value of social interaction and talking, as identified in policy documents and research. Later, there was evidence that this concern transformed into Stage 3 Management concerns where she was focused on managing the learning environment and accepted a noisy and chaotic classroom at times. Throughout the study, Leonie expressed inconsistent glimpses of Stage 4 Consequence, focused on the effects of new tools or teaching strategies on her students. These Stage 4 Consequence concerns were evident in some PLGs as well as a journal entry where she wrote:

> The impact the PLG meetings are having on my mathematics teaching is significant; especially in the way I view the children in the mathematical process. I reflect on, and try to pick up on, how they are feeling. It has made me realise how important the reflection process is for all students and myself and not only in mathematics but all [learning] areas (Journal 5).

However, higher-level concerns appeared inconsistent and not representative of the majority of concerns evident in her interviews, PLG sessions or classroom observations.

In summary, overlaying the theoretical lens of AT to the case study of Leonie and her two dominant concerns, revealed how Leonie‘s engagement in discussion, debate, and reflection whilst participating in the PLG supported her to expand her teaching practice in her mathematics classroom. In particular, rather than simply identifying the stage of concern, the use of AT enabled the researcher to analyse and theorise the related tensions that may extend beyond the direct control of teachers in their classroom. From this position, the focus could turn to investigating with participants the possible ways to resolve these tensions and facilitate further insights and improvements to teaching practice. The addition of AT with the SoC was very powerful and valuable during the data analysis and discussion as it revealed new insights into promoting PPA and challenges that would not have been possible without the use of the two lenses in combination. The SoC provided ways to identify concerns that were associated with tensions in teachers‘ practice and the use of AT enabled the exploration of the nature of these tensions. AT enabled the researcher to customise the PLG meetings to meet the individual needs of the group as a collective.

Tensions in Leonie’s Activity System

There are three tensions evident in Leonie‘s Activity System related to the two dominant concerns identified above.

The first dominant concern has two related tensions in Leonie‘s Activity System. The first tension in Leonie‘s Activity System is between Leonie (the Subject of her activity system), the Tools and the Object of promoting PPA (Figure 2, Tension 1). This tension originates from the agreement of the PLG about the shared Boundary Object. As a consequence, Leonie struggled to use different tools in ways that could achieve the Object but _fit‘ within the time constraints of the school day. The implications of this tension suggest that Leonie would benefit from support to identify which aspects of the school day and activities need to be her priorities and which aspects she could combine or remove. Alternatively, there may be aspects in her practice that could be better organised and managed to make it more time efficient. For example, time taken for journaling and the use of story books to promote PPA in mathematics could also be considered literacy activities. This is not an aspect that is currently addressed in the existing range of tools recommended to teachers, however it is a topic well suited to PD and PLG discussions located at the school level.

The second tension in Leonie‘s Activity System related to the same concern is between the Subject, Tools and Boundary Objects for both Leonie and the PLG (Figure 2, Tension 2). The PLG discussions were often guided and informed by teaching strategies or tools recommended in the literature. This tension suggests that the tools identified in the literature to reach the Object of promoting PPA (including surveys and journals), were designed by researchers for teachers, resulting in teachers being consumers of the tools. This resulted in some teachers feeling uncomfortable in their use of the tool resulting in some resistance to adopt and ‗fit‘ the tools designed by others into their classroom teaching or their individual activity system. This suggests that further research is needed to investigate if PPA tools created by teachers for teachers would be more readily adopted in classroom practice.

The third tension in Leonie‘s Activity System relates to the second dominant concern, and it is between the Subject and the Community (Figure 2, Tension 3). As Leonie attempted to change her mathematics teaching practice, this tension appeared to have originated from outside of her classroom. Specifically, Leonie (Subject) was aware of expectations from parents and staff (Community) for mathematics to be ―done in a particular way” and for the classroom to appear well managed with students seated quietly at their desks. Yet Leonie explained that in her view, there is benefit for students to make ―noise” because dialogue and talking are important affective elements to include in the mathematics classroom. The identification of this tension suggests there is an opportunity for parents and staff in general to review and challenge the perceptions of the characteristics of learning environments that promote PPA. Again, this aspect is not currently addressed by the tools recommended to teachers. It is a well suited topic for investigation as part of teacher PD and PLG discussions at the school level.

The three tensions in Leonie‘s Activity System are represented in a Third Generation Activity System in Figure 2. The first tension is labelled with a number 1, the second tension is labelled with a number 2 and the third tension is labelled with a number 3.

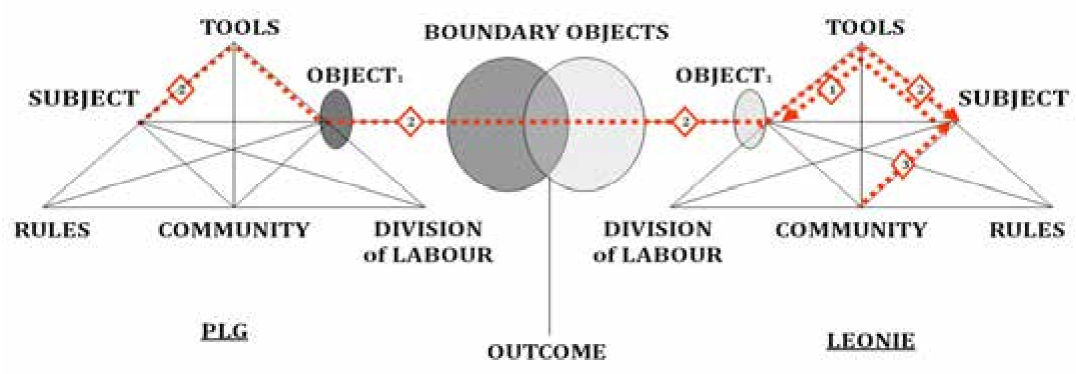

Figure 2. Leonie’s activity system and tensions.

In combination, the use of the theoretical lens of AT and SoC reveals how Leonie‘s engagement and participation in the PLG supported her to address her concerns and expand her teaching practice in her classroom. In particular, the SoC provided a lens to understand where these concerns were situated in the change process when attempting to implement a new innovation and make a change in practice. The use of AT focused the researchers‘ attention on tensions within and beyond the direct control of the research participant in their classroom.

Implications and conclusion

The addition of the SoC with AT proved to be a powerful and valuable combination during the data analysis and discussion stage of this research as it revealed unexpected insights and challenges related to the promotion of PPA that may not have been possible with the use of only one theoretical construct. Specifically, the SoC served as a valuable construct to monitor, describe, and quantify the affective part of teachers‘ implementation of tools for promoting PPA while AT facilitated the examination of the promotion of PPA in terms of a complex system. Together, these constructs supported the researcher to look beyond a narrow focus on tools used by teachers in classrooms and to value teachers‘ voices to theorise the dialectical relationship between teachers‘ perceptions, affective responses, motives and actions that support and constrain pedagogical change. This paper has drawn attention to the value of the SoC and AT as a combination of theoretical tools with which to understand and analyse teachers‘ concerns and tensions in relation to the promotion of PPA in mathematics classrooms.
The use of the SoC and AT in combination together with PLGs has potential use in further research as a means of investigating and understanding teachers‘ tensions and concerns related to teachers implementation of tools and innovations towards the promotion of powerful positive affect in the mathematics classroom.

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IDENTIFYING MATHEMATICS IN CHILDREN'S LITERATURE: YEAR SEVEN STUDENT'S RESULTS

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Using children's literature in mathematics is not a new idea. Although resources have been produced to support teachers in using literature in their mathematics programmes, there is little research to show this approach is successful. One debate associated with using children's literature in mathematics teaching/learning is how much support is required for children to recognise the mathematics in the literature. The research that is available has focused on very young children interacting with stories being read to them or identifying adaptations needed to the text and/or illustrations to allow children to recognise the mathematical information inherent in the story. This paper presents the results of a study that used book reviews as a tool to identify the extent Year 7 students could identify the mathematics in children's literature.

Background

The use of children's literature[1] in mathematics is an idea that has been promoted since the 1970s (Whitin & Wilde, 1992). This combination of reading and mathematics allows children to use their strengths in one subject to support their learning in the other and worthy of including in classroom programmes (Thraikill, 1994). In integrating literature and mathematics, the challenge is in keeping the integrity of both curriculum areas (Perger, 2004). The ability to focus on the mathematics without losing the enjoyment of stories would seem to depend on the skill of the reader and the extent to which the mathematics can be identified. To support this, books have been published to aid teachers in identifying the mathematical possibilities within specific examples of children's literature, for example *Books You Can Count On* (Griffiths & Clyne, 1988). Reference to children's literature can also be found in teacher publications such as The National Council of Teachers of Mathematics' (NCTM) *Teaching Children Mathematics* journal and in the teacher's edition of some mathematics textbooks. Here

[1] *Note:* The definition of children's literature used in this paper is that of Anderson (2006). She identifies children's literature as all books written for children excluding comics, joke and cartoon books as well as non-fiction or reference books that were not intended to be read from cover to cover such as dictionaries or encyclopaedias.

the children's literature is used to support the mathematical learning focus of the chapter, for example *Elementary and Middle School Mathematics: Teaching Developmentally* (Van de Walle, 2004).

With a renewed focus on literacy and numeracy, publishers have produced books linked to both curriculum areas (e.g., the MacMillan *Side-By-Side* series). These books are often levelled to children's reading ability, yet they rely on children using their knowledge of mathematics to gain meaning of the story. Although there are a variety of resources available that encourage teachers to use children's literature in their mathematics programmes there is little research that identifies the benefits of combining children's literature and mathematics (Anderson, Anderson & Shapiro, 2005; Hong, 1996).

One issue relating to the use of children's literature in mathematics is the extent to which the mathematics needs to be outlined for it to be a useful tool for mathematical learning. Whitin and Whitin (2004), although supporters of using children's literature in mathematics programmes, considered the task of identifying the mathematics a time consuming one for teachers. To help teachers with this process they developed a set of criteria for teachers to use when assessing the quality of children's stories they may wish to use. The criteria they developed identified four aspects they consider mathematics related children's literature book should demonstrate. The four aspects are "mathematical integrity", a "potential for varied responses", an "aesthetic dimension", and "ethnic, gender, and cultural inclusiveness" (2004, p. 4). In developing these criteria Whitin and Whitin referred to the NCTM's *Curriculum and Evaluation Standards for Mathematics* (1989) and the *Principles and Standards for School Mathematics* (2002) as well as standards set by the National Council of Teachers of English (NCTE) and the International Reading Association (IRA). The development of these criteria would indicate the responsibility for identifying the mathematics is that of the teacher. Whitin and Whitin believed that when teachers choose children's books carefully and introduce them effectively not only will mathematics learning be enriched but there is the potential for learning in other curriculum areas as well.

Schiro (1997) also recognised a benefit in using children's literature in teaching mathematics, although he believed that the mathematics could easily be missed or appear confusing to the reader if not made explicitly obvious. In a study to identify literature appropriate for mathematical learning, Schiro (1997) developed a criterion for assessing children's literature. This criterion differed from Whitin and Whitin's (2004) in that it focused on how explicit the mathematics was presented rather than the quality of the mathematics and the story. His explanation required the mathematics to be made explicit in both illustrations and text if it were to benefit mathematical learning. He considered the responsibility of making the mathematics clearly that of the author and illustrator. Schiro recommended that algorithms be included in text and that numerals be presented in digit form to support the mathematical word. Schiro stated that mathematical information also needed to be incorporated into the illustrations if the reader is to understand the mathematics inherent in the story. For example the illustrations for the story *Ten in the Bed* should include the mathematical equation $10 - 1 = 9$ written on the end of the bed. In the story *One grain of Rice* (Demi, 1997), for example, tables could be incorporate into the illustrations so as to demonstrate the increasing value of the numbers. This inclusion of numeral data into texts and illustrations would seem to indicate that the reader needs this support to identify the mathematics within children's literature. Yet in doing so is the opportunity for mathematical thinking being taken away from the reader?

Perkins (2001) states that if we believe mathematics is everywhere we should be able to easily find it represented in children's literature, whether the books have been written with a mathematical focus or not. Supporting this idea, young children have been observed using mathematical language when discussing a story being read to them (Van den Heuvel-Panhuizen & Van den Boogaard, 2008; Anderson et al., 2005). Van den Heuvel-Panhuizen and Van den Boogaard's (2008) study in The Netherlands showed that young children are able to identify the mathematics in children's literature without the need for adaptations or teacher direction. They identified examples of mathematical related thinking during the reading of a picture book to four young children. They found that five-year old children used mathematical language during discussing the illustrations with peers. When analysing the children's talk they discovered that half their utterances were mathematically related.

The importance of illustrations in promoting mathematical discussion when children are read to was also identified in a Canadian study. Anderson et al. (2005) used videoed sessions of 39 parents reading specific examples of children's literature with their four-year old children. During these sessions parents and children were viewed engaging in mathematical dialogue to co-construct the meaning of the text. Size, number, and shape were the concepts discussed, in order of frequency. Many of these discussions were child initiated. The evidence of these studies would indicate that even young children are able to recognise mathematics within children's literature without the adaptations Schiro (1997) considered necessary.

Using children's literature in a mathematics programme can enhance the learning of mathematical concepts through giving children the opportunity to talk about mathematics. Van den Heuvel-Panhuizen and Van den Boogaard (2008) found that providing an opportunity to talk about a story not only contributed to the understanding of mathematical concepts but also helped develop a positive attitude towards mathematics. Griffiths and Clyne (1991) recognised that children's literature was able to play a larger role in mathematical learning as it provided a model, illustrated a concept, posed a problem or stimulated an investigation.

In summary, the effectiveness of using children's literature as a tool for teaching/learning mathematics would seem to hinge on the reader identifying the mathematical possibilities within the story, but the extent to which the mathematics needs to be made explicit to the reader is debatable. At one end of the argument Schiro (1997) states mathematical information needs to be presented to the reader in digit and/or equation format in both text and illustrations. At the other end of the scale Van den Heuvel-Panhuizen and Van den Boogaard (2008) believed that young children are able to identify the mathematics even when the teacher has given no clear indication that it is present.

Given this debate as to how much support is required for readers to be able to identify the mathematics, a group of Year 7 students were asked to complete two tasks. These tasks were designed to allow students to demonstrate the extent to which they could recognise the mathematics in children's literature. This paper presents the findings of that study.

This study

This study focused on two questions:

- Could students identify the mathematics in a storybook without being given any indication that mathematical concepts / opportunities where present?
- What mathematics do students identify in a story, when there is no indication of mathematical content, and when they are told mathematics is present?

To answer these questions, a group of Year 7 students (11 year olds) approaching the end of their first year at intermediate school were given two tasks. Tasks were completed as part of their normal class programme separated by an interval break. Students had been exposed to a variety of children's literature throughout the year although never in their mathematics programme. Thirty books were provided for students (see Appendix A) to choose from. These all contained opportunities for mathematical learning although in some the mathematics was more obvious than others. There was a range of instructional reading and mathematical concept levels. Each student chose a different book to review for each task.

The first task students completed was to select a book and write a book review. Added to the book review, students were asked to identify how a teacher might use the book in their classroom programme. Students were not given any indication which areas of the curriculum or age level the books could be used for. All students present in the class at the time the task was presented participated. 16 students of mixed ability (in both mathematics and reading) completed a book review.

The second task was set once students returned to class after a twenty-minute interval break. This time, 19 students participated, again of mixed mathematics and reading ability. They included 16 students who completed the first task and 3 others. The same selection of books was used. For this task students were informed that the books they could choose from were purchased to be used in a mathematics programme. The task required students to identify the mathematics that children could learn from reading or being read the books. Each student chose one book and listed the mathematical learning that could be achieved through the use of the book selected.

Results and discussion

Book review task (Task 1)

Like the children in Van den Heuvel and Van den Boogard's (2008) study the majority of Year 7 students in this study were able to identify the mathematics even when not altered to its presence. 13 of the 16 students recognised mathematical learning possibilities for a teacher using the book they had reviewed (see Table 1)

Table 1: Mathematical concepts identified in book review task

	Number	Measurement	Geometry
Concepts Identified	Counting Place Value Addition Multiplication Division Knowledge of Numbers	Length (cm, m) Weight Angles - degrees Time Area of a Circle Size	Shape

As in Anderson et al. (2004) number, measurement (size) and geometry (shape) were the areas of mathematics identified. The students noted that angles (degrees) and the area of a circle could also relate to concepts in the geometry.

When presenting learning possibilities students stated the mathematics clearly and concisely. For example,

> If I were a teacher, the children in my class would learn how to divide in half.
> *(The Great Divide, Dodds, 2000)*

> If I were a teacher children will learn how to multiply, add and learn more about maths.
> *(Anno's Mysterious Multiplying Jar, Anno & Anno, 1982)*

> If I were a teacher the children would learn how to find the area of a circle and how to measure. *(Sir Cumference and the Dragon of Pi, Neuschwander, 1999)*

Two students elaborated further on the mathematics children could learn if a teacher used the book they had reviewed by listing the concepts children could learn in more detail. For example,

> Children could learn how zero's make a number even bigger and numbers never end. They could also learn how to count from one to a googol; which has 100 zeros! You can also learn the names of other huge numbers.
> *(Can You Count to a Googol? Wells, 2000)*

Another student included detail of a task he would set the class.

> If I were a teacher I would get my students to find out how many humpback whales and dogs would fit in the class room and how many peas would fit in a bowl. The students would learn about measurements like metres, centimetres, and weight.
> *(Counting on Frank, Clement, 1990)*

These students recognised only mathematical learning possibilities in the books they reviewed. The mathematics identified by these students in their book review task indicated that they were able to recognize mathematical possibilities within children's literature. The specific links to the mathematics and/or detail they were able to provide, as to the learning possibilities, indicates that the mathematics was neither obscure nor confusing. The adaptations to text and illustrations Schiro (1997) recommended were not required for these children to recognise the mathematics inherent in the books they reviewed.

When identifying mathematical possibilities for a teacher, four students were confident enough to also recommend an age group for the book they reviewed. All the age levels identified were appropriate for the story and the mathematics they had selected. Three of these students justified the age group through links to other curriculum areas or interests. These justifications included an interest a child at that age group might have, i.e. cats for the story *Six Dinner Sid* (Moore, 1990), or other learning such as learning to read which they saw as focus of year level. The ability for books with a strong mathematical content to enrich learning in other curriculum areas was something Whitin and Whitin (2004) and Griffths and Clyne (1991) stated to be an advantage of using children's literature in a mathematics programme. Five students in this study identified learning in both mathematics and other areas of their lives. These areas included other curriculum areas such as reading, language (narrative text), and science. Students also made links to real life learning that could eventuate from the story reviewed such as "love and romance" and "wisdom". Here students identified the links Griffths and Clyne, as well as Whitin and Whitin, had assigned to teachers. The students' ability to identify mathematical learning, specific concepts, possible activities and learning in other curriculum areas, and as well learning associated with real life, would support Griffths and Clyne's (1991) observation that children's literature is able to play a role in mathematical learning.

Three students failed to identify any mathematics. Two of these students had chosen books where the mathematics was not obvious, although these books could have provided motivation for mathematical investigations. One example of children's literature where obvious mathematics was not identified was a counting book. Of course, the inability to identify the mathematics may relate to an individual's mathematical or reading ability, but individual participant's ability in reading and mathematics was not identified in this study.

Identifying the mathematics (Task 2)

The second task students were assigned a more mathematics-focused task. It was to list possible mathematical learning in a book chosen from the same selection as in task one. For those students who completed both tasks, a different book from the one they had used for task one was chosen for task two. For this task students were alerted to the fact that the books had been purchased for use in mathematical programmes. No information about what mathematics the books contained was given.

All students were able to identify appropriate mathematical learning possibilities. Once again students identified mathematical concepts associated with number, measurement and geometry. See Table 2 for mathematical concepts identified during Task 2, in order of frequency mentioned.

Table 2: Mathematics identified in children's literature

	Number	**Measurement**	**Geometry**
Concepts Identified	Division	Volume/Capacity	Shapes
	Multiplication	Weight	Tangrams
	Addition	Distance	
	Subtraction	Height	
	Counting	Area of a Circle	
	Reading Big Numbers	Time	
	Place Value	Cooking	

The Year 7 students in this study linked 11 books to number, 7 to measurement and 3 to geometry. The frequency of mathematical concepts has changed from the measurement (size), number, and geometry (shape) noted by Anderson et al (2004) in that the identification of number has overtaken that of measurement. This could be due to a stronger focus on number as students reach higher levels of the school curriculum.

The mathematics identified in this second task was more detailed than in the first. For example, in the first task when no indication of mathematical content was given, the mathematics associated with the book *The Dot and the Line* (Juster, 1963) was noted as "some mathematics shapes". In the second task when students are altered to the mathematical content the specific mathematical learning was identified as "children could learn shapes—squares, triangles, hexagons, parallelograms, rhomboids, polyhedrons, trapezoids, decagons, tetragrams as well as angles". The two books where students failed to identify the mathematics in Task 1 (one where the mathematics was not obvious) had appropriate mathematical content identified in this second task. This difference could be attributed to different students completing the book review. This would indicate teachers could play an important role in enhancing the mathematics in examples of children's literature through the way they introduce it the story (Whitin & Whitin, 2004).

Although students were only asked to identify mathematics in the second task, some students made links to other learning as well. These included riddles and rhymes, history, reading and science. One student also made links to more general aspects of mathematics such as problem solving ("being able to answer mathematics problems"). Another student who seems to agree with Perkins' (2001) belief that if mathematics is everywhere we should be able to find it represented in children's literature linked mathematics to everyday life. In reference to the book *Maths Curse* (Scieszka, 1995) she concluded her list of possible mathematical learning with the sentence "Children can learn that maths is all around us and mathematics has real life applications and is very important".

Conclusion

The year 7 students in this study showed that they could identify opportunities for mathematical learning in samples of children's literature. Mathematics was the predominant curriculum area identified even when no indication of mathematical possibilities was provided. Without adaptations to text or illustrations, or teachers' input, these students identified appropriate mathematical learning, often linking it to a wider field of knowledge— both other curriculum areas and life skills. It is possible that if the adaptations recommended by Shiro (1997) had been evident in the text or illustrations of these books, the mathematical opportunities may have been limited to those of the author and students may not have made the wider links to other curriculum areas or the life skills they did. When altered to the presence of mathematical content the student's descriptions of the mathematical possibilities was even more detailed. This would indicate that although students can independently identify the mathematics in children's literature, the input of a teacher could further enhance learning opportunities. With a student's ability to recognise the mathematics and a teacher's careful selection and introduction of books, the use of children's literature could be a powerful tool in both motivating and consolidating mathematical knowledge.

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Appendix one: Children's literature used in the study

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EARLY CHILDHOOD NUMERACY LEADERS AND POWERFUL MATHEMATICS IDEAS

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During 2009-2010, the South Australian Department of Education and Children‘s Services implemented a mathematics learning and teaching program for educators in preschools and the first year of school in response to national curriculum agendas. The Early Years Numeracy Pilot Project (EYNPP) used an inquiry model of professional education led by early childhood educators who were designated as numeracy leaders. This paper analyses the impact of EYNPP on the knowledge, skills, confidence, and pedagogical approaches of these numeracy leaders and on the development of powerful mathematical pedagogy in their own settings and those of the colleagues whom they were leading.

Introduction

Evidence of the importance that the Australian government sees for the mathematics education of young children can be found in the injection of $540 million from the *National Partnership Agreement on Literacy and Numeracy* (Council of Australian Governments [COAG], 2008) and the development of both the *Early Years Learning Framework* (Commonwealth of Australia, 2009) and the *Australian Curriculum—Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010).

Within this context, the South Australian Department of Education and Children‘s Services (DECS) implemented the Early Years Numeracy Pilot Project (EYNPP) to enhance mathematics outcomes for children and educators in preschools and the first years of school. This paper reports outcomes from the 2009-2010 implementation of EYNPP. In particular, it considers the professional learning of four ‗numeracy leaders' who were integral to the project and the impact of this learning on the development of powerful mathematical pedagogy in their own settings and those of the colleagues whom they were leading.

[1] I wish to acknowledge the following people who have inspired me to write this paper and without whom it would not exist. In very many ways they are co-researchers in EYNPP and co-authors of the paper. They are: Di Hogg, Tammy Mann, Tanya Pojer, Sandy Warner, Elspeth Harley, and Noel Thomas.

Background

Early childhood mathematics

Recognition of the importance of mathematics in the early years of children‗s lives is well recognised (Hunting et al., 2008; Kilpatrick, Swafford, & Swindell, 2001; Lee & Ginsburg, 2007; Perry & Dockett, 2008; Sarama & Clements, 2009; Thomson, Rowe, Underwood, & Peck, 2005). The Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) state that

> … all children in their early childhood years are capable of accessing powerful mathematical ideas that are both relevant to their current lives and form a critical foundation to their future mathematical and other learning. Children should be given the opportunity to access these ideas through high quality child-centred activities in their homes, communities, prior-to-school settings and schools (AAMT & ECA, 2006, p. 1).

The *Early Years Learning Framework for Australia* (Commonwealth of Australia, 2009) provides a list of what these powerful mathematical ideas are:

> Spatial sense, structure and pattern, number, measurement, data, argumentation, connections and exploring the world mathematically are the powerful mathematical ideas children need to become numerate. (p. 38)

Early childhood educators

Educators‗ mathematical knowledge and dispositions are key to effective mathematics learning in early years settings (Anthony & Walshaw, 2007; Sarama & Clements, 2009). However, it is often clear that ―low levels of content knowledge and the resulting lack of confidence about mathematics limit teachers‗ ability to maximise opportunities for engaging children in the mathematical learning embedded within existing activities" (Anthony & Walshaw, 2007, p. 47). The need for ongoing professional learning in mathematics education for early years educators is emphasised by Perry & Dockett (2008):

> At this time when children‗s mathematical potential is great, it is imperative that early childhood teachers have the competence and confidence to engage meaningfully with both the children and their mathematics. (p. 99)

Professional learning and mentoring

The concept of mentoring has been applied to professional learning in many different walks of life, including education where ―[t]he central premise of mentoring as a form of professional learning stems from the belief that individuals may best learn through observing, doing, commenting and questioning, rather than simply listening" (Nicholls, 2002 cited in Onchwari & Keengwe, 2010, p. 312). Landry, Anthony, Swank, and Monseque-Bailey (2009, p. 449) suggest that ―[a]n advantage of mentoring is its ability to individualize professional development to the needs of the learner, which may be particularly important for early childhood teachers who vary in education and training".

Recently, the mentoring approach has been applied in Australia with the advent of ‗numeracy coaches‗ in many jurisdictions. Initial findings concerning the learning needs of these coaches suggest the need

> … to include a focus on developing knowledge for teaching mathematics, including content and pedagogical content knowledge (Ball, Thames, & Phelps, 2008). It is important that coaches have strong mathematics content and pedagogical content

knowledge to support teacher development and ultimately achieve the policy imperatives of improving student learning. (Anstey & Clarke, 2010, p. 51)

The Early Years Numeracy Pilot Project 2009–2010

The EYNPP was undertaken during 2009–2010. It was designed to:

- build on previous work that had led to the development of a numeracy matrix linking powerful mathematical ideas to developmental learning outcomes (Perry, Dockett, & Harley, 2007);
- develop and trial reflective continua[2] based on the numeracy matrix and the use of learning stories (Carr, 2001) to guide mathematics teaching and learning in the early years (children aged 3–8 years);
- develop mathematics content knowledge and pedagogical content knowledge among early childhood educators and numeracy leaders;
- enhance the knowledge and skills of a small number of numeracy leaders so that they could lead their colleagues to improve young children‘s mathematics learning outcomes; and
- ensure that all material and learning emanating from EYNPP reflected the philosophy, approach and content of the *Early Years Learning Framework* (Commonwealth of Australia, 2009).

Methodology

Participants

The participants in EYNPP were 45 early childhood educators from prior-to-school settings and schools in four clusters in South Australia. In each cluster, one educator was chosen to act as a Numeracy Leader/Mentor. Three of these Numeracy Leaders (NLs) were early primary school years teachers and one was a preschool educator. All had more than 10 years of teaching experience.

Professional development for Numeracy Leaders

From July, 2009 until December, 2010, regular professional learning meetings were held for the four NLs. These meetings were led by two university-based researchers and covered mathematical content knowledge such as the powerful mathematical ideas and links to both the *South Australian Curriculum, Standards and Assessment Framework* (Department of Education, Training and Employment, 2001) and the *Early Years Learning Framework* (Commonwealth of Australia, 2009). Input concerning pedagogical content knowledge centred on the previously developed numeracy matrix and the development and implementation of the reflective continua for each of the powerful mathematical ideas[3].

Material was also provided concerning the role of numeracy leaders and the development of collaborative partnerships with colleagues. The task of the NLs was challenging, as each was responsible for engaging and guiding both prior-to-school and

[2] The Reflective Continua highlight progression and engagement with the relevant powerful mathematical ideas. Four levels of development are used. Student‘s work samples are provided to illustrate how each level might present in preschool and the first years of school.

[3] While the numeracy matrix and reflective continua are major artefacts arising from EYNPP, the focus of this paper is on the professional learning of the NLs. Hence, these artefacts are not described in this paper.

school educators in their clusters. This involved NLs working with colleagues at each site to identify an inquiry learning project and associated inquiry questions, and supporting staff as they implemented and evaluated this project. Ongoing support from each other and the research team were critical elements of the NLs' program.

Professional development for site participants

Each of the NLs devised a collaborative approach to working across the sites in their cluster. In most cases this involved regular weekly or fortnightly meetings with colleagues on each site. As well, meetings at the cluster level and across clusters were held. The final professional learning experience for all participants in EYNPP was a Celebration Day, where each site outlined their learning journey to their colleagues.

Data gathered in EYNPP derive from a number of separate sources, including questionnaires completed by participants, presentations from each site team and reports of the impact of the project from participants, NLs, and leaders in the prior-to-school and school settings in which the project was implemented. As the aim of this paper is to report the impact of EYNPP on the knowledge, skills, confidence, and pedagogical approaches of the NLs and their influence on the development of powerful mathematical pedagogy in their own settings and those of their colleagues, data used in this paper are drawn from reports prepared by the NLs, a focus group interview conducted with the NLs at the conclusion of the project, and interview data from prior-to-school and school leaders.

Results and discussion

Knowledge, skills, confidence, and pedagogical approaches of the Numeracy Leaders

One of the aims of EYNPP was to develop the NLs as genuine leaders in both mathematical content knowledge and pedagogical content knowledge. There are many examples in the data that point to such growth.

One NL listed the following outcomes in her final report:

- having the opportunity to build leadership skills in a supported program;
- improved ability to speak in front of large groups and facilitate training sessions;
- enormous learning in the area of mathematics education through the opportunity to work with so many educators and professionals in this area;
- improved numeracy education practice;
- using more authentic assessment methods;
- a hunger to learn more about the development of children's mathematical thinking skills; and
- improved communication skills.

Another suggested that as well as learning about talking with educators from both preschool and school levels, she ―was inspired and stimulated to change my pedagogy ... and I have been teaching for longer than 20 years". The principal of the school at which another NL was based commented on the impact of EYNPP:

> She recognises considerable growth in her own understanding of working in a leadership role and working to organise things for other staff and other schools, acknowledging different cultures in other sites, acknowledging that people come with various levels of defensiveness about exposing their practice or working on their practice with other

> people. ... She has spoken of enormous growth in her understanding of numeracy and also in her conviction that the [reflective] continuum ... does have a great deal to offer teachers.

The preschool-experienced NL was enthusiastic about her involvement in EYNPP:

> [It‘s] sharpened me up again and given me a new passion about my teaching. The old reasons are still there—the children come first and all that sort of stuff—but now I‘ve really got bones to actually articulate that which I never had before to the degree I‘ve got now.

The following comments from the NL focus group highlight some of the growth the NLs themselves had achieved in both mathematics content and pedagogical content knowledge.

> It was really positive for me because it did raise my thinking about mathematical theory, about what numeracy was and what it meant for me.

> We just got to know so much more about how to help these students and what they were doing and where they were working.

> I‘ve done things that I never would have done before and would never have felt comfortable doing before. It‘s put me right out of my comfort zone and I‘ve really loved it. I‘ve loved the challenge.

All of the NLs reported that they were using the artefacts of EYNPP, particularly the numeracy matrix and the reflective continua in their own settings. As well, they transferred the inquiry approach advocated by EYNPP for professional learning to their own teaching.

> In order to promote the real-life links, we planned a maths trail around the community, ensuring the tasks we designed were open-ended and involved working through problems. We really promoted it with the children as a way of sharing their maths learning and encouraged all family and community members to attend—parents, grandparents, aunties, uncles, siblings, neighbours, etc ...
> On the day of the Numeracy Trail, the children were buzzing with excitement. Including the children in the two classrooms, we had over a hundred people embark on this trail, ranging from age 3 months to 74 years—it was amazing! Groups worked solidly for over two hours and most groups ran out of time to complete all problems.

All other numeracy leaders had similar stories about how their involvement in EYNPP had resulted in their reflecting on their teaching practices and changing where they felt it was needed.

> In my class now, because you know we had that whole argumentation thing at the beginning and that really stuck in my head. ... We were going through some stuff and I was asking them ‗What answer did you get‘ and the kids would say what the answer is. Except now they say the answer and then they go ‗And I know that because …‘ and then they just start with the whole explanation of what they did. And then someone else will put their hand up and say ‗Oh I got the same answer but I did it this way‘ and whatever. [A visitor who was in the class said] ‗Far out, like these kids are really good at talking about what they know!‘

Powerful mathematics pedagogy, cluster colleagues, and children

Cluster colleagues benefited from the collaborative learning experiences that were established by the NLs. For example, in one cluster, the participants suggested that, as a result of their participation in EYNPP, they were now:

- developing new techniques for assessing the children and extending their mathematical thinking;
- taking risks with the children in a positive environment;
- letting children solve problems without intervening too quickly;
- recognising the value of modeling to the children that it‘s OK to make mistakes;
- recognising the importance of making connections between home and centre;
- recognising the importance of using correct mathematical language and terminology with children; and
- providing an environment in which parents and other family members feel comfortable and valued.

In another site, educators listed the following child outcomes resulting from the educators‘ participation in EYNPP:

> Observations of our children at our site have shown that students now;
> - have strategies to work together;
> - can break down problems;
> - know how to express how they worked it out;
> - are questioning;
> - are using collaborative strategies and are more accepting of each others‘ input;
> - are valuing and encouraging each others‘ opinions;
> - are spending more time working through problems; and
> - can recognise that there is often more than one way to get to an answer and each should be valued.

Conclusion

The EYNPP NLs have built positive relationships with their colleagues; modelled, observed, commented, questioned, and collaborated in the development and implementation of inquiry-based pedagogies; and introduced and explained the artefacts of EYNPP. The data presented in this paper demonstrate that EYNPP has had a positive impact on the NLs as well as on the other participants, including the children in the participants‘ learning groups. Key aspects of EYNPP—especially the inquiry and reflective approaches—have provided much impetus for change. This is epitomised by the following unsolicited note from one of the EYNPP participants.

> I just wanted to thank you sincerely for the opportunity to be involved in the Early Years Numeracy Pilot Project. For many years I have craved the opportunity to be challenged in my thinking and professional practice. I have been constantly trying to do this myself but it is difficult without a structure and time for reflection and professional dialogue. This project, and in particular the reflective continua have provided the most wonderful opportunity for me to receive this challenge to my professional practice. It has been wonderful to receive positive feedback on what we are doing but more importantly it has been an awesome scaffold for that professional dialogue and also that self/professional reflection. I believe that I am a better practitioner as a result and will continue to strive to better myself.

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PLAYING WITH MATHEMATICS: IMPLICATIONS FROM THE EARLY YEARS LEARNING FRAMEWORK AND THE AUSTRALIAN CURRICULUM

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After an introduction to the current conceptions of play in early childhood settings, we consider what The Early Years Learning Framework and the Australian Curriculum say about play and mathematics learning in the home and preschool, and the early years of school. We analyse similarities and differences in the two documents with regard to their philosophies about play as pedagogy for the learning of mathematics. We use the construct of a Numeracy Matrix to illustrate how playing with mathematics can be utilised to provide curriculum and pedagogical continuity between preschool and school.

Introduction

The current context of early childhood education in Australia is one of social, political, and educational change. At a time of unprecedented political focus on early childhood education and growing awareness of the importance of high quality early childhood education for children, their families and communities, two national curriculum documents that will shape the nature of early childhood education for some time to come have been introduced.

Early childhood education in Australia has recently embraced *Belonging, Being and Becoming: The Early Years Learning Framework for Australia* [EYLF] (Department of Education, Employment and Workforce Relations [DEEWR], 2009). The EYLF advocates play-based learning, supported by quality teaching, as the basis for promoting children's learning and development. At the same time, a national school curriculum, the Australian Curriculum, is being developed and implemented (Australian Curriculum, Assessment and Reporting Authority (ACARA, 2010). This curriculum is organised across distinct subject areas, and has a focus on curriculum content, rather than pedagogy.

Both documents emphasise the importance of children's learning and note some specific outcomes for learning in the early childhood years. However, each document reflects a different focus on that learning. The EYLF, in keeping with the consideration of young children aged birth to five years, reflects a holistic approach to learning and development, embedded within play-based environments and includes broad learning outcomes. The Australian Curriculum is focused much more on specific learning outcomes, associated with discrete subject areas and definite years of schooling. Partly, this relates to the purpose and intention of each document: the EYLF is a curriculum framework, with emphasis on pedagogy, principles, and practice; the Australian curriculum is much more focused on the curriculum content.

Clearly, the documents serve different purposes and reflect the different nature of the educational settings for which they are designed. However, educators across early childhood settings and schools are required to work with both documents, and the associated expectations, in order to promote continuity of learning and positive educational outcomes for all children. This paper explores one means of facilitating such continuity in the area of mathematics, through pedagogy based on contemporary conceptualisations of play. It provides a summary of current conceptualisations of play and links these to learning through the construct of a Numeracy Matrix. This matrix and the data reported in the paper are drawn from the *Early Years Numeracy Pilot Project,* conducted with a total of 130 preschool and school teachers in South Australia (Perry, Dockett, & Harley, 2007; Perry, Dockett, Harley, & Hentschke, 2006) as they explored ways to enhance the mathematical opportunities and experiences for their students.

Play-based pedagogy

Early childhood education has a long tradition of play-based pedagogy. Play has been regarded as both a vehicle for learning and as an opportunity for children to demonstrate their knowledge, skills, and understandings (Johnson, 1990). Traditional approaches to play have emphasised the child-initiated and directed nature of play, relegating adults to the roles of stage managers and onlookers (Bennett, Wood, & Rogers, 1997). Recent reconceptualisations of play have moved away from these notions, referring instead to the social nature of play and the opportunities afforded through play for children to engage with important others in many of the routines and interactions important within their social and cultural contexts (Rogoff, 2003). Rather than casting play and learning as opposite elements of children's lives—where play is something that is child-initiated and learning is adult-initiated (Pramling-Samuelsson & Asplund Carlsson, 2008)—recent critiques of play note the importance of adult and child interaction within play, particularly in situations of sustained shared thinking (Siraj-Blatchford, 2009) and scaffolding (Arthur, Beecher, Death, Dockett, & Farmer, 2008). Current play-based pedagogy recognises the complexity, as well as the potential of play to contribute to learning. It also acknowledges that not all play is either productive or likely to lead to positive learning outcomes. Along with this, it emphasises active roles for participating adults as they co-construct meaning through strategies such as inviting children to elaborate on their ideas and play, clarifying ideas, offering alternative views, speculating and modelling thinking (Siraj-Blatchford, 2009).

Young children's play often includes a great deal of mathematics (Greenes, Ginsburg, & Balfanz, 2004; Seo, 2003). Sometimes, this is identified and extended by educators; at other times educators' own understandings of mathematics may limit the identification and response to mathematics within play (Sarama & Clements, 2009). The potential of play to facilitate children's mathematical thinking depends largely on educators' ability to "seize on the teaching opportunities in an adequate way" (van Oers, 1996, p. 71). This ability requires mathematical knowledge; understanding of the nature of children's play, particularly the characteristics of play that promote mathematical learning and thinking; and awareness of the role of educators in promoting both play and mathematical understanding.

What about mathematics?

Preschool educators tend, at least in Australia, to reject the divided, content-based approach to mathematics curriculum that is often used in schools (Australian Association of Mathematics Teachers and Early Childhood Australia, 2006). There is, however, general agreement that all children in their early childhood years are capable of accessing powerful mathematical ideas that are both relevant to their current lives and form a critical foundation for their future mathematical learning, and that children should be given the opportunity to access these ideas through high-quality child-centred activities in their homes, communities, and preschool settings (Lee & Ginsburg, 2007; Hunting et al., 2008; Perry & Dockett, 2008).

Recognising young children's competence may mean that educators introduce a range of curriculum content and promote children's learning around a set of agreed learning outcomes. Both the EYLF and the Australian Curriculum adopt such an approach. However, there is tension between the two documents around both the nature of mathematics curriculum for young children and appropriate pedagogies to deliver this. Part of this tension involves resistance to 'push down' academic curricula from preschool educators, who argue strongly for early childhood curriculum that is play-based and child-centred, rather than curriculum that is subject driven. Also contributing to the tension are moves for greater accountability for teachers and schools, and growing emphasis on national and international testing. While children in Australia do not engage in national testing until Year 3, there is certainly anecdotal evidence that teachers in the first year of school, and even preschool, feel pressure to start preparing children early for such assessments and that this influences their pedagogy. In this context, how can educators work together, utilising the curriculum documents that are prescribed for their settings, to build on children's existing understandings and promote positive learning outcomes for all children?

Connecting curricula

The EYLF outlines five broad learning outcomes, each with several key components. While it is possible to align these outcomes with broad curriculum areas, it is argued that they represent integrated, rather than subject specific, learning outcomes. These outcomes are:

1. Children have a strong sense of identity.
2. Children are connected with and contribute to their world.
3. Children have a strong sense of wellbeing.
4. Children are confident and involved learners.
5. Children are effective communicators.

There is potential for mathematics to be an integral part of each of these outcomes. However, the last two are particularly relevant for addressing mathematics learning.

Material developed to support the implementation of the EYLF includes reference to a recent survey of Australian early childhood educators which concluded that young children were capable of working with mathematical ideas that could be attributed to the areas of number, algebra, geometry, measurement, data analysis, and probability

(Hunting et al., 2008). While these terms are not used in the support material, the importance of mathematical thinking for young children is reflected in the inclusion of a range of these same areas within the EYLF, which refers to the importance of sharing and clarifying thinking and ideas, developing understanding of measurement and number, experimenting with ways of expressing ideas, recognizing patterns and relationships, and using symbols to represent meaning. The Australian Curriculum for the Foundation Year of school also reflects these areas, although it formalises them into the content strands of number and algebra, measurement and geometry, and statistics and probability. These strands include a number of powerful mathematical ideas identified in mathematics teaching and research (see, for example, Greenes et al., 2004; National Council of Teachers of Mathematics [NCTM], 2000; Perry & Dockett, 2008).

Continuity in pedagogy

In a quest for pedagogical continuity across preschool (represented by the *Early Years Learning Framework for Australia*) and early school (represented by the *Australian Curriculum—Mathematics*) settings, the construct of a Numeracy Matrix linking the two curriculum documents has been explored. The current version of the numeracy matrix (DEEWR, 2010) provides direct links between the learning outcomes from the EYLF and these powerful mathematical ideas, many of which match closely the strands of the Australian Curriculum. The links are made through 'pedagogical inquiry questions' that ask educators in both settings what they might do to promote both the learning outcomes and the powerful mathematical ideas. These 'pedagogical inquiry questions' are about pedagogical approaches designed to lead educators to reflect on their pedagogical practice based on knowledge of their children's learning and the mathematics that they are endeavouring to develop in these children. Hence, the Numeracy Matrix provides a guide to the mathematics that might be developed by preschool educators—which is not highlighted in the EYLF—and a guide to the pedagogies which might be developed by early years of school educators—which are not highlighted in the Australian Curriculum.

Tables 1 and 2 illustrate the nature of the numeracy matrix and its potential to link the EYLF and the Australian Curriculum through pedagogical inquiry questions.

Table 1. Numeracy Matrix Cell—Example 1.

	Australian Curriculum—*Mathematics* Number and algebra
Early Years Learning Framework Children are confident and involved learners	What opportunities do we provide for each child to accept new challenges, make new discoveries and celebrate effort and achievement? What do we do to encourage children to use symbols and different representations of their mathematics?

Table 2. Numeracy Matrix Cell—Example 2.

	Australian Curriculum—*Mathematics* Statistics and Probability
Early Years Learning Framework Children are effective communicators	How do we encourage children to collect, analyse and represent data? How do we encourage children to begin to recognise, discuss and challenge unfair attitudes and actions?

The Numeracy Matrix can be used to link approaches taken in the preschool settings and the early years of school by having the early childhood educators in both of these sectors ask the pedagogical inquiry questions. While the answers may be quite different in each of the sectors, the asking of similar questions can provide opportunities for continuity across the transition to school, something which is known to benefit children in the early years of school and later (Wood & Bennett, 1999).

Using the Numeracy Matrix to promote pedagogical continuity

As part of the South Australian Early Years Numeracy Pilot Project, early childhood educators from both preschools and schools have developed a collection of work samples that illustrate each of the cells of the Numeracy Matrix. These work samples show how the same pedagogical inquiry questions can be used in both preschools and schools to help children develop their mathematical ideas through play.

For example, consider the Numeracy Matrix cell details in Table 1. Figures 1 and 2 below provide examples from children in the first year of school who were answering the question "How many legs do 10 chickens have?" in the context of farmyard play.

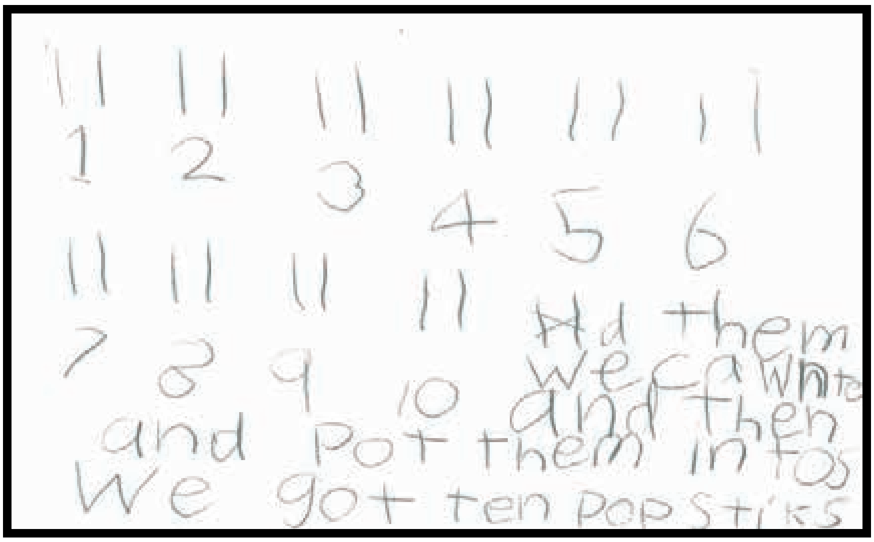

Figure 1. Tracey's justification of her solution to the challenge and use of symbols.

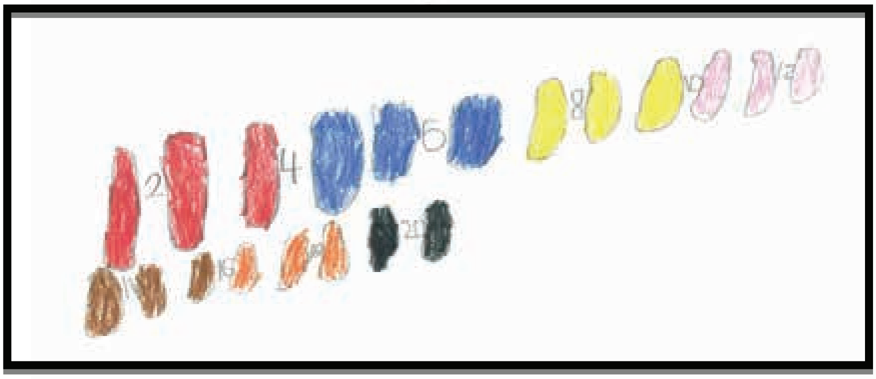

Figure 2. Emily's response to the challenge and use of patterning.

There are many ways in which young children might be challenged in number and algebra to demonstrate their use of symbols and show that they are confident and involved learners. Blair (Figure 3) is a 4-year old preschooler, and his preschool educator wrote the learning story.

Blair was sitting playing at the play dough table and I went to have a chat with him about making a number game. Blair said to me, 'I know a really big number—a million.' I asked Blair if he knew how to write one million in numerals and as he didn't I showed him. After briefly looking at the numbers I had written down (1,000,000) Blair said, 'Now I get it, a million is six zeros. A thousand is a one and three zeros. A hundred, one and two zeros. If you took three zeros away (from a million) it would be a thousand.' I asked Blair what the number would be if I replaced the one with a six and he told me it would be six million. Blair then said that he knew an even bigger number, a fillion! I said that there was not a number called a fillion but there was a billion (with nine zeros) and a trillion (with twelve zeros). He was very impressed by the number of zeros in these numbers.
Following our conversation Blair decided to paint a picture. He painted numbers from zero to fourteen on his paper. I asked Blair why he had stopped at number fourteen and he said that fourteen was his favourite number, he just likes the four.

Blair is able to initiate, explore, listen, and respond. He is curious and can classify and order, having a wonderful understanding of numeracy concepts. He uses language to express his thoughts and understands the function of print.

Figure 3. Blair knows a big number.

Conclusion

Clearly, the Australian Curriculum and the EYLF differ in the ways that curriculum is organised and delivered. Partly, this is related to the different philosophies and approaches underpinning the different documents and sectors. Early childhood educators who work across the sectors, including those involved in transition programs, need to be aware of these differences and the ways in which they can be navigated. While the children will not be aware of the pedagogical continuity provided by the Numeracy Matrix, it does provide educators in the preschool and school settings with a common language and format that can be used to discuss the children's learning and the educators' pedagogy. Educators who have used the Numeracy Matrix have been able to provide such guidance through the linking of the learning outcomes from the EYLF and the content strands in the Australian Curriculum—Mathematics.

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REACTING TO QUANTITATIVE DATA: TEACHERS' PERCEPTIONS OF STUDENT ACHIEVEMENT REPORTS

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As part of an investigation into statistical literacy for the teaching workplace, this research paper uses a framework for professional statistical literacy to examine teachers' perceptions of the complexity and value of such reports. Although teachers identified aspects of the data as useful for their work, many features were described as being difficult to understand. Even tertiary educated adults may not be well prepared for dealing with quantitative data in their workplace. There are lessons, too, for the presentation of statistical information.

Introduction

Since the 1990s there has been increasing recognition of the importance of statistical literacy, or the statistical understanding needed for everyday life as an informed citizen. However, statistical literacy for the workplace may mean more than this. The project reported here focuses on the needs of the education workforce and presents some preliminary work examining teachers' perceptions of the complexity and value of one statistical report of the kind received by teachers. The report was chosen because it has elements typical of those prepared by the Victorian NAPLAN Data service and provided to schools. In reporting our findings we will first review key literature related to statistical literacy and then propose a framework for "professional" statistical literacy. This is followed by details of the current study and the results for the chosen data report. Finally we consider the implications of these findings for both school mathematics and for pre-service and in-service teachers' professional learning.

Background

In education—as in other workplace sectors—quality control, accountability, and forward planning are informed by statistical data. The technological revolution has supported the collection, analysis, and sharing of vast quantities of data. Australia, for example, has developed a *Measurement Framework for National Key Performance Measures* (Ministerial Council on Education, Employment, Training and Youth Affairs, 2007) to monitor and advance outcomes from school education. Governments expend significant resources on collecting such data from the education sector via, for example, the National Assessment Program: Literacy and Numeracy (NAPLAN) involving students in Years 3, 5, 7, and 9 from all states and territories, with these intended to inform planning and practice. In Victoria the Victorian Curriculum and Assessment Authority (VCAA, n.d.) provides reports to schools. Despite the expectation that performance data be used to improve teaching and learning (e.g., Boudett, City, & Murnane, 2005), the extent to which this occurs is not clear. A pilot survey of Victorian mathematics teachers (Pierce & Chick, in-press) found low engagement, but never-the-less an expressed desire for guidance on using data well.

Statistical literacy for the workplace

Reading and interpreting statistical reports requires more than conventional literacy: it requires statistical literacy. Analysing and interpreting quantitative data in the context of a school setting—or any workplace—is not a trivial task. The concept of statistical literacy has been well encapsulated by Gal (2002) as the ability to interpret and evaluate statistical information from diverse contexts, and discuss the meanings of, implications of, and concerns about such data and conclusions. For the education workplace this definition encompasses the expectation that teachers should be able to interpret national testing data (being data "encountered in diverse contexts").

Issues surrounding teachers' capacity to interpret and use statistical reports have been noted internationally, as illustrated by three examples. Matthews, Trimble, and Gay (2007), writing from their Georgia, United States experience, expressed concern that teachers need to be able to interpret data in terms of the local context. An Organization for Economic Cooperation and Development (2004) report on the improvement of education in Chile discussed the introduction of national testing in that country. It also found that constructive use of data seemed to be restricted by teachers' lack of capacity to interpret the reports they received. Finally, and locally, a pilot study with junior secondary mathematics teachers and junior secondary English teachers (Pierce & Chick, in-press) suggested that some of these Victorian teachers felt that Australian testing data were difficult to understand.

Framework

In order to analyse "professional statistical literacy" generally, and for education settings in particular, we propose a framework for considering the elements of statistical thinking that are important for those who must engage with workplace data. Other frameworks already exist that address parts of the issue, but they are focused on children's learning rather than the tasks faced by professionals. Curcio's 1987 study of graph comprehension in Year 4 and Year 8 students highlighted the ideas of "reading the data" (read direct factual information on the graph), "reading between [or within] the data" (attend to two or more data points on the graph, often for comparison), and "reading beyond the data" (extend, predict, and infer from the data). More recent work of Shaughnessy and colleagues (1996, 2007) suggests an additional category, "reading behind the data", which addresses the context from which the data arise. Watson (2006) also emphasised the place of context in the interpretative process. The first tier of her three-tiered statistical literacy hierarchy involves understanding of basic terminology, and then the second tier requires "an understanding of probabilistic and statistical language and concepts when they are embedded in the context of wider social discussion" (p. 16). The third tier concerns the ability to challenge and question statistical claims. The statistical knowledge base posited by Gal (2002, p. 10) also indicates the importance of knowing why data are needed, having familiarity with basic terms, and understanding how statistical conclusions are reached.

Our proposal for a framework to encapsulate *professional* statistical literacy is shown in Figure 1. The professional—the teacher in the case of this study—needs to be able to examine the data at several levels, each more complex than and dependent on the lower levels (indicated by overlapping circles). *Reading values* requires a technical understanding of labels, scale, data type (e.g., numerical, categorical) and things like percentage versus percentile. *Comparing values* requires an awareness of relative and absolute differences, early informal inference, and low-level statistical tools. *Analysing the data set* involves being able to consider the data as a whole: observing and interpreting variation, observing and interpreting trends, observing and interpreting changes with time or other variables, and attending to the significance of results.

All statistical data are numbers in context, represented here by the surrounding context that impacts on the data and which should be considered in the teachers' interpretation of the outcome of their examination of the data. First, the *Professional Context* involves knowledge of information recognised within the whole profession and needed for the data set (e.g., meaning of special terms such as "band", "like schools", "VELS level"). Finally, the *Local Context* comprises contextual understanding that may be known by individuals about the specific data set but is not evident in the data set alone (e.g., knowledge of local school situation, knowledge about timetabling issues affecting class composition). The boundary between the two context components may not be distinct, as indicated by the dashed line.

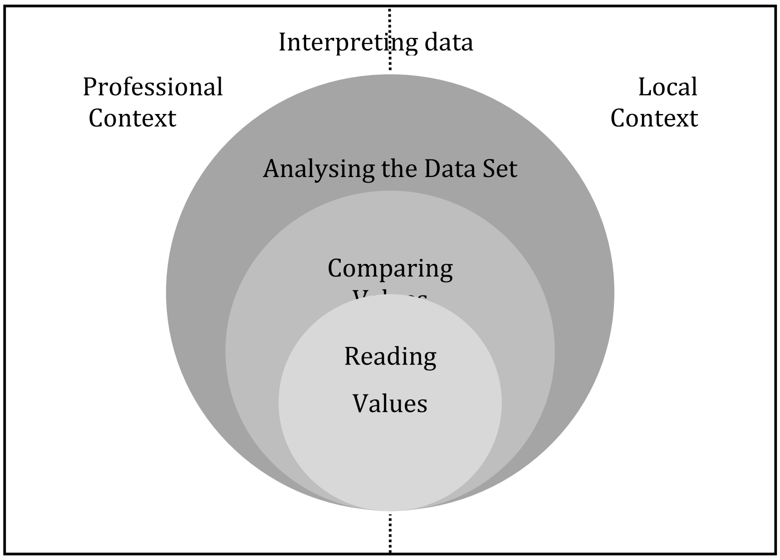
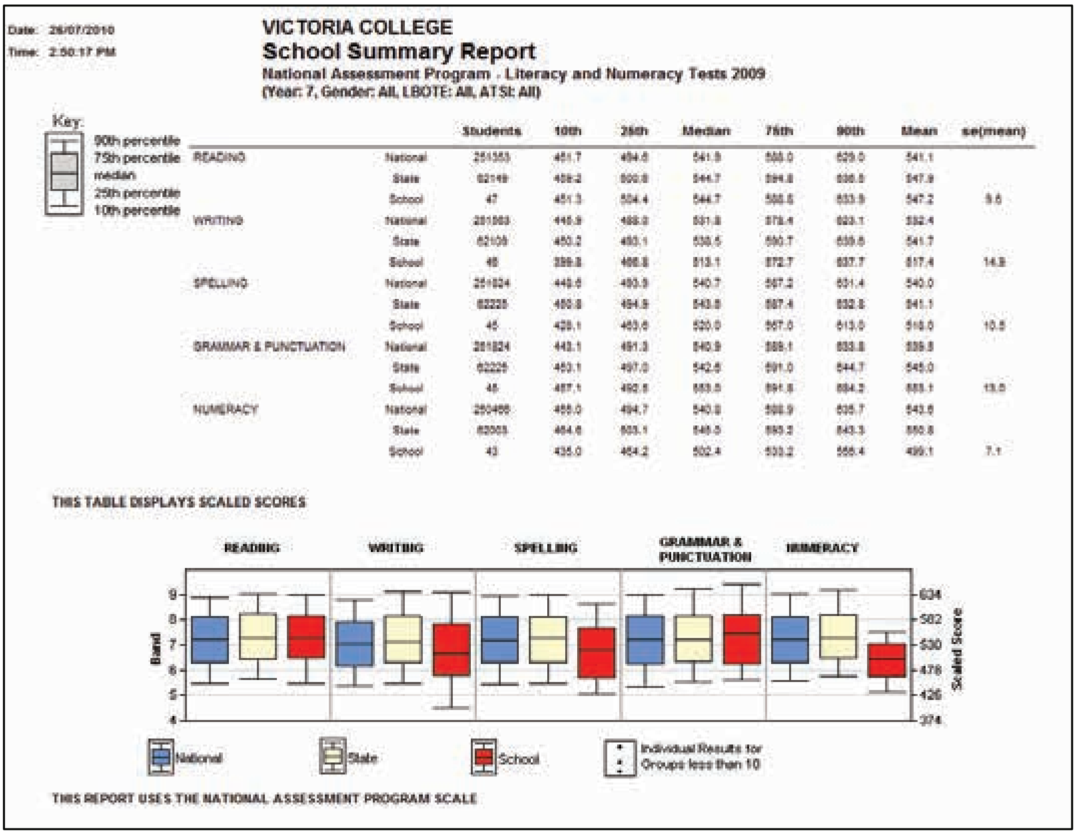


Figure 1. A framework for considering professional statistical literacy.

The study

This study was conducted with teachers from Victorian government schools. The Department of Education and Early Childhood Development (DEECD) operates through a structure of regions. A cluster sample of 20 schools—10 primary and 10 secondary—was obtained by first randomly selecting one network from each of the 4 metropolitan and one of the 5 non-metropolitan regions, then randomly selecting 2 primary and 2 secondary schools from those networks. The school principal (or their "data expert" nominee) together with 7 randomly selected teachers from each school were asked to participate by completing a questionnaire. The first part of the questionnaire probed demographic background, information about the use of statistical reports in each school, and attitudes and beliefs about statistical reports. The second part examined statistical literacy.

This paper reports data from part of one item in the statistical literacy section of the questionnaire, concerning the NAPLAN report shown in Figure 2. It was chosen for this study because it presents both graphical and tabular information, showing school, state, and national data. Three prompts (see Figure 3) probed teachers' affective and cognitive responses by asking about the aspects of the report that teachers thought they would make *use of* and those that were *hard to make sense of*. These prompts focus on teachers' perceptions of the usefulness and difficulty of the NAPLAN report; the resulting data set provides indirect information about the teachers' statistical literacy.



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VICTORIA COLLEGE
School Summary Report
National Assessment Program - Literacy and Numeracy Tests 2009
(Year: 7, Gender: All, LBOTE: All, ATSI: All)

Key: 90th percentile, 75th percentile, median, 25th percentile, 10th percentile

		Students	10th	25th	Median	75th	90th	Mean	se(mean)
READING	National	251353	451.7	494.6	541.9	588.0	629.0	541.1	
	State	62149	459.2	500.8	544.7	594.8	636.8	547.9	
	School	47	451.3	504.4	544.7	588.6	633.9	547.2	9.6
WRITING	National	251563	446.9	488.0	531.8	578.4	623.1	532.4	
	State	62105	450.2	493.1	538.5	590.7	639.6	541.7	
	School	46	399.8	466.8	513.1	572.7	637.7	517.4	14.9
SPELLING	National	251624	449.6	493.9	540.7	587.2	631.4	540.0	
	State	62225	450.8	494.9	543.8	587.4	632.8	541.1	
	School	46	428.1	463.6	520.0	567.0	613.0	516.0	10.8
GRAMMAR & PUNCTUATION	National	251624	443.1	491.3	540.9	589.1	633.8	539.8	
	State	62225	453.1	497.0	542.6	591.0	644.7	545.0	
	School	46	487.1	492.6	553.5	591.6	664.2	553.1	13.0
NUMERACY	National	250466	455.0	494.7	540.8	588.9	635.7	543.6	
	State	62003	464.6	503.1	546.5	593.2	643.3	550.8	
	School	43	435.0	464.2	502.4	533.2	556.4	499.1	7.1

THIS TABLE DISPLAYS SCALED SCORES

THIS REPORT USES THE NATIONAL ASSESSMENT PROGRAM SCALE

Figure 2. *Victoria College's School Summary Report created by VCAA for NAPLAN Data Service.*

> A. Please identify any aspects of this report which you think *might be of use to you as a teacher* by *circling it/them* and annotating the relevant aspects to indicate what is helpful and why.
>
> B. Please identify any aspects of this report which you think *are hard to make sense of* by *drawing an arrow* to it/them and annotating the relevant aspects to indicate what may cause difficulties and why.
>
> C. Any other comments about this particular report:

Figure 3. *Excerpt from survey questions.*

The teachers' responses were pooled, with annotations in some cases to capture key points and any highlighted material, and the resulting data were examined to identify major themes associated with the teachers' views of the value and accessibility of the statistical information in Figure 2. The themes were analysed in the light of the "professional statistical literacy framework". Although statistical skills were not directly examined by prompts A to C, the data provide evidence of how the participants' statistical literacy might drive their reactions about the usefulness of the data. For the purposes of this analysis, no distinction is drawn between principals (or nominees) and teachers.

Results

Data were received from a total of 150 teachers: 41 males and 109 females. The professional and statistical backgrounds of those in the sample were diverse. The secondary teachers came from the full range of disciplines and the primary teacher participants included language and music specialists. Subject, year level, curriculum, and student welfare co-ordinators were also present in the sample. The teachers' mean number of years of teaching was 13.7 (SD=11.2). Their statistical backgrounds also varied widely, with 23.3% claiming never to have studied statistics formally as a subject or topic, 13.3% only having such study prior to or in Year 10, 26% having studied some statistics at Year 11 or 12, and 37.3% having done statistical study beyond Year 12. Sixty percent of respondents indicated they had attended professional learning program(s) related to student achievement data/school system data.

There were two versions of the questionnaire, the second reversing the order of most of the statistical literacy items. This structure meant that our focus report (Figures 2 and 3) was one of the last items on the questionnaire for about half of the teachers, and so not all teachers may have had time to attempt it. Of the 150 teachers in the study, 143 responded to at least one part of the item that included three statistical literacy questions and prompts A, B, and C (Figure 2). Thirty-eight teachers did not respond to any of the A, B, and C prompts, so the data here are from 112 primary and secondary teachers.

Four major themes emerged from the data: (i) technical, statistical issues related to tables; (ii) technical, statistical issues related to graphs; (iii) knowledge or understanding of statistical terms and measures (notably lack of knowledge of "bands", "scaled scores", and "se(mean)"); and (iv) reactions based on personal preferences. The first two themes relate largely to reading and comparing values; the third theme relates largely to the professional context, but also incorporates understanding of the more advanced statistical skills required to analyse the data set; and the fourth theme is influenced not only by technical issues but also by local context. Rather than organise the discussion by the themes themselves, we have incorporated them within the categories of the framework for professional statistical literacy, along with a section on "reactions". The details will be discussed below, illustrated by quotes from the teachers' responses to prompts A, B, and C. In general it will be clear from the content of the quote whether the teacher was responding to A (useful aspects of the NAPLAN report) or B (difficult aspects of the NAPLAN report).

Reading and comparing values — Dealing with graphs and tables

First, it was clear from the data that the teachers wished to be able to read the data to gain some idea of the spread or variation in the scores of the students they need to cater for in teaching. Many of them wrote about the school data and what that meant for the individuals within the school's classes. Second, it was evident that they wished to make at least broad comparisons in order to assess their school's results with those of the state and note differences in the students' performances on different tests.

Turning now to specific aspects of the report deemed useful or difficult, the graphs were commonly mentioned as helpful and easy to read (e.g., quotes 1-3) while the table was more commonly associated with difficulties (e.g., quotes 4-8)

1. [Graph] Graphical representation [is helpful]. It's good to be able to compare the different areas and see the spread of results.
2. [Graph] Boxplots makes it much easier to make comparisons when compared to tables.
3. [Graph] Easy to compare school with state and national results; mean as well as "spread" of student results. Easy.
4. [Table] Hard to make sense of — too many numbers, don't understand layout.
5. [Table] Unsure of 10th, 25th etc [percentiles]; too many figures.
6. [Table] I could not understand the table. I would need someone to explain it to me.
7. [Table] Too many figures and comparisons.
8. [Table] These statistics don't make sense to me.

However, these views were not held by all respondents. In contrast, some found graphs difficult (e.g., quotes 9-10) or tables helpful (e.g., quotes 11-13).

9. [Graph] Hard to read this type of graph [boxplot]. I don't get it.
10. [Graph] I find this report a bit difficult to interpret as I struggle to decipher boxplots.
11. [Table] Good for specific info [Graph] Quick.
12. [Table] Good to show mean here where it is not shown in the graphs below.
13. [Table] The median and mean scores are helpful for determining how our school is performing in comparison to other schools.

There were two difficulties related to reading values that were commonly mentioned, associated with not understanding a specific technical term. The standard error of the mean (appearing in the table of Figure 2 as se(mean)) was specifically highlighted by 17 of the teachers in response to prompt B (e.g., quotes 14–15), and the "scaled scored" or "bands" on the graphs were also mentioned frequently (e.g., quotes 16–18). Although se(mean) is a standard statistical term, it is a concept that might not be considered a necessary part of statistical literacy for good citizenship and is not covered in the compulsory years of schooling. Here, however, understanding se(mean) is necessary for these teachers' "professional statistical literacy". The "scaled scores" and "bands", in contrast, are not so much technical statistical terms, but arise from the professional context of the way in which the NAPLAN test results are processed. This will be discussed further below.

14. [se(mean)] Not sure what this column is?
15. What is se(mean)?
16. Scaled scores or bands - these numbers mean nothing to me.
17. "Bands" aren't descriptive. What classifies a band?
18. [Scaled scores] I'm not exactly sure what these scores mean.

Other issues related to reading the data were noted by the researchers but not by the teachers, and reveal aspects of the teachers' statistical literacy. It is of concern that, while many teachers noted that having a key was useful not all teachers noted the details in the key. The boxplot, as is usual and as the key states, shows the median, not the mean mentioned by some teachers (see quote 19). The key also reveals that the boxplots being used in the graphs are not typical of most boxplots, with the whiskers truncated at the 10^{th} and 90^{th} percentiles. Consequently it does not show the top or bottom students' results (see quotes 20 and 21). Some teachers also have difficulty understanding that boxplots represent percentiles of the student cohort not numbers of students (22).

19. [Graph] The means and range of student outcomes in all graphs gives me a general whole school indication.
20. The whisker showing how top and bottom students are performing [is useful].
21. [Graph: top and lower tails] Highest 10%, lowest 10%, tell us they may need support ...
22. [Graph - school writing] the long tail means that there is a big group of students who need extra support.

Analysing the data set

Most analysis was at the simple level of noting the variability or spread of students' results. Several teachers commented that they would like to be able to see trend data, with one teacher actually trying to get a better picture of the whole data by deriving some additional information (quote 23).

23. [The teacher created a new row in the table noting increased differences between the school and state results for the 10th, 25th, etc., percentiles] As we went up the scale the difference between us and the state became greater—weaker children catered for, top half not?

One teacher wondered if the size of the school group should be taken into consideration when analysing the data. While the teacher's question highlights some lack of statistical literacy, it also reveals appropriate thinking about issues that may need to be considered when analysing the data.

24. Do the huge difference in numbers for each group skew the results? How can 251353 [State] be compared to 47 [School] as sample numbers?

Considering local and professional contexts

Some teachers expressed difficulty or lack of familiarity with details that are part of the Australian education context, i.e., their own professional context. The vertical scales on the graphics show "bands" at the left and the related "scaled scores" on the right. The table shows statistics (to one decimal place) based on "scaled scores". These scores, scaled in theory from 1 to 1000, are produced by the Australian Curriculum, Assessment and Reporting Authority (ACARA), which then divides these scores into "bands" and sets national minimum benchmarks for each year level tested. It is a complex process but it is part of teachers' "professional" statistical literacy to at least be familiar with the parts of the scale that apply to their students (see quote 25). Several teachers commented on the confusion between "bands" and the more familiar numbered levels associated with the Victorian Essential Learning Standards (VELS) (e.g., quotes 26 and 27). Some teachers were not familiar with the acronyms ATSI (Aboriginal or Torres Straight Islander) and LBOTE (Language Background Other Than English) (quote 28), despite the fact that these are now standard acronyms used in schools.

25. Scaled scores or bands - these numbers mean nothing to me.
26. I don't believe anyone understands national benchmarks or its comparison to VELS.

27. Suggest a graphic indicating comparison to VELS levels would be more useful.
28. [Points to the acronym ATSI] What does this mean?

Local context requires knowledge of details of the school and student cohort referred to in the report. This was impossible since the report summarises data from a fictitious school, so teachers made little attempt to explain the findings but suggested generic responses that would be appropriate (quotes 29–32).

29. [Circled sections on school boxplots below the median] What can be done to improve the results; targeted intervention. [Referring to numeracy plot] As a year 7 teacher you would plan to go back to the primary curriculum to give students a greater understanding.
30. [Grammar & Punctuation plot] Why are the scores so high at the top compared with state/national top?
31. [School writing plot] Why such a spread? [Top whisker of Grammar & Punctuation plot] Why so well? [Numeracy plot] Why so low?
32. School appears to be focusing on Reading and Writing but little focus on maths/numeracy. – Hard to judge without previous years' figures to see changes.

Reactions

It was clear that some teachers were overwhelmed by what they perceived as the complexity of the report (e.g., quotes 33-39). Some teachers expressed the view that they did not intend to engage with such reports for a variety of reasons (e.g., quotes 40–42), including sheer cynicism regarding statistics (quote 43).

33. I found this whole sheet confusing.
34. This would be useful if I knew what it referred to.
35. … not keen on tabled data—prefer visual. I would prefer one system:VELS. I don't believe everyone understands national benchmarks or its comparison to VELS.
36. Too many figures and comparisons.
37. Top half of the report: figures don't make sense to me.
38. I need some PD on how to interpret box-and-whisker.
39. This only works for colour photocopiers.
40. I would not use this report to inform my teaching.
41. As the LOTE teacher in the school, I don't feel that this data does a great deal for me.
42. As an English teacher I don't respond well to numbers and tend to dismiss them.
43. ... Still one can make stats say anything, can't one.

Implications and conclusions

The results provide an important snapshot of the way that teachers might respond to the school assessment data that they receive. Their reactions range from those verging on the statistics-phobic (e.g., responses 33 and 42), through to deep engagement with the issues. The contrasts in the reactions of teachers to different types of representations of data (tables versus graphs) was interesting, and has important consequences for those who prepare data for schools. Although there was a marked preference for graphical representations, these were still problematic for some, and others appreciated the detail provided within the tables. Many teachers reacted strongly about the overwhelming complexity of the data, with quotes 33–37 being but a sample of the 50 or so teachers who expressed uncertainty or confusion over some or all aspects of the data.

Although this part of the research project did not target specific skill-based competence within statistical literacy, the teachers' responses to prompts A, B, and C still revealed specific areas of difficulty, particularly with general boxplot reading skills. The non-standard presentation of the boxplot data may contribute to this, especially since other reports (not shown here) are different again.

The results point to a strong need for professional development in the area of professional statistical literacy, and also has implication for pre-service courses. They also alert us to important issues related to teaching statistics at the secondary level, such as preferences for graphical or tabular presentations of data, and difficulties with reading, comparing or interpreting data. Although boxplot representations provide a concise summary of data, many teachers appear to be in need of more experience with interpreting data in this form. The extent of teachers' difficulties and ways of developing their fluency in interpreting such data is an issue for future research.

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STUDENTS' EMERGING INFERENTIAL REASONING ABOUT SAMPLES AND SAMPLING

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This paper investigates students' emerging inferential reasoning about samples and sampling through observation of 13- to 14-year-olds, challenged to infer aspects of an unknown population in an inquiry–based environment. This paper reports on how students working with *TinkerPlots* focus on changing aspects of the samples as the sample size grew larger. Students made connections to key statistical concepts during the process of growing samples and quantified the level of confidence about their informal statistical inferences. They generally recognized the relationship between the sample size and the confidence interval for a given confidence level.

Introduction

Over the past decade there has been an increasingly strong call for statistics education to focus more on statistical literacy, reasoning, and thinking. The Australian Curriculum and Reporting Authority [ACARA] (2010) advocates the broadening of probability and statistics in the school curriculum. "Statistics and probability initially develop in parallel and curriculum then progressively builds the links between them" (p. 2). In particular, these two topics are connected in the study of inferential statistics, in which one makes inferences that are based on data and qualified using probability. The curriculum anticipates that "students recognise and analyse data and draw inferences. They develop an increasingly sophisticated ability to critically evaluate chance and data concepts and make reasoned judgements and decisions, as well as building skills to critically evaluate statistical information and develop intuitions about data" (p. 2).

Related to this, by the end of primary school students are expected to "develop an understanding of sampling" (p. 32) and "consider the need of sampling and recognizing when a census of an entire population is not possible or necessary, and identifying examples of sampling in the media" (p. 33). Year 10 students are expected to be able to "evaluate the appropriateness of sampling methods and sample size in reports where statements about a population are based on a sample" (p. 48).

Developing a sophisticated reasoning about sample data, and sampling "may be associated with developing literacy and social reasoning skills rather than developing numeracy skills" (Watson, 2004, p. 279) because the target reasoning is the cornerstone of drawing conclusions about populations in our society. Such reasoning is embedded in decision-making under uncertainties in different contexts and fields.

Literature review

Selecting samples of data and using samples to draw inferences about unknown populations lie at the heart of statistics. The concepts of “sample” and “sampling” are structurally complex and require the coordination of several concepts including graph interpretation, spread, distribution, randomness, and likelihood (Ben-Zvi, Makar, Bakker, & Aridor, 2011).

Although, literature is replete with research on college students’ conceptions of sample size and representativeness (Tversky & Kahneman, 1971), limited research was undertaken until recently on school students’ conceptions of samples and sampling.

Research on students’ conceptions of sampling by Watson and Moritz (2000a, 2000b), has shown that children as young as 8- and 9-years-olds have relatively naïve conceptions about samples. According to Watson and Moritz, the children of this study were typically comfortable drawing conclusions about a population based on small samples without recognizing any potential problems of bias. Early middle school students (age 13–14) understood the concept of samples in real world situations, but they had difficulties making the transition to the formal statistical meaning and using appropriate associated terminology. Watson and Moritz (2000b) showed that older students (age 14–15) were concerned about potential errors arising from small samples. The observations in the research study of Watson and Moritz show the importance of making explicit the differences between taking a small sample from a homogeneous entity (for example, a small sample of blood) to make generalisations about the larger entity from which it was drawn, with taking a sample from a heterogeneous population that has much variability (for example, a sample from a population of students) to estimate a specific characteristic such as weight. The ideas inherent in sampling from homogenous entities do not generalize to the notion of sampling variation and the need for large samples in making inferences from data. Watson and Moritz have also emphasized the importance of the notions of variation and representativeness when students engaged in a sampling related task.

Watson (2004), who summarizes outcomes of research on reasoning about sampling, notes that students often pay attention to fairness and distrust random sampling methods as a process producing unbiased samples. According to Watson, students prefer biased sampling methods, such as voluntary samples. Other researchers have documented that students and teachers often have difficulties in distinguishing samples from populations when working with data (Pratt, Johnston-Wilder, Ainley, & Mason, 2008; Pfannkuch, 2008). In response, there has been a recent research effort to understand how better to approach the topic from a pedagogic perspective. One response has been *informal statistical inference*, characterised as a process of drawing generalised conclusions expressed with uncertainty from data, which extend beyond the data collected (Makar & Rubin, 2009). Two international research forums on statistical reasoning, thinking and literacy (SRTL-5 and SRTL-6) have been dedicated to the study of how students might make sense of informal inferential processes and reason about inference related tasks. In particular, the definition of *informal inferential reasoning* provided for SRTL-6 in 2008 was “the cognitive activities involved in drawing conclusions with some degree of uncertainty that go beyond the data and having empirical evidence for them”. Three fundamental principles of informal inference were provided: generalising beyond data, using data as evidence of generalisations, and expressing the degree of certainty (due to variability) for the generalisation. The main types of generalisations indicated were predictions, parameter estimates, and conclusions. Making such inferences informally, gives students the sense of the power of statistical techniques in making reasoned judgements and decisions about data from real-world contexts.

Another response has been *Growing Samples*─an instructional idea suggested by Konold and Pollatsek (2002), but then developed by Bakker (2004) and used by Ben-Zvi et al. (2011). Bakker helped eighth grade students who engaged with a sequence "of growing samples" activities to see stable patterns generated by larger samples, thus students understood that larger samples are less variable and better represent a population. Bakker suggested that asking students to make conjectures about the growing samples build students' reasoning about sampling in the context of variability and distribution. Such an approach is helpful in supporting coherent reasoning, based exclusively on the integration of key statistical concepts such as sampling, data, distribution, variability, and tendency. Ben-Zvi et al. used data from a design experiment in Israeli Grade 5 classrooms to show how 11 year-olds develop inferential reasoning about sampling while working with *TinkerPlots*. This research was in line with the literature of growing samples beginning from a sample of size eight from their class (including themselves), and moving to a bigger sample (a whole class) and then to the whole grade in the school. The students not only experienced the limitations of small samples when making inferences about a larger population, but also an emerging quantification of confidence in making such inferences, interconnections of concepts of sampling, and informal statistical inference with key concepts such as spread, distribution, likelihood, randomness, average, and graph interpretation

In this paper, the focus on informal statistical inference and children's reasoning about sampling, emerges out of aspects of the work of Ben-Zvi et. al (2011). Two of the questions for future research as suggested by Ben-Zvi, et. al guide this research study. First, this research investigates how ideas about sampling in relation to informal statistical inference can be further developed in the next stage. It is expected through asking this question that some insights might be gained into the conceptual struggle that needs to take place for 13- to 14-year-olds to engage in inferential reasoning about samples and sampling. In doing so, a constructivist stance is used to search for naïve conceptions that might serve as resources in deploying more sophisticated strategies. Second, this might shed some light on how the instructional idea of growing samples can be further improved and used.

Method

This research study falls into the category of design experiments (Cobb, Confrey, diSessa, Lehrer, & Shauble, 2003). Typically, design experiments require several iterations. This article reports on a pilot study that examined students' exploration of a dataset using *TinkerPlots* (Konold & Miller, 2005).

The learning sequence was built around two sessions of extended data investigations of a student-administered survey from Years 7−9 in the previous school where the researcher taught. The survey gathered information about students' weight and weight of students' backpacks[1]. Afterwards, the weight of a student's backpack was divided by the student's weight, and the calculated percentages were compared with the doctor's recommendations.

Students used *TinkerPlots* to analyse the data collected from the student-administered survey for approximately two hours. Sliders and filters which control the increase and decrease of the sample size and formula-defined attributes were implemented in order to allow the students to have more control over the sample size that they select from the dataset. During the activities, students observed how the animated plots they were studying varied when the sample size slider was used to add cases to the graph.

The design of activities evolved around the idea of growing samples, starting from a sample of size 10, moving to about 30, then 100, and finally the entire population. Using a sequence of "growing sample" activities was a pedagogical design conjecture to help students understand that larger samples better represent the population, progressively developing their inferential reasoning about samples, and a level of confidence students place in their inference.

The researcher conducted clinical interviews with small groups 13- to 14-year-old students, in Year 8 of an Australian secondary school. The researcher worked with students from one class, covering a range of attainment. Students worked in pairs.

While the students were working, Camtasia software was used to video record the computer screen output and audio record the students' voices. The data collected were analysed using progressive focusing (Robson, 1993). At the first stage, the audio recordings were simply transcribed and screenshots were incorporated as necessary to make sense of the transcription. Subsequently, the transcript was turned into a plain account with no explicit interpretation other than through selection of the most promising sections. The less interesting sections were replaced with discursive descriptions of what happened. At the third stage, an interpretative account was written. Episodes were selected to illustrate students' evolving informal reasoning when making inferences about samples and sampling.

The findings are presented below through the case of Rafael (Ra) and Gina (G). Analysis of the data from other students is ongoing.

Results

Stage 1: First investigation with 10 data points

Rafael and Gina expressed dissatisfaction with working from only ten data points (Figure 1) and formed an initial reaction:

[1]The activity was inspired by a report written by students at Hermantown academy, available at www.ga.k12.pa.us/Academics/LS/5[TH]/Backpck/Index.htm

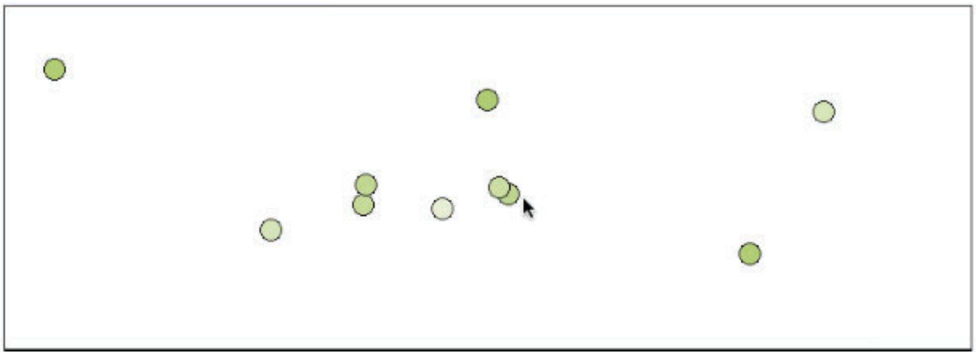

Figure 1. Distribution of 10 data points.

1. G: I don't think we can draw any conclusions about this data because we do not have enough data.
2. Ra: The data we have are too spread out to make inferences about all the students' backpacks.

The students then began to look more closely at the distribution of the data trying to identify an apparent pattern. Rafael characterised the data as too spread out to make inferences about all the students' backpacks (line 2).
Rafael and Gina organised the ten data points to spread across eight categories of backpacks weights, with most categories having zero to three points (Figure 2).

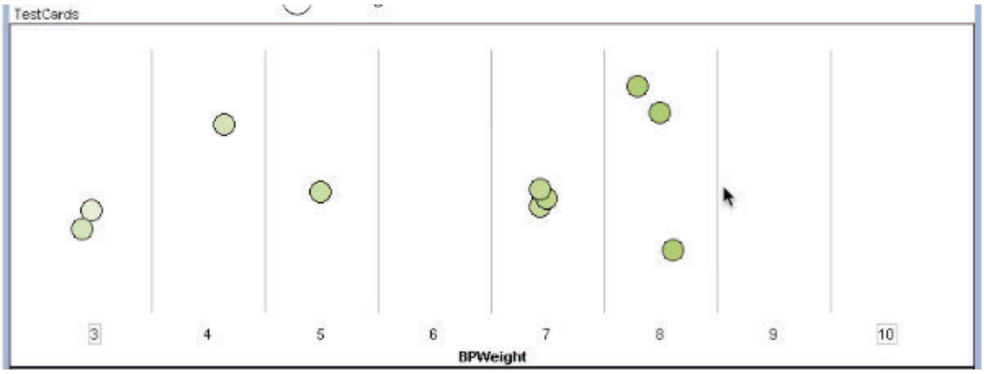

Figure 2. 10 data points spread across 8 categories.

When exploring more carefully the weights of backpacks:

3. Ra: I can see just two packs weighing 3 lbs, one 4 lbs, one 5 lbs, three 7 lbs, three 8 lbs… Most of the packs are 7-8 lbs.
4. Re: Do you think if we weight all the backpacks from all the students from Years 7–9, we will be able to draw this conclusion?
5. Ra: I do not think so. Maybe.
6. Re: Can we talk about all the backpacks from looking only at this data?
7. G: No, there are not enough backpacks to say this represents the entire school.
   I want to see the weights of more backpacks.

Rafael seemed to be able to draw conclusions from numerical data (line 3). The relatively high frequency of backpacks weighing seven to eight lbs attracted Rafael's attention but he seemed to be uncertain whether he could base any inferences about the weights of all the backpacks of all the students from Years 7–9 upon the current sample of ten (lines 2, 5). Similarly, Gina appeared to be reluctant to draw any conclusions about the population from this sample size (line 7), so she suggested investigating the weights of more backpacks.

Stage 2: Second investigation with 30 data points

Students were given data for a class from Year 7 (30 students). Gina and Rafael's immediate reaction was to engage enthusiastically with the investigation of the data points (Figure 3).

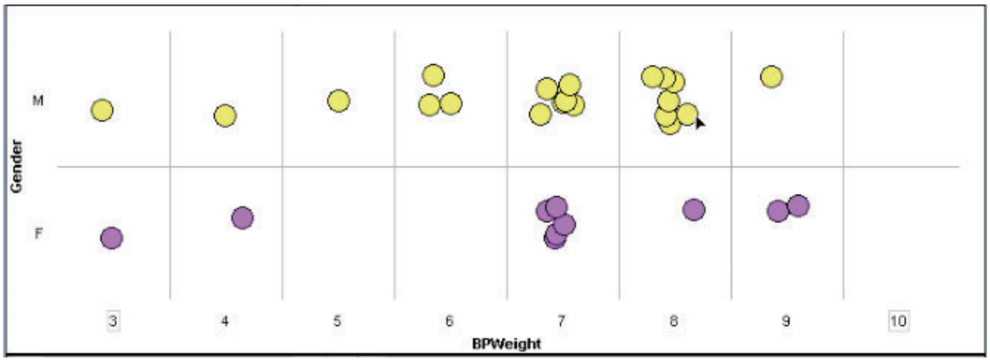

Figure 3. Distribution of 30 data points.

8. Ra: My initial observation still seems true. Most of the packs are heavy. There are more packs weighing seven to eight lbs then there are three-six lbs. I see that most of the boys carry packs that weigh seven to eight lbs.
9. G: This is interesting. We can see clearly that most of the packs are still on the heavy side. We only have one boy that his backpack weighed nine lbs. But we have two girls that carry pack weighing nine lbs…

Rafael's immediate reaction was one of "surprise" when he realised that the prediction he made earlier for the sample of ten hold true more broadly (line 8). A recurring feature of students' investigation was their focus on the changes or similarities that occurred in the appearance of the distribution of data as they compared the new data (sample size 30) with those from the previous investigation (sample size 10).

On the one hand, Rafael did not seem to experience any kind of conflict when drawing conclusions from a small sample. The researcher wished Rafael could see that the small sample size was a flaw in the validity of his inference due to the vagaries of sample variability. On the other hand, Gina found it "interesting" because she did not anticipate that her conclusions would be similar to those she made for the sample of ten. Even though Gina recognised the unexpected similarities (line 9), she did not seem to understand the reasons underlying them.

10. R: Do you think that in general the boys carry heavier packs which weigh 7 to 8 lbs more than girls?
11. Ra: I guess so.
12. G: I cannot tell. The backpacks of 30 students cannot represent all the backpacks of students, but we can better draw conclusions now about bigger samples than when we were given the data for a sample of 10.

Rafael seemed to be more certain than Gina (line 11). On the contrary Gina expressed her lack of confidence in drawing conclusions from only 30 data points although she recognised that the increase of the sample size gave a better basis for their inferences (line 12). When asked to informally quantify their level of confidence about their conclusions from the current sample of thirty:

13. R: Can you give me an interval from 1 to 10 how certain you are the distribution of the data points will remain the same if we carry out the investigation with more data?
14. Ra: 7 to 9.
15. G: This is too high. I'm giving 5 to 6.

Rafael's confidence in his inferences was stronger than Gina. Gina's expressed a lack of confidence in making inferences about the population because she was expecting to observe even small changes on the distribution of data.

Stage 3: Third investigation with 100 data points

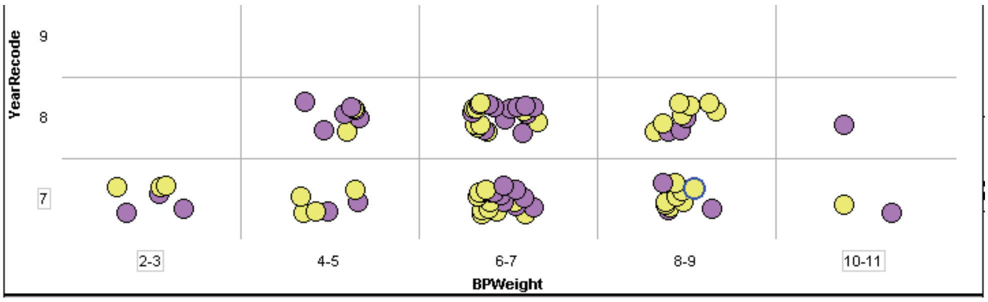

Figure 4. Distribution of 100 data points.

In the third investigation, students were given data for the entire Year 7 and Year 8.

16. Ra: The data points are spread out from four to ten lbs, the data points for Year 7 are closer together, between 4–11 lbs and Year 8 is spread out evenly from 8–14 lbs
17. R: So, do you think these results shows what is happening in the whole school?
18. Ra: Yes. I am giving an 8–9 interval of how certain I am.
19. R: Why not 9-10?
20. Ra: I can not be so sure. You see … The more backpacks we weigh, there will be less room for mistakes or unknown backpacks weights.
21. G: I'm not quite sure too, it is better than what it was before, but I'm not sure if I can talk about all the students in the school. The bigger the sample is, the better the results are.

The students engaged in interpreting the graph (line 16) and tried to focus on even small changes. Of course the process of growing samples produced consecutive images of the distribution of data points that allowed Rafael and Gina to gain some sense of the sample/sample size and population relationships (lines 18-21). Rafael articulated that "the more backpacks we weigh, there will be less room for mistakes or unknown backpacks weights". He seemed to acknowledge the importance of large samples and the uncertainty caused by unexplained variation in the weight of backpacks (line 20).

However, there is perhaps another way to think about what the students were expecting to experience: that is the idea of an ideal sample that can perfectly represent the parent population. This might have proved very convincing, regardless of the size of the sample. Gina suggested:

22. G: Why don't we try something else?

23. R: What do you want to try?
24. G: I want to see how many students' backpacks we need to weigh in order to decide the size of the sample upon which we can draw valid conclusions.
25. R: What do you want to try?
26. G: I want to gradually move the slider of the sample size from 0 to 180 (size of the population) and then move it back from 180 to 0.
27. R: Why do you want to do with that?
28. G: I want to see how spread out the data are for Years 7−9 and how their distributions change as more data values are added or taken out.

The students then spent time adjusting the slider, moving it forward and backwards and looking simultaneously at the representations of data. As they progressed, they expressed their preference to engage in data investigations that involved activities of growing and reducing the size of samples (lines 26–28).

Discussion

The above analysis of students' excerpts sheds some light on the developmental process of students' inferential reasoning about samples and sampling issues. The findings demonstrate that the two students placed highest emphasis on the distribution of sample data to make inferences about the population. It is also evident that students forged new connections about the interplay of sample size and population, and they further linked those concepts to other statistical fundamental concepts during their investigations, such as spread, distribution, (explained) variation in data, unexplained variation, uncertainty, randomness and graph interpretation.

In this paper, there is evidence that the students perceived the importance of large samples (line 21). It is likely that they had a global resource such as the Law of Large Numbers available to them. Nevertheless, students felt comfortable to explore the impact of the sample size on data representations when they engaged in data investigations which involved activities of growing and reducing the size of samples. This shows that the students needed to have a broader experience of the interrelationship of sample size and data representation. The emergent new idea of "reducing the size of samples" or "shrinking the size of samples" needs to be further improved and elaborated.

This paper gives some light into students' emerging quantification of confidence intervals in making informal inferences. The above activity demonstrates students' changes in thinking towards a situated abstraction, which was schematised as "I am giving an 8–9 interval of how certain I am … I can not be so sure … The more backpacks we weigh, there will be less room for mistakes or unknown backpacks weights" (lines 18–20). Rafael seemed to acknowledge how variation (in the weights of backpacks) arises, and the uncertainty caused by unexplained variation in the weight of backpacks (line 20) such as measurement errors. Such understanding of variation in a real situation is prerequisite in making informal statistical inferences.

As mathematics educators, we need to ask ourselves about the level of confidence students place in their informal inferences. It would be interesting to explore the level of confidence students place in drawing informal conclusions about a population based on sample data. Should we be satisfied with increasing our understanding of how such decisions are made or should we consider this evidence as a pedagogic challenge to find ways to support changes in our students' thinking towards an abstraction, which might be schematised as “the bigger the sample we have, the more confidence we could place in our informal inferences”? This is potentially an unusual question for the mathematics curriculum.

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REFLECTING ON PARTICIPATION IN RESEARCH COMMUNITIES OF PRACTICE: SITUATING CHANGE IN THE DEVELOPMENT OF MATHEMATICS TEACHING

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The relationship between the development of the teaching of mathematics and the classroom teaching of mathematics is of considerable interest to teachers and university academics. This article reflects upon the nature of the participation of teachers and university academics as they participate in research communities of practice that use inquiry as a tool to engage with change and development. Conclusions are drawn in terms of the nature of the relationship of university academics to classroom teachers within a research community of practice.

Introduction

For decades educational researchers such as Brown and Duguid (2000) have maintained that the professional development of teachers is best situated within their own practice and best supported by local communities of practitioners. In this paper we critically reflect on this claim by examining the shifting identities of a teacher of mathematics and a university academic as they participate together in different communities of practice – a local research community of practice composed mainly of classroom teachers and an international research community of practice composed mainly of university academics. We suggest that the differences and tensions created by maintaining membership in both these communities create possibilities for professional growth and new insight in mathematics education that transcend the local community level.

The notion, ‘community of practice’, has been described as “a set of relations among persons, activity, and world, over time” and as being “an intrinsic condition for the existence of knowledge …” (Lave & Wenger, 1991, p. 98). It is through participating in the practice of the community that members learn what it means to be a competent practitioner and how they can contribute to emerging practices (Brown & Duguid, 2000).

The notion of ‘participation’ that is deployed in this paper arises from the work of Vygotsky and contemporary sociocultural theorists. According to Vygotsky (1987), learning has its origins in mediated social action and the deployment of tools such as language and other mnemonic systems. Learning results from ongoing engagement in social contexts and is mediated by ways of knowing, doing, and valuing that are socially situated. It is through participating in situated practice that an individual is initiated into the culture of a particular community of practice. In this process a person's relationships to others, to activity, and to the world are transformed over time to show both congruence with and critique of the ways of knowing, doing and valuing of a community (Lave & Wenger, 1991).

In elaborating forms of initiation into the culture of mathematics, van Oers (2002) maintains that discourse plays a pivotal role in raising people's awareness of and engagement with the practices and ideologies of mathematical communities. According to van Oers (2002), it is through the mathematical 'attitude' displayed by community members, for example, classroom teachers and university academics, that aspiring new members of the community, for example, prospective teachers and early career academics, are afforded or constrained in becoming autonomous, critical, and authentic participants in the practice of the community. Attitude as defined here refers to the personal stance manifested by an individual participating in discourse (van Oers, 2002). When this attitude is in accord with a socially accepted genre of mathematical discourse the individual may be said to be employing the 'voice of mathematics'. Bakhtin's (1986) notion of 'voice' provides a mechanism for situating the personal and the social within a particular 'community of practice'. For us (see Renshaw & Brown, 2007), meaning making is simultaneously developed dialectically in accord with Vygotsky's (1987) 'general genetic law of cultural development', and dialogically in accord with Bakhtin's (1986) theory of language. To develop deep understanding of a cultural knowledge, such as mathematics, individuals necessarily have to adopt different voices when speaking within a community of practice and when speaking to different audiences. This continuous dialogical interplay between speakers and audience where ideas and viewpoints are proposed by members of the community allows for the notion of 'voice' to seen as being compatible with the sociocultural approach used in this paper.

Method

The first research community of practice that is the focus of this article was formed within a larger research project designed to examine classroom teachers appropriation of a theoretically informed and research based pedagogy (Collective Argumentation) for teaching mathematics in the middle years of schooling (Years 6 to 9). Collective Argumentation (Brown & Renshaw, 2000) is a sociocultural approach to teaching and learning mathematics that presents students with mathematics tasks that require them to individually represent a solution, compare, explain and justify this solution within a small group of peers and then come to an agreement with this group on a solution to the task that the group can present to the broader class of students and the teacher for discussion and validation.

The large research project employed a sociocultural methodology, based on a 'design-experiment' (see Schoenfeld, 2006). Professional development sessions were used to assist classroom teachers and university academics to reflect upon and assess the nature of their activity as teachers of mathematics. Each session, based upon what van Huizen, van Oers, and Wubbels (2005, p. 273) refer to as the "basic principles of a Vygotskian paradigm for teacher education" oriented participants towards 'ideal forms' of teaching mathematics using the principles of Collective Argumentation. Each session provided opportunities for participants to learn through providing reports of their own classroom practice and through interacting in discussions about each other's performance in the mathematics classroom.

The second research community of practice is an international group of mathematics education researchers comprised of university academics and classroom teachers who meet annually to present and discuss refereed research papers, to view and discuss posters of works in progress, and to engage in symposia about mathematics education. *The classroom teacher*, Sam, who is the focus of this article, is an experienced teacher of mathematics in the middle (students ranging in age from 11 to 14 years of age) and senior phases of schooling (students ranging in age from 15 to 18 years of age) and a long-term member of both research communities of practice. *The university academic*, Ray, an author of this article, is a mathematics educator in a large metropolitan university and a long-term member of both research communities. (For an elaboration of Ray's mathematics teaching journey see Brown, 2009).

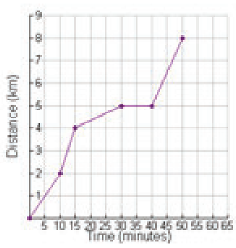
The corpus of data that is the focus for reflection in this paper comprises two data sets. The first set of data relates to a professional development session (the 8th of 12) held towards the end of the second year of the large study. The session, attended by seven classroom teachers and three university academics, focused on inviting teachers to report on the teaching and learning of mathematics in their classrooms. During this session Sam provided a 20 minute report on a mathematics activity that took place in his Year 6 classroom (see Figure 1).

The second set of data relates to an interview Sam provided in the last year of the study whilst he was attending an international mathematics education conference. This interview was conducted by a Research Assistant after Sam had presented a refereed paper (co-authored by Sam and another classroom teacher). The format of the interview comprised 22 open type questions designed to elicit Sam's perceptions of the four interrelated components of knowing and doing within a community of practice as elaborated by Wenger (1998) - 'meaning', 'practice', 'community', and 'identity'.

Analysis and discussion

Sam's report to teachers and university academics

Sam's report focused on the activity reproduced in Figure 1. In the analysis that follows, italics have been used to identify Sam's and others' actual words.



a. You are to write a plausible story for this graph.
b. Suppose there is need to return back to the starting position by 65 minutes, how can this be shown on the graph? What speed would you need to be travelling to allow this to occur? Do you think it is likely that it will be possible to achieve this goal?

Figure 1: Task sheet as presented to a Year 6 class.

In describing the lesson to teachers and university academics, Sam commenced by situating the activity of this class of Year 6 students within a problem-solving context that allowed students to *“give me some information about how they are going in developing … understanding”*. Adapting a textbook activity so that it *“allowed them to use collective argumentation”* and so that the students could go *“away and have a bit of a play”*, Sam’s purpose in the lesson was about eliciting *“a variety of responses”* from students.

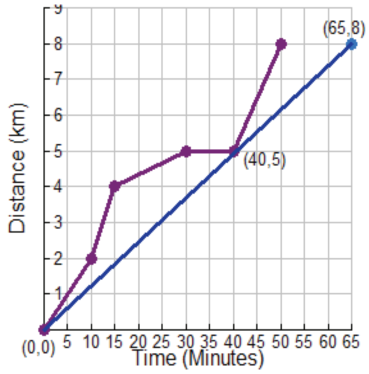


Figure 2: A novel response to a ‘straightforward’ task.

According to Sam, this group of students had translated the problem in terms of the concept of ‘speed’, a translation that Sam considered to be *“pretty cool”*. However, what was *“really cool”* was the return journey where the students “*found the average speed. So they said I want to get home in sixty-five minutes, so this is where home is* [see Figure 2 (0,0)], *so what they found was they drew a point from here* [see Figure 2 (65,8)] *to here* [see Figure 2 (0,0)], *because this* [points to the coordinates (65,8)] *is how far out they are, so they are eight kilometres away and they want to get back home in sixty-five minutes… So that’s the speed he’s got to travel to get back home, and they (the group) found the equation to that particular line*.” Sam then went on to say that he had “*never actually thought about it (the task) like that*” and that “*for them (the group) to interpret it (the task) that way, it’s really cool*”.

Sam’s peers (Julie and Jay) then engage him in discussion about this group response to the task. As the discussion progresses and the teachers struggle to understand this group’s interpretation of the task, it becomes clear that these teachers have adopted a stance within the discourse that privileges thinking about and understanding the mathematics generated by students. The verbal interactions between Julie, Jay and Sam evidence an ‘attitude’ directed toward using students’ representations of solutions to tasks as ‘cultural tools’, that is, as thinking devices that may explain and generate understanding. This ‘attitude’ is given voice by Jay when she states that the student solution “*is a realistic way*” of addressing the task, by Julie when she states that even though “*…we always do it (read graphs) left to right…*”, the group response “*is right*”, and by Sam’s statement that “*…in terms of what I expected* (the students to do) *and what I got, I thought that* (the group response) *was pretty cool*”. By adopting this stance, these teachers, whilst struggling to interpret this novel group response within the conventions used for reading graphic representations, are, at the same time, gaining insights into their own practices as teachers of mathematics. However, this stance does not appear to be the one adopted in the discussion by the university academic, Ray.

During the discussion, Ray questions the novelty of the student response by stating that the solution posed seems a "*pretty natural* (everyday) *thing to do*" and by questioning whether Sam is "*reading too much*" into the thinking behind this response to the task. This questioning is challenged by Sam who states that a "*natural* (conventional) *solution to the task would be to drop a line from the point* (See Figure 2 [50,8]) *to the point* (See Figure 2 [65,0])". Later in the discussion, Ray calls into question the sophistication of the student response by implying that an alternate response to the task that interprets the question as requiring the bicycle trip to be completed from beginning to end in 65 minutes is more sophisticated than the novel group response being discussed where the students have interpreted the task as requiring the home leg of the bicycle trip to take 65 minutes. However, this questioning is challenged by Julie who implies that an unusual response like the group response is "*more interesting*", therefore more sophisticated, than a "*normal* (conventional) *response*" to the task.

These interactions with the university academic provide some evidence of what is being valued in the discussion of Sam's report. For the teachers, privileging thinking about and understanding the mathematics generated by the students seems vital to the teaching learning process. For the university academic, privileging more conventional interpretations of mathematics tasks over student interpretations seems paramount. For the teachers, valuing the "realistic", the "cool", the "reinterpreted", the "unusual", and the "interesting", is highlighted in the discussion of Sam's report. For the university academic, valuing not "reading too much" into a solution, the "sophisticated" and the conventional is highlighted.

According to the teachers, as expressed in their statements during this discussion, student presentations are not just about the representation of correct answers, but about using representations to enhance meaning. That is, using mathematical representations to (a) show a "realistic" way of solving a task (Jay and Sam), (b) "reinterpret" a task construct such as time (Julie), and to (c) show a solution to a task so that others could "understand what their interpretation was and why they did it" (Sam). This view of the function of student representations is in line with Schoenfeld's (1988) expanded notion of mathematics instruction. According to Schoenfeld, teachers may assist students to think mathematically by using appropriate mathematical notations to make conceptual connections explicit and by applying formal mathematical knowledge to problem situations in a flexible and meaningful manner. As such, it may be said that within the discussion of Sam's report, Sam, Jay and Julie have adopted a stance which privileges thinking mathematically and that they are giving voice to an expectation that teachers need to engage in thinking mathematically with the class during mathematics lessons.

However, in the same discussion, the university academic seems to be privileging 'mastery' as he focuses on whether the group of students have mastered the conventions associated with representing information graphically and with due regard to the authority voiced in the problem text. This voice is displayed by the university academic when he poses the question to Sam - "*So are you reading too much into this?*" This utterance places the university academic in the position of the evaluator of Sam's contribution to the discussion. We can assume from Julie's statements, where she focuses on the convention of reading graphical representations from left to right and questions "*why they* (the group) *are going that way* (right to left)", that the university academic's statements are a signal to the teachers that Sam's interpretation of the student response is now 'old' information and that the discussion is now ready for 'new' perspectives on the group's solution to be expressed. However, during the discussion of the report, Jay, Sam and Julie assert and maintain control of the discussion stating that the group solution is a "different interpretation", "pretty cool", and "interesting" and position the role of expertise firmly within the teachers participating in the discussion.

According to Wenger (1998), privileging a practice such as 'thinking mathematically' within a communal context requires privileging the ability to create new meanings. In turn, privileging this ability entails relations of power, in other words, what legitimacy and efficacy does an interpretation of a task have to ourselves and to others within a community of practice. Communities of practice are important sites for the legitimisation of meaning because they define socially accepted ways of knowing and doing (Wenger, 1998). From the analysis of the text of the discussion that accompanied Sam's report of his own practice, it appears that the university academic is a legitimate, but 'peripheral participant' (Lave & Wenger, 1991) in this discussion, despite his attempts to identify himself with classroom teachers when he states "*Alright now just explain to an … old primary school teacher …*" why the group response being discussed was 'cool'. The interactions between the statements of the university academic and those of the teachers imply that the teacher participants in this discussion view the classroom teaching and learning of mathematics as being their domain of expertise.

This positioning of 'power' within this research community of practice raises what Sullivan (2006, p. 307) refers to as interesting "complexities" in the teaching of mathematics. For the university academic it seems important that an alternative solution to the task that interprets 65 minutes as being the total time from the beginning to the end of the bike ride, be a preferred response. Yet Sam, Jay, and Julie have privileged a response that may or may not be acceptable depending on a person's point of view. The nature of this complexity is, perhaps, given voice in the silence of the other four teachers and university academics present during this discussion. However, for the university academic who did contribute to the discussion, the complexity arises from the tension between his commitment to ensuring that the voices that are privileged within the community are organised around criteria that promote systematic, critical and non-contradictory inquiry and his commitment to valuing Sam's reported 'enacted instruction' practices (Herbel-Eisenmann, Lubienski, & Id-Deen, 2006). The nature of the factors that shape Sam's practice is evidence in an interview conducted after his presentation of a paper about his classroom practice to delegates at a mathematics education research conference. Due to the focus of this paper only those responses that refer to Sam's identity are referred to in the analysis below.

Sam's interview at a mathematics education research conference

In response to the question "What role do you think researchers play in the way you teach in your classroom?" Sam made it clear that he values research with references to using *"Teaching for Understanding"* (Perkins, 1992), *"Collective Argumentation"*, *"mathematical modelling"* and *"technology"* when teaching mathematics. However, Sam also made it clear that the value of research lies in its provision of tools to teachers so that they may assist students to *"build understanding"*. For Sam this utilitarian relationship between classroom teaching and research is given voice when he referred to teachers giving theoretical ideas about the teaching of mathematics *"a go"* to see if *"they fit"* or *"they don't fit"* with your *"own way of doing things*". However when asked about the relationship between classroom teachers and university academics the stance adopted by Sam in his response was very much based on the insider-outsider distinction (Smith, Blake, Curwen, Dodds, Easton, McNally, Swierczek, & Walker, 2009). To Sam, university academics are peripheral participants in the community of classroom teachers of mathematics, participants who enter classrooms to *"look"*, *"probe"* and to *"try some things to see if they make a difference"*, but then *"go away"* leaving the teacher to appropriate what they will from the encounter. In the end, Sam voices the conclusion that university academics *"can't necessarily have a big impact on a school or on groups of kids or on teachers because you are always looking in …"*. This *"looking in"* is problematic for Sam because it is sometimes conducted by people who are too far removed from the classroom, who sometimes *"really don't know what"* teaching mathematics in the classroom is like, who *"dabble here"* and *"dabble there"* and who do not effectively communicate to teachers the findings of their research – *"…what happens in between? Who knows?"* Throughout the interview, Sam adopts the stance that classroom teachers are in an *"interesting place"* because they can *"have more of an impact on what can happen in the classroom"*.

Conclusions

The above analyses provides interesting insights into the notion of teacher identity as a teacher of mathematics and a university academic participate in different communities of practice – a local research community of practice composed mainly of classroom teachers and a more global research community of practice composed mainly of university academics. Through the implementation of pedagogical approaches such as Collective Argumentation, Sam allowed students to make their thinking visible. Representations of student thinking were used by Sam in his classroom as a catalyst to compare, to explain and to develop the mathematical understandings of students. In Sam's communities of practice (the local and the more global) representations of student thinking were used to model his teaching of mathematics so that others (teachers and university academics) could explore his teaching in order to consider its effectiveness in developing student understanding. As such, the voice of Sam acting in his research communities of practice is imbued with the meanings, intentions, and accents of mathematics lessons that he has taught and is presently teaching in his local school environment. In this sense, the ways in which meaning is represented, explained and developed, between the communities of practice (the local community focused on professional development and the more global focused on research) is, for Sam, 'critically aligned' (Jaworski, 2006) to his classroom practice.

Within this alignment, difference is privileged and seen as 'interesting', 'novel' and as contributing to the understanding and development of mathematics. Hence, an essential insight gained from this research is that learning about the teaching of mathematics occurs most productively when the professional audience is diverse and includes both local community members of teachers and others, such as university academics, whose taken for granted perspectives suggest novel ways of 'seeing' and interpreting the local practices. In this way, theories of teaching and learning, for example Collective Argumentation, may inform and become interwoven with teachers' everyday classroom practice and teacher education may be presented to teachers as a 'process of becoming' (Wenger, 1998).

Sam's 'process of becoming' as evidenced in the above analyses is of interest in itself as it foregrounds a dialectical tension between his classroom community of practice and the more global mathematics education research community as represented by university academics. The research interests of mathematics university academics are concerned with developing and communicating principled, theory based pedagogies that are well researched and evidenced based. However, this is not the central concern of Sam who draws upon a wide range of approaches to teaching mathematics and tries them out for what they are worth to him as a teacher of mathematics. It is as if Sam is saying that, as a teacher I use various approaches to teaching mathematics and am not constrained by the preoccupations of university academics, so the notion of an 'ideal form' of teaching mathematics seems less important to him.

However, improving student understanding of mathematics, as his report and interview responses attest, is important to him. He, himself, is a researcher who has presented evidence of the effectiveness of his practice to an audience of researchers; however, he remains committed to a local community of research practice. As such, there is a dialectical tension within the 'attitude', within the mathematical voice, Sam displays within his interview, a tension that portrays future zones of proximal development that university academics may negotiate with Sam and others so that talk about the practice of teaching mathematics and how to improve it may become accessible to those beyond the local district. What these future zones of contact may be and where they may be negotiated provides fertile ground for further study.

In turn, Ray, the university academic's process of becoming is of interest. The complexity of providing teachers with opportunities to report on and discuss their own practice of teaching mathematics in the classroom, and, at the same time, provide opportunities for teachers to use the products of research to systematically guide the development of classroom practice and to critically look at its effectiveness, needs to be coped with. Too much emphasis on the former may result in teachers adopting an 'ad hoc' accumulation of 'things that fit' with their ways of teaching mathematics, too much emphasis on the latter may reduce teacher participation in a local research community of practice. Perhaps achieving this balance lies within the provision of opportunities to classroom teachers to discuss and compare their classroom practice with 'ideal' forms as practiced and modelled by university academics in their teaching of mathematics. This tension between the 'ideal' (evidenced based approaches to teaching mathematics) and the more local, contextualised approaches to teaching is necessary to the development of a competence that may extend the local research community of practice beyond its borders (Wenger, 1998).

A second important insight gained from this research is that university academics and teachers need to work collaboratively to build on each other's understanding and to develop knowledge networks that encompass more global educational communities of teachers and university academics. In this way, teachers and academics may encounter a diversity of practices and a diversity of audiences that lead to productive tensions and new insights. As such, the relationship between the teachers and university academics in these communities of practice needs to be reciprocal in nature, a relationship not based on replacing one practice with another but on interweaving (Renshaw & Brown, 2007) classroom practice with the practice and products of research.

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SOME LESSONS LEARNED FROM THE EXPERIENCE OF ASSESSING TEACHER PEDAGOGICAL CONTENT KNOWLEDGE IN MATHEMATICS

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For the past three years, the authors have been using questionnaire items to assess the pedagogical content knowledge (PCK) of primary teachers involved in a multi-faceted professional learning program in Catholic schools in Victoria. We will describe the challenges of developing and coding items which assess PCK in mathematics, levels of performance on various items, and the extent to which change over time was evident. We will also share insights about areas for which professional learning programs might give greater emphasis, arising from the data.

Introduction

Shulman (1986) first introduced the notion of pedagogical content knowledge, which he described as "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (1986, p. 9). Since this time, researchers have attempted to conceptualise and measure the mathematical knowledge needed for teaching (Ball & Bass, 2000; Chick, 2007; Hill, Ball, & Schilling, 2008; Hill, Sleep, Lewis, & Ball, 2007). Barton (2009), in reflecting on the phrase *pedagogical content knowledge* with respect to teaching mathematics suggested that it "includes knowledge about how mathematical topics are learned, how mathematics might best be sequenced for learning, having a resource of examples for different situations, and understanding of where conceptual blockages frequently occur, and knowing what misunderstandings are likely" (p. 4).

In studying teacher knowledge, some researchers developed frameworks (Ball, Thames, & Phelps, 2008; Chick, Baker, Phum, & Cheng, 2006). Ball and her colleagues proposed a model with several categorises encompassing Shulman's Subject Matter Knowledge and Pedagogical Content Knowledge. They include all of these under the domain of *mathematical knowledge for teaching.* Also Ball, Hill and Bass (2005) differentiated between two types of mathematical content knowledge: "We defined mathematical content knowledge for teaching as composed of two key elements: "common" knowledge of mathematics that any well-educated adult should have *and* mathematical knowledge that is "specialised" to the work of teaching and that only teachers need to know" (p. 22).

Some researchers have investigated teachers' PCK associated with a particular domain of mathematics, such as proportional reasoning (Watson, Callingham, & Donne, 2008), area and perimeter (Yeo, 2008), fractions (Watson, Beswick, & Brown, 2006), chance and data (Watson, 2001) and decimals (Chick, et al., 2006), while utilising different instruments of assessment (e.g., multiple choice items, open response items, interviews, and classroom observations).

Background to CTLM

The Contemporary Teaching and Learning of Mathematics Project (CTLM) is a professional learning and research project that will involve 82 Catholic primary schools in Victoria (Australia) between 2008 and 2012. Each school participates in a two-year program with Australian Catholic University (ACU), consisting of 10 to 12 full days of teacher professional learning (including workshops, professional reading, and between-session activities), along with in-classroom support from the research team. Each year a new cohort of schools begins their first year of the project with ACU (i.e., Intake 1 in 2008; Intake 2 in 2009; Intake 3 in 2010 and Intake 4 in 2011). This cycle of professional learning continues until 2012 when the final intake completes their second year. One of the project aims is to enhance teacher pedagogical content knowledge, prompting a need to measure improvement in PCK over time.

The PCK framework

Prior to constructing PCK items, the authors developed a framework that would underpin their construction. There were three considerations that were helpful in this: frameworks for mathematics PCK developed by other researchers; the mathematical content focus of the CTLM Project for the respective cohorts in the given year (whole number, rational number, structure, measurement, space, chance and data); and some of the key aspects of the teacher role on which the project was to focus.

In light of the three considerations above, our current framework has the following components:

Pathways: Understanding possible pathways or learning trajectories within or across mathematical domains, including identifying key ideas in a particular mathematical domain.

Selecting: Planning or selecting appropriate teaching/learning materials, examples or methods for representing particular mathematical ideas including evaluating the instructional advantages and disadvantages of representations or definitions used to teach a particular topic, concept, or skill.

Interpreting: Interpreting, evaluating and anticipating students' mathematical solutions, arguments or representations (verbal or written, novel or typical), including misconceptions.

Demand: Understanding the relative cognitive demand of tasks/activities.

Adapting: Adapting a task for different student needs or to enable its use with a wider range of students.

The authors stress that the framework was not intended to be exhaustive, and clearly is not as broad as some others mentioned previously.

The questionnaires

We developed six questionnaires each year, one each for teachers of Grades Prep–2 (teachers of 5 to 8 year-olds), 3–4, and 5–6, for each cohort. Typically, during the teachers' first year of the project the questionnaires contained between four and six items focusing on Whole Number, (and Rational Number and Structure in years 5-6) and three items in the second year focusing on Measurement, Space, and Chance and Data. Each questionnaire involved items intended to reflect the broad content focus of the CTLM professional learning program, but also provide data on teachers' capacities in each element of the framework, with several items addressing more than one component of the framework. To date, 774 teachers have completed the same questionnaire twice in a single year (119 in 2008, 321 in 2009 and 334 in 2010). 632 teachers completed their first questionnaire this year.

Constructing PCK items

Apart from the considerations above, several decisions were made in our attempt to construct items that assessed teachers' PCK and these were:

- It was intended that the items reflect some classroom scenario to which the teacher had to respond hopefully illuminating knowledge used in the practice of teaching.
- In order to show change over time, most items were chosen to be relatively challenging for teachers early in the year.
- Initially, items were scored out of three points, but as will be discussed later, this was changed in 2010 to scores out of six.
- The questionnaire was intended to take no longer than 40 minutes to complete.

Challenges of constructing and coding PCK items

In order to highlight that this process was difficult and evolving, we note that in the period 2008 to 2010, 39 "new" items were developed and administered. Seven items were administered only once and then dropped. Fifteen items were administered twice (in the same year) and then dropped. Some items had minor changes to wording or format and were not deemed "new" items, while others had substantial rewrites and were considered "new" items. Some difficulties faced in creating PCK items and coding teachers' responses are now discussed.

Highly rated responses early in the year

A reason for some items being administered only once (usually in February/March) was the high quality of responses, indicating teachers were generally proficient in this area and retesting would not show growth or positive change (for example, see Fig. 1). In respect of our framework we categorised this task as related to the Interpreting and Pathways elements of our framework.

In March 2008, most teachers in P-2 who were given the item called "Molly's Method" were able to identify a less sophisticated response, e.g., counting all by ones; and a more sophisticated response, e.g., multiplying 3 x 4.

Molly's Method

Molly, in Year 2, is asked this assessment question:
Here are three cherries in one bunch. How many cherries would there be in four bunches?
Molly says: "There would be 3, 6, 9, 12. That's 12 cherries altogether."
Describe 2 **other** ways a Year 2 child might correctly solve this problem; one **less** mathematically sophisticated and one **more** sophisticated than Molly's.
Less sophisticated:
More sophisticated:

Figure 1. Molly's Method (P-2 questionnaire 2008).

Assessing content only

For some content areas, in particular Space, we had difficulty devising an item that was not an assessment of *common content knowledge* (CCK) "in disguise". (e.g., Fig. 2). However, with some alterations we chose to continue with the use of items like these for Space for four reasons: results from 2009 indicated that many teachers had difficulties with the terminology and attributes of 2D and 3D shapes and we felt an improvement in their content knowledge might have a positive effect on their ability to provide worthwhile classroom examples; the items could be consistently coded; this content *per se* was a key feature of some of the professional development days; and the item could be easily adjusted for the different questionnaires for the various year levels.

Can you do it?

Mr Magoo's Year 6 students were having fun trying to describe shapes and solids which actually don't exist. Mr Magoo made a list of some of his favourites.

(a) Place one tick in each row to indicate whether it is "possible", "impossible", or "I'm not sure."

	Possible	*Impossible*	*I'm not sure*
A trapezium with no lines of symmetry			
An equilateral, right-angled triangle			
A rectangle that is not a parallelogram			
A cone that is a prism			

(b) For one response which you labelled "impossible", please explain why.

Figure 2. Can you do it? (5-6 questionnaire, 2009).

Disadvantages of multiple choice items

The item "Decimal Diversity" (Fig. 3) was dropped after being administered twice in 2008 to teachers in Years 5-6, for three reasons. It was deemed to be assessing only mathematical content and not PCK; teacher knowledge related to decimals was assessed better in 2009 by another item called "Ordering Decimals"; and most importantly, we made a decision at this point not to use pure multiple choice items where the teachers were not asked to explain their reasoning (see reasons below).

Decimal Diversity

Mrs Mason asked her Year 6 students to give her another name for this number. **0.125**
Her students came up with a variety of answers; some correct, some incorrect.
Please circle all of the following which are **really the same** as 0.125. (More than one is correct).

A. One tenth, two hundredths and five thousandths
B. One hundred and twenty-five tenths
C. One eighth
D. Twelve hundredths and five thousandths
E. 0.125%
F. 12.5 percent
G. One hundred and twenty-five thousandths
H. 1 tenth and a quarter of a tenth

Figure 3. Decimal Diversity (5-6 questionnaire, 2008).

We know that multiple-choice items for students without the opportunity to elaborate their decision making are not very useful and can provide deceptive information about understanding (see, for example, Clements & Ellerton, 1995). In our opinion, this is also a major weakness of much of the work of Ball and her colleagues with respect to teachers (e.g., Ball & Bass, 2000). In 2010, 202 teachers of Prep–6 were asked "Is this shape a rectangle?" (see Fig. 4). We categorised this task as Pathways, in particular the "key ideas" aspect of this element of the framework. In February, 33.0%, 45.1% and 64.2% of teachers in P-2, 3-4, and 5-6 respectively, indicated "Yes". However, only 2.7%, 4.9%, and 3.0% of Prep–2, 3–4 and 5–6 teachers respectively could provide an appropriate explanation for children. Some examples of inappropriate explanations provided by teachers who circled "yes" were:

A rectangle is a square by definition therefore a square is also a rectangle
A rectangle is a four sided shape therefore a square is technically a rectangle
It has two sets of parallel sides
A rectangle has four straight sides with opposite sides of equal length

During class discussions, there was much debate about whether this shape (with equal length sides and equal angles) is a rectangle.
Is this shape a rectangle? …………….Yes / No ***[please circle one]***
Please say how you would explain your reasoning to children.

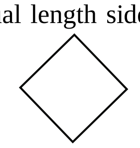

Figure 4. Is this shape a rectangle? (P–6 questionnaires 2010).

It is clear that if those teachers had been scored according to their correct answer (without any elaboration expected), the data would have been misleading.

Limitations of pencil and paper assessment

While we believed it was important to have teachers explain their thinking or justify their choices, sometimes these responses were not well articulated. This sometimes led the coder into making inferences about the correctness of a response or the potential of a described activity about which they were not fully comfortable, making the coding difficult and at times unreliable and ultimately the item unusable. It could be argued that a teacher's written response may not match their intended practice and it is sometimes difficult to know how the teacher intended to enact their idea, without observing the teaching or at least interviewing them to enable elaboration of their response.

We acknowledge that providing a written test to assess PCK is not ideal and that a more comprehensive view of a teacher's knowledge would be obtained through a series of assessments such as observing teacher's practice in the classroom and using interview protocols to supplement questionnaire data. These data would more likely take the form of a case study approach which provides information about a small number of teachers, a small number of lessons, limited mathematical content and possibly only a few aspects of a PCK framework. However, while not as comprehensive as possible, our data provide information about a large cohort of teachers and some aspects of their content knowledge and related PCK.

Constructing items for Chance and Data

Overall we had the most difficulty coming up with items we felt were appropriate for assessing elements of our framework for Chance and Data. In particular, we had difficulty creating challenging items for teachers of P-2, while focusing on Chance and Data content for those grade levels. In 2011 we chose not to include items that assess Chance and Data.

Creating scoring rubrics and applying them

The two main difficulties in creating a rubric for each item were creating a rubric that could be consistently applied across coders (see previous discussion) and one that captured evidence of change over time, if it existed. In 2009, each item was coded out of three and the data from that year indicated limited change over time. It was decided that this *may* be partially due to the coarse rubric. In 2010, we decided to code most items out of six, as we hoped a more fine-grained rubric for the PCK items would help us identify changes that were not evident in the coarser 3 point rubric. Depending upon the item, the results were mixed as to whether the fine-grained rubric was helpful.

For most items, one coder who was proficient with the rubric scored each teacher's responses to items. Where there was any confusion or uncertainty about assigning a code, a second coder was used to confer or check codes. For some items all responses were checked by a second coder. Where two researchers (one of the authors and a PhD student) coded all items independently, they obtained 96% agreement, a very high level of inter-rater reliability (Roche & Clarke, 2009).

Teachers' improved performance on PCK items

Although space does not allow us to provide detailed data for all questionnaires, Figure 5 provides a sample of the results from 2010. These data indicate the mean results per item, for February and October, for Year 5-6 teachers in 2010. As anticipated, the scores for items administered early in the year were generally low and for most items the improvement was also small.

For each task, the element(s) of the framework we judged to be reflected in the item were as follows: Comparing fractions (Selecting and Pathways); Up the garden path (Interpreting); Division stories (Selecting and Pathways); Ordering decimals (Interpreting and Pathways); and Closed to open (Selecting and Interpreting and Adapting).

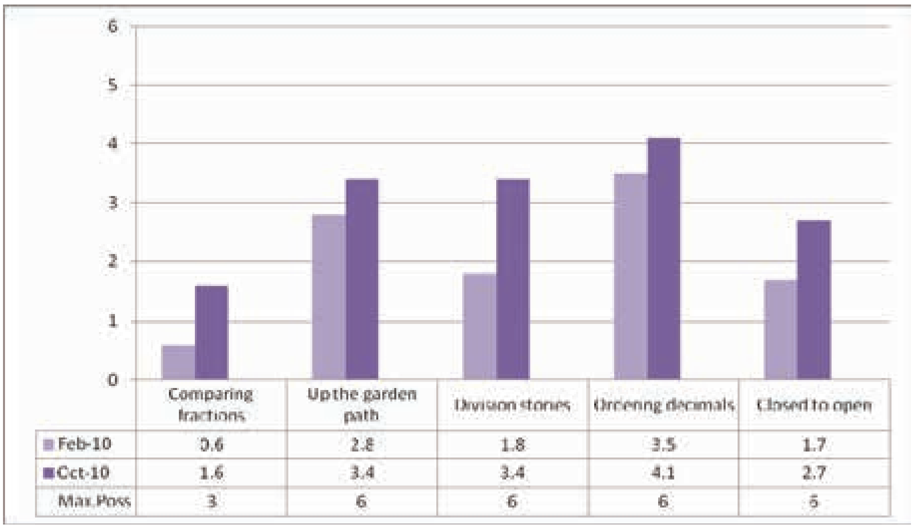

Figure 5. Mean score per item for Year 5/6 teachers (Feb and Oct 2010).

The item for which there was the greatest mean improvement for teachers in P–2 and 5–6 was Division Stories (see Roche & Clarke, 2009 for a description of this item and the scoring rubric). The teachers were required to name the two forms of division, draw a simple picture and write a story problem that represented 12 ÷ 3 for each form of division. The results showed that teachers were more familiar with partition division than quotition division and were generally unaware that quotition division was helpful for making sense of division by a fraction less than one.

In 2010, we added a question within the Division Stories item whereby teachers were asked to solve 8 ÷ 0.5. While the added question is assessing *common content knowledge (CCK)* and not PCK, we would argue that improvement in teacher knowledge about the two forms of division might improve a teacher's content knowledge. Table 1 provides the percentage of teachers who, in February and October, could correctly name quotition division as the form of division most helpful for making sense of 8 ÷ 0.5 and could explain why. It is reasonable to argue that the professional learning program was responsible for the teachers' improvement in their knowledge of the two forms of division.

Table 1. Percentage of teachers who were successful in naming and describing why quotitive division is helpful for division by a fraction less than one (P-6 questionnaires 2010).

	Feb	*Oct*
P-2 (n = 63)	8.7%	30.4%
3-4 (n = 33)	6.6%	23.0%
5-6 (n = 36)	5.5%	30.9%

Table 2 shows the percentage of teachers who could correctly solve 8 ÷ 0.5 in February and October in 2010. It should also be noted that some teachers appeared to use the "invert and multiply rule" to solve 8 ÷ 0.5 as evidenced by their scribbles next to the calculation. This apparent need to use a rule may be one reason why the percentage of success in Table 2 is much higher than for Table 1. Not surprisingly, teachers of Year 5-6 were more successful with this content than teachers of lower grades.

Table 2. Percentage of teachers who solved 8 ÷ 0.5 correctly (P-6 questionnaires 2010).

	Feb	*Oct*
P-2 (n = 63)	44.4%	60.3%
3-4 (n = 33)	57.6%	72.7%
5-6 (n = 36)	77.8%	83.3%
P-6 (n = 132)	56.8%	70.5%

Possible areas of emphasis in professional learning programs arising from our data

After slightly more than three years assessing teachers' PCK using our classroom scenarios approach, there have been several themes that have emerged, which point to possible extra emphasis in professional learning programs. We note the difficulty many teachers have in changing a closed question into an open one, being able to interpret student-invented alternative algorithms; in articulating the nature of a very high quality response to a given mathematics task; understanding alternative methods and solutions in Structure (early algebraic thinking); and creating a story problem to match a particular equation (e.g., 12 ÷ 3 in Division stories task). We have already acted to embed a greater emphasis on these aspects in our 2011 professional learning program.

Conclusion and recommendations

The process of creating questionnaires related to our framework, while challenging, has helped us to think about what we value in PCK and more importantly reflect on our professional learning program and to make adjustments as necessary along the way. It has highlighted the very complex nature of teacher knowledge and the difficulties in defining and assessing all those elements that constitute the act and art of teaching.

We have acknowledged some of the difficulties in measuring teachers' mathematical PCK, such as the limitations of pencil and paper items; designing items that we believed assessed faithfully a teacher's PCK for mathematics; creating rubrics that could be applied consistently; making choices about on which content to focus; and ultimately finding evidence of change over time, if it exists. We note that it may be difficult for teachers to show greater improvement, given the breadth of content of the professional learning program and the limited ability to make an impact on such a large group of teachers. However, we also acknowledge that some improvement was evidenced by our PCK items and that improving teachers' PCK appeared to improve teachers' CCK.

Acknowledgement

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VALUE OF WRITTEN REFLECTIONS IN UNDERSTANDING STUDENT THINKING: THE CASE OF INCORRECT SIMPLIFICATION OF A RATIONAL EXPRESSION

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Encouraging students to articulate their thinking when doing mathematics is a means by which teachers ascertain understanding. Reported here are the results from a content analysis of the written reflections of 67 undergraduate students who incorrectly simplified a rational expression. Although asked to write about the thinking that led them to their solutions, most did not. Instead, they recounted what they had done or had not done. Of those who did write about their thinking, most wrote of their confusion or uncertainty; only a few provided a rationale for the procedures they used. Nevertheless, insights into student thinking were gleaned.

Reflection, silent or articulated, undertaken individually or with others, scaffolded or unaided, has a place in the teaching and learning of mathematics. Carpenter and Lehrer (1999, p. 22) state that ―reflection involves the conscious examination of one‘s own actions and thoughts". In the cognitive science literature, reflection has been described as a metacognitive activity. Sjuts (1999) describes metacognition as ―knowing and thinking about one‘s own cognitive system as well as the ability to control and check this system" (p.76). He explains that while reflection can be seen as a metacognitive process, the subject of the reflection involves cognitive processes, such as learning, remembering, understanding, thinking, and knowing.

Carpenter and Lehrer (1999) argue that communication itself can be a reflective act:

> Articulation involves the communication of one‘s knowledge, either verbally, in writing or through some other means like pictures, diagrams, or models. Articulation requires reflection in that it involves lifting out the critical ideas of an activity so that the essence of the activity can be communicated…. in fact, articulation can be thought of as a public form of reflection. (p. 22)

The benefits of incorporating one form of articulation, written reflection activities, into student learning experiences have been documented for both the school context (Goldsby & Cozza, 2002; Lim & Pugalee, 2004) and the university context (Borasi & Rose, 1989; Parnell & Statham, 2007). Learning benefits have been found in both the cognitive and affective domains. Written reflection can improve students‘ problem solving, mathematical content knowledge, and understanding. It can also provide therapeutic value. For the teachers, student written reflections can inform their pedagogy. It may provide explanatory data about student misconceptions that shed light on worked solutions and verbal responses. Although Payne and Squibb (1990, p. 445) argue that “important insights into the nature of cognitive skill and its acquisition can be gained by examining errors”, making inferences from worked solutions alone has limitations.

An area of mathematics in which student errors and, to a lesser extent, student thinking has been probed is the manipulation of rational expressions. For example, research into students‘ struggle with simplifying rational expressions, also referred to as algebraic fractions, has had a long history (Grossman, 1924; Guzmán et al., 2010; Storer, 1956). In 1924, Grossman wrote

> Every teacher of experience knows that a great many of his algebra pupils all the way from the first year in high school up to college continue with almost comical regularity to make strange mistakes in the subject of “cancellation” in fractions—mistakes that show clearly that the essence of the matter has escaped them. (1924, p. 104)

Almost ninety years later, there exists an extensive literature that classifies the “strange mistakes” students make in simplifying rational expressions, theorises the thinking that may be causing the errors, and makes recommendations for pedagogy. Yet, students at school and in higher education continue to make errors when simplifying rational expressions. The research reported here adds to this body of knowledge in two ways.

The paper explores the merit of post-solution written reflection, a form of “reflection-on-action” (Schön, 1987, p. 27), for collecting explanatory data on student thinking on this topic amongst undergraduate students. This method of generating explanatory data has been rarely used in this context. The literature suggests that spoken reflection through interviews has been the primary means by which researchers have explored student thinking when working with rational expressions (Guzmán, Kieran, & Martínez, 2010; Nishizawa, Matsui, & Yoshioka, 2002). Secondly, undergraduate students‘ understanding of rational expressions does not appear to have attracted the same interest as that of school students.

The research reported here is part of a larger study (Ruhl, 2011) investigating student learning, in particular student errors, in the algebraic component of an undergraduate preparatory mathematics subject at an Australian university. The study analysed three sets of data, namely, the worked solutions to a test students sat upon completion of the algebraic unit, the confidence levels they expressed for each question of the test, and the written reflections on the questions they answered incorrectly. The test, the reflection activity, and the preparation leading up to both were part of the teaching and learning experience of all students in the cohort.

This paper focuses on the written reflections that students who volunteered for the study generated for one question in the test. The question asked students to simplify a rational expression in one variable in which the denominator was already factorised. The solution required factorising the binomial expression in the numerator prior to cancelling the one factor common to the numerator and the denominator.

The question was selected because of the high error rate (86% of study participants simplified incorrectly) and the high level of false confidence. Of those who indicated “I am confident I am right”, 94% were wrong. Similarly of those who chose ‘I am fairly confident I am right‘, 82% of the responses were incorrect.

Method

An algebra test of twenty questions was administered to a cohort of students enrolled in an undergraduate preparatory mathematics subject at university. The subject is equivalent to a secondary school mathematics subject that prepares students for entry into disciplines at the tertiary level where knowledge of calculus is required (such as engineering, or the natural sciences). A range of students enrol in the subject; some have not satisfied mathematics prerequisites on entry to the university, while others are enrolled in degree programs that have no mathematics prerequisite for entry and are required to study this level of mathematics during their degree.

Students sat for the test after having completed the five week long algebra component which comprised approximately the middle third of the subject. It is assumed that students enrolled in this subject do not have any prior algebraic knowledge.

The students had sat for a similar test at the commencement of the algebra course, the results of which had been used for teaching purposes. That test, which had also been administered to other cohorts, served as a pilot to the final modified test.

The test, taken under formal exam conditions, was worth 15% of the total assessment. Students were directed to show all their working for each question attempted.

Ten days after sitting the test, in a 50 minute lecture timeslot, the marked papers were returned to the students. As well as providing a mark, the examiners highlighted for most questions the parts of the responses where the errors had occurred. A set of written solutions for the test questions was also distributed to the students.

Upon receiving their papers, students were invited to write reflections for the questions they had answered incorrectly. Most students spent 30 to 40 minutes on the task writing on average more than 10 reflections.

The reflection task was scaffolded. In addition to the cognitive prompts of errors being highlighted and the provision of worked solutions, there was also the metacognitive prompt asking students to recall the thinking they experienced at the time of responding to the question. The directions given orally and in writing for the written reflective task included the following:

1. If you have an error highlighted in yellow, compare your answer to the worked solution. Note that not all errors are highlighted.
2. Describe the mathematical thinking you were doing that led you to respond to the question in the way you did.

Students had been encouraged in the intervening tutorials and via email to attend the written reflection session. The benefit stressed was that reflection would help maximise their learning from the test in preparation for their final exam. There was also the incentive of gaining up to 2.5% bonus marks for having written reflections.

Experience in writing reflections on their solutions had been included in the five tutorials leading up to the test. Students experienced a range of reflection activities that included individual and group tasks and oral and written tasks. The ongoing constraint the tutor encountered was the lack of time to develop reflection skills; much of the tutorial time involved re-teaching of the mathematics presented in lectures. Students had little or no experience articulating their mathematical thinking and the reflection tasks in the tutorials met with some resistance.

The target question for which the analysis of the reflections is reported in this paper was written as follows:

Simplify the following rational expression completely.

$$\frac{b^3 + 6b}{3b}$$

The solution for the target question given to the students is reproduced in Figure 1.

Q6	Simplify the following rational expression completely. $\frac{b^3+6b}{3b}$	Common factors can't be cancelled, unless the numerator and denominator is factorised first.
	$=\frac{b(b^2+6)}{3b}$	1. Factorise the numerator.
	$=\frac{\cancel{b}(b^2+6)}{3\cancel{b}}$ $=\frac{b^2+6}{3}$	2. Cancel the common factor of b.

Figure 1. Solution for the simplification of the rational expression.

Of the 160 students enrolled in the subject, 151 students had volunteered for the study; of these, 133 sat the test and had provided a response to the target question. One hundred and fifteen of the 133 students (86%) produced incorrect responses to the question; of these, 68 (51%) wrote reflections regarding their response. One reflection was unusable, leaving 67 for analysis.

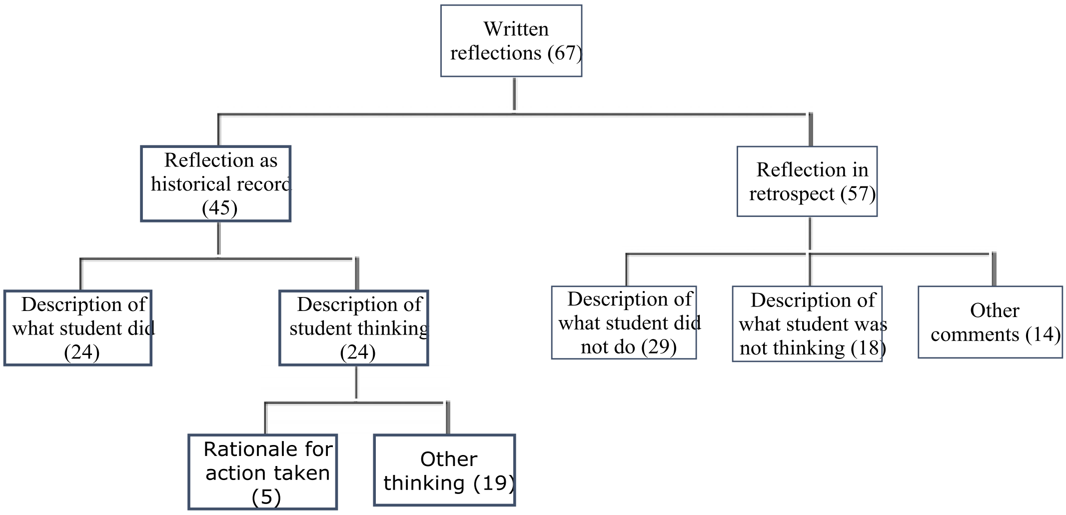
The written reflections were subsequently typed, coded, and recorded using the Nvivo 9 qualitative data processing software. Content analysis was used to categorise or code the reflections in a two part process. On Patton‘s (2002) inductive-deductive continuum along which he places qualitative research methodologies, the analysis used in this study would best be described as inductive but with some tentative pre-existing conceptual guidelines. Using the analogy of a category as being a ―bin” in which data are placed, Miles and Huberman (1985) note that ―any researcher no matter how inductive in approach knows which bin to start with and what their general contents are likely to be”(p. 28).

After the three researchers had read and reread the reflections, the first ―bin” or category became ―Did the students do the task required i.e., did they describe the ‗mathematical thinking‘ that had led them to their response, and if not, what did they do?” This led to categorising the reflections according to type of reflection. The second phase of analysis focussed on how the reflections contributed to understanding the student thinking that led to the specific errors evident in their solutions. The errors in the responses had been categorised prior to analysing the reflections (Ruhl, 2011). This second process required analysing the reflections with reference to the student worked solutions.

The analysis of the data took place the semester following the delivery of the subject. As the main researcher (Ruhl) was the sole tutor for the subject, ethics required that the names of the students who had volunteered for the study would be available only after the results for the subject were released.

Results and discussion

The results of the first phase of analysis in which the reflections were coded according to type are summarised in Figure 2. The number attached to each category indicates the number of reflections coded in that category. The first categorisation used a temporal dimension distinguishing between the reflections that were a historical record, for example, “Tried to cancel the *b* with top and bottom”, from those that were written from the perspective of hindsight or in retrospect, for example, “I should have factorised”. Of the 67 reflections coded, 35 were coded in both categories.



Note: Some reflections were coded in more than one category.

Figure 2. Reflections coded according to type.

The second level of coding subcategorised each of the two sets in terms of whether the reflections referred to “doing” (see examples above) or “thinking”. Two “thinking” examples were “Thought I could cancel because the base letters were the same” and “I forgot to factorise”. The “reflections in retrospect” required a third category called “Other comments” which included reflections such as “A lot more study needed perhaps”.

The third level of coding further categorised the five categories from the second level of coding. However, the figure includes only the coding done for the “description of student thinking” category for the reflections that recorded what students believed they were thinking at the time of simplifying the rational expression. The two most important findings from this phase were the large number of reflections that focussed on “doing” rather than “thinking” and the very small number of reflections that were a rationale for simplifying the rational expression in the chosen manner. The reflection types for each error type identified in students‘ solutions are shown in Table 1.

Table 1. Summary of reflection type for error type in student solutions.

Error			Reflection		
Type	Example	No [1]	Description of what student did[2] (No)	Description of student thinking (No)	
				Rationale	Other
Simple cancellation	$\frac{b^3 + \cancel{6b}2}{\cancel{3b}} \Rightarrow b^3 + 2$	30	18	3	9
Cancellation by subtraction of like terms	$\frac{b^3 + \cancel{6b}3b}{\cancel{3b}} \Rightarrow b^3 + 3b$	6	3	1	1
Cancellation by division of coefficients retaining the variable	$\frac{b^3 + \cancel{6}2b}{\cancel{3}b} \Rightarrow b^3 + 2b$	2	0	1	1
Conjoining error	$\frac{b^3 + 6b}{3b} \Rightarrow \frac{6b^4}{3b}$	1	0		1
Other (e.g., computational errors, and uncodeable solutions)	$\frac{b^3 + 6b}{3b} \Rightarrow \frac{\cancel{3b}(b^2 + 2)}{\cancel{3b}}$	5	1		5
Working absent		2	2	0	2
Total		46	24	5	19

Note 1. The total is 46 rather than 45 (Figure 2.) because one student made two errors.
Note 2. Some reflections were categorised in more than one category.

Of the 24 reflections that recorded what students did in simplifying the rational expression, 15 referred to “cancelling” (14) or “eliminating” (1). Thirteen of the 15 were reflections for solutions where students had made a “simple cancellation error”. A breakdown of these 13 reflections showed that

- 5 referred to cancelling or eliminating “common factors”
- 2 referred to cancelling “variables”
- 2 referred to cancelling “numbers”
- 3 made no reference to what was cancelled
- 1 cancelled “some properties of the expression”

It is possible to infer from these reflections that students realised that simplification of the rational expression requires cancellation of “something” common to the numerator and the denominator with the “something” being described in various ways. However, their understanding of what constitutes a common factor appears to be that it is either a number or a variable that is found in a term in the numerator and in a term in the denominator. Hence none saw the need to factorise the numerator.

Not all cancellation errors however, seem to be associated with failing to factorise. The student script reproduced in Figure 3, for example, indicates that the student knew to factorise the numerator. In this instance, the component of the written reflection analysed stating “took $1b$ from $3b$” reinforces the categorization of this error as a “cancellation by subtraction of like terms” error.

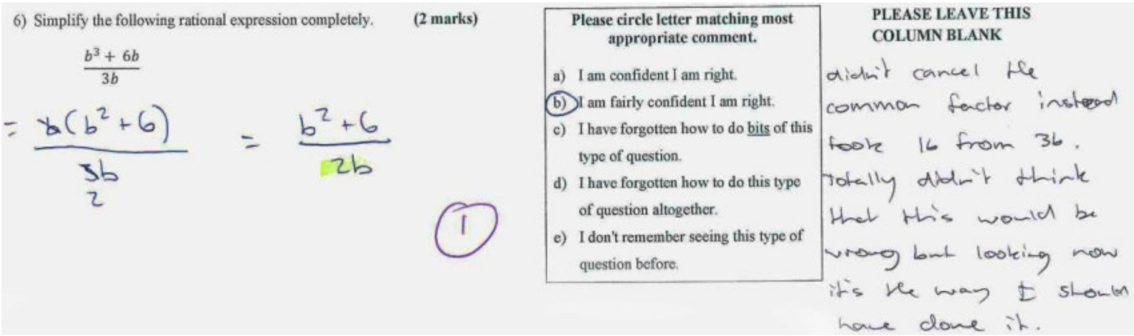


Figure 3. Student script for a cancellation by subtraction of like terms error.

Of the 19 thinking reflections categorised as “other thinking”, all, with one exception, expressed confusion or uncertainty. Examples include, “I got confused” and “Was unsure how to do it”. The exception was that of a student who had performed a “simple cancellation” error. Her script is worth commenting on (Figure 4) because, unlike the previous case, the reflection does not appear consistent with the worked solution.

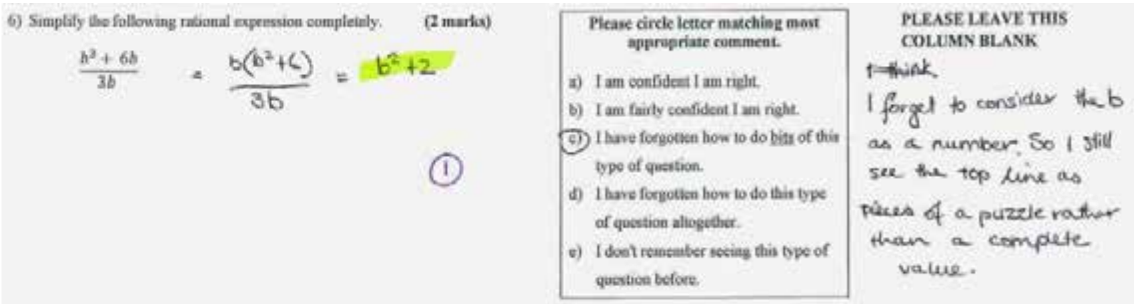


Figure 4. Student script for a simple cancellation error.

Apart from the “simple cancellation” error, the worked solution appears to indicate that the student appreciates the need to factorise the numerator before cancelling. The common factor *b* also appears to have been cancelled successfully. Yet the reflection indicates that perhaps the student does not understand why she manipulated the expression in the way she did. She states, “I still see the top line as pieces of a puzzle rather than a complete value”. This image suggests that the student sees the elements of the numerator as discrete pieces that can be lifted and discarded when their match is found on the denominator.

Finally, the category that explicitly described the thinking that led students to their solutions contained five reflections. These are reproduced below. Four of the set provide the opportunity to see that conceptually similar reflections need not mean similarly worked solutions.

> “I simplified the 6 and the 3 by 3 because they were both common factors of each number.”
> “I believed you could cancel if the numerator and denominator had same letters/symbols.”
> “I was thinking that because 6b and 3b are like terms I could just cancel.”
> “Thought that I could cancel because the base ‘letters’ were the same.”
> “I tried to divide by 3b, because the question said to simplify, I looked for like terms. 6b÷3b=2b.”

The first three reflections were written by students who made the ―simple cancellation” error; the fourth was written by someone who made the ―cancellation by subtraction of like terms” error and the fifth was by a student who made the ―cancellation error involving the division of coefficients while retaining the variable”. Apart from the first reflection, the remaining four seem to share an understanding that cancellation in a rational expression involves the cancellation of ―like terms”. Notwithstanding the common ground amongst the reflections, the corresponding solutions displayed different errors and different end results. The result corresponding to the second and third reflections were both b^3+2; the result corresponding to the fourth reflection was b^3+3b; and the result from simplifying the rational expression that corresponded to the last reflection was b^3+2b.

The results produced in this study were influenced by a number of conditions that may have limited the quality and the quantity of the reflections. Firstly, time constraints meant that learning how to reflect mathematically had been limited. Secondly, the time allocated to writing the reflections was limited; students may have sacrificed depth for breadth. Thirdly, the scaffolding, in particular, the worked solutions, may have strongly influenced the nature of the responses and provides at least a possible partial explanation for the large number of reflections that focussed on the teacher‘s solution as their point of reference.

Conclusions and implications

In conclusion, the findings from this study have implications for pedagogy in the algebra unit of undergraduate preparatory mathematics subjects. Using written reflections to generate explanatory data about student thinking has the benefit of accessing a large number of students in a time efficient way. However, it does not allow for the teacher/researcher prompts that dialogue offers which can lead to richer reflections. Notwithstanding this limitation, the study showed that written reflection provides insights into the student thinking, including its contradictions and anomalies, that contributes to incorrectly simplifying rational expressions as well as revealing the difficulty that students have with writing about their thinking.

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IMPROVING SELF-CONFIDENCE AND ABILITIES: A PROBLEM-BASED LEARNING APPROACH FOR BEGINNING MATHEMATICS TEACHERS

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This paper draws from a pilot study about a teacher education program that focused on building preservice primary teachers' confidence and abilities in teaching and learning mathematics. The cohort involved on-campus [*n*=82] and off-campus [*n*=420] participants. The qualitative study was based on developing three aspects of mathematics teacher education: (1) Content knowledge; (2) Pedagogical knowledge; and (3) Knowledge of the learner. A problem-based learning environment was created to build students' self-efficacy and to encourage the beginning teachers' willingness to engage in the unit content by providing authentic teaching contexts, and to develop a richer conceptual and procedural understanding of mathematics.

Introduction

For many preservice primary teachers, learning to teach mathematics can be a challenging and, at times, a fearful undertaking. Many researchers (Black, 2007; Jorgensen, Grootenboer, & Sullivan, 2010) have discussed the nature of preservice mathematics education, and in particular, how a social constructivist approach can enhance a productive disposition and willingness to engage in learning mathematics. Student-centred learning offers a pedagogical approach for mathematics education in the 21st century where the educational paradigm shifts from traditional, teacher and textbook-centred approaches, to situations where the learner is personally challenged and engaged in a social construction of knowledge.

This paper describes an ongoing project that seeks to investigate a productive learning environment for first-year preservice primary teachers taking an initial mathematics education unit of study. During the first stage of the project, the focus was on the plausibility of a problem-based learning (PBL) approach for enhancing productive dispositions with preservice teachers to teaching and learning mathematics.

Background

Many preservice primary teachers have demonstrated negative feelings and attitudes to learning mathematics (Cady & Rearden, 2007). In addition to poor attitudes, mathematics educators are often faced with teaching students with low mathematical content knowledge and a history of mathematical experiences that are predominantly teacher-centred (Tobias, Serow, & Schmude, 2010). To complicate the situation further, it has recently become necessary to broaden the scope of tertiary teaching and move beyond lecture-plus-tutorial and 9-to-5 approaches, as well deliver units online and via mixed modes. Whilst face-to-face and even mixed mode strategies enable real-time contextual experiences in social situations, replicating this is in an online environment where the preservice teacher experiences the multiple facets of student-centred teaching, is a hurdle that many tertiary educators are facing as we move to a more global classroom environment.

Teachers' work is often described as working within the union of different domains of knowledge. Lappan and Theule-Lubienski (1992) provide a visual model for teacher education that defines at least three kinds of knowledge that a teacher must have in order to teach effectively. These domains are represented visually in Figure 1.

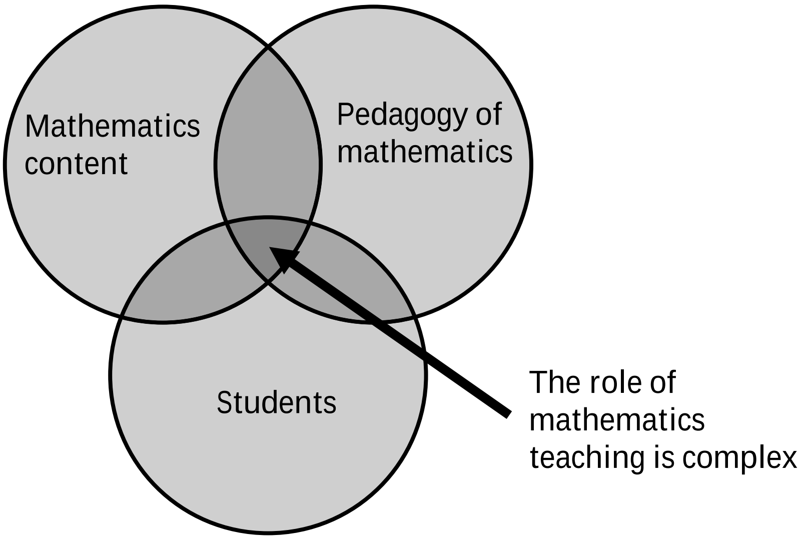

Figure 1. Knowledge domain framework for mathematics teacher education (Lappan & Theule-Lubienski, 1992, p. 253).

It has been previously said that many teacher education programs only teach students these domains of knowledge in isolation from each other (Lappan & Theule-Lubienski, 1992). The lack of integration between these three key areas of knowledge can create divisions between these different aspects of teacher education, and leaves the student without the appropriate experiences and skills needed to reason and analyse their teaching and students (Lappan & Theule-Lubienski, 1992).

Figure 1 depicts effective teaching as the intersection of these three domains of knowledge and identifies the inherent complexity in good teaching. Cooney (1994) appreciated the value of this mathematics education framework and that the task for the teacher was more than imparting knowledge about content and processes. However, Cooney also recognised the complexity of the task for effective teacher education. "The problem is that these different domains are neither mutually exclusive nor clearly defined, thereby making the nature of teacher education anything but a well-defined process" (p. 609). While this paper does not seek to clarify or clearly define these domains of knowledge, it does recognise the benefit that the Lappan & Theule-Lubienski (1992) framework offers, by illustrating the interplay of the different types of knowledge needed for effective mathematics teaching. However, the intention of the research study is to enunciate how the three domains of knowledge interact and are utilized as a model for "effortful" mathematics teaching. It is argued that if we want our preservice teachers to have a positive attitude and enhanced teaching practices when they graduate, then it seems essential that, during their tertiary studies, they need to have authentic and engaging experiences that incorporate the complex nature of mathematics education.

One approach that lends itself to working closely with many interrelationships in domains of knowledge is problem-based learning. This pedagogical approach has been identified over many decades as a successful way to educate students in medical education (Azer, 2007). Since its extensive use in the education of medical students at McMaster University in Ontario, Canada, which began in the 1960s, problem-based learning has spread to many other fields of education including law, engineering, psychology, and architecture (Gijbels, Dobchy, Bossche & Segers, 2005; Peters, 2006).

However, problem-based learning has not been used extensively, thus far, in teacher education. As the problem-based learning approach is designed to use and promote student-centred learning, it appears to have the potential to embrace and place the preservice teachers in the complexity that is inherent in teaching by providing authentic, ill-defined problems that need resolution.

The early pioneers of problem-based learning were Howard S. Barrows and Robyn M. Tamblyn. Barrows and Tamblyn (1980) observed that medical students, who had passed a number of courses in basic medical knowledge, using a non-problem-based learning approach, were not able to sufficiently transfer their knowledge when applying it to the assessment of a patient's condition. This was evident when Barrows and Bennett (cited in Barrows & Tamblyn) investigated medical students as they performed an inquiry on a simulated patient. For the most part, the students would gather data procedurally and try to combine it together later, or make a diagnosis based on a single symptom or sign, without looking deeper for other possibilities.

Barrows and Tamblyn (1980) felt at this time that the current use of problems in the curriculum was misplaced. Problems were often given to students to solve only after they had been given the facts, concepts and principles, either as an example to highlight the importance of the knowledge they had just been given, or as an opportunity to apply this knowledge. However, Barrows and Tamblyn believed a complex problem should be introduced before the facts were known, as a focus for the study to be carried out. Problem-based learning has certain broad characteristics with the central one being that "the problem is encountered first in the learning process" (Barrows & Tamblyn, 1980). They believed that the application of this knowledge helps enthuse students, teach problem solving skills, and aid in retention, and assert that knowledge used is better remembered.

It is important to note from the outset, as does Savin-Baden (2000), that not all learning that involves some kind of problem is problem-based learning. Eng (2000) mentions that with the "explosion" of interest in problem-based learning, concern has arisen that the concepts of problem-based learning will be confused with any educational approach that uses the word "problem", which may then be seen as applying a problem-based learning model. This concern has given rise to the question of what actually qualifies as problem-based learning. Many have asked what characteristics does the learning process need to have in order to be considered a genuine problem-based learning approach. To what extent does the use of problem-solving have to be included in a course to have a genuine problem-based learning status?

Many researchers (Eng, 2000; Savin-Baden, 2000) agree that the characteristics of problem-based learning laid out by Boud (1985) are key features. These are:

- the presentation of a problem occurs at the beginning of the learning process, and that this process is in response to the problem;
- an emphasis on students taking the initiative and responsibility for their own learning;
- more scope for the crossing of boundaries between disciplines;
- a focus on processes rather than products of knowledge attainment;
- a more collaborative relationship between students and teachers;
- an appreciation and accommodation of a student's knowledge and experience at the beginning of the learning process;
- a greater attention to the communication and interpersonal skills so that students understand that in order to relate their knowledge, they require skills to communicate with others; and
- tutors/lecturers are not used as significant sources of content, but rather as facilitators of the learning process, achieved through guiding and questioning.

Whilst it appears in theory that problem-based learning has much to offer mathematics preservice teacher education, the approach has had little investigation using the key features outlined by Boud. This paper reports on the findings of a pilot study that required a four-week problem-based learning intervention as a precursor to assist in the development of a semester long problem-based learning unit in mathematics education. The pilot program implemented is described in the following methodology.

Research questions

The following research questions were used to guide the pilot study and to establish whether a student-centred approach could positively influence preservice mathematics teachers' dispositions to learning and teaching mathematics:

- How do preservice teachers respond to a problem-based learning approach to learning?
- What are some of the implications of applying a problem-based learning approach in teacher education?

Methodology

The problem-based learning approach was undertaken in the initial stages of Semester Two 2010, with 82 (67 female, 15 male) first-year primary preservice teachers, undertaking the Bachelor of Education course at The University of New England in Armidale, NSW. The preservice teachers were enrolled in a first-year, semester-length mathematics education unit of study. All participants are described as continuing students who arrive at university immediately or within a few years of completing secondary school education. This intervention is a pilot study to inform a larger project investigating problem-based learning in the mathematics education context.

The problem-based learning program

The problem-based learning program was conducted over a 4-week duration at the commencement of a semester-long unit (11 weeks). The intervention focussed on early Number using the Count Me In Too framework (NSW Department of Education and Training, 2002). Each week involved a 2-hour tutorial, followed by a 1-hour content lecture. The tutorials were broken into two parts: The Open and The Close.

The Open involved preservice students being presented with a scenario of a student engaged in mathematical tasks, and involved opportunities to determine the student's level of understanding. The scenario was typically divided into two or three packets of information that were released throughout the tutorial time. These packets would often describe the context of the scenario and student work samples in varied forms, such as paper artefact, a description, or video of the student doing a task. The pre-service teachers were expected to discuss, analyse, and critique the information, as well as hypothesise possible educational implications. Once the information had been exhausted, another packet of information was provided. This new information was usually more comprehensive and revealed further detail about the scenario's context and insight into the student's mathematical situation. It was expected that while preservice students were working through the scenario they would identify areas where they believed their knowledge was inadequate to deal with the situation if they were the teacher. These items of need were called Learning Targets. Each person in the group was assigned a Learning Target to study, and asked to report the findings back to the group at the beginning of the next tutorial.

The Close began with students sharing what they had found about their Learning Target. This was usually followed by a whole group discussion to bring the scenario to a conclusion.

At the conclusion of the 4-week PBL intervention, participants were invited to complete an online questionnaire concerning their experiences of learning in a problem-based learning environment. From the sample of 82 participants, 48 participants elected to complete the post PBL survey. The questionnaire comprised of multiple-choice responses and open-ended responses. The goal of the survey was to collect the participants' subjective feedback, as well as their practical experiences of learning mathematics education in a problem-based learning environment. Examples of questions relating to their experiences were "What three things have you most valued about learning through a problem-based learning?" and "What has been challenging about learning using the problem-based learning approach?" The responses to the online survey were analysed qualitatively to identify emerging themes (Miles & Huberman, 1994).

Results

The following results report on the responses to three questions from the online questionnaire. The data highlight some interesting themes from the first-year cohort's reflections of their experiences during the problem-based learning section of the unit. These are presented in tabular form and are a result of a thematic content analysis of the qualitative responses received in the questionnaire.

Table 1 includes the themes identified in the students' qualitative responses when asked "What three things have you most valued about learning through a problem-based learning (PBL) approach?" Table 2 shows the themes of the students' qualitative responses when asked, "What has been challenging about learning using the PBL approach?" Table 3 shows the themes from the student responses when asked, "What could be improved to assist learning using the PBL approach?"

Table 1. Students' most valued aspect about the PBL experience.

Theme	Frequency
Real life/practical	29
Group work	24
Learned teaching strategies	23
Independence/self-directed/own responsibility	14
Structure of the PBL (tutorial/scenario first, then lecture)	11
Creating own learning targets	10
Lectures	8
Discovering resources	3
Logical sequence of unit content	1
Problem solving	1

Table 2. Students' themes of the most challenging aspects of the PBL experience.

Theme	Frequency
Group members not doing work	15
Finding relevant information/knowing what to look for	14
Being inexperienced 1st years	10
Didn't know the answers	8
Group dysfunction	7
Content	2
Repetitive scenarios	2
Lecture spoiled Close (gave the answers)	2

Table 3. Students' advice of what could be improved in the PBL experience.

Theme	Frequency
The Close (in general)	17
The Close (need for tutor/class summary)	13
Accountability	12
Need for more direction, such as question/goals	7
Create a final product/presentation/portfolio	7
Nothing	6
Better explained Close	6
Provide more resources	4
Not every week/too repetitive	3
Glossary/terminology	2
Have lecture first, before tutorial	1
Smaller groups	1
More PBL (pilot too short)	1

Discussion

The results from the post-intervention questionnaire revealed a number of aspects of the problem-based learning experience that the students clearly valued. These highly-regarded attributes can be closely aligned to the general features of student-centred learning, such as collaboration, autonomy in their learning and working on authentic tasks that are relevant to the students. This is an encouraging sign, because experiencing and valuing student-centred learning is one of the goals the researchers set out to achieve. It is hoped that this will assist the students to reform their view of pedagogy from a teacher-centred approach to a student-centred approach.

The most highly valued aspect was the authentic or "real-life" nature of the scenarios. The majority of the cohort appreciated this element of their problem-based learning experience. An example of this appreciation can be seen in the following student's comment.

> It was really good seeing real-life situations. Seeing how things don't work out all of the time … Like we've watched videos in Drama, and everything works out perfectly. The class did everything correctly. But with these [scenarios], you are working on problems, which is what teachers do, they have to work out problems.

The students generally appeared to value the actual goals of the scenarios, such as analysing the mathematical work of the student. They engaged in exploring strategies and ideas that could help develop the student's understanding of mathematics.

> [I valued] how to improve students learning by being able to recognise where the students are having difficulties and as a teacher, what steps to take to help the students succeed.

> By using real-life problems and seeing them occur, it makes it much easier to understand and learn how to fix the problems rather than just being taught about different approaches.

Autonomy and self-directed learning was also seen as a positive aspect to the problem-based learning experience.

> I like how it is a peer-directed option but still have the tutor there to help out, and how we feel in more control of our learning.

> I like the idea of having a problem and having time to locate the answer for ourselves, then being able to check our ideas with the lecturer.

Approximately half the cohort mentioned that they valued the group work and collaboration. Interestingly, group work was also mentioned as one of the greatest challenges they faced while working in the problem-based learning environment.

> [I valued] discussion with group members, to bounce ideas off each other and come up with ideas you would not normally have thought of.

There were, however, areas that need to be significantly improved in order to implement a successful problem-based learning unit. A clear weakness of the pilot program identified by students was the second part of the tutorials, The Close. Many students saw it as ineffective for a variety of reasons. Partly this dissatisfaction with the Close was attributed to the students' belief that it was lacking a clearly defined structure and had limited direction. This was evident in student responses that offered advice on what could be done to improve the problem-based learning experience.

> The Close part of the PBL might need to be more organised and structured, possibly to get more out of it and to come to a final conclusion about the strategies that should be put into place, to help the students.

The Close was also seen as ineffective by a number of students, owing to the lack of contribution by a few group members. It was revealed that some students believed members of their group were not contributing sufficiently, which was due to a lack of accountability, and this resulted in dysfunction within the group.

> It would be good if there was someone to ensure that all group members were doing their share of the work, as it was really focused on everyone being involved. Everyone in the group relied on others to learn certain areas and when they wouldn't do it and you spent a lot of time doing yours, it gets quite irritating.

Group dysfunction has been identified as a very common cause of impeded learning in a problem-based learning environment. However, if the facilitator is only working with a single group, this dysfunction can usually be resolved (Azer, 2007). This raises a challenge that needs to be addressed when a single lecturer is working with multiple groups in a problem-based learning environment, as was the case in this pilot study.

With respect to the inherent complexity of teaching described in the Lappan and Theule-Lubienski framework (1992) and offered in a problem-based learning environment, a small number of students commented that they appreciated the complexity and challenge of the scenarios.

> I personally liked being chucked in the deep end, because then you have to sink or swim. Because then you know that if this happens to me in real life, I know I can do it. Whereas I'd rather have the choice to sink or swim now, than be in a job and don't have a choice.

The results of the questionnaire reveal a number of implications for incorporating a problem-based learning approach in mathematics teacher education. Barrows and Tamblyn (1980) mention that problem-based learning was originally designed specifically for use in medical education. This raises a number of issues for educators wanting to incorporate this approach in areas such as teacher education, which has significantly less resources, and if it is to be used in other modes of education, such as distance learning. Problem-based learning was designed for face-to-face learning with a facilitator for each group of eight students. This is a significant use of resources that are simply not available in current times in teacher education, resulting in a number of practical implications evidenced in this pilot study.

Conclusions

This paper reported on a pilot study used to inform and assist in the development of a much larger main study, which is to be undertaken in Semester 2, 2011. A significant development in the main study will be the inclusion of approximately 400 online students as well as a cohort of 100 on-campus students. Consequently, this significantly broader environment will provide a larger collection of data, including pre- and post-tests looking at attitudinal and pedagogical change of the preservice mathematics teachers. It is anticipated that the main study research evidence will lead to teacher educators gaining greater insight into how preservice mathematics teachers construct their pedagogical understandings in an interactive and technologically rich environment.

Problem-based learning has been an effective pedagogical approach in medicine, architecture and engineering for over forty years. There has been a surge in popularity in the last few decades and it has been used in many other areas of education to enable students to develop their skills and understandings in an authentic and personally meaningful manner. Curiously, the PBL approach is yet to be used extensively in teacher education and is rarely reported in mathematics teacher education. This ongoing investigation offers a potentially powerful means of modelling with preservice teachers an effective student-centred approach for an inherently complex and challenging mathematics education environment.

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STUDENTS' ATTITUDES TOWARDS HANDHELD COMPUTER ALGEBRA SYSTEMS (CAS) IN MATHEMATICS: GENDER AND SCHOOL SETTING ISSUES

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This paper reports on a recent research study that investigated Victorian year 10-11 mathematics students' attitudes and beliefs on the impact of handheld CAS calculators on students' mathematics achievement. Students were surveyed using the Mathematics and Technology Attitudes Scale, which was used to monitor five affective variables relevant to learning mathematics with CAS. Principal component analysis, t-tests, correlations, and MANOVA were used for the analysis of responses. Students' responses indicated that there is a positive correlation between their attitudes towards CAS and their prior knowledge and experience. The results also reflected the common finding that boys express greater confidence than girls in technology use in mathematics learning.

Introduction

The aim of the study was to investigate year 10-11 mathematics teachers' and their students' attitudes and beliefs towards the impact of handheld Computer Algebra System (CAS) calculators on students' mathematical outcomes in relation to gender. This paper focuses only on the students' beliefs.

Computers, graphing calculators and handheld CAS calculators have been used in secondary schools for the learning of mathematics in Australia and overseas for more than two decades. Their use has been supported and advocated through schools' mathematics curriculum and government initiatives (Australian Association of Mathematics Teachers, 1996; National Council of Teachers of Mathematics, 2000; Victorian Curriculum and Assessment Authority, 2005, 2007). Burton and Jaworski (1995, cited in Vale, 2002) expressed concern that the use of computers and other technologies in mathematics might erode advancements made toward gender equity in mathematics. Furthermore, Vale (2002) claimed that the research about gender and computers illustrates the concerns raised by mathematics education researchers about the cultural influence of computers in mathematics and hence the need to carefully examine what is happening for girls in these learning environments. Also, the research into mathematics teachers' and students' attitudes and beliefs about teaching and learning contexts established a series of systematic associations linking teachers'

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attitudes and approaches with their students' attitudes, learning approaches, and outcomes (Prosser & Trigwell, 1999).

An explanation of these associations is therefore important in understanding the significance of investigating mathematics teachers' and their students' attitudes and beliefs of teaching and learning using handheld CAS calculators in mathematics classrooms. Handheld CAS calculators are currently mandatory in senior secondary mathematics classrooms in Victoria, Australia. Thus, it is becoming important for educators and mathematics teachers to know students' perceptions if they want students of both genders to be more successful in mathematics classes. The purpose of the study reported in this paper was to investigate students' attitudes and beliefs about handheld CAS calculators in mathematics learning and to determine if males' and females' views differ.

Literature review

The brief review of the literature that follows explores the studies and findings of previous research on gender differences in mathematics outlined by Ruthven (1995), Fennema (2000), Forgasz (2002, 2003) and others on their analysis of gender and technology in mathematics education.

A major goal of the research on gender issues in relation to technology is to increase our understanding of how gender differences develop and relate to technology in mathematics. However, with regard to gender and technology, the small number of studies, particularly those addressing Victorian secondary mathematics students, gave conflicting results to students' attitudes towards computers and graphics calculators.

Previous research studies on gender differences showed how different ways and methods have been used to minimise the gender gap, not only in mathematics teaching, but also in many fields of study especially in science, engineering and technical fields.

Much research focused on how students' attitudes towards mathematics tended to influence their performance in the subject as well as their future careers involving mathematics (Clifford, 1998; Fennema, 2000). Also, the interactive nature of technology could provide the opportunity for girls, especially, to work independently and become more confident in their learning of mathematics.

In her study focussing on gender and attitudes towards computers in mathematics learning, Forgasz (2002) found that:

> Compared to males, females are generally reported to be less positive about computers, like them less, perceive them as less useful, fear them more, feel more helpless around them, view themselves as having less aptitude with them, and show less interest in learning about and using computers; females are also less likely than males to stereotype computing as a male domain, to have received parental encouragement, to use computers out of school or to own one. (p. 369)

However, research on graphing calculators by Ruthven (1995) found that the performance of upper secondary female students using graphing calculators was clearly superior to that of their male counterparts on items that required visual-spatial abilities. Similarly, Forster and Mueller (2001) suggested that girls are not disadvantaged in mathematics, as often suggested, where the use of graphing calculators is an integral and important part of the teaching and learning and when assessment questions and tasks are completed using graphing calculators.

In Victoria, the 2006–2009 Mathematics Study Design (VCAA, 2005) further extended the use of CAS in the other Units 3 and 4 subjects, allowing handheld CAS calculators into the assessments of Further Mathematics, Mathematics Methods (CAS), and Specialist Mathematics. This introduction and implementation of CAS calculators has resulted in changes to existing curricula, assessment and teaching styles because it challenges the algorithmic algebra and graphing that form the central thread of secondary mathematics (Asp & McCrae, 2000). As mathematics classes in Victoria are on the cusp of a new era in handheld CAS calculators, it seems reasonable to research mathematics teachers' and their students' beliefs and attitudes towards the impact of handheld CAS calculators on teaching and learning mathematics and on the mathematics curriculum, particularly in the Victorian context.

The literature presented here suggests that with regular calculator use male and female students show significant improvement in their mathematical understanding and skills when dealing with mathematical problems. However, this is dependent on the nature of their experiences, including the classroom culture and the teaching and learning activities set by their mathematics teachers.

Research method

The participants were 520 Year 10-11 students from 15 Catholic secondary schools across Victoria. Invitations to participate in this research were sent to 85 coeducational, and single-sex Catholic secondary schools in Victoria. There were 268 (51.5%) students from metropolitan and 252 (48.5%) students from non-metropolitan Catholic schools. Of the 15 schools that participated in the study, three were from high, six from medium, and six from low socioeconomic areas.

In order to investigate the relationship between the students' mathematics confidence, confidence with handheld CAS calculators, attitude to learning mathematics with CAS calculators, affective engagement and behavioural engagement, achievement, gender and year level, the Mathematics and Technology Attitudes Scale (MTAS) (Pierce, Stacey & Barkatsas, 2007) was administered. Five subscales were developed by Pierce et al. (2007), which allowed the researchers to monitor the following five variables:

1. Mathematics confidence (MC): Students' perceptions of their ability to attain good results and their assurance that they can handle difficulties in mathematics.
2. Affective engagement (AE): How students feel about mathematics.
3. Behavioural engagement (BE): How students behave when learning mathematics.
4. Confidence with CAS technology (TC): Students' confidence in using handheld CAS calculators.
5. Attitude to the use of CAS technology to learn mathematics (MT): Students' interaction with CAS.

These variables were selected because they were constructs required to measure students' competence and confidence when using handheld CAS calculators in the mathematics classroom. The instrument consists of 20 items. A 5-point Likert-type scoring format was used for the four subscales MC, AE, TC and MT listed above. Students were asked to indicate the extent of their agreement with each statement, on a five point scale from strongly agree to strongly disagree (scored from 5 to 1). A different but similar response set was used for the Behavioural Engagement (BE) subscale. Year 10-11 students were asked to indicate the frequency of occurrence of different behaviours. A five-point system was again used: Nearly Always (NA), Usually (U), About Half of the Time (Ha), Occasionally (Oc), Hardly Ever (HE), and these were scored from 5 to 1 respectively.

A t-test was used to determine any differences that existed between boys' and girls' responses.

Data analysis and discussion

Factor analysis

The twenty survey items of the MTAS were initially subjected to a Principal Component Analysis (PCA- extraction method: Maximum Likelihood), using SPSS Version 18.0. The five components that were extracted were identical to the five components of the original MTAS by Pierce, et al. (2007*): Mathematics Confidence [MC], Confidence with Technology [TC], Attitudes to Learning Mathematics with Technology [MT], Affective Engagement [AE], and Behavioural Engagement [BE].* Prior to performing the PCA, the suitability of data for a PCA was assessed. Inspection of the correlation matrix revealed the presence of many coefficients of .3 and above. The Kaiser-Meyer-Oklin sampling adequacy value was .87, exceeding the recommended value of .6, and the Bartlett's Test of Sphericity was statistically significant ($<.001$), supporting the factorability of the correlation matrix (Pallant, 2009, p. 197).

The PCA using data from 520 students' responses to the twenty items forming the MTAS indicated that the data satisfied the underlying assumptions of the PCA and that together Principal Component analysis revealed the presence of five components with eigenvalues greater than 1, explaining 29.7% (component 1), 15.3% (component 2), 7.9% (component 3), 6.9% (component 4), and 5.4% (component 5) of the variance respectively. An inspection of the scree plot revealed a clear break after the fifth component. The five components that were extracted were identical to the five factors of the original MTAS survey (Pierce et al., 2007), and those reported by Barkatsas, Kasimatis and Gialamas (2009).

Reliability analysis

Reliability analyses yielded satisfactory Cronbach's alpha values for each subscale of (MTAS) indicating a strong or acceptable degree of internal consistency in each subscale. The lowest value was that of the MC subscale (0.69), however, according to Hair, Anderson, Tatham and Black (2006), the generally agreed upon lower limit for Cronbach's alpha is 0.70, although it may decrease to 0.60 in exploratory research.

Further statistical analyses

In order to explore gender differences in the set of dependent variables, a multivariate analysis of variance (MANOVA) was conducted. Five dependent variables were used (MC, TC, MT, AE, and BE). The independent variable was gender. Preliminary assumption testing was conducted to check for normality, linearity, univariate and multivariate outliers, homogeneity of variance-covariance matrices, and multicollinearity, with no serious violation noted (Wilk's Lambda = .88, $F(5, 177) = 4.57$, $p<.001$). There were statistically significant differences between males and females in two subscales TC (p<.05) and MC (p<.05). Gender differences are examined in the next section.

Gender differences

This section reports results on the five subscales by gender. Only responses from the six Catholic coeducational schools are considered in this section, in which the boys and girls have experienced the same mathematical learning environments. One hundred and eighty four (87 boys and 97 girls) completed all the items of the survey. Background characteristics of the student sample are listed in Tables 1 and 2 below.

Table 1. Students' characteristics by gender in coeducational schools.

	Frequency	Percent	Valid percent	Cumulative percent
Male	87	47.3	47.3	47.3
Female	97	52.7	52.7	100.0
Total	184	100.0	100.0	

Table 2. Students' characteristics by year level in coeducational schools.

	Frequency	Percent	Valid percent	Cumulative percent
Year 10	42	22.8	22.8	22.8
Year 11	142	77.2	77.2	100.0
Total	184	100.0	100.0	

The breakdown of these scores by gender, illustrated in Figure 1 below, revealed that boys have statistically significantly higher scores than girls for subscales TC (t=2.78, df=180, p< .01) and MC (t=3.01, df=180, p<.01) indicating significant gender differences. No statistically significant gender differences were found for the BE (t=-.657, df=182, p=.512), MT (t=.044, df=182, p=.965) and AE (t=.25, df=182, p=.801) subscales. These results reflect the common finding that boys express greater confidence than girls in technology and mathematics, as shown in the respective MC and TC distributions of scores in figure 1, and are similar to those of Pierce et al. (2007) who found gender differences on variables corresponding to TC (Confidence with Technology) and MC (Mathematics Confidence), and less difference on variables MT (Attitudes to the use of CAS technology to learn mathematics) and AE (Affective Engagement).

As reported earlier, *no* statistically significant differences were found for the BE subscale. These results contrast with those of Vale and Leder (2004) who found gender differences only on their variable corresponding to MT. They found that boys view computer-based mathematics lessons more favourably than girls. Vale and Leder (2004) viewed students' attitudes to computer-based mathematics as being defined by the students' perceptions of their achievement in mathematics. They noted differences in boys' and girls' behaviours in mathematics lessons when computers were used: "girls viewed the computer-based learning environment less favourably than boys and boys and girls thought differently about the value of computers in their mathematics lessons" (Vale & Leder, 2004, p. 308).

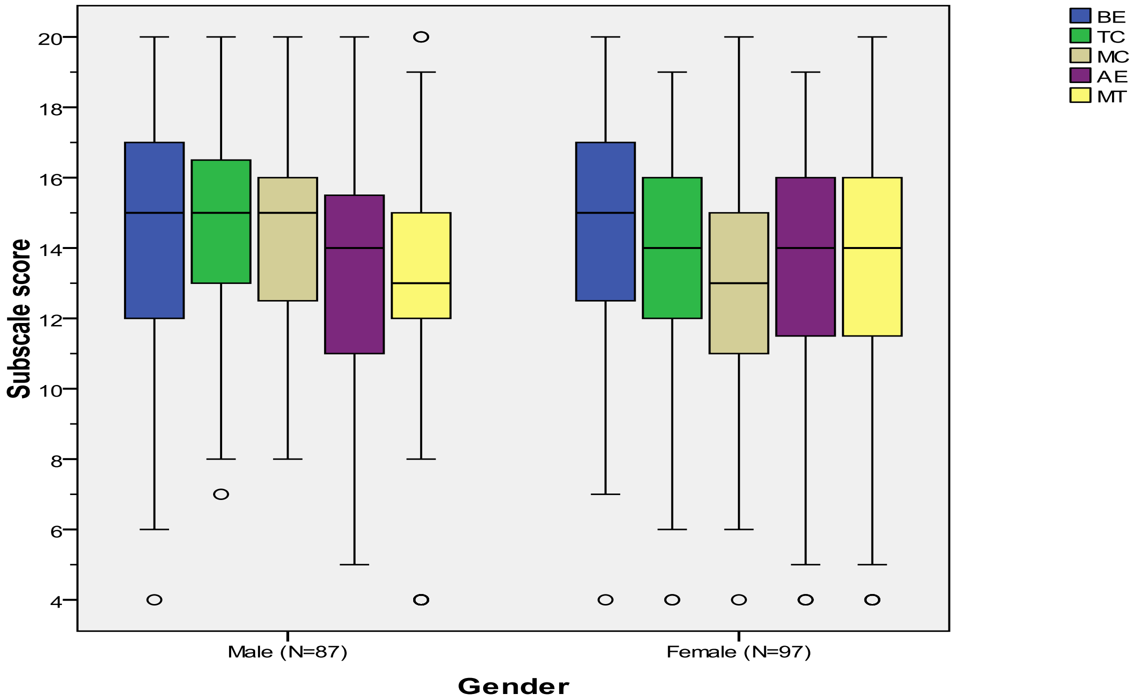

Figure 1. MTAS scores for subscales by gender.

School setting differences

This section reports results on the five subscales by students' school type. Responses from the nine (4 single-sex boys, and 5 single-sex girls) schools are considered. Three hundred and thirty-six students (145 boys and 191 girls) completed all the items of the survey.

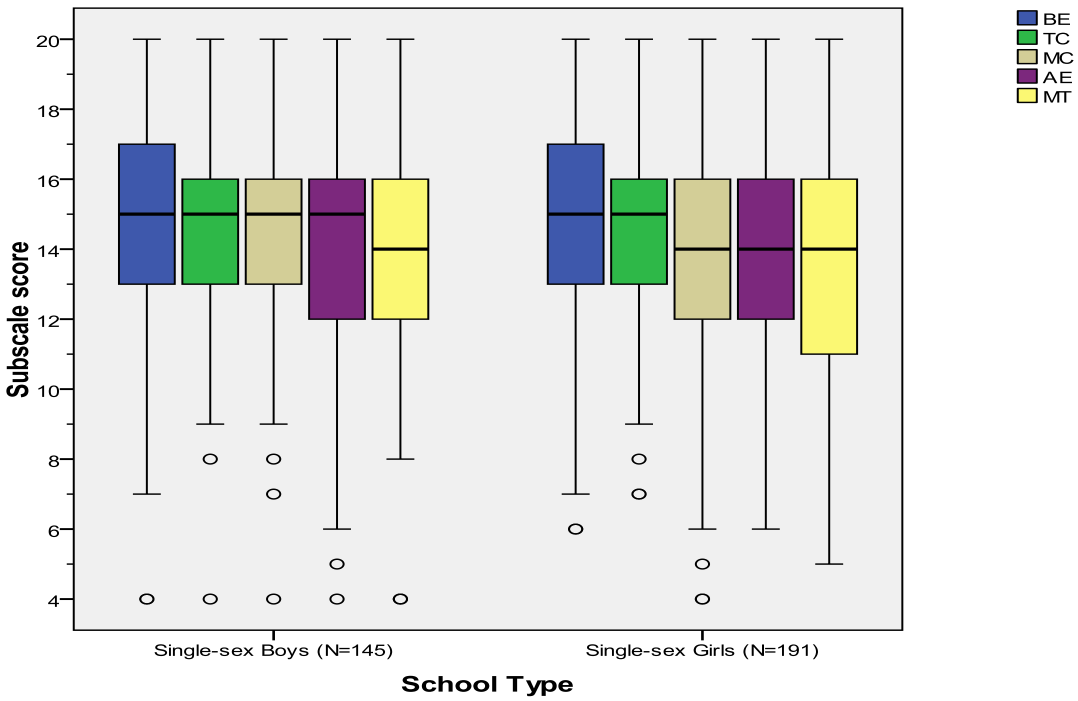

Figure 2. MTAS scores for subscales by school type.

As shown in Figure 1 above, the median, the upper quartile and the maximum value in the MT distribution of scores for girls are all greater than the respective values in the boys' MT distributions of scores, indicating that not all the students with negative attitudes for learning with CAS are girls, and that boys and girls valued using CAS in mathematics lessons. The breakdown of the scores by school type, illustrated in Figure 2 above, revealed that single-sex boys' schools have statistically higher scores than single-sex girls' schools for the MC and the AE subscales. No significant differences between single-sex boys' schools and single-sex girls' schools were found for the BE, TC, and MT subscales.

Conclusions

In this paper we investigated Victorian secondary students' attitudes towards handheld CAS calculators in mathematics learning. The *Mathematics and Technology Attitudes Scale* (MTAS) was used to examine student engagement, attitude, and confidence in learning mathematics with CAS.

The findings revealed that there are statistically significant positive correlation (weak, moderate or strong) between all parts of scales BE, TC, MC, AE, and MT for males and females for the 184 students from the six Catholic coeducational schools. We have two explanations for this positive correlation: 1) there is a strong tendency for year 11 girls and boys who feel confident about mathematics to value using handheld CAS calculator for learning mathematics; and 2) boys and girls are experiencing the learning of mathematics more positively, simply because the use of handheld CAS calculators is currently mandatory in years 10 and 11 in all Victorian Catholic secondary schools, and students value it because they feel it has the potential to compensate for self-perceived shortcomings (Pierce et al., 2007).

The results of the study also indicated that boys in single-sex boys' schools were more confident about their ability to attain good results and also could handle difficulties in mathematics (MC) better than girls in single-sex girls' schools. However, no differences were found in students' confidence in using handheld CAS calculators (TC) or attitudes to the use of CAS technology to learn mathematics (MT). These results are similar to those reported by Forgasz (2008), who analysed the VCE mathematics results for 2007. This analysis revealed a clear pattern of male dominance among the highest achievers in all of the subjects examined, and the proportions of high achieving males far exceeded their proportions of enrolments in the various subjects. The study also revealed that students in single-sex schools, particularly in boys' schools, were highly over-represented among the highest achievers in all three VCE mathematics subjects.

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METAPHORS USED BY YEAR 5 AND 6 CHILDREN TO DEPICT THEIR BELIEFS ABOUT MATHS

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Student beliefs about mathematics are difficult to access and categorize. This paper discusses one method used in an attempt to mitigate this issue. As part of a larger study into Year 5 and Year 6 students' beliefs about the nature of mathematics as well as their self-beliefs about the domain, a subgroup of 185 students completed a drawing task. The metaphors used in these drawings are explored as a way of accessing, grouping, and understanding the range of beliefs held by these students.

Theoretical framework

This study of student beliefs about the nature of mathematics and how they view themselves in terms of mathematics—their self-beliefs about maths—is part of ongoing doctoral research into the beliefs of Year 5 and 6 children, aged between eight and eleven. “Beliefs about knowledge and knowing have a powerful influence on learning, and deepening our understanding of this process can enhance teaching effectiveness” (Hofer, 2002, p. 13). Further, students' beliefs about mathematics relate to their interest and motivation in the subject (Kloosterman, 2002).This paper examines some of the beliefs depicted in a drawing task that was implemented in order to address several of the challenges inherent in accessing and interpreting children's beliefs about mathematics.

Beliefs

Frank Lester (2002) defines belief as “a special form of knowledge—namely, personal, *internal knowledge,*” in contrast to “*external knowledge* which is knowledge resulting from the consensus of some community of practice” (p. 351). He maintains that teachers need to be aware of their students' beliefs because each individual's internal knowledge “directs her or his actions and subsequent learning” (p. 351). Even though the mathematics education community recognises the importance of researching and understanding beliefs about mathematics, questions remain about how to access and interpret these beliefs. Traditionally, beliefs data have been collected either by asking individuals about their beliefs through the use of questionnaires and/or interviews, or by observations. Both of these methods have inherent problems: inferring beliefs from classroom observations is controversial (Lester, 2002) as it is extremely difficult, perhaps impossible, to interpret what is an internal, private belief from external behaviour alone; and self-report measures are also problematic because individuals may respond in ways they think the researcher expects (Creswell, 2003). Moreover, young respondents may not know what they believe or may not be able to articulate their beliefs about mathematics (Young-Loveridge, Taylor, Sharma, & Hawera, 2006). One of the solutions to these problems is to collect data by using multiple methods (Lester, 2002; McDonough, 2004); for example, a combination of various self-report measures and observations at different times. A challenge, here, is to ensure the methods are of interest to the participants as well as allowing them control over how much of their experiences and beliefs are shared with the researcher (Christensen & James, 2008).

Metaphors

Studying beliefs about mathematics is difficult because they are not easy to categorise or analyse. One solution is to explore the metaphors used by the students in their drawings. A metaphor, a device for trying to make meaning of one thing by comparing it to something else (Chapman, 2002; Gauntlett, 2007; Lakoff & Johnson, 2003), is not merely a literary figure of speech but is fundamental to human understanding and to describing and making sense of experience: ―[M]etaphor is pervasive in everyday life, not just in language but in thought and action" (Lakoff & Johnson, 2003, p. 3). Metaphors are used both in language and visual representations (Gauntlett, 2007; Lakoff & Johnson, 2003). Lim and Ernest (1999) describe images of mathematics that encompass both the cognitive and the affective by including ―all visual or metaphorical images and associations, beliefs, attitudes, and feelings related to mathematics and mathematics learning experiences", some of which they classify as myths such as ―mathematics is just computation" and others as metaphorical images of a journey, a skill, ―daily life experience" or a game (p. 44). Picker and Berry (2000) also use the term ―image" when looking at drawings that included metaphors associated with mathematics and mathematicians such as ―maths as coercion" (p. 75), ―the foolish mathematician" (p. 79) and ―mathematicians with special powers" (p. 84). Young-Loveridge et al. (2006) discuss students' beliefs about the nature of mathematics in terms of perspectives, some of which are described through metaphors of utility and problem solving.

Literature review

For the purposes of this research, it was decided to access children's beliefs through a drawing task because images are ―a rich source of understanding the social world and for representing our knowledge of the social world" (Freeman & Mathison, 2009, pp. 109–110). Drawings are a vehicle for researchers to access children's lived experiences (Anning & Ring, 2004; Golomb, 1992; Hubbard, 1989; Veale, 2005) and image-making is one of the ways children make meaning of the world. Drawing is often viewed as an enjoyable activity that children choose both in and out of the of the classroom as a medium through which to communicate experiences, feelings and beliefs (Christensen & James, 2008; Veale, 2005).

Within the classroom, it can be unclear what individual children understand and believe about a topic or an area of study, particularly in situations where they have problems with articulating exactly what they know or mean. Because drawings or other image-making tasks involve a different sign system, however, children are given an alternative way to communicate (Sidelnick & Svoboda, 2000), an alternative medium for explaining concepts and experiences that are difficult to put into words (Golomb, 1992; Veale, 2005). A drawing task is also a familiar activity that is easy to administer in the classroom.

Recently, image-based data have been used in education research (Kilpatrick, Carpenter, & Loma, 2006; McDonough, 2004; Sidelnick & Svoboda, 2000) and in health studies (Backett-Milburn & McKie, 1999; Horstman, Aldiss, Richardson, & Gibson, 2008; Veale, 2005); however, few authors address the issue of how to analyse drawings in a systematic way (Rose, 2007).

As with any other one-off method of data collection such as questionnaires and single interviews, the researcher or classroom teacher needs to be aware of the extent to which the data are influenced by the context of the moment, bound both by time and place. As a result, it is important to use information from additional sources such as observations as well as written and spoken responses when interpreting drawings. Thus for this research, the contents of the drawings have been coded and interpreted by looking at the written text many of the students chose to include with their drawings, and in terms of classroom and school context observations as well as some interview content.

This use of mixed methods is not new. Picker and Berry (2000), for example, analysed images of 476 students in five countries in conjunction with questionnaire responses to identify images of mathematics and mathematicians as coercive (mainly domineering teachers), foolish, and overwrought, as well as brilliant, and possessing –special powers” (p. 75). Picker and Berry noted the words children included in drawings and writing about the drawings (particularly questions), the size of elements, and features of the characters and other aspects of images such as classrooms. They analysed common themes and concluded that –there is more agreement than disagreement across countries about mathematicians among pupils at the lower secondary age” (p. 91).

Research design

In contrast to Picker and Berry's (2000) research, the participants in this research were primary students and more varied types of data were collected. The participants were 823 New Zealand primary school students from 17 schools who answered a mathematics beliefs questionnaire that included a combination of open- and closed-questions. In addition, a subsample of 185 students at two focus schools completed a belief drawing task, mathematics classes were observed, and video and audio-recordings were made in two focus classrooms. A year later, nine students and nine teachers were interviewed.

The data were analysed by the first-named researcher using a combination of quantitative and qualitative approaches. The data from these drawings were coded in terms of mathematical content, metaphors used, affect (Goldin, 2002) and utility, and entered into SPSS. Initial codings were discussed with teachers from the focus schools. Final coding decisions were checked with a colleague who is using a similar method for analysing data from children's drawings for his research. The choice of the first three categories was based on the frequency of appearance (after Glaser & Strauss, 1967), while utility was included both because of its prevalence at the Blue School and the Young-Loveridge et al (2006) findings. Cross-tabulations enabled a comparison of the frequencies in terms of gender, ethnicity and school.

The beliefs that were of particular interest in the analysis of questionnaire and interview responses as well as the drawings were the participants' epistemological beliefs about mathematics. This was both in terms of the nature of knowledge and truth, as well as the mathematics self-beliefs that individuals use to predict or explain how well they achieve in a specific domain (Schunk & Pajares, 2002). Some of the questions, for example, asked participants to describe the nature of mathematics and how they viewed the world of maths; others asked them how good they thought they were at mathematics, or how they saw themselves as engaging and achieving within this world. For the drawing task, students were asked to draw ―what maths or doing maths means to you". Both the nature of mathematics and doing maths were included because of Lim and Ernest's (1999) findings that participants have difficulty in discriminating between the two aspects.

The results section below focuses on the drawings, interpreted in terms of their content but also through background information from the some of the participants' written responses, interviews, and classroom behaviour.

Findings

All of the students chose to complete the drawing task; in addition, many of the participants wrote more on their drawings than they had on a written task about the nature of mathematics. A very brief quantitative summary is included first. In the following subsection, a qualitative summary of selected metaphors will be presented.

Quantitative results

Most of the drawings (90%) included some depiction of the content of mathematics: in particular, number and basic operations (83%), geometry (25%), measurement (22%), and algebra (11%). However, this information was communicated in very different ways. Overall, 67% of the students included metaphors in their drawings to explain ―what maths or doing maths" means to them, 59% included some aspect of affect, and 13% used metaphors of ―maths as useful". Fewer girls than boys used metaphors (61% as opposed to 73%), or included affective elements (53% c.f. 65%), but more girls than boys included notions of utility in their drawings. Asian and Pakeha students more frequently included metaphors and affective elements in their drawings than did Maori and Pasifika students. Under affective elements, students included images, metaphors and words to indicate concepts like ―maths is fun", ―maths is exciting", ―maths is boring", ―maths is terrible": 70% of these suggested positive feelings, 40% negative, and for 42% a combination of positive and negative feelings. There was a marked difference between the drawings at the two schools, which is summarised in Table 1.

Table 1. Percentages of students who include metaphors, affective elements, and utility, compared by school.

School	Metaphor	Affect	Utility
Blue: N= 42	74%	26%	45%
Red: N=143	65%	69%	4%

Although more students at the ―Blue School" seem to include metaphors in their drawings (74%) than at the ―Red School" (65%), if the numbers are adjusted by removing metaphors that only refer to the utility of mathematics, then only 45% of Blue drawings include metaphors. This suggests that the students at the Blue School view mathematics in a much more utilitarian way than the students at the other school do (45% and 4%). This difference, as well as the difference in including affective elements, suggests that there are major differences in how the students at these schools view mathematics.

Reading the metaphors

One of the problems with a quantitative analysis of drawings is that the complexity of the metaphors and the distinctness of the individual voices tend to get lost. If metaphors are viewed as an essential to understanding of concepts, then it is important to explore them as a way to access what individual students believe and understand. All of the drawings were analysed, but, because of space constraints, only a small sample of the students' metaphors can be included here.

The nature of mathematics

A wide range of beliefs about the nature of mathematics, from the extremely narrow to a universal view, appeared in the drawings. For some students, number and/or the utility of maths were important: Ella[1], for example, includes a long explanation on the back of her drawing, and incorporates images of money, a person working as a ―cashier", a calendar and ―a teacher [who] is teaching the child so she can get a job involving numbers."

Other children depicted mathematics in universal terms, ―as life", as something that underpins all of existence. For instance, Zach's picture included a volcano, the sky, sun, and fishes in the sea with a sprinkling of algorithms and symbols. He explains, ―Well, you know maths is everywhere. It's in the sky, in the volcano, and under the sea".

Katia used the sea to reflect her understanding of the never-ending universality of mathematics. She also views mathematics as a separate culture or world with its own language and symbols.

Other students use geographic metaphors like ―Numberland" or ―Mathsland".

Self and mathematics

In many of the drawings, especially the more complicated ones, notions of the nature of mathematics become entangled with the individual students' views of themselves and mathematics. Tom (Figure 1) views himself in ―Mathsland" as if on a quest, with words and concepts reflecting metaphors associated with computer gaming. ―It is raining numbers in Mathsland. I almost fall into a new equation." He leaps over ―the hurdle of maths … A new strategy comes flying at me. I get to know it later on". There is a sketch of Tom patting a purring strategy, and there is the ―Evil Textbook" to avoid. Other students also use gaming imagery by portraying themselves as figures of power, the holder of knowledge, a king, or ―Plus Man" in the world of maths.

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[1] All names of students have been changed, usually to an alias of their own choosing.

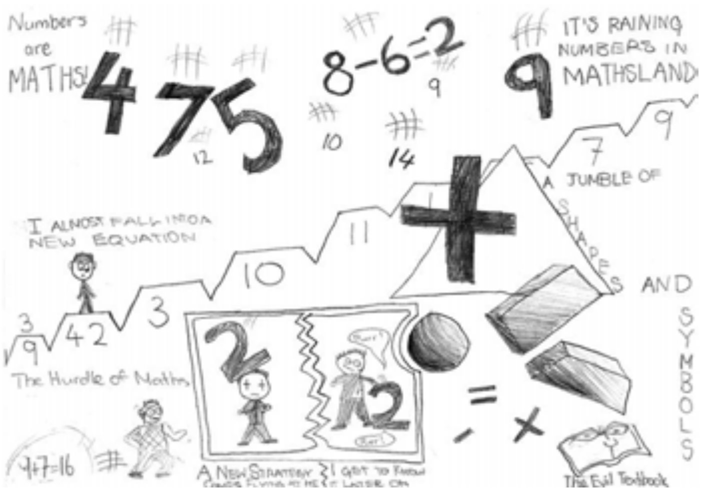

Figure 1. Tom's drawing.

Heads and brains

One of the most common metaphors—albeit used in different ways—is of the head or brain. The first way of expressing this indicates thinking, learning, knowledge, and improving one's intellectual capacity. More complex drawings include actual algorithms, concepts, and notions either within the head or streaming out. In contrast, negative drawings depict brain-burn and stress (Figure 2), with drawings of the tops of heads hinged off and numbers spewing out, or flames leaping from heads—and accompanying legends like Jason's, ―Maths gives me brain burn," or Lyle's, ―Kill me now with numbers".

Feelings positive or negative

A range of feelings is portrayed through metaphors in the drawings from maths is fun and exiting, through boring, to stress-inducing, as well as a mixture of feelings. Positive feelings are displayed through smiling faces, hearts, flowers, words, and other happy images. For Chloe, it is a combination of fireworks, bombs, and ―Maths makes me feel as good as an icecream tastes". Lucy has positive brains, smiling girls, tunes, and a bottle of maths pills: ―Dosage: Take a lesson a day to get you going happily." Hamish includes ―fun with games", but ―boring as in a subject without colour", and a tombstone inscribed with ―R.I.P Famous Mathmatician dude". Hazel uses black and red to show how much she finds maths boring, ―Hates it". and that it gives her a headache. Harry (Figure 2), one of the brain-burn artists, has students hanging from the light fixtures, throwing up, or calling for ―Mummy".

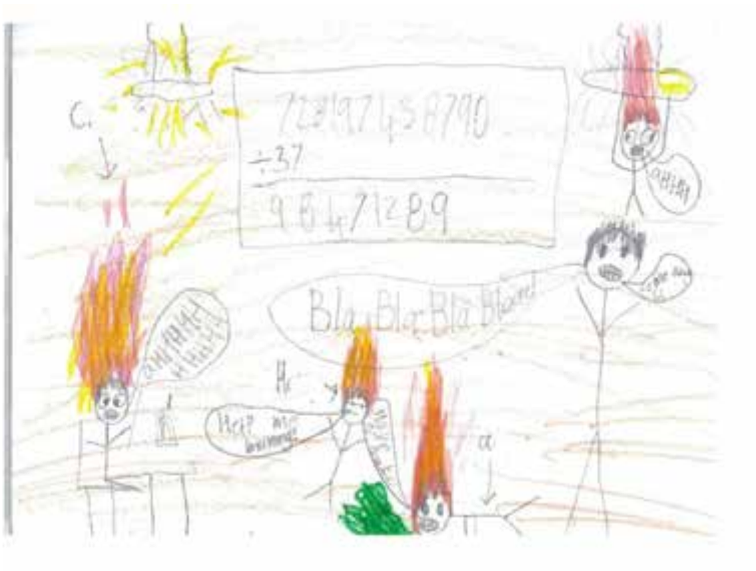

Figure 2. Harry's drawing.

Discussion and implications

This drawing task allowed students to portray their idiosyncratic beliefs and experiences through a medium that they viewed as fun and non-threatening (Christensen & James, 2008; Veale, 2005). It permitted those with weaker literacy skills to present as much information as their more literate peers. Although many of the students view mathematics in terms of number and basic operations (as found by Young-Loveridge et al., 2006), they communicated this view of the nature of mathematics through a variety of metaphors in their drawings: for some, numbers and symbols; for others, notions of usefulness for future employment, measurement, the ability to use money for shopping; or in terms of the fabric of life, a more universal approach.

The majority (70%) of the students who include affect position themselves as belonging to a world of fun, excitement and challenge, while the remainder feel boredom, hate the subject, or seem stressed by their experiences of the mathematics classroom.

Even though the students include a great range of metaphors in their drawings, it seems that the greatest difference between responses can be explained by school rather than by gender or ethnicity. In particular, a greater percentage of children from the Blue School included metaphors of utility, and affect metaphors were more prevalent in the Red. Teachers, the school context, as well as socio-economic status (Blue School middle, Red high) seem to have influenced the responses (Hattie, 2009). For example, Mr R's class[2] (Blue School) represented utility most frequently, which reflects his beliefs about mathematics as useful[3]. Ms McG's class (Red School), an accelerated group, had the most complex representations of affect and content which were probably influenced by the teacher, their high decile school, and perhaps their ability; although in no other group did ability seem to account for differences in metaphors.

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[2] Blue School.
[3] Based on a questionnaire and interview.

Conclusion

In summary, it was found that the majority of students picture mathematics largely as number and basic operations. Metaphors and affective elements were common in their drawings. It seems that schools, and specific teachers in particular, affected these primary students' understanding of the utilitarian nature of mathematics, and this finding has implications for practice, suggesting that it is possible to convey to young children the usefulness of what they are learning and its applications in many aspects of their environment and lives. Further, the children expressed a range of strong feelings about mathematics as well as both positive and negative images of its nature, and it seems that these can also be influenced by schools and/or teachers—a point that would need to be researched more deeply.

Lakoff and Johnson (2003) and Gauntlett (2007) discuss the notion of metaphor as conceptual, essential for abstract thought, and based on the individual's experience. It is clear from this research that by looking at the metaphors children use, teachers may gain valuable insight into what their students believe and understand, which in turn could help explain differences in engagement and learning in mathematics. They may also become aware of what their students feel about mathematics and/or about students' experiences of learning mathematics. This information has the potential to assist teachers in making decisions about classroom practice (McDonough, 2004) in terms of the next steps for individual students as well as for groups. However, it is important to interpret the content of drawings in terms of additional information such as other tasks, discussions, or interviews (Freeman & Mathison, 2009; Lester, 2002; McDonough, 2004), as well as to recognise that teaching behaviours and biases—as well as classroom and school contexts—may affect what students portray. Despite these cautions, the use of a drawing task to access students' beliefs and understanding about mathematics proved an effective means of collecting complex and varied data from a large group of students.

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TEACHER CAPACITY AS A KEY ELEMENT OF NATIONAL CURRICULUM REFORM IN MATHEMATICS: AN EXPLORATORY COMPARATIVE STUDY BETWEEN AUSTRALIA AND CHINA

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This exploratory study involving Australian and Chinese teachers seeks to characterise teachers' capacity to help students connect arithmetic learning and emerging algebraic thinking. The study is based on a questionnaire given to Australian and Chinese teachers, comprising seven students' solutions of subtraction sentences. Teachers' responses to the questionnaire were analysed in terms of four categories: knowledge of mathematics, interpretation of the intentions of the official curriculum documents, understanding of students' thinking, and capacity to design appropriate instruction in the short and long term. These four categories form the basis of our construct of teacher capacity.

Curriculum focus

In many countries, official curriculum documents now endorse the building of closer relationships between the study of number in the primary school and the development of algebraic thinking. Algebraic thinking is not the same as the use of algebraic symbols. It is about identifying generalisations and structural relations in number sentences and operations. This is very different to what in the past was seen as the study of arithmetic.

Australian Curriculum in Mathematics (ACARA, 2010) presents Number and Algebra as a single content strand for the compulsory years of school. In its overview statement to this strand, ACARA (2010) website states that:

> Number and algebra are developed together since each enriches the study of the other … They (students) understand the connections between operations. They recognise pattern and … build on their understanding of the number system to describe relationships and formulate generalisations. They recognise equivalence and solve equations and inequalities … and communicate their reasoning.

This statement echoes important ideas that have been present for at least five years in related State curriculum documents, for example, in linking Number, Structure and Working Mathematically in the Victorian Essential Learning Standards (VCAA, 2007); and in other official curriculum documents such as the Mathematics Developmental Continuum (DEECD, 2006).

The *Chinese Mathematics Curriculum Standards for Compulsory Education* (Ministry of Education, 2001), also present a single strand entitled Number and Algebra. In Stage 2 which covers Years 4 to 6, two "teaching objectives" refer to the importance of considering the inverse properties of calculation and to investigating the properties of equivalent sentences. Objective 5 on “operation of numbers” refers “to experience the inverse relation between addition and subtraction, as well as that of multiplication and division in the process of concrete operation and solution on simple practical problems.” (p. 21). Objective 3 on “sentences and equations” (p. 21) refers “to understand the property of equal sentences and enable to solve easy equations with the property of equal sentences (e.g. $3x + 2 = 5$, $2x – x = 3$)”. Several Chinese researchers, such as Xu (2003), emphasise that closer alignment is needed between the study of number and number relationships in the primary school and the study of algebra in the secondary school in this curriculum reform.

Official documents in both countries clearly endorse a more coherent treatment of number sentences and operations and the development of algebraic thinking in the primary and early years of secondary school; and we argue that teacher capacity is a key dimension in realising that goal. However, the implementation of curriculum change is never simply from the top down. Teachers’ interpretations and responses at the level of practice are never simple reflections of what is contained in official curriculum documents. While curriculum documents set out broad directions for change, any successful implementation of these “big ideas” depends on teachers’ capacity to apply subtle interpretations and careful local adaptations (Datnow & Castellano, 2000). Teachers’ professional insight and agency in translating these ideas into practice must frame any definition of teacher capacity (Smyth, 1995). Moreover, simply focussing on enactment as the defining feature of capacity tends to place any teacher opposition to reform in an entirely negative light.

Research focus

In examining the importance of teacher capacity in building a bridge between number operations and algebraic thinking, our mathematical focus is on students’ ability to read and interpret number sentences as expressions of mathematical relationships, rather than seeing them exclusively as calculations to be performed. Specifically, we draw attention to the importance of assisting students to use ideas of *equivalence* and *compensation* to solve number sentences involving subtraction. These methods, Irwin and Britt (2005) have argued, may provide a foundation for algebraic thinking (p. 169). Jacobs, Franke, Carpenter, Levi and Battey (2007) use the term relational thinking to refer to these kinds of strategies. These authors agree that there is still room for debate whether relational thinking in arithmetic represents a way of thinking about arithmetic that provides a foundation for learning algebra, or is itself a form of algebraic reasoning. They argue strongly that “one fundamental goal of integrating relational thinking into the elementary curriculum is to facilitate students’ transition to the formal study of algebra in the later grades so that no distinct boundary exists between arithmetic and algebra” (p. 261).

The research instrument and some results

Teachers in both countries were invited to complete a written questionnaire based on a “scenario” where some researchers had visited their school and gave students (either in Year 6 or Year 7) the following number sentences, asking them to write a number in the box to make a true statement, and in each case to explain their working briefly.

These two questions, according to the scenario, had been accompanied by similar questions dealing with addition, and were intended to see how students interpret and solve number sentences involving different operations:

> For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.
>
> 39 – 15 = 41 – ☐
>
> 104 – 45 = ☐ 46

The Australian and Chinese teachers were then presented with seven responses selected from actual responses by Australian and Chinese students in a study reported by Stephens (2008). In the Australian sample (see Appendix A), two Australian students, A and B, correctly found the missing number by calculating the result of the subtractions 39 –15, and 104 – 45, and then used these results to calculate the value of the missing numbers on the right hand side. Student C refrained from calculating, attempting to use equivalence, but compensated in the wrong direction to get answers of 13 and 103 respectively (or mistook the operation for + instead of -). Two students, D and E, successfully argued that since 41 is two more than 39 the missing number has to be two more than 15 to keep both sides equivalent. They applied similar reasoning to the second problem. Student F used arrows connecting the two related numbers (e.g. 39 and 41), and also connecting the other number (15) to the unknown number. Above the arrows Student F wrote +2 for the first problem and +1 for the second problem, obtaining correct answers. Finally, student G placed the letters A and A1 beneath 39 and 41, and B and B1 beneath 15 and the unknown number, and found correct values for the unknown numbers using an explanation based on equivalence and compensation. While the answer to the first problem is correct, Student G's written explanation contained a small error.

The Chinese sample contained parallel examples as far as possible. Students A and B gave calculated solutions that were almost identical to their Australian counterparts. Students C and D, in the Chinese sample, gave correct and clearly articulated relational explanations. Student E used a diagrammatic representation almost identical to Student F in the Australian sample. Student F in the Chinese sample used compensation correctly in the first problem, but in the wrong direction for the second problem (like Student C in the Australian sample) giving an answer of 103. Student G in the Chinese sample also used compensation in the wrong direction in the first problem, obtaining an answer of 13. However, in the second problem Student G gave the missing number as 59 which is the result of calculating 104 – 45.

Teachers were then asked three key questions, with each question on a separate A4 page. Firstly, teachers were asked to comment briefly on each of the seven samples. Secondly, Australian teachers were asked how they would respond specifically to Students A, B and C if they were students in their class. They could respond to other students if they wished. Chinese teachers were asked to respond specifically to Students A, B and F. Thirdly, all teachers were asked: "In planning your teaching program, how do you want to move students' thinking forward in regard to these and related questions? How will you develop the kind of mathematical thinking that students need to solve these kinds of number sentences? You can talk about a short design of one or several lessons, or a longer plan over the year."

The sample

Both samples used in this exploratory study were convenience samples. The Australian sample consisted of 20 Numeracy coaches working in Victorian government schools who were participating in an extended professional development program. All 20 were school-based with time release to support mathematics teaching in their home school or in other local schools. 17 were based in primary schools. Two of the three coaches who were based in secondary schools were not mathematics specialists, although all were teaching mathematics. The Chinese sample of 20 teachers was randomly selected from a larger group of more than 100 specialist mathematics teachers who had agreed to complete the questionnaire (Chinese version) during several teacher professional development programs in Nanjing, Whenzhou and Chongqing. All Chinese teachers were teaching Mathematics across several grades; and 18 were teaching in elementary schools.

The analytical framework: Four criteria

Teacher capacity to build effective bridges between the teaching of number and thinking algebraically about number sentences using equivalence and compensation is defined in this study in terms of four criteria: *Criterion A:* Knowledge of mathematics; *Criterion B:* Interpretation of the intentions of official curriculum documents; *Criterion C:* Understanding of students' thinking; and *Criterion D:* Design of teaching (See Table 1, over). This construct of teacher capacity is similar to the construct of mathematical knowledge for teaching elaborated in two important papers by Ball, Thames and Phelps (2008) and by Hill, Ball and Schilling (2008). Our Criterion A is intended to capture their category of *Specialized Content Knowledge*; our Criteria B and C are derived from their category of *Knowledge of Content and Students*, that is, knowledge that combines knowing about students and knowing about mathematics; and our Criterion D gives emphasis to their category of *Knowledge of Content and Teaching*, which combines knowing about teaching and knowing about mathematics. Our construct of teacher capacity differs from the construct of mathematical knowledge for teaching in giving a greater emphasis to knowledge of official curriculum documents.

Qualitative analysis

Each criterion of our analytical framework was expressed in terms of four specific indicators (see Table 1). In the case of the first two criteria, these indicators expressed how well teachers' responses indicated a clear understanding of the mathematical thinking that the two problems were intended to examine; and in the second criterion how this thinking reflected key ideas of current official curriculum documents in the respective countries. Indicators of capacity associated with the third criterion looked specifically at how well teachers could describe and interpret key features of performance expressed by the seven students, and how well they could respond to what the students had done in terms of immediate classroom teaching. Finally, those for the fourth criterion looked at how well teachers could plan for teaching that fostered a deeper appreciation of the mathematical thinking embodied in these and related tasks, especially in fostering ideas of equivalence and compensation.

Table 1. Four criteria and associated indicators.

Criterion A – Knowledge of relevant Mathematics: (1) Does the teacher recognise that there are two mathematical approaches to solving these kinds of problems – using calculation; or using equivalence and compensation for the operations of subtraction or difference? (2) Does the teacher recognise that students need to attend specifically to subtraction or "difference" when using equivalence? (3) Does the teacher refer specifically to mathematical terms such as "equivalent difference" or "difference unchanged"? (4) Does the teacher understand that equivalence using subtraction is compensated differently from addition, and/or that the key idea of equivalence also applies to the other operations?	**Criterion C – Understanding of students' thinking:** (1) Does the teacher recognise that Australian students D, E, F & G (or Chinese students C, D & E) were correctly using relational thinking although expressed in different ways? (2) Does the teacher identify the typical error (compensating in the wrong direction) shown in solution C of Australia sample (or solutions F(2) and G(1) of China sample)? (3) Does the teacher recognise the importance of identifying those students who can *only* use calculation? (4) Do Chinese teachers see that solution G(2) suggests a deeper misunderstanding; or do Australian teachers recognise that student G has a clear understanding of equivalence although makes a small error in the explanation for question 1?
Criterion B – Interpretation of the intentions of official curriculum documents: (1) Does the teacher realise that "Mathematical Thinking" should be treated as an important consideration whilst calculation remains valued? (2) Does the teacher understand and support the intention of the curriculum to link number learning and algebraic thinking? (3) Does the teacher show in his/her descriptions of children's responses, an awareness of the key curriculum goal of moving students from reliance on calculation to using equivalence in number sentences, here with respect to "difference" or subtraction? (4) Does the teacher's response use terms, words or expressions that are found in official curriculum documents?	**Criterion D – Design of teaching:** (1) In designing teaching, does the teacher focus on the important aspects of mathematics to be taught and fostering mathematical thinking, not on general strategies? (2) Does the teacher have a short-term teaching plan to respond to selected students in the next lesson? Does the teacher recognise that it is *more* important to let students who can think relationally explain their thinking to the whole class, but not so important for those who used calculations? (3) Does the teacher have a longer-term teaching plan to move students' relational thinking forward? How well does this plan reflect knowledge of students' thinking (Criterion C)? (4) Does the teacher give teaching examples or use teaching with variation to help students' learning and thinking?

Qualitative evidence of demonstrated teacher capacity

Criterion A: Knowledge of Mathematics

Chinese teacher 1 commented: "If the same number is added to both minuend and subtrahend, the difference represented by the number sentence will be unchanged … this is also called the *law of difference unchanged*." Similarly, Australian teacher 11 said: "Although the process is the same with both + and - , the students often misunderstand whether they have to add or subtract to get the equivalent value on both sides of the equals sign."

Criterion B: Interpretation of the intentions of official curriculum documents

Australian teachers 3 and 4 referred to *Key Characteristics of Effective Numeracy Teaching P-6* (DEECD, 2009). Teacher 4 pointed to the need to:

> engage students in identifying and using arithmetic relationships within number sentences to solve problems without calculating and teach a repertoire of strategies – (using) guess-guess-check (systematic trial and error), logical arithmetic reasoning and inverse operations to solve a wider range of number sentences.

Chinese teacher 15 said: "In the elementary teaching of number and algebra, integrity and coherence need to be embodied".

Criterion C: Understanding of students' thinking

> Despite their different responses, students C, D, E, F & G were all using the relations to solve the questions which is different from students A & B. This is a better method and to be encouraged because it is closer to the structural thinking that students need when learning algebra. These number sentences have been carefully chosen to make this method better. Student C spotted the relationships between the numbers being used in both algorithms (addition and subtraction) …. s/he has added to one of the numbers, (whereas) s/he needed to subtract from the other. (Australian teacher 5)

Chinese teacher 13 said, "It is not easy to judge whether A and B solve it through calculation, or through the reverse principle between addition and subtraction," noting the importance of distinguishing between those students who can only use calculation. Chinese teacher 8 says that "After students' agreement on Type 2 (relational thinking), further explain the rationale of type 2 to help students understand it."

Criterion D: Design of teaching

One Australian teacher 16 gave a well-designed five-stage plan to move students thinking forward with each stage reflecting a different level of mathematical thinking.

Chinese teacher 7 suggested: "Explore variations, change the '–' in both sides into '+' or have the change in one side and leave the other unchanged." Teaching with variation is used effectively in the following teaching examples:

1. Fill in ">" , "<" or "=" in ○.	2.Fill in "+" or "–" in ○, and numbers in □.
45 – 36 ○ (45 + 3) – (36 + 3)	87 – 45 = (87 ○ □) – (45 - 3)
45 – 36 ○ (45 – 3) – (36 – 3)	98 – 36 = (98 – 5) - (36 ○ □)
198 – 42 ○ (198 + 2) –(42 + 2)	184 – 56 = (184 ○ □) – (56 + 8)

A weak or inappropriate response to Criteria B and D

Australian teacher 7 said: "I am not familiar with working in this area of the school I would need to consult the Maths (Developmental) Continuum ... I need further help as I was probably looking in the wrong progression point."

A dissenting response to Criteria A and B but strong on D

Chinese teacher 7 showed a clear understanding of the mathematical elements of the questions and designed very clear teaching examples to help students develop relational thinking. However, teacher 7 had very strong resistance to fostering mathematical thinking other than ensuring students' correct calculations:

> The solutions of Students A & B need to be energetically popularized (to the whole class), because most students can master them ... The deep thinking of Student C

deserves praise, *but* it shouldn't be introduced, because it is not very good and some students may be confused by it and cause mistakes like that of Student F.

An exploratory quantitative analysis

By assigning a score of 1 if one of the four indicators was evident in a teacher's response, and 0 if it was omitted from their response or answered inappropriately, it was possible to construct a score of 0 to 4 for each criterion, and hence a maximum score of 16 across the four criteria. While the four listed indicators for each criterion are, in our opinion, the most relevant in terms of reflecting teacher capacity, they are not the only possible indicators. We allowed for the possibility that teachers' might provide convincing alternatives to the four indicators that we had listed.

For the Chinese sample, the highest score was 15 and the lowest score was 5, with a median score of 9. For the 20 Australian teachers, the highest score was 14 and the lowest was 2, with a median score of 10. The respective mean scores were 9.05 (Chinese) and 8.9 (Australian) with standard deviations 2.31 and 3.54 respectively. In the Australian sample, four teachers scored less than 5, whereas a score of 5 was the lowest for the Chinese teachers. Table 2 shows means that were calculated for each of the Criteria, and a global mean score calculated across all four Criteria.

Table 2. Means for each criterion and global means.

Sample	Criterion A	Criterion B	Criterion C	Criterion D	Total
Chinese	3.0	2.2	1.75	2.1	9.05
Australian	2.3	2.45	2.1	2.05	8.9

An initial classification of Teacher Capacity

Those teachers who scored between 11 and 16 were classified as demonstrating High Capacity; those scoring between 6 and 10 were classified as having Medium Capacity; and those scoring less than 6 as having Low Capacity. An initial classification of the two samples is shown Table 3.

Table 3. Classifications of teacher capacity.

Capacity	Chinese	Australian
High	4	6
Medium	15	10
Low	1	4

Discussion and conclusion

Among Chinese and Australian teachers, High Capacity to make an effective bridge between the teaching of number and fostering of algebraic thinking was demonstrated by teachers' clear understanding of the mathematical nature of the tasks students had been engaged in; their capacity to relate these tasks to relevant curriculum documents; their high interpretative skills when applied to each of the seven samples of students' work; and their extensive range of ideas for designing and implementing a teaching program to support the development of students' mathematical thinking. Medium

Capacity was shown by other teachers who, while possessing knowledge and skills supportive of these directions, clearly need to increase their current levels of professional knowledge and skills. In both samples, Low Capacity was evident in a minority of teachers who appeared unable to express a clear articulation of the mathematical nature of the tasks, or what differentiated the seven responses used in the questionnaire. These teachers were unable to point to how the tasks related to what is contained in official curriculum documents, or to describe how they would plan a program of teaching to foster these and related mathematical ideas.

Chinese and Australian teachers in the sample appeared to perform similarly on Criteria B (Interpretation of the intentions of official curriculum documents) and D (Design of teaching). However, Chinese teachers appeared to perform better than their Australian counterparts in elaborating the mathematical thinking embedded in the tasks that the students were asked to work on. On the other hand, Australian teachers appeared slightly better than the Chinese sample in responding to Criterion C (Understanding of students' thinking). These apparent differences call for further investigation. An initial pair-wise comparison of the four criteria shows a significant correlation at 0.05 level between Criteria A & B, and A & C; and at 0.01 level between Criteria A & D, B & C, and B & D. Similar analysis at the level of indicators should also be explored, and a factor analysis could also be used.

As a basis for a study involving a larger sample of teachers in both countries, with a more carefully stratified sample with respect to specific mathematical training, years of experience and location, this exploratory study has been successful in several respects. The questionnaire using the seven samples of students' work, and its three key questions, was effective in eliciting teachers' responses. In turn, teachers' responses were able to be used as a basis for examining teacher capacity in terms of teachers' mathematical knowledge, knowledge of official curriculum documents, understanding of students' thinking – that is, ability to analyse and interpret students responses and to frame appropriate responses to individual students – and to design credible sequences of teaching to foster the underlying mathematical ideas. These interpretations and professional dispositions go well beyond the "big ideas" or "general statements of intent" that are typically expressed in official curriculum documents. These subtle interpretations and the ability to frame immediate and longer term instructional responses are pre-requisites of any successful implementation of the "big ideas". In this paper we have elaborated a definition of teacher capacity firmly based on these characteristics. We feel confident that the conceptual and analytical framework of this exploratory study is robust enough to guide a larger study examining teacher capacity and curriculum reform in China and Australia.

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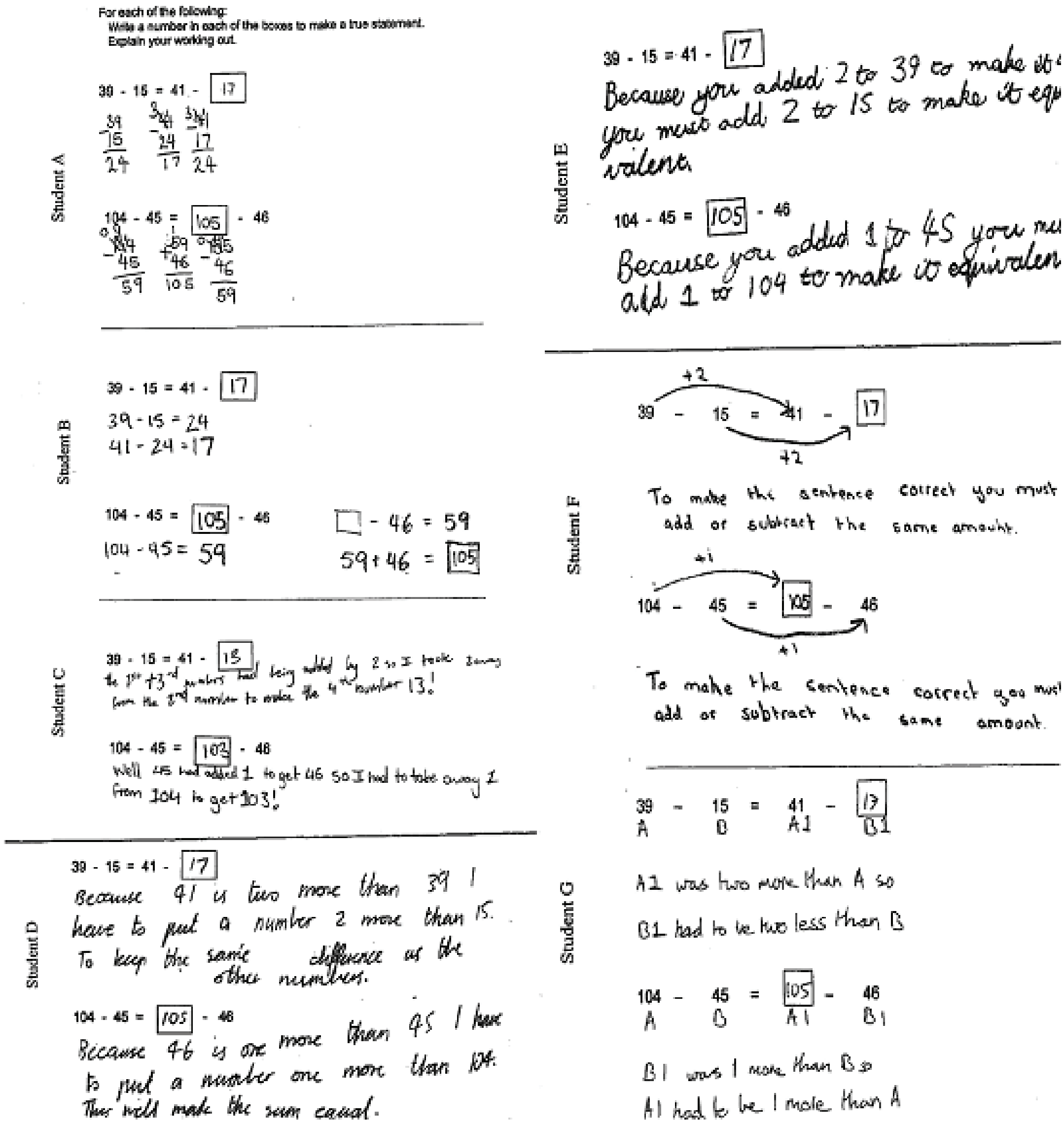
Appendix A



‘GET DOWN AND GET DIRTY IN THE MATHEMATICS’: TECHNOLOGY AND MATHEMATICAL MODELLING IN SENIOR SECONDARY

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Applications and mathematical modelling have been a distinctive part of the senior secondary curriculum in Queensland for over two decades. Findings related to technology use from an on-going longitudinal study of this initiative are reported. Twenty-three teachers and curriculum figures from across the state were interviewed and artefacts related to technology use were collected from teachers. Teachers’ understanding of the nature of modelling and the potential for technology to be used at various junctures in the modelling cycle affected the extent of technology use in teaching and assessment. The culture of the classroom was perceived as being very different by teachers who made significant use of technology during modelling. Technology was also seen as being essential for the future successful teaching of applications and modelling.

Introduction

With the wisdom of hindsight it seems obvious in 2011 that a plethora of technological devices is relevant to the teaching of applications and mathematical modelling at all levels of schooling but particularly at the senior secondary level. The use of technology appears relevant whether modelling is seen as a vehicle for teaching other mathematics or as part of mathematical content to be taught and learnt in its own right. Both of Hußmann’s “central tasks of the technology that supports [sic] independent concept formation” (2007, p. 348) are relevant to either approach—”the *function of construction* by contributing to building ideas, and on the other hand, … the *function of irritation* by initiating a change of concept” (p. 348). Indeed we have found both operating in modelling classrooms where technology rich teaching and learning environments were being researched (Stillman, in press; Stillman, Brown, & Galbraith, 2010). Nevertheless, the question needs to be asked what is the reality across the spectrum of classrooms in a context where applications and mathematical modelling have been promoted at an educational system level for a considerable time? As an example of what has transpired in everyday classrooms we consider the implementation of applications and mathematical modelling within senior secondary mathematics curricula in Queensland, where the initiative was first introduced in 1989 (e.g., Queensland Board of Senior Secondary School Studies [QBSSSS], 1989).

Background

A long lasting ideological legacy of the 1960s and 1970s which saw marked changes in many countries in the Western world has been a desire among young people to be convinced of the efficacy of any activities in which they are asked to engage rather than being expected to be willing participants who follow directions given by others in authority (Niss, 1987). At the secondary and tertiary levels of education students began questioning the relevance of the mathematics they were studying; "and right from the beginning relevance was interpreted by students, teachers and educationalists as *applicability*" (Niss, 1987, p. 491). At the same time there was employer dissatisfaction with mathematics departments of universities (see McLone, 1973) because of the scarcity of mathematics graduates who appreciated the applicability of mathematics in other fields and who could model real problems and readily communicate results to non-mathematical clients. Educational reforms such as those flagged in *Everybody Counts: A Report to the Nation on the Future of Mathematics Education* (National Research Council, 1989) identified modelling as one of the "distinctive modes of thought" (p. 31) offered by mathematics and mathematics was said to play a special role in education because of "its universal applicability" (p. 31). From this milieu of influences came the impetus to change the Queensland senior mathematics syllabuses from purely abstract approaches to teaching and content to ones incorporating an emphasis on applications and mathematical modelling as a distinctive characteristic. According to the current Mathematics B syllabus, mathematical modelling is "the act of creating a mathematical model, which may involve the following steps: identify assumptions, parameters and/or variables; interpret, clarify and analyse the problem; develop strategies or identify procedures required to develop the model and solve the problem; investigate the validity of the mathematical model" (Queensland Schools Authority, 2010, p. 44).

The advancement of technological devices and the beginnings of the manufacture of such devices for dedicated teaching purposes in school and university settings serendipitously coincided with the development of the new syllabuses (Stillman & Galbraith, 2009). However, the importance of technological devices to the work of applied mathematicians who engage in mathematical modelling of real situations and to teachers and students teaching and learning through applications and mathematical modelling quickly became apparent. "These devices provide not only increased computational power, but broaden the range of possibilities for approaches to teaching, learning and assessment" (Niss, Blum, & Galbraith, 2007, p. 24). Niss, Blum and Galbraith also warned of the possibility of "associated problems and risks" if these devices were not used and incorporated in the teaching/learning environment in an appropriate manner.

The use of technological devices as tools to carry out repetitive or difficult processes in the solution of a mathematical model has been recognised for some time but several researchers (Confrey & Maloney, 2007; Galbraith, Stillman, Brown, & Edwards, 2007) have seen the potential for technology in the inquiry/reasoning processes that occur throughout the modelling cycle. Recently, Geiger, Faragher, and Goos (2010) confirmed that "student-student-technology related activity takes place during all phases of the mathematical modelling cycle" and that, in particular, technology plays a role in "the conceptualisation of the model" not just the solving process (p. 64). This is consistent with what happens in workplaces where modelling is conducted. Ekol (2010), from a study of 10 applied mathematicians teaching in university but also working as modellers in industry, concluded that “technology plays a big role in fostering exploration towards discovery, also in sustaining interest in the modeling process” (p. 196). In particular, his interviewees believed that modellers needed to be able to make the appropriate choice of which technology to use and when to use it and also to use technology in a playful way during modelling “for meaningful exploration” (Ekol, 2010, p. 194) of the situation being modelled and the mathematics being applied.

Research methods

Queensland syllabus and review documents from the late 1980’s up to the latest syllabuses implemented in 2009 were examined. In addition semi-structured interviews were conducted with 23 interviewees. Samples of 5 key curriculum figures [QKCG] (e.g., non-teacher members of expert advisory committees, curriculum officers of the state education department, or statutory board or authority officers overseeing syllabus implementation), 6 secondary mathematics teachers in key implementation roles [QKTG] (e.g., state or district review panel chairs or state review panel members), and 12 secondary mathematics classroom teachers [QCTG] were purposefully selected (Flick, 2006) as being relevant to the purposes of the study (Richards, 2005, p. 41). These teachers were representative of several school districts and of the state, Catholic and independent schools systems. A series of interview questions covering the period of introduction, and later periods of widespread implementation and modification were asked. In addition, practising teachers provided artefacts, usually in the form of tasks, which typified their use of real world applications and modelling in teaching and assessment, and their use of technology in these contexts.

In order to identify emergent themes within the interview responses, and the teaching and assessment artefacts, these data were entered into an NVivo 8 database (QSR, 2008) and analysed through intensive scrutiny of the data from a particular interviewee and across the corpus of the data from all interviewees to develop and refine categories related to these themes (Richards, 2005). Specifically this paper will address emergent themes related to responses to the following interview questions:

1. To what extent have you incorporated the use of technology when exploring real-life situations that require investigative, modelling or problem-solving techniques?
2. The syllabuses require a balanced assessment plan that includes a variety of techniques such as extended modelling and problem-solving tasks and reports. What types of task do you use in your alternative assessment? To what extent do these use real world contexts? To what extent do they also incorporate the use of technology? How?
3. How is the culture of the classroom influenced by the presence of technological devices in a classroom environment promoting both technology and applications and mathematical modelling?
4. What possible implications does technology have for the future successful teaching and assessment of applications and modelling within upper secondary mathematics?

Findings

Implications of developing understanding of nature of modelling and potential of technology use

As the affordability and quality of technology allowed it to be freely able to be used in the classroom, it was soon realised by some teachers that a classroom rich in technology would serve to facilitate the implementation of the syllabuses particularly those aspects pertaining to mathematical modelling and applications. It allowed the messiness of real world data to be dealt with as this teacher points out:

> I think that has been a big driving thing, the fact that you have the technology that you can then explore real-life situations and the kids can actually get down and get dirty in the mathematics rather than everything being really nice and neat because up until that stage, like in the old syllabus, because they didn't have that facility, everything was always pretty much nice. (QKTG3)

This potential has not been realised in all schools, however, with the uptake of technology being described as "patchy" by some (QKCG4; QKCG5) especially with respect to the extent of how it is used mathematically in exploring real world situations.

> Some people use technology really well and all the time and others, because the syllabus says you have to use it, they will use it just to do more calculations or just to draw graphs or things like that. (QKCG4)

Extent of technology use in teaching

As teachers' understanding of (a) what mathematical modelling entailed increased, (b) how it differed from mathematical applications became clear and (c) what technology offered to teaching and learning, the necessity for modelling and technology to be an integral part of the teaching/learning environment became more accepted. Thus, modelling and technology came to enjoy a symbiotic relationship in the classrooms of these teachers where technology is "just natural, you don't even think about it that it is there. Kids pick it up and just use it" (QKTG2). Although technology was seen as ideal for demonstrating by the teacher, it also had a pivotal role to play in the hands of students who were allowed to play and explore models and emerging ideas when modelling.

> I think you need to be able to engage people more immediately in what's going on there so I think technology being used to demonstrate and for students to play with as well as illustrate mathematical concepts generally I think is very important and also for modelling and problem solving as well. (QCTG12)

Technology allowed timely access to modelling or exploring of situations for which students were yet to learn more sophisticated mathematics to model.

> So at this stage in the course [end of semester 2 in year 12] I am actually revisiting the same problem and employing the algebraic approach and differentiating and saying, "Okay, that's how we do that at this stage in the course" even though earlier in the course we were prepared to let the calculator do most of the work for us. (QCTG2)

Others saw this as a means to extend the sophistication of the modelling their students were able to do with one teacher stating: "The increase in technology we can get our hands on now means we can tackle increasingly sophisticated modelling" (QCTG7). This was seen as an underpinning reason for using technology.

> Models become quite sophisticated quite quickly and then kids can't take them any further but technology offers you an opportunity of scaffolding around that. (QKTG2)

Not all teachers used technology to a significant extent in teaching about applications or modelling, with some reserving it "mostly [for] alternative assessment" (QCTG5) although they were "quite happy to go to the computer labs" and work on computer investigations from textbooks.

Extent of technology use in alternative assessment

"Assessment techniques other than traditional written tests or examinations" (QBSSSS, 1992, p. 40) became known as alternative assessments. These were required to be included in a school's assessment program at least twice yearly. Some teachers spoke of using technology almost exclusively in their alternative assessment although some, but not all of these, also used technology in teaching when exploring and investigating real world situations. For many the motivation was not that they believed using technology when exploring real situations to be good pedagogy or essential but rather it was "because it is mandated" (QCTG5).

> In assessment, well we can't use computers in exams so we try to see if we have their alternative assessment task, their one per semester, try to have something there where they would be using the computer. … (QCTG1)

How students used the technology seemed to resonate with the teachers' view of modelling. Those teachers who saw modelling as no different from mathematical application designed assessment tasks that provided opportunity for using technology only as a tool in solving.

> It is just making use of the technology to do the number crunching more than anything else and then being able to interpret what you have at the end of that. (QCTG5)

Others saw alternative assessments as providing the ideal forum to show evidence of meaningful technology use when assessing applications and modelling.

> We look to our assignments as the main evidence that our students use technology because in the supervised exams they certainly use technology to draw graphs, to do calculations, find mathematical models…but what is the proof of it really but it is evident in the assignments. (QCTG7)

Classroom culture in an environment promoting technology and application and modelling

Most teachers who had embraced technology spoke of their classroom culture being "very different to what we used to do way back in time. Absolutely we couldn't do the sorts of things that we do if we didn't have the technology" (QKTG3). This was partly in response to teaching a generation of students who are technologically knowledgeable in certain respects reacting in quite different ways to students of the past:

> I think having an internet generation has meant that the way that students interact with each other has obviously changed and [as] learners has become different and I think students need more immediate gratification these days. They need to see a dynamic situation happen in front of them. They don't have patience to sit there and graph things manually. (QCTG12)

Elements of the classroom culture that were said to be enhanced were also elements of what researchers have identified as integral to conducting modelling successfully in the classroom such as the technology rich environment becoming a "vehicle for opening up ideas" (QKTG1) and "more discussion amongst them" (QCTG4; QKTG1). The increased discussion was seen by some teachers as helping students' mathematical understanding (QKTG1). The classroom was also seen as becoming "a little bit more collaborative" (QKTG1) with a "bit more [group work] because even though they have got their own [calculators] they still compare" (QCTG4).

Several teachers pointed out that it was not just having the access to the technology that was the key to the changed culture. It was very much dependent on the approach taken to teaching modelling.

> Oh, yeah, very definitely changes the way you teach because the tedium of the algebra or whatever it is, the calculation is taken away and the answer … to that stage will come up very quickly and the kids are more interested, much more engaged. It's not just technology but yes it does help. … it depends on what you do in the classroom too. (QCTG1)

Some acknowledged that an enquiry approach was called for.

> It is not just the fact that technology is there. It is the way it is used. And it is the way the teacher uses it and the type of culture they build themselves. So if they build a culture that is about enquiry and mathematical modelling and all that sort of thing and incorporate technology into that then you can really kick on. (QKTG2)

However, these teachers were still limited in their view of the potential of technology in a modelling environment as technology was seen as being of assistance only in the solving phase of the modelling cycle and not as a means of enabling model conceptualisation or decision making at all phases throughout the modelling.

> If you're going to introduce technology in there it is just likely going to be used as a number cruncher and not much more. So you have to build in the other stuff as well and it is not just technology alone that does it. (QKTG2)

Implications for the future for successful teaching and assessing of applications and modelling

Some teachers saw technology as essential to successfully implementing the intentions of the syllabuses in the allowable time.

> I think it is essential because I think you have got to, for the limited time that we have that we can spend in assessment you can't have them not using the technology. It is too time consuming to do all that without the technology… as long as they know what the technology is doing and I think that is the idea. (QCTG1)

Others spoke of it enriching the whole experience that was the perceived intention of the syllabus with technology playing an essential role in exploration of real life situations mathematically enabling students to confirm their own understandings.

> I don't think you can teach mathematics successfully without technology to be honest with you. You can teach mathematics but you can't build an understanding of those real life situations. (QKTG3)
> I just think it is enriching the whole process, the whole experience. It is giving kids other ways of confirming the learning that they have. (QCTG9)

One of the key curriculum figures took a futuristic "learning community" approach considering the classroom as borderless with students being willing to share ideas with others within their classroom and across classrooms which could be co-located or geographically distant.

> I am interested in Rudd's idea of providing every student with a laptop … What it seems to me is that it would provide the opportunity for kids to form learning groups and to share things and to see how other people work on things. Now I think this would be more powerful than anything, if a teacher here who was working on mathematical modelling, one of their [groups] could some how or other share what they were doing and let the others see what they were doing and thinking about and how this group was thinking about it. You would get a lot of "Ahas". What I am saying is the technology if it could provide that sort of networking then you could really pick up the pace in the mathematical modelling side of it. …The learning community stuff is still pie in the sky, I suppose, but it is still exciting even to someone who is past exciting. (QKCG3)

Discussion and conclusion

With respect to the responses of participants in relation to the extent of technology use in teaching and alternative assessment involving real world contexts some teachers clearly had welcomed the opportunity to expand their repertoire of teaching and assessing practices with respect to applications and modelling that technology brought. Others saw technology providing little more than a computational device to remove the tedium and potential inaccuracies of repetitive calculations or graphing associated with the solution of a mathematical model. In the latter instance this usually was related to a view of modelling as being no different from using mathematical applications and opportunities for use of technology being more prominent in assessment than in teaching.

In classrooms where technology was said to play a significant role in teaching applications and modelling the classroom culture was said to be very different as the "internet generation" was more engaged by immediate feedback and dynamical displays available by teaching with technology. The constructive function of technology in concept formation (Hußmann, 2007) was acknowledged by these teachers. Hußmann's "function of irritation" was less obvious in the responses but could perhaps be inferred as being present in communities of inquiry or when students were said to be using the technology to confirm their learning. Exploration, sustaining interest and engagement, and playing with the mathematical ideas and the situation being explored as identified by Ekol's (2010) applied mathematicians were all mentioned as elements of the classroom culture where technology was readily available and expected to be used. Again the teacher's view of modelling limited the perceived potential and promoted use to the solving phase or expanded it to pervade the modelling cycle along the lines promoted by Confrey and Maloney (2007).

Finally, some saw the use of technology as essential to successfully fulfilling the intentions of the syllabuses with respect to modelling. Even though several saw this as clearly enriching the whole teaching/learning experience as intended by the syllabus writers, there was mention of the unfulfilled potential of a borderless learning community providing networking amongst modelling groups across distances and geographical boundaries further enriching that experience.

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A STRATEGY FOR SUPPORTING STUDENTS WHO HAVE FALLEN BEHIND IN THE LEARNING OF MATHEMATICS

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Given the diversity of achievement in most classes and the other pressures on teachers, it seems unrealistic to assume that class teachers, as part of their everyday teaching, can provide whatever support is needed by students who have fallen a long way behind. The following is a report of a specific initiative aimed to investigate the potential of an out of class student support intervention, the goal of which is to prepare students for the mathematics lessons they will experience subsequent to the support.

Introduction

One of the consistent conclusions from international comparisons is that while Australian students overall are doing well, there is a long tail of underachievement (Thompson & De Bortoli, 2007). This is no surprise to teachers of mathematics. A Year 9 mathematics teacher sent us the following story. He was revising some recent work, and the students were working on this problem.

> You earn $12 per hour for 22.5 hours. You pay 26% of your earnings in tax. How much tax will you pay?

A girl, Emma, wanted help.

Mr T:	Do you have a job?
Emma:	Yes
Mr T:	How much an hour do they pay you?
Emma:	I don't know—I just started.
Mr T:	Let's say you earn $12 an hour and you work for 3 hours. How much is that?
Emma:	I don't know. Do you divide?
Mr T:	No. Think about earning $12 an hour. You work one hour, and then another, and then another. How much have you earned?
Emma:	I don't know.
Kylie (sitting nearby):	Is it $36?
Mr T:	Yes. Good.
Emma (to Kylie):	God, you're smart.

This story highlights a number of critical issues for mathematics education: How has this student survived so long in the system without action being taken to address her inability to answer such questions? What benefit is she gaining from her Year 9 mathematics classes?

However it seems that this is not an isolated case. Consider the following two NAPLAN items taken from the 2009 Year 9 assessment in which students do not use a calculator. One item was presented as follows:

> Steven cuts his birthday cake into 8 equal slices. He eats 25% of the cake in whole slices. How many slices of cake are left?

This was a "write in" answer and 85% of Victorian students did this correctly. Even though it is a very straightforward question, there were 15% who gave the wrong or no answer. Attributing difficulties to the reading does not explain the number who could not do it. All Year 9 work in mathematics classes would be much more difficult than this question.

Another item was as follows:

> A copier prints 1200 leaflets. One-third of the leaflets are on yellow paper and the rest are on blue paper. There are smudges on 5% of the blue leaflets. How many blue leaflets have smudges?

The students could choose from four options: 40, 60, 400, or 800. There were 59% of the Victorian students who chose the correct answer. Recognising that this item involves three steps after reading the question, there are 41% who could not choose the correct response from the four options. We suspect that those students would have substantial difficulty in comprehending most of their Year 9 mathematics. We also suspect that their difficulties started well before that stage.

Given the diversity of achievement in most classes and the other pressures on teachers, it seems unrealistic to assume that class teachers, as part of their everyday teaching, can provide whatever support is needed by students who have fallen such a long way behind. The following is a report of a specific initiative aimed to investigate the potential of an out-of-class student support intervention, the goal of which is to prepare students for the mathematics lessons they will experience subsequent to the support.

Theoretical framework

One aspect of the rationale for this approach is derived from cognitive load theory (Bransford, Brown, & Cocking, 1999). Pegg (2010), for example, outlined his perspective on maximising learning based on this perspective. The theory suggests that all information is processed in working memory and then stored in long term memory. The idea is to have the information that is stored in long term memory efficiently chunked so that it can be readily retrieved. The initial processing of information and preparation for this chunking happens in working memory, which is of limited capacity.

While Pegg (2010) focused on developing fluency in calculation as a way of reducing the load on working memory, this intervention program focuses on the ways that students attend to stimuli around them and the key information that they select for processing. In all situations, and especially in classrooms, there is more happening than can be effectively attended to, so it is necessary to select from among the sensory experiences.

Students who have fallen behind have greater difficulty in selecting the appropriate information and so the instruction, the task, the language, and even what the other students are saying and doing becomes confusing. The hypothesis is that if the attention of such students can be focused on key information, they can select more appropriately what is likely to help them learn. An example of the way this works is in mathematical language. If students do not know what is meant by terms such as parallel, right angle, index, remainder, or average, then instruction using those terms will be confusing and ineffective since so much of their working memory will be utilised trying to seek clues for the meaning of the relevant terminology.

Without necessarily drawing on this particular theory, many educators have based their approaches to instruction on the same principles. Tzur (2008), for example, argued that instruction should begin with what the students already know and are confident with and then move to content that is unfamiliar, rather than what he claims is the common approach of starting with unfamiliar content.

A second rationale for this approach is that classrooms are social and students prefer to participate positively, thereby satisfying a need for social connectedness (see Hannula, 2004). Of course, sometimes some students do not seem to reflect that need in their behaviour. This can be explained by Dweck's (2000) notion of describing students as either seeking mastery of the content or affirmation of their performance from the teacher (or someone else). Elliot (1999) explained that students who have a performance orientation but who see the risk of failure as high will actively avoid participation, which is commonly manifest in them threatening classroom order.

Approaches to supporting students experiencing difficulty

There are a number of existing programs designed to support students who have fallen behind in their learning. Gervasoni (2004) for example, argued that low achieving students can lose confidence in their ability, and develop poor attitudes to learning and to school. One outcome is that the gap grows between the knowledge of these children and of other children and that the typical learning experiences provided by the classroom teacher for the class do not enable each child to participate fully and benefit. Ginsburg (1997) concluded that "as mathematics becomes more complex, children with mathematics learning difficulties experience increasing amounts of failure, become increasingly confused, and lose whatever interest and motivation they started out with" (p. 26). Gervasoni (2004) outlined the *Extending Mathematical Understanding* program that involves professional development for teachers along with dedicated time in small groups with students experiencing difficulties. Gervasoni presented evidence that students who experience the program's structured intervention improve. A similar program, *QuickSmart*, also results in impressive improvement in students who complete the program (Graham, Bellert, Thomas, & Pegg, 2007).

Programs such as these are clearly successful in what they seek to achieve and this particular initiative is seeking to extend these in a particular direction.

The *Getting Ready* intervention

The intervention reported below was supported by the Wyndham Network of Schools in the Western Metropolitan Region of Melbourne[1]. The participating schools agreed to release tutors to work with selected students in small groups, with the goal of providing preliminary information on the upcoming topic to those students, prior to their participation in the classroom mathematics lesson.

Initially the tutors met with the second author on three occasions to consider appropriate models for working with students in the tutoring sessions. The advice offered to the tutors was that they should:

- highlight and familiarise students with the vocabulary of their next mathematics lesson;
- use questioning to focus the students' attention on the relevant concept(s) and to 'resurrect' any prior knowledge of the concept that the students may have; and
- briefly model the sorts of activities to be undertaken in the next classroom lesson.

It was emphasised to tutors that they should not seek to teach the content, because the goal is that the students prepare to learn in the lesson, as distinct from removing the need for them to concentrate when they get to the lesson.

Year 3 students were selected for inclusion in the program by the tutors on the basis of the annual teacher judgement data, Early Years Interview results, and 'On-demand' testing data. Year 8 students were chosen on the basis of their NAPLAN results from the previous year, with students appearing in the bottom 20% of the applicable data being eligible for selection to the program. Where this number proved too many, the tutors conferred with the classroom teachers to select from the identified cohort, those students whom it was believed, would most benefit from inclusion in the program.

The data collection

There are three types of data presented: the first and major section uses the response of participants to elaborate details of the intervention; the second section is a brief description of insights from a videotape record of a tutoring session; and the third section is some analysis of pre- and post assessments including comparisons of the results of those who were tutored and those who were not.

From the participants

As a preliminary evaluation of the initiative, participants were interviewed at the end of the first phase, some six months from commencement of the project. In particular, the interviews were intended to explore:

> What are the organisation and administrative challenges in implementing this initiative?
> What was the experience of the tutors, the class teachers and the participating students?

Sixteen people were interviewed at the end of phase one: three tutors (one from each school, designated as Schools A, B, and C); six teachers (four from School A, and one each from Schools B and C); and seven students (five from School A and two from School B). The following presents selected representative responses on aspects of the initiative.

[1] The program was supported by Sharon Taylor, managed by Lucy Glover and Steve Boyle, and involved the participation of a range of energetic and professional tutors and teachers.

One of the interview questions sought participants' perspectives on the aims of the intervention. The aim of the program being implemented before mathematics lessons was to give students "pre-knowledge" based on relevant language, according to two tutors and one teacher. Particular comments were:

> What my hope is, is that I'm giving them the pre-knowledge, so front-loading the kids, so when they actually go to the numeracy lesson ahead they will have an idea of what is going to happen. In that way, that will free up some learning space so they're not still behind the eight ball. So they come in, they know what's expected and then they can gain more out of the numeracy lesson than they normally would have. (Tutor, School C)

The response of other teachers were similar. It seems that their perspective on the aims and intention of the initiative were aligned with the goals of the program.

The interviews also sought insights into what the respondents saw as the benefits of the program. One tutor discussed the benefits to the students with regard to how they think about their learning.

> I can see the benefits to being one step ahead rather than always being on the back foot trying to continuously catch up. I think it's a benefit even if it's just seeing the main word and then that's a word that they're familiar with so when they go into the grade they say "Oh, I remember that word, I know what it is". And that's what I'm actually finding when I go back to revisiting. "Do you remember yesterday when we talked about this—can you tell me about it?" So it's going back so we can go forward, making constant connections. It's helped me hone my teaching skills, and then I'm relaying it back on to the teachers to say "We need to work on this". Because it's just the three of us we need to focus on what they're actually doing, instead of with 20 kids. (Tutor, School B)

These benefits are indeed those that were hypothesised.

Another of the consistent messages from teachers was that they found that students gained in confidence when they came back to the classroom.

> I think it's because they feel more confident about the topic we're learning about because they've already had a bit of work with [the tutor], so I think that helps them a lot. It's sort of a confidence thing where they can participate and they're willing to participate in discussions. (Teacher, School B)
>
> Definitely. Confidence. Before if I was questioning them they would never put their hand up where as their hands are up straight away trying to tell me what they've been learning with [the tutor]. They want to tell me everything they've been learning. They do have a greater confidence to be put into a discussion, so it gives them the confidence to contribute in the class where before they would just sit on the floor, not really put their hand up, but now I can see the connection between working with [the tutor] and when they come in here. They are familiar with the words, they are familiar with the vocab, and different areas like that. (Teacher, School C)

While confidence is not the end goal, it is clearly advantageous and is likely to lay the foundation for changed approaches to participation.

Building on the development of students' confidence, there were examples of how the intervention transformed the experience of some students when they returned to their class. The following is an illustrative example:

> My thought was that we are targeting those students that just need that extra bit to give them that shove, and the biggest thing I've noticed is their confidence. They are coming back in and these kids are putting up their hand and they are getting the answer right. I do a lot of language in mathematics before I start anything, so that's constant reinforcement. They are not just hearing it from me, they are hearing it from [the tutor] as well and that

> has made the biggest difference. I got so excited yesterday because one student who is part of the program has come from being so quiet—she doesn't like to speak a lot, she's almost mute and ESL—and she now puts her hand up for everything. … And another child in my class in the program, he is now so positive. I say "Maths" and he goes "Yeah, I've done this with [the tutor], and he gets so excited because he knows what he's doing. I think it's ace. (Teacher, School A)

These comments are very powerful and indicate the potential of this approach in transforming willingness of students to participate in class.

A somewhat unexpected outcome was that the program seemed to impact on classroom teaching. The following is a comment by one of the teachers:

> Yes. Even with my own teaching – I've been able to keep the lessons flowing rather than having to stop and start. This way it's been easier to go through all the topics. (Teacher, School A)
>
> It's benefited my kids dramatically, but I don't know whether it's just my kids and the fact I've taken it on so completely that might have been what's made the difference. (Teacher, School A)

It is interesting that this should happen, and perhaps might be one of the major benefits of this approach to intervention over others, in that it potentially improves the learning of all students.

Participating students were also interviewed and the following are some representative comments:

> [The tutor] helps me know maths very well and it's very fun to do maths. She teaches me how to skip numbers and it's easier for me to skip numbers so I don't have to count by ones [for subtraction]. (Student, School B)
>
> [The tutor] helps me practice my writing. She's helped with multiplication and ladders and vertical. She has helped me with division because we've been doing that (in class). (Student, School B)
>
> It made it easier. 'Cause first I didn't know it [division] and then with (the tutor). I learnt something. (Student, School A)
>
> Because you get to learn how to do them and also sometimes I get confused about it. She explains things and tells us to do them in our scrap books. (Student, School A)
>
> It is because then I can understand and I know what to do. I always answer questions. (Student, School A)

The students clearly feel that they have learned, and see the connection to class participation.

Lessons from the video records

One of the video records shows three Year 8 boys participating in a tutoring session. The boys start the session very restless, paying little attention to the task, and potentially threatening order even in the small group. The tutor on the video progressively engages the students in reviewing what they know about the topic and clarifying any language issues that may have been relevant.

The tutor also models the action expected in the class, which was using a protractor and compass. The boys became progressively more engaged in the task, and at the end were fully engaged in listening and watching. Subsequent reports indicate that the students returned to their class and participated well and appropriately in the full class learning experience.

The inference is that an outcome of the tutoring is that students are more able to participate in the social experience that is the classroom. Another insight is that allowing such students to watch, as distinct from merely listening may be a potentially useful strategy in both tutoring and whole class sessions.

Some comparisons of assessment scores

All schools used a form of assessment at the start of the year and then again near the end of Term 3, using VCAA on demand testing. The schools were quite different in the way that the results were recorded, so they are presented separately.

Table 1 presents the results for the four primary schools that participated. The scores relate to the VELS levels. Notionally each year students should improve 0.5 of a level. In each case the gains in the means of the tutored students are compared with the students who did not participate in the tutoring.

Table 1. Comparison of gains for tutored and non-tutored Year 3 students.

Name	Tutored students' gains (n)	Not tutored students' gains (n)
Primary School A	.44 (12)	.30 (12)
Primary School B	.40 (12)	.32 (26)
Primary School C	.38 (22)	.40 (81)
Primary School D	.30 (11)	.50 (55)

In two primary schools the tutored students improved more than the others, and in two schools the reverse is the case. Therefore it is not possible to make judgments about effectiveness for tutored students from these data. It is noted though that the assessments measured knowledge over broader content than was covered by the tutoring program, indicating that any learning of students being tutored seemed to apply beyond those topics taught. Table 2 presents results for Year 8 students in two secondary schools.

Table 2. Comparison of gains for tutored and non-tutored Year 8 students in two schools.

School	Group	N	Mean gain	Median gain
Secondary school A (N=168)	Tutored	24	.45	.55
	Not tutored	144	-0.03	.08
Secondary school B (N=111)	Tutored	21	.48	.50
	Not tutored	90	.29	.40

In both cases, the gains for the tutored students were greater than for the not tutored students, and in one case much greater. Again, given that the comparison is on more than the taught topics, this improvement is outstanding.

Important considerations/issues that evolved during the pilot

There were a number of organisational considerations that became apparent during the trials and constitute learning from the experience of developing the initiative interactively with the teachers and tutors.

- *Group size:* Initially the tutors worked with groups of varying sizes however over time it became apparent that the optimum group size was 2 or 3 students. A common characteristic of the selected students was a lack of confidence when working in numeracy and this manifested itself as a reluctance to ask and/or answer questions, to offer suggestions or take risks during the sessions. It became apparent that the more students in the group, the greater the opportunity for each student to continue these practices. At the same time, the tutors could recognise a real benefit in having at least two students per group because this provided opportunity for students to discuss the mathematics in a familiar way, share strategies, support each other to take risks and finally to remind and prompt each other during their whole-class mathematics lessons.

- The tutors gradually all reduced their groups to either two or three students. Careful attention was paid to the mix of students in each group to avoid personality clashes. Similarly each group comprised students from the same class to ensure they would be able to support each other back in the classroom. Some tutors opted for groups of 3 to accommodate the frequent absenteeism that appears to be linked to low achievement, particularly in the secondary school setting. A group of three students meant that even if one student was absent, the remaining two students could have worthwhile dialogue during the session.
- *Absenteeism:* Initially, when a student was absent, some classroom teachers substituted other students into the tutoring session to ‘fill the gap’. As it happened, this upset the balance of the sessions and became a source of frustration to the tutors. The short sharp timing of the session became lost when the model needed to be explained to the non-tutoring student, the reflection on past lesson success/challenges was difficult, the confidence of the tutored student sometimes regressed and the relationship between tutored student and tutor became difficult. It was therefore decided that in the case of an absenteeism non-G.R.I.N. students would not be substituted into the G.R.I.N. session.
- *Withdrawal from class:* We recommended to schools that students not be withdrawn from the same lesson repeatedly. By the same token, students with low confidence in numeracy may demonstrate confidence in other fields and should be given an opportunity to demonstrate this where possible. Tutors sought to spread the lessons from which a student was withdrawn across a range of subjects on different days and at different times during the week.
- *Adequate time for tutors and teachers to meet:* It was an ongoing challenge for tutors and teachers to meet regularly to share information about future lessons and discuss student progress. Some tutors are also classroom teachers and have ready access to team planning sessions, while others are forced to rely on teachers’ work programs, casual conversations in the staffroom or chats on the run in corridors.
- *Professional learning of tutors:* All tutors met together on a monthly basis for the purposes of sharing ideas and professional learning. During these sessions the structure of the tutoring sessions was re-visited and refined as required and common tools for data collection were developed. Tutors also participated in professional development about effective questioning and ‘wait time’. More recently tutors worked as a professional learning team and gave feedback to each other on the basis of video clips that they have taken of themselves delivering tutoring sessions. These video clips have become a powerful tool for focusing on

the structure and intervention model, the purpose for each section of the structure and the tutor's role within that structure.

Conclusion

It is reasonable to conclude that the intervention was extraordinarily successful. The tutoring initiative has a clear rationale, it was received positively by students, tutors, and class teachers, and there is evidence of positive broader learning gains from the students.

There are two important aspects to the initiative. First, participation in the tutoring does indeed prepare students to be able to participate in their usual classes by reducing their cognitive load in the class. Second, since learning and classrooms are social settings, it allows tutored students to participate normally in the social settings thereby changing the way that they see themselves.

It is noted that there are other advantages in this approach. The resources required are not much greater than the cost of providing the tutor. There is a need for teachers to plan effectively and to articulate their plans to the tutors. There is a need for close collaboration between teachers and tutors. While there is some requirement of education of tutors and teachers, these are minimal and the whole process is readily sustainable. There are multiple stories of transformational change in the behaviour of some students. The approach clearly has potential.

There is one further issue. Sometimes commentators suggest that some students cannot learn mathematics whatever we do. This project demonstrates that this is a false assumption and that students can learn if given appropriate opportunity.

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STUDENTS' WAYS OF USING HANDHELD CALCULATORS IN SINGAPORE AND AUSTRALIA: TECHNOLOGY AS MASTER, SERVANT, PARTNER AND EXTENSION OF SELF

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Students' ways of using handheld calculators were investigated and compared on a sample of 964 Singaporean and 176 Victorian (Australia) senior secondary students. A survey instrument was developed based on four metaphors of technology use proposed by Geiger (2005): Master, Servant, Partner, and Extension of Self. Factor analysis found three factors: Master, Servant, and combined Partner and Extension of Self. Victorian students were found to have significantly lower scores on calculator as Master and as Servant, compared to Singaporean students. Males in both regions exhibited higher fluency of calculator use, compared to females.

Background

Handheld calculators such as the graphics calculator (GC) and calculators with computer algebra system (CAS) play an important role in secondary mathematics education (Wong, 2003). The GC and CAS calculators have been used in high stakes examinations at senior secondary levels in different parts of the world. In Victoria, Australia, the GC has been allowed in mathematics subjects in the Victorian Certificate of Examinations (VCE) since 1997 (Routitsky & Tobin, 1998), and currently CAS calculators are allowed in some examinations for year 12 VCE mathematics subjects (Victorian Curriculum and Assessment Authority [VCAA], 2010). In Singapore, GC use has been implemented in all the mathematics subjects at General Certificate of Examinations Advanced-level curriculum since 2006 (Wong & Lee, 2009). With the large number of senior secondary students taking mathematics examinations each year, research on handheld calculators is crucial in benchmarking and investigating the impact of the technology on mathematics teaching and learning.

The theoretical framework, instrument developed, and a report of the analysis and findings are described in the following sections.

Theoretical framework

While there have been a number of instruments developed to find out about students' attitudes towards the use of technology in mathematics education (e.g., Pierce, Stacey, & Barkatsas, 2007), there are few that measure students' ways of using technology. In order to enable investigations into students' use of technology in a broad and systematic manner, without tying the technology use to specific topics or specific instructional strategies, an instrument was developed which drew upon the four metaphors framework originally developed by Goos, Galbraith, Renshaw and Geiger (2000). Grounded in socio-cultural models of learning, Goos et al. (2000) theorised that technologies are cultural tools and their use is "actively re-shaping human interactions and interactions between humans and the technology itself" (p. 318), thereby transforming the learning process. They posited four roles for technology in the teaching and learning context: technology as Master, Servant, Partner, and Extension of Self (MSPE). Geiger (2005) further refined the metaphors into subcategories with representative student descriptions. These descriptions were then modified into the survey items (see Tan, 2009) used in this study. An outline of the metaphors is shown in Table 1. Geiger (2005) noted that while the MSPE metaphors represent increased levels of complexity of technology use, they correspond to an expansion in the "technological repertoire where an individual has a wider range of modes of operation available to engage with a specific task" (p. 370), and not to a hierarchy of stages of use where an individual abandons one level to progress to another. Hence a student who is proficient in using technology as a Partner might use technology as a Servant for certain mathematical tasks such as menial computation, when required.

Table 1. MSPE framework of technology use by students.

Metaphor	*Description*
Technology as Master	The student is subservient to the technology – a relationship induced by technological (limited operations used) or mathematical dependence (blind consumption of whatever output generated, irrespective of accuracy and worth).
Technology as Servant	Technology is used as a reliable timesaving replacement for mental, or pen and paper computations. Student "instructs" the technology as an obedient but "dumb" assistant.
Technology as Partner	Students often appear to interact directly with the technology, treating it almost as a human partner that responds to their commands – for example, with error messages that demand investigation. The calculator acts as a surrogate partner as students verbalise their thinking in the process of locating and correcting such errors.
Technology as Extension of Self	Students incorporate technological expertise as an integral part of their mathematical repertoire. Technology is used to support mathematical argumentation as naturally as intellectual resources.

Adapted from Geiger (2005, p. 371)

Research questions

1. How are Singaporean and Victorian students using handheld programmable calculators, with respect to the MSPE framework?
2. Are there differences among students from the two regions?
3. Are there any gender differences?

Methods

A survey instrument was developed based on the MSPE framework and piloted with 178 Singaporean senior secondary students (Tan, 2009). For the main study, a different group of Singaporean mathematics students (N=964) and 176 Victorian students taking the VCE mathematical methods subject participated.

An online survey was created using SurveyMonkey (http://www.surveymonkey.com) in two versions; the words "calculator" referring to "GC" for Singaporean students and to "CAS calculators" for Victorian students. Recruitment was carried out through schools via invitation emails in 2009–2010. There were three phases in the collection of Victorian data: (1) 110 schools from the Government, Catholic and Independent sectors were invited to participate; only two independent girls' schools and one Catholic boys' school participated; (2) 20 Independent schools (3 girls', 2 boys', 15 co-ed) were invited to forward invitation emails to their students; (3) an advertisement was created using Facebook (http://www.facebook.com) to invite more Victorian students to participate in the study (Tan, 2010).

The instrument consists of 12 positively worded items using 5-point Likert response formats, ranging from 1 (Strongly disagree) to 5 (Strongly agree). The items are shown in Table 2. Factor analysis was conducted using the software PASW Statistics 18.0 (SPSS) to investigate the underlying factors, based on the MSPE theory for both data sets.

Analysis and discussion

There were 964 Singaporean students (37.1% males, 62.9% females), and 176 Victorian students (31.3% males, 68.8% females). Higher percentages of females than males responded to the online survey in both regions. For the Victorian data, more independent girls' (n=6) than boys' (n=3) schools were invited to participate in the study in phases 1 and 2 as there were more Independent girls' (24) than boys' (14) secondary schools in Victoria (http://www.independentschools.vic.edu.au/); this partially explains the higher percentage of female participants. Similar numbers of female (29) and male (30) Victorian students responded through Facebook. In contrast, all four participating Singaporean schools were co-ed. In 2009, there were more female (55.5%) than male senior secondary students in Singaporean junior colleges (Ministry of Education, 2010), likely to be replicated among the participating schools. Research also indicates that girls are more likely than boys to respond to invitations issued via schools (e.g., Porter & Whitcomb, 2005).

It must be noted that the small sample size of Victorian data (< 300) and the high percentage of Independent school students (73.9%) limit the generalisability of the Victorian findings.

Factorability of the data was assumed since the Kaiser-Meyer-Olkin (KMO) measures of sampling adequacy (KMO_{Vic}= 0.680 and $KMO_{S'pore}$= 0.763) were more than 0.6 (Tabachnick & Fidell, 2001), and Bartlett's tests of sphericity yielded significance (χ^2_{Vic} (66)= 404.5, $\chi^2_{S'pore}$ (66)= 3202.8; p<0.001). Initial factor analyses using the Kaiser criterion of eignenvalues > 1 (Pallant, 2001) resulted in a four-factor solution for Victorian data and a three-factor one for Singaporean data. Inspection of the scree plots justified the use of three-factor solutions to explore both data sets. Principal factors extraction (principal axis factoring) with Varimax rotation and Kaiser normalisation was performed on the items specifying three-factor solutions, accounting for 43.9% of total variance for the Victorian data and 46.8% for the Singaporean data. The rotated factor matrices with factor loadings less than .3 removed (Pallant, 2001) are shown in Table 2.

As seen in Table 2, the factors matched the MSPE metaphors, with items for Partner and Extension of Self combined as one factor (henceforth referred to as "Technology as

Collaborator"), consistent with the pilot study results (Tan, 2009). The factors are labelled Tech_Master, Tech_Servant and Tech_Collaborator. The cross-loadings in the Victorian data for items M1 and P2 are still consistent within the theoretical framework of increasing levels of sophistication of technology use.

Table 2. Rotated factor matrices for Victorian and Singaporean data.

	Victorian			Singaporean		
Factors	1	2	3	1	2	3
(M1)* I do not know why sometimes the calculator does not give me the answer that I want.		.40	.50			.60
(M2) I usually just follow the steps taught when using the calculator to solve problems, and do not really understand the maths involved.			.75			.68
(M3) I find calculators confusing because it uses different conventions and symbols than normal maths.			.72			.71
(S1) I use the calculator for basic calculations because it is more accurate than working by hand.		.59			.71	
(S2) I use the calculator for calculations because it is faster than working by hand.		.70			.89	
(S3) I use the calculator to look after large calculations and tedious repetitive methods.		.72			.55	
(S4) I copy the graphs on the calculator in my answers because they are more accurate than drawing by hand.		.42			.30	
(P2) I use the calculator to help me simplify steps in a complex problem.	.68	.33		.59		
(P3) I use the calculator to help me look at the same problem or concept in different ways (e.g., using graphs and tables to understand the process of differentiation in addition to algebraic method).	.62			.75		
(P4) The calculator helps me understand concepts better.	.53			.68		
(E1) I often use the calculator to explore maths even before the teacher tells me to.	.43			.63		
(E2) The calculator allows me to expand my ideas and to do the work my own way.	.64			.76		

* Items developed according to MSPE framework (Tan, 2009).

Cronbach–α values were calculated to assess the internal reliability of the items for each of the three subscales: Tech_Master (α_{Vic}= 0.686; $\alpha_{S'pore}$= 0.714), Tech_Servant (α_{Vic}= 0.703; $\alpha_{S'pore}$= 0.699), and Tech_Collaborator (α_{Vic}= 0.735; $\alpha_{S'pore}$= 0.819). Although for the Victorian data the Cronbach–α value was less than the ideal of 0.7, it was still reasonable (Pallant, 2001). For the two data sets, performing the same factor analysis procedures produced the same factor solution consistent with the theoretical framework. This confirms the stability of the factors, and the validity and reliability of the instrument, allowing for comparisons between the two groups of students to be undertaken.

Subscale scores were calculated using the average score of all items within each factor, reduced to the range 1 to 5 for ease of interpretation. Table 3 shows the results of comparisons of the mean subscale scores by region and gender, using t-tests (except where otherwise indicated).

Table 3. Regional and gender comparisons: N, mean scores, standard deviations, test statistics, and p-values.

Factor	*Region*	*Mean (SD)*	*test statistic, p value*	*Gender*	*Valid N*	*Mean*	*SD*	*test statistic, p value**
Tech_Master	Singapore	3.189 (0.801)	t (1051) = 5.341, p<0.001	Female	586	3.235	0.775	t (930)=2.308, p<0.05
				Male	346	3.110	0.840	
	Victoria	2.771 (0.860)		Female	98	2.871	0.840	U=751.5, p<0.05
				Male	23	2.348	0.832	
Tech_Servant	Singapore	3.777 (0.659)	t (140.2) = 2.793, p<0.01	Female	585	3.750	0.657	NS
				Male	349	3.821	0.661	
	Victoria	3.563 (0.807)		Female	98	3.625	0.836	U=745.5, p<0.05
				Male	22	3.284	0.599	
Tech_ Collaborator	Singapore	3.034 (0.733)	NS	Female	579	2.964	0.691	t (641.9)= -3.781, p<0.001
				Male	341	3.152	0.786	
	Victoria	3.058 (0.766)		Female	99	3.022	0.724	NS
				Male	23	3.218	0.934	

* Mann-Whitney U test was used for Victorian gender comparisons.

Comparisons between Victoria and Singapore

As seen in Table 3, Singaporean students generally scored significantly higher for Tech_Master ($\bar{x}_{\text{S'pore}}$=3.189, $\bar{x}_{\text{Vic}}$=2.771) and Tech_Servant ($\bar{x}_{\text{S'pore}}$=3.777, $\bar{x}_{\text{Vic}}$=3.563) than Victorian students. This suggests that the Victorian students had higher levels of fluency with handheld programmable calculators than Singaporean students.

There are various possible explanations for these differences, for example a socio-economic factor suggested by the high percentage of Independent school students in the Victorian sample. School-sector differences in student performances have been reported in Australia (e.g., Marks, 2009).

Another explanation may be the differences in the school systems in the two regions:

- With the use of GC allowed in the VCE since 1997, Victorian mathematics teachers may have more experience with teaching the use of programmable calculators and might be better able to mediate students' learning with calculators than Singaporean teachers.
- Most Victorian senior secondary students learn in a classroom structure, using published textbooks, whereas most Singaporean senior secondary students learn in a lecture-tutorial structure, using lecture notes provided by their teachers.
- Victorian secondary schools usually encompass grades 7-12, whereas most Singaporean senior secondary schools consist of grades 11-12 only.

These differences may have advantaged Victorian students with better quality or more consistent teaching and increased exposure to the use of programmable calculators. Since CAS calculators and GCs share a number of similar functionalities and syntax, Victorians may be less likely to use their CAS calculators at the Master level than Singaporean students use their GCs.

For both regions, the mean scores for Tech_Collaborator were not significantly different from the neutral value 3 (S'pore: $t(919)=1.394$, Vic: $t(120)=0.831$; $p>0.1$). This suggests that students use the calculators at this highest level only some of the time, consistent with Geiger's (2005) findings.

Gender differences in how students use calculators

Figure 1 shows the box plots for the three subscales by region and gender. The skewness in the distribution for Tech_Servant for male Victorian students, possibly due to the small sample size, is evident. Hence the non-parametric Mann-Whitney U test was used for the Victorian data (Pallant, 2001) – see Table 3. As shown in Table 3, males had significantly lower mean scores than females for Tech_Master in both regions (S'pore: $\bar{x}_M=3.110$, $\bar{x}_F=3.235$; Vic: $\bar{x}_M=2.348$, $\bar{x}_F=2.871$). Singaporean males also had significantly higher mean scores for Tech_Collaborator than females ($\bar{x}_M=3.152$, $\bar{x}_F=2.964$), with no significant difference for the Victorians.

For Tech_Servant, no significant gender difference was found for the Singaporeans, whereas Victorian males' mean scores were significantly lower than females' ($\bar{x}_M=3.284$, $\bar{x}_F=3.625$). This suggests that Victorian males may be less reliant than females on calculators to replace mental or pen-paper computations; this finding may partially explain the higher percentages of males than females scoring top grades in the calculator-free VCE mathematics examinations (Forgasz & Tan, 2010).

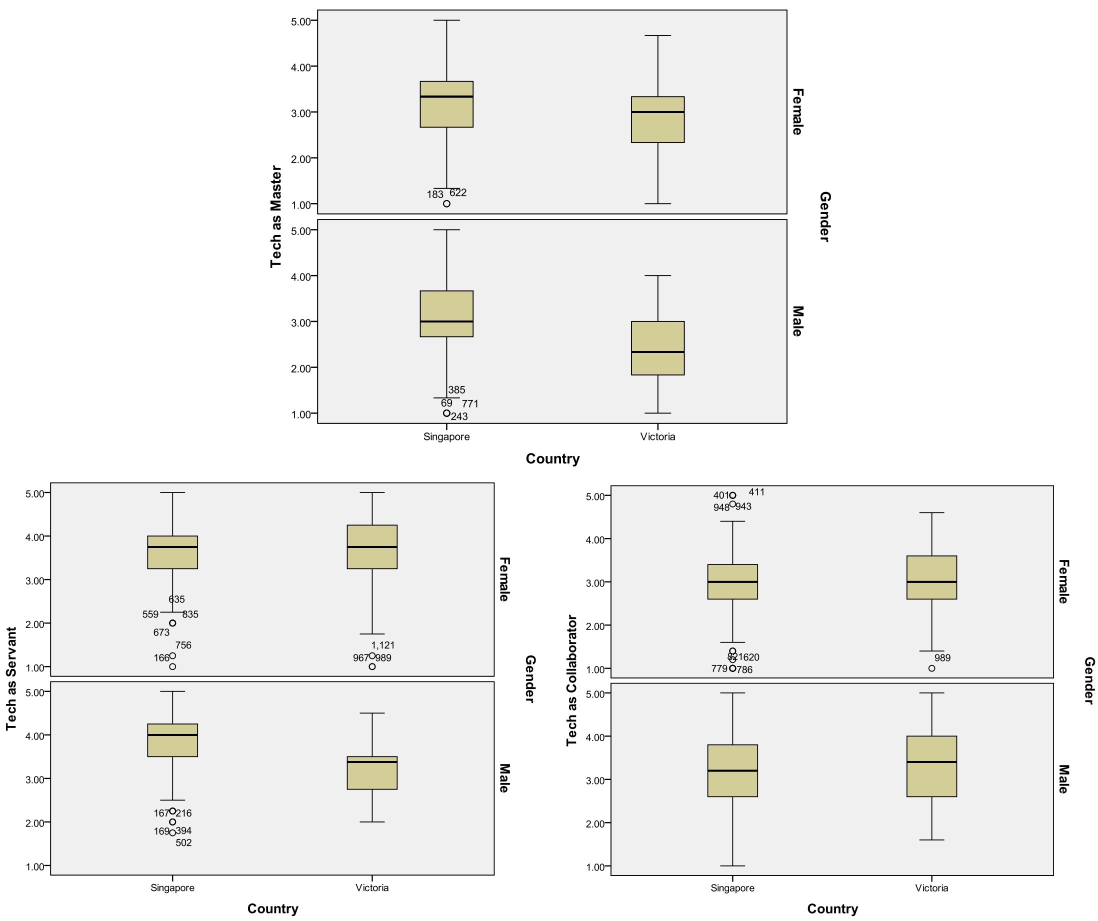

Figure 1. Box plots for Singaporean and Victorian data, grouped by gender.

Conclusion

In conclusion, Victorian students appeared to have greater fluency than Singaporean students with sophisticated calculators, despite the finding of no significant differences at the highest level of calculator use (Tech-Collaborator). For greater generalisability for Victoria in particular, more research is needed with larger samples and broader school sector representation. Gender differences were consistent with past research in that males showed greater mastery of the calculators than females, in both regions. Given that the calculators were used in high-stakes mathematics examinations where the results affect entrance into university courses, these findings call for further research into assessment and instruction to address these gender differences.

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USING ASSESSMENT DATA: DOES GENDER MAKE A DIFFERENCE?

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Since 2000 gender differences in mathematics achievement in Australia have reappeared. In this paper we report on the achievement outcomes of girls and boys in a longitudinal study of reform in low economic school communities. Analysis of student data to inform teaching was one element of student centred approaches implemented by teachers. Teachers targeted students' next point of learning and more girls than boys participated in mathematics intervention programs. Growth in achievement was greater for boys than for girls in the primary years, and so the achievement gap that favours males widened. It is concluded that student centred approaches need to be gender inclusive.

Introduction

For the past decade, researchers have observed the re-emergence of gender inequality in mathematics achievement, participation and affect in Australia at all levels of schooling (Vale, 2010; Vale & Bartholomew, 2008). These outcomes represent a reversal of a trend toward gender equality in achievement and participation observed during the 1990s (Forgasz, Leder, & Vale, 2000) and can be attributed a lack of focus on gender equity in educational policy in general and on the educational needs of boys in particular (Vale, 2010). The focus of the current federal government education policy on equity and socio-economic equity in particular (MCEETYA, 2008) has provided an opportunity to refocus the attention of education systems and teachers on equity issues in education. Australian government initiatives for school reform include programs designed to build capacity of educational leadership and teachers; promote whole school approaches, the use of data, and student centred teaching; develop intervention programs for students; and support the engagement of parents and community (DEEWR, 2009).

In this paper we examine gender issues in low socio-economic (SE) school communities using data gathered during a longitudinal study of teacher practice and student achievement in schools that participated in a school reform project jointly funded by the Victorian government (DEECD, 2009) and the federal government, under its *Smarter Schools Pilot* program (DEEWR, 2009).

Background

Various feminist theories have informed the struggle for gender equity in mathematics education through policy initiatives, curriculum development, and pedagogical approaches since the 1980s (Vale, 2010; Vale & Bartholomew, 2008). It is generally agreed that equitable practice is responsive to students' learning needs, intellectually challenging, and inclusive (Anthony & Walshaw, 2007; Jorgensen, Grootenboer, & Sullivan, 2010). Such practice, it is argued, is student centred.

The capacity to respond to students' learning needs depends on teachers knowing their students well. Analysis of students' work samples and formative assessment practices, also known as assessment for learning, has been driving reform of mathematics teaching in Australia since 2000 as a result of research projects such as *The Early Years Numeracy Project* (Clarke, et al., 2001) and *Scaffolding Numeracy in the Middle Years Project* (Siemon, Izard, Breed & Virgona, 2006) conducted in Victoria. By analysing students' mathematical reasoning teachers are able to target their teaching to address students' misconceptions or challenge them within their "zone of proximal development" (Vygotsky, 1978).

Critical theory supports transformative pedagogies that go beyond addressing student learning needs which, from the school improvement policy context, are often perceived from a deficit perspective. Transformative pedagogies connect with students' cultures, involve reciprocal learning and develop respect (Atweh, 2009). Boaler (2008) believes that transformational practice also involves equitable relations in diverse classrooms with students "acting" equitably and "treating each other with respect and considering different viewpoints fairly" (p. 168). These approaches shift student centredness from a constructivist perspective to a social-constructivist perspective where teachers also design tasks for students organised in mixed achievement-level groups and classes.

Findings from international assessment studies (TIMSS and PISA), Australian national benchmarking, and particular research studies for the period 1995 to 2007 are reported and summarised by Vale (2010). Studies of affect consistently report gender differences favouring males at all year levels and this has remained unchanged since the 1980s. Since 2000 males typically out perform females in the early years of schooling (for example, Horne, 2004) as 9-year-olds in TIMSS and 15-year olds in PISA. Higher proportions of females achieved the national benchmark in Years 3, 5, and 7 however the proportion of females performing below expected benchmark increases with year level. Studies by Forgasz (2006) and Leder and Forgasz (2010) show that female participation in senior secondary mathematics is falling in the subjects required for continued study of mathematics beyond schooling and that males are more highly represented among the top performers at all levels of schooling. Studies also reveal that gender differences in mathematics achievement are mediated by other equity factors such as individual and school socio-economic level, indigenous status, language background other than English, and degree of remoteness (Thomson, De Bortolli, Nicholas, Hillman, & Buckley, 2010).

For some years accountability has been driving educational policy and interventions in Australia and internationally. All schools in Victoria are required to develop annual strategic plans that aim to improve the proportion of students achieving the national achievement benchmarks. To date, gender equity is not given prominence in current government policy, and therefore schools are not called upon to develop targets and strategies for gender equity.

Australian government initiatives, as indicated above, are now providing resources and small amounts of funding for low SE and indigenous school communities to support reform. The suite of reforms that these schools and their leaders and teachers are expected to adopt include the use of assessment data to inform school planning and classroom teaching, student centred approaches to teaching, and appropriate intervention programs. In this study we report briefly on the way in which schools and teachers who participated in one of the Literacy and Numeracy Pilot programs (DEEWR, 2009) implemented these strategies, and the achievement outcomes for girls and boys from low SE school communities.

The study

The 76 government schools in this study belong to four networks of primary and secondary schools located in metropolitan Melbourne and regional Victoria. The Department of Education and Early Childhood Development (DEECD) selected these networks of schools for participation in the Victorian Pilot because of the low SE of the school communities and the general underperformance of these networks, overall and individual schools, when compared with other networks in Victoria. Some schools in these networks also have high proportions of Koori students, students who are new arrivals in Australia, refugees, or students meeting the criteria for learners of English as a second language (ESL).

The study used a mixed methods design incorporating quantitative assessment of student mathematics outcomes and collaborative practitioner research methods (Cherednichenko, Davies, Kruger, & O'Rourke, 2001). Principals, numeracy leaders, numeracy coaches, regional network leaders, and other regional project staff from all schools in the Pilot were invited to respond to three open-ended questions (personal accounts). Other qualitative methods including observations of meetings and classrooms and analysis of documents were used for in-depth case studies of nine schools. Schools also completed a questionnaire about the numeracy intervention program(s) implemented at their school and provided student identification numbers of the students who participated in these intervention programs.

Student mathematics achievement data were collected using online assessment tools provided to schools by the DEECD. Data were collected four times at six-monthly intervals during the study: March and September, 2009, and March and September, 2010. The *Mathematics Online Interview* (MOI) adapted from the *Early Years Numeracy Interview* (Clarke et al., 2001) was used to gather assessment data for students in years P–2 and results are reported in "growth points." The *On Demand Adaptive Test for Number* (VCAA, 2009) was used for students in Years 3–10. This test is designed to assign items to the student based on their relative success with a beginning set of items at a level indicated by the classroom teacher. Results are recorded to one decimal place using the *Victorian Essential Learning Standard* (VELS) score (VCAA). Individual student results for each assessment period were paired. Growth in student achievement for each six-month period (March 2009 to September 2009, September 2009 to March 2010, and March 2010 to September 2010) was calculated. Analysis of variance was used to compare achievement and growth by gender.

Findings

Using student data, student centred approaches and interventions

At last year's MERGA conference we reported on the student centred approaches implemented by teachers in the pilot study (Vale, Weaven, Davies, & Hooley, 2010). We showed that the schools and teachers adopted a constructivist interpretation of student centred approaches since they focussed on the children's "point of need" to differentiate teaching and learning. These approaches were typically more evident in the practices of primary teachers than secondary teachers in the pilot. Teachers used a range of data to identify children's learning needs. These included analysis of student responses to MOI, the *Number Fluency Interview* (Montgomery & Waters, n.d.), and NAPLAN test items, along with samples of students' class work. School Numeracy Leaders and Numeracy Coaches supported analysis of these data, often taking responsibility for compiling results and responses in formats that made interpretation of data for individual students and classes of students easier for teachers. Analysis of these data enabled teachers to identify students at various levels of risk of under-achievement, and hence target students for particular intervention programs implemented at schools. Intervention programs included both in class and withdrawal programs. Some were individual; others were for small groups of students. The data analysis practices and intervention programs are presented in more detail elsewhere (Vale, Davies, Hooley, Weaven, Davidson, & Loton, 2011).

Gender differences in the early years (P–2)

Mean scores and growth in achievement for place value and addition and subtraction for female and male students for March 2010 to September 2010 is recorded in Table 1. Male achievement is significantly greater than female achievement for both place value and additive thinking ($F=19.411$, $p<0.05$ and $F=4.361$, $p<0.05$ respectively). While growth in achievement is greater than the equivalent ENRP benchmark for six months (0.56 for place value and 0.82 for addition and subtraction (Clarke et al., 2010), the effect of the Pilot has been to widen the achievement gap between males and females. The gap widens from 0.09GPs to 0.15GPs for place value and 0.05GPs to 0.1GPs for addition and subtraction from March to September 2010. While these changes appear to be small they are statistically significant ($F=5.454$, $p<0.05$ and $F=4.260$, $p<0.05$ respectively). The gaps in achievement are illustrated in Figure 1.

Table 1. Achievement and growth in place value and addition and subtraction for students in Years P–2 (MOI growth points), March 2010 – September, 2010.

Domain	Month	Females (N=2664)		Males (N=2937)		Mean difference
		Mean	SE	Mean	SE	
Place Value	March	0.984	0.021	1.075	0.020	-0.091
	Sept	1.675	0.023	1.823	0.022	-0.148*
Mean growth		0.691		0.748		-0.057*
		Females (N=2652)		Males (N=2930)		Mean difference
Addition & Subtraction	March	1.423	0.028	1.468	0.027	-0.045
	Sept	2.316	0.030	2.426	0.028	-0.110*
Mean growth		0.893		0.958		-0.065*

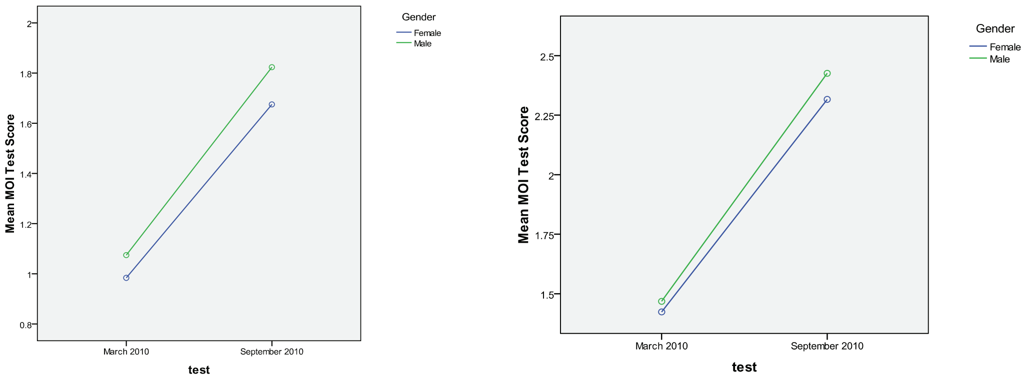

Figure 1. Mean scores for Place Value and Addition and Subtraction for female and male students in Years P-2, March to September 2010(MOI growth points).

Gender differences in the middle years (3-10)

Mean scores for female and male students in Years 3, 4, and 5 in March 2009 and Years 4, 5, and 6 in September 2010 are recorded in Table 2 and illustrated in Figure 2. Overall growth in achievement is significantly greater than expected (0.75 VELS points in 18 months). Gender differences in achievement favour males and are statistically significant. Over the period, the gap in favour of males doubles (0.06 VELS points in March 2009 to 0.13 VELS points in September 2010) and is statistically significant ($F=3.868$, $p<0.05$).

Table 2. Achievement and growth for primary students, March 2009 – September, 2010 (VELS).

		Females (N= 667)		Males (N= 697)		Mean difference
Year	Month	Mean	SE	Mean	SE	
2009	March	2.690	0.030	2.749	0.029	-0.059
	Sept	3.018	0.031	3.111	0.030	-0.093
2010	March	3.185	0.033	3.247	0.033	-0.062
	Sept	3.509	0.033	3.639	0.033	-0.130*
Growth Mar09–Sept10		0.819*		0.890*		-0.071*

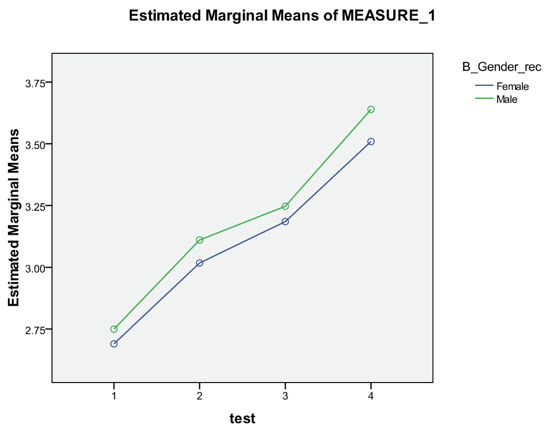

Figure 2. Mean number scores for female and male primary students March 2009 – September 2010.

Mean scores for female and male students in Years 6 and 7 in March 2009 and Years 7 and 8 in September 2010 are recorded in Table 2 and illustrated in Figure 3. Growth in achievement is well below the expected level (0.75 VELS). The gender difference is negligible at the beginning and end of the 18-month period.

Table 2. Achievement and growth for secondary students, March 2009 – September, 2010 (VELS).

		Females (N=453)		Males (N= 496)		Mean difference
Year	Month	Mean	SE	Mean	SE	
2009	March	3.807	0.042	3.833	0.040	-0.026
	Sept	4.075	0.042	4.081	0.041	-0.006
2010	March	4.178	0.044	4.222	0.042	-0.044
	Sept	4.351	0.044	4.353	0.042	-0.002
Growth Mar09– Sept10		0.544		0.520		0.024

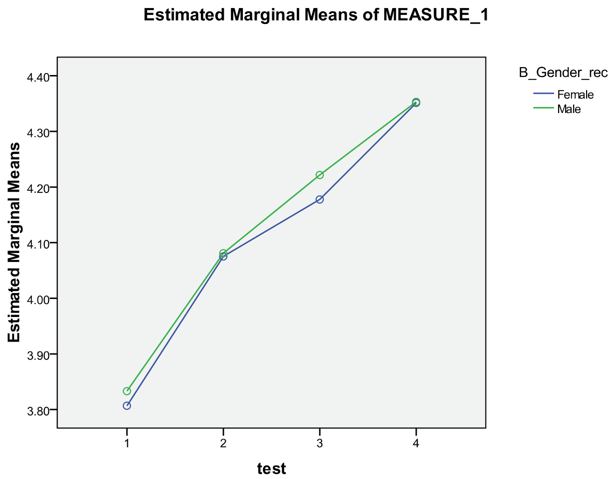

Figure 3. Mean number scores for female and male secondary students March 2009 – September 2010.

The different rates of growth for male and female students indicate the way in which classroom approaches support their learning and how they are affected by the summer slow-down. Growth in number achievement is higher for primary males than primary females from March to September and lower over the summer months. The opposite is the case for secondary students. Growth in number achievement for secondary females is higher than for males during Terms 2 and 3 and lower in Terms 4 and 1.

Numeracy intervention

There were more female than male primary students participating in numeracy interventions as expected, given the difference in achievement favouring male students in the primary years (see Table 4). In secondary schools there were more males than females who participated in numeracy intervention programs. The primary numeracy intervention programs especially benefited male students as their growth in achievement was significantly greater than the expected level (0.25 VELS for 6 months) and greater than the growth achieved by female students participating in these programs, though this gender difference was not significant. Students participating in secondary numeracy intervention programs recorded growth in achievement at the expected rate for 6 months and there was no difference between males and females. Hence while the primary numeracy intervention programs supported these students to achieve growth at above the expected rate they did not make an impact on closing the gender gap in achievement, and may have contributed to widening the gap.

Table 4. Growth in number achievement for students in numeracy intervention programs from March to September 2010.

	Primary (Year 3–6)		Secondary (Year 7–10)	
Cohort	N	Mean Growth	N	Mean Growth
Females	77	0.301	32	0.287
Males	51	0.428	38	0.283

Conclusion

Students in primary schools benefited from the student centred differentiated pedagogical approaches of the Pilot, since growth in achievement was greater than the expected level. However the stereotype of male mathematics hegemony was not challenged as the gender gap widened for students in all primary year levels. This was despite the fact that more females participated in numeracy intervention programs. The different effect of the summer slow-down on female and male primary and secondary students requires further investigation, however it is clear that student centred approaches must involve more than differentiated tasks if we are to close the gender gap in primary settings. It seems to us that a transformative approach that embraces the socio-constructivist perspective of learning is required if we are to address the intransigent gender differences in affect in mathematics and the persistence of gender differences in achievement.

Acknowledgement

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LEARNING FROM A PROFESSIONAL DEVELOPMENT DESIGN EXPERIMENT: INSTITUTIONAL CONTEXT OF TEACHING

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In this paper we report our learning as researchers from a 5-year professional development design experiment. At its completion, we identified five strands of support as being essential to mathematics teachers' learning. However, when planning the design experiment based on prior research, we only explicitly considered two of these strands—*Building Mathematical Competence* and *Focus on Student Reasoning*. The significance of three more strands of support became evident during the course of the experiment. We document the emergence of one of these strands, *Understanding the Institutional Context of Teaching,* by focusing on *pivotal episodes* from the experiment.

Introduction

Effectively supporting mathematics teachers' professional learning is a complex endeavour (Ball & Cohen, 1999; Goos, Dole, & Makar, 2007; Little, 1993; Simon, 2000). Research indicates that effective professional development (PD) programs should have a *longitudinal*, ongoing character as well as a *focus on content* (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Carpenter et al., 2004) in order to support significant, generative teacher learning. However, detailed analyses of the means of support used in longitudinal PD programs are largely missing (Little, 2002). By reporting such analysis, our goal is to contribute to teacher development theory (cf. Cobb, Zhao, & Dean, 2009) relevant to supporting the learning of teachers within high-stakes accountability environment.

The case for our discussion is a 5-year PD program[1] developed as part of a PD design experiment (cf. Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), conducted with a group of middle years mathematics teachers from a USA school district[2] with a high-stakes accountability program[3]. The PD goal was to "help teachers develop

[1] The PD program included a two-day summer institute and three one-day work-sessions during the first year of the study, a three-day summer institute and six one-day sessions during each of the subsequent four years, and a concluding three-day summer institute.
[2] In the USA, school district is an important, independent, administrative unit whose policies can have a significant influence on teachers' instructional practices.
[3] The presented study was a part of a larger research project. The research team included Paul Cobb, Kay McClain, Chrystal Dean, Teruni Lamberg, Melissa Gresalfi, Lori Tyler, Jana Visnovska, and Qing Zhao. In addition to the authors' analyses, this paper draws on dissertation analysis developed by Dean (2005). The preparation of this paper was supported in part by The University of Queensland under NSRSU grant No. 2009002594.

instructional practices in which they induct their students into the ways of reasoning of the discipline by building systematically on their current mathematical activity" (Cobb & McClain, 2001, p. 207). At the beginning of the design experiment and based on prior research in the field, Cobb and McClain (2001) outlined the initial conjectured trajectory for the teachers' learning (cf. Simon, 1995) and the means by which this learning would be supported. Two key strands of support were initially identified: *Building Mathematical Competence* and *Focus on Student Reasoning.* While the first directly addressed the need for PD to focus on mathematical content, the second aimed at supporting mathematics teachers' pedagogical reasoning and practices (e.g., Fennema et al., 1996; Franke & Kazemi, 2001a). As we argued elsewhere, both these strands proved to be critical in supporting the teachers' learning (Dean, 2005; Visnovska, 2009). However, additional directions of support were instrumental.

In this paper, we document how a specific new strand of support, *Understanding the Institutional Context of Teaching*, emerged in working with the teachers[4]. We first explain that the research team conceptualised teacher learning as situated within institutional context of teachers' schools and the district from the outset (Cobb & McClain, 2001; Cobb, McClain, Lamberg, & Dean, 2003), yet did not view institutional context as an explicit strand of support in facilitating changes in teachers' views of mathematics teaching. We then present *pivotal episodes* (cf. Cobb, Stephan, McClain, & Gravemeijer, 2001) from PD sessions that, in retrospect, provided insight into the emergence of this strand of support and its importance.

Data and method of analysis

The data consisted of video-recordings of all PD sessions, field notes of these sessions, copies of the teachers' work, and a debriefing and planning research log. We analysed the data using an adaptation of constant comparative method described by Cobb and Whitenack (1996) that involves testing and revising tentative conjectures while working through the data chronologically. As new episodes are analysed, they are compared with conjectured themes or categories, resulting in a set of the theoretical assertions that remain grounded in the data[5].

Initial focus on institutional context: Framing the PD design experiment

In planning the PD design experiment, Cobb and McClain (2001) conceptualised teaching mathematics as a distributed activity, shaped by the types of tools that were made accessible to teachers as well as the institutional context in which teachers worked. Rounds of data collection were conducted to document the institutional context of the teachers' work. These included interviews with the teachers and the key school and district administrators, and were used to understand (a) how the activity of teaching

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[4] The additional 2 strands of support that emerged from our work with teachers were *Building Teacher Community* and *Focus on Student Engagement* (for partial analysis see Visnovska & Cobb, 2009). Importantly, all 5 strands were interrelated and mutually reinforcing in supporting the learning of teacher group.

[5] Given the scope of this paper, we include representative teacher comments and interactions where possible as we build our argument. These examples do not provide a complete evidence base for the presented claims. References to our published work and dissertations indicate where more systematic evidence for the claims can be found.

mathematics was accomplished at the school and district levels and (b) what supports and constraints the teachers experienced in their work (Cobb, McClain, et al., 2003).

Oriented by this conceptualisation of teaching, in the initial PD sessions we included PD activities, in which teachers shared their views of their institutional context with us. We used these conversations to tune the initial PD design, by better understanding how the teachers reasoned about mathematics and mathematics teaching at the time. For instance, the persistent pressures for improving achievement on standardised tests helped us understand why it was reasonable for the teachers to focus on students "getting" the solution methods that lead to correct outcomes. Only much later in our collaboration with the teachers, through the analyses of the actual learning of the teacher group, did we realise that this initial attention to institutional context and the conversations we had with the teachers facilitated a number of changes that proved essential to teachers' learning. Specifically, through these conversations, the teacher group was supported to (a) "deprivatise" their teaching practices, that is, start to publicly discuss and critique their teaching, and (b) come to view changes to their current ways of teaching as both necessary and, more importantly, feasible in their schools. In addition, the group recognition of institutional context as a means to understand and transform how mathematics was taught in the district later contributed to the process of (c) inducting new members to the group, thus supporting the continuation of group learning. In the ensuing sections, we discuss how each of the three changes was realised in our work with the teachers, and build an argument for considering institutional context strand of support when designing PD programs.

Deprivatising teachers' instructional practices

The institutional context in which the teachers worked was characterised by high-stakes accountability and lack of formal and informal professional support (Cobb, McClain, et al., 2003). As a result, the teachers worked in almost complete isolation. When they initially participated in the PD sessions, it was both alien and uncomfortable for them to talk about their teaching openly without feeling they were being judged and their professional status threatened. However, in order for the teachers to engage productively in PD inquiries into classroom teaching (Ball & Cohen, 1999; Borko, 2004), it was imperative that they deprivatised their teaching practices.

In retrospect, the explicit conversations about institutional context that we had with the teachers were instrumental in the deprivatisation process. At the time, we included these conversations to deepen *our* understanding of how the schools and the district organised for mathematics teaching. The retrospective analysis revealed a pivotal episode that took place in year 1, session 3. During this work session, the teachers brought in their students' written work from a statistical task on life span of batteries and were asked to investigate how these students reasoned statistically. This, for the teachers, appeared to be a high-risk activity, perceived as a way to evaluate their instructional practices. As they carefully treaded the terrain, issues pertaining to the institutional setting dominated the discussion.

Naomi: … we were doing this [statistical task in my classroom] yesterday, my principal came in, she saw me at the overhead and the room was kind of dark and the kids were talking about batteries. And she is looking at me like "[Standardised tests] and you are talking about batteries?"

Amy: My principal took flack because the superintendent came in to my room and I was teaching Roman Numerals and they are not on the [standardised test]. I don't care.

Significantly, while expressing frustrations about pressures they felt, the teachers started to ask each other for advice. This was the *first* instance in which the teachers openly discussed events from their classrooms.

Rachel: [to Amy]…you were saying that you would give a kid a half an hour to get a kid to discuss something that you asked them. I agree with that totally, but … well my principal would say, "You are not covering all your topics". I agree, I want kids to explain things, but administrators would say, when they come in to observe your class, and I have had several to observe my class, they say "You are taking too long on this. You should ask them, maybe wait two or three minutes and then move on". So sometimes you can't get into that deep discussion because of time limits, because of behavior.

Amy: Part of it is the fact that I have been at this a lot longer than you and I know they ain't gonna bother me.

[Teachers laugh, some express agreement with Amy.] …

Rachel: Well how do you like, if you are talking to one particular student, for example, you are talking to me and I am hesitant about talking to you, how do you keep the rest of the class engaged? Because sometimes if I am focusing on one particular student, the rest of the students are like, okay ...

Amy: Simply the force of my personality to a certain extent. They know, that in this class everyone has a right to speak and everyone has a right to make a mistake. And everyone has a right to an opinion. And by God, if I am going to listen to yours, you are going to listen to his. It is just a matter of directing them...

In retrospect, this and similar conversations in a number of subsequent PD sessions helped teachers realise that they had experienced similar challenges and frustrations in their classrooms, and that these were related to the institutional context in which they worked. In a sense, teachers too began to view teaching as distributed. This allowed them to feel less judged when opening up their classrooms for discussions of their teaching, as they no longer felt the responsibility for failures to be solely theirs.

Cultivating a sense of feasibility of change

In our view, effective PD programs should proactively cultivate teachers' "reason and motivation to want to change the way they teach mathematics" (Cobb & McClain, 2001, p. 208). In our own and others' prior work, the teachers were successfully supported to develop such need by engaging in activities in which they realised that what their students understood mathematically as a result of their instruction was different from what was intended (e.g., Fennema et al., 1996). This led the teachers to question the teaching practices responsible for such learning and motivated them to work on improving these practices.

In contrast, the teachers in the PD program reported here initially considered it impossible to alter the ways they taught because, in their experience, the ways they taught were mandated by their schools[6]. Even after the teachers established that there was a contradiction between teaching for understanding and *content coverage* approach (for which they were accountable to their principals), they did not come to believe that it was feasible to change their practices and did not become interested in scrutinising them. To the teachers, institutional pressures of their work appeared to be given and not susceptible to change. From our perspective as researchers, it became critical to cultivate both teachers' motivation and sense that it was feasible to change their teaching practices.

This led us to introduce PD activities in which we proactively challenged these teachers' views. In retrospect, two episodes were pivotal and we introduce one of them here. At the end of session 5 in year 2, we proposed a possible future project for the group: generating evidence to show school leaders that covering content does not help students learn mathematics. The teachers picked up the proposal and, in a quick progression, brainstormed ideas for getting the principals involved in thinking about mathematics teaching and learning more deeply.

Wesley:	I just had an idea: think about it. The middle school principals are going to be here on the 19th. Maybe if they are here for food, maybe we could be in here with them to convince them we are doing something good.
Ruth:	It is a small group of them. But they are going to be looking at the schools. …
Naomi:	So maybe we should be doing an activity while they are here and invite them to come see the activity.
Muriel:	Or with the kids?
Researcher:	Or what the kids are doing.
Muriel:	Yeah, I'd like for them to see what the kids are really thinking [mathematically, like when we interviewed students in last PD session]. …
Naomi:	I bet they would be surprised.
Researcher:	I bet they would… That idea might have merit … Letting them know that the 6th grade teachers are doing what [principals] are telling them: they are covering the material, they are reviewing, but [in 7th grade students need to learn it again anew].
Muriel:	I would like for them to see it and then hear the discussion afterwards.

In the subsequent months, the teachers proactively pursued opportunities to engage with the school leaders, continued to plan for the joint PD activity, and framed these efforts as an avenue to justify to the school leaders the need for resources (e.g., time to collaborate) to improve students' mathematics learning and performance.

While five school leaders eventually attended PD session 6 in year 3[7], the changes in teachers' perceptions of feasibility of changing how they taught mathematics were obvious form the very beginning of year 3. Despite the fact that there were no discernible changes in institutional context and the teachers continued to be dissatisfied with the situation in their schools, they no longer merely shared their complaints.

[6] The analysis of the institutional context (Cobb, McClain, et al., 2003) corroborated the teachers' reports. The school leaders viewed teaching mathematics as a straightforward endeavor and responded to accountability pressures of state-mandated achievement tests by monitoring teachers' content coverage. Some of them conducted daily drop-in visits in teachers' classrooms to check whether appropriate objectives were being covered. As a result, the teachers viewed themselves as having little control over both goals of mathematics instruction and how these goals should be accomplished in their classrooms.

[7] It is indicative of the institutional context that it took the teachers and district mathematics coordinator more than one year to succeed in securing the school leaders' participation.

Instead, they planned what they could do in order to change their school leaders' views. The teachers came to realise that they collectively had a better understanding of how mathematics should be taught compared to their school leaders and were thus better positioned to guide instructional improvement. This motivated the teachers to foster their professionalism by scrutinising their practices and developing more effective ways of teaching for mathematical understanding.

Supporting the continuation of the teacher group

Realising the importance of explicit attention to institutional context and how it shaped the work of teaching mathematics, both the researchers and the teachers made institutional context an explicit topic of conversations when new teachers were recruited to join the group[8]. The continuing teachers named the *working on issues related to institutional context* among the 4 goals[9] of the PD work when they introduced PD goals to the newcomers during an orientation session in year 3. Sharing the group history, they also clarified that discussions of institutional context helped to build trust between teachers and researchers at the beginning of the collaboration.

> Amy: [The researchers] have provided us with a [soundboard]. … like at the very beginning, I knew we were supposed to do statistics, [a researcher] came to the first one [PD session], and we were sitting there for four hours and she listened to us complain about every single solitary thing that ever crossed our minds as we've been teaching. And I was [thinking] "When is she going to tell us to shut up, that that's not what we are here for?" And she never did. So they've always sat around and listen. They wanna know what is important to us whether it is on their agenda or not.

Continuing teachers contrasted the context of PD sessions to institutional context in their schools when they talked about the collaborative nature of the PD group, its non-threatening culture, and highlighted how this difference helped them to open up their practices to the group.

> Marci: I guess we are all comfortable with each other, and not just that, but comfortable with having people to come in and not criticise you based on what you taught, not on what their idea of teaching math is. … It is different from when the administrators may come in or even for new teachers, when a mentor is coming to observe. Because you feel that you are looking for something in particular to criticise their way or their method of teaching mathematics.

They also demonstrated the deprivatised nature of their practices by bringing their students' work and classroom video to sessions, and by talking openly about difficulties that they faced in their teaching[10].

The stories told by continuing teachers and their actions had face validity for the newcomers. While deprivatisation of teaching practices initially took more than one

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[8] A group of ten teachers participated in PD program in first two years. In the remaining years, some of the teachers left the district and the PD group and others were recruited to join. For details on membership in the PD group and conceptualisation of group learning across its changes see Visnovska (2010).

[9] The other three goals teachers named were: (a) understanding students' thinking, (b) "redoing" textbook units on statistics, and (c) learning about improving lessons over time like in Japanese lesson planning (Visnovska, 2009).

[10] Two situations in which the old-timers commented on their classroom difficulties spontaneously occurred in session one, one in session two, and others occurred with a similar rate throughout the year.

year (Dean, 2005), all newcomers opened up their practices for scrutiny within their first four PD sessions (Visnovska, 2009). They also actively engaged in working on improving their teaching and shaping their institutional context, and raised no doubts about feasibility of these efforts. Successful initiation of the newcomers enabled the teacher group to continue working towards its goals across changes in the group membership (Visnovska, 2009).

Conclusions

diSessa and Cobb (2004) clarify that productive design-based theorising includes "hypothesizing and developing explanatory constructs, new categories of things in the world that help explain how it works" (p. 77). We propose that—along with longitudinal character of PD support, focus on content and on students' mathematical reasoning—attending to *institutional context of teaching* is important in both understanding and effectively supporting teachers' generative growth (cf. Franke & Kazemi, 2001b).

To substantiate this claim, we discussed three practical problems that occurred in our PD collaboration that had to be overcome for the PD program to be effective. Firstly, *deprivatising* teachers' practices was necessary if these were to become a subject of inquiry in PD sessions. Secondly, coming to see changes as *feasible* within the institutional environment was instrumental in developing teachers' genuine need and motivation for improving their teaching. Lastly, establishing *continuation* of the group learning across changes in its membership was important as the district in which we worked had relatively high teacher mobility. We have illustrated how attending to *institutional context of teaching* in the PD activities helped in addressing these practical problems.

The research team was initially unaware that conversations about institutional context would be influential in the group learning. Presented results are thus a product of genuine research team learning enabled by the retrospective analysis of the PD collaboration. We suggest that the results are most relevant to PD designers and facilitators working with teachers in similar institutional settings, and to teachers who would benefit form effective PD programs.

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AN EXPLORATION OF YOUNG STUDENTS' ABILITY TO GENERALISE FUNCTION TASKS

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The *Early Years Generalising Project* involves Australian students, Years 1–4 (age 5–9), and explores how the students grasp and express generalisations. This paper focuses on the data collected from clinical interviews with Year 3 and 4 cohorts in an investigative study focussing on the identification, prediction and justification of function rules. It reports on students attempts to generalise from function machine contexts, describing the various ways students express generalisation and highlighting the different levels of justification given by students. Finally, we conjecture there are a set of stages in the expression and justification of generalisations that assist students to reach generality within tasks.

The *Early Years Generalising Project* (EYGP)[1] is a series of cross-sectional studies of cohorts of students from Year 1 to Year 4 (age 5 to 9) that aims to build theories regarding young students' ability to grasp and express generalisations, the two components of the act of generalisation in terms of Radford, 2006). Each cross-sectional study covers a particular context and form of generalisation (e.g., growing patterns and pattern rules, equivalence and equation principles, operations and arithmetic processes and structures). Each study has two stages: (a) exploration—an initial stage where a small sample of students (n=5) from each Year level participate in one-on-one clinical interviews; and (b) validation—a final stage where, as a result of these interviews, conjectures were posed and tested in one-on-one semi-structured interviews conducted with a further cohort of 20 students from each Year level, selected to represent a wide range of academic abilities and cultures.

This paper presents a single aspect of the project; an exploration of how Year 3 and 4 students (age 7 to 9) express and justify generalisations for the context of input-output changes using function machines and the form function rules. It covers two year levels of the initial stage of the cross-sectional study on function machines.

Context

For EYGP, mathematics consists of relating and transforming things (numbers, shapes, variables) with relationships and transformations being two ways of looking at the same idea (Scandura, 1971), and the power of mathematics being the way relationships and

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transformations on their own or together give rise to generalisations (Warren, 2005). Functional thinking emerges from the transformational perspective but can be understood in relationship terms, and is the ability to identify the rules that relate two or more varying quantities (Smith, 2008).

There are some studies that suggest that young students can think functionally and generalise in functional situations. For example, Blanton and Kaput (2005) found that students can engage in co-variational thinking as early as Kindergarten and use t-charts and express rules in Years 3 to 5, while Cooper and Warren (2008) found that Years 3 and 4 students can generalise rules for function contexts. However, there is still little known about how young students' identify and generalise function rules. Most studies of functional thinking have focused on middle years' students and explored functions represented as growing patterns (e.g., Lannin, 2005, Radford, 2006). These studies require students to coordinate two variables where one is explicitly represented (e.g., the visual representation of the growing pattern) and the other variable is more abstract (e.g., the position of each term). By focusing on function machines and input-output changes, this paper explores the question that, if we represent both variables and the function action more explicitly, does this assist students to reach more explicit generalisations?

Studies with older students with the focus on growing patterns have identified the different approaches students use when completing generalisation tasks. Harel (2001) identified two approaches: (a) results generalisation where a generality is developed from a few examples usually involving trial and error; and (b) process generalisation where a generality is developed and justified when considering progression across many steps. This classification is supported by Radford (2006) who has labelled the two approaches as naive induction and generalisation and Lannin (2005) who has labelled them non-explicit and explicit. To investigate this classification in younger students, this paper also explores the extent to which young children can justify their generalisations.

Theoretical framework

Underpinning this research project is the theoretical perspective of semiotics. Mathematics has been depicted as an intrinsic symbolic activity which is achieved through communicating using oral, bodily, written and other signs (Radford, 2006). The discipline of semiotics is based on perceivable signs that assist understanding of the mathematics processes of thought, symbolisation and communication. Of particular importance to this paper is the use of body and language, seen best through the physical activity of students as they interact with artefacts (Sabena, 2008). Additionally, studies have noted that cognition is strongly related to the use of the body (Lakoff & Núñez, 2000). It was this framework that drove the construction of the activities and framed the data analysis.

Method

Ten students from Years 3 and 4 (4 males and 6 females with an average age of 8.5 years) were selected to be interviewed in the initial exploration stage of this study. The students were from a middle socio-economic school in the outer suburbs of a major city and had a range of academic abilities and cultural backgrounds. The interviews consisted of 6 tasks; two having a language focus, one having a geometry focus, and three having a number focus. The aim of the tasks was to probe students' understanding of functions. The interview was video recorded and was of approximately 20 minutes duration. The students were presented with activities involving concrete materials and whole body movement starting from unnumbered situations and moving to numbered situations. Table 1 presents the six tasks, each tasks function rule, and an example of the input and output values for each rule.

Table 1. Example of tasks given to students.

	Unnumbered situations situations				Numbered		Numbered situations					
Task	Language (1)		Language (2)		Shape (3)		Number (4)		Number (5)		Number (6)	
Rule	Add 'ip'		Add 'ap'		Make it thinner and smaller		Add two		Subtract three		Double	
Example	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
	T	Tip	M	Map	Red, large, thick triangle	Red, large, thin triangle	5	7	10	7	4	8

Initially, students were introduced to a cardboard box function machine called Rosie. The input and output values were presented on cards or as physical shapes. The interview began with the first language task—Language (1). Each student was shown a letter and asked to place it into Rosie's ear (input) and then the researcher produced the output card from the opposite ear (output). This occurred for three input numbers. Then they were asked to predict the output value for given input values. Each student was then asked to identify the rule.

The questions posed were contingent on the responses given by the student. After the first question, depending on their responses, students were either given further examples or were asked to predict output values for given input values. They were asked then to predict input values for given output values and to identify the reverse rule. The researcher asked students to justify their answers and express the rule and its inverse in general terms. This process was repeated for each task. In practice, the process mirrored an "acting out" of input-output tables (t-tables) and identifying the relationship between the corresponding pairs of values in the table. From a semiotic perspective, the signs were the cards and kinaesthetic movement.

All video recordings were transcribed with attention paid to both the students' verbal responses and their manipulation of the concrete materials, in particular how students engaged with the signs and interpreted these signs as they identified the function. The data was analysed by two researchers and member checks were performed. Semiotics has been used throughout the research project to analyse the data. Within this particular study, the data sets have emerged out of the semiotic analysis conducted. The interpretation of actions are not included in this paper, but if interested please refer to Warren, Miller, and Cooper (2011).

Results and discussion

The data associated with each task is organised into four sections, namely, the student's ability to correctly predict: (a) output values from given input values, (b) the function rule, (c) input values from given output values, and (d) the inverse function rule.

Table 2 presents the tasks together with the frequency of students who were successful in each section.

Table 2. Frequency of student's correct responses to six tasks.

Tasks	Language tasks		Shape task	Number tasks		
	(1)	(2)	(3)	(4)	(5)	(6)
Rule	Add 'ip'	Add 'ap'	Make it smaller & thinner	Add 2	Subtract 3	Doubling
Predict output	8	8	6	10	9	8
Identify output rule	8	8	8	9	8	8
Predict Input	10	8	7	8	8	8
Identify Input (inverse) rule	7	7	6	8	7	4

The results indicate that students could predict the output card for given input cards when asked. The shape task was the only task in which students appeared to have difficulty and this pertained to their inability to describe the attributes of the particular shape (colour, size, thickness).

At least 80% of students could identify the rule Rosie was using to create the output value. However, students were not always able to identify the input rule (inverse rule). This was particularly so for the last number task (doubling) as students did not appear to have the mathematical language to describe the action of halving or dividing by two.

The students were then asked three questions to explore their ability to generalise the three number tasks.

- First, they were asked to pick the largest number they knew as an input value and identify the corresponding value that would come out of the function machine. For the purposes of this study this has been labelled a quasi generalisation (Cooper & Warren, 2008, adapted from Fujii and Stephens', 2001, notion of quasi-variable).
- Second, the students were given a fictitious number (e.g., finky) as the input value and asked to predict the output value.
- Third, their ability to inverse the process was also probed by asking them if 'finky' came out what value would they put in the machine.

These questions were included to determine if the student could generalise the rule beyond the use of numbers and move to a more abstract understanding that entailed the use of variables.

Table 3, below, identifies students' responses to each of these three questions for each of the tasks. The tick indicates that their quasi-generalisation was correct and the written text identifies the rules they predicted for the fictitious number 'finky'.

Table 3. Student's success in quasi generalisation and generalising the number tasks.

Student	Plus 2 (number task 4)			Subtract 3 (number task 5)			Double (number task 6)		
	Quasi	Finky in	Finky out	Quasi	Finky in	Finky out	Quasi	Finky in	Finky out
S1	✓	Take out the inky	nr	✓	nr	nr	✓	nr	nr
S2	✓	2Finky	Finky	✓	00Finky	3finky	✓	2finky	nr
S3	✓	2Finky	-2finky	✓	-3finky	3finky	✓	Finky2	nr
S4	✓	Finky2	Finky-2	✓	finky-3	finky3	✓	Double finky	Half finky
S5	nr	nr	nr	nr	nr	nr	nr	nr	nr
S6	✓	It will turn into a 2	Finky-ky	✓	nr	nr	✓	nr	nr
S7	✓	Finky add 2 letters	Finky take 2 letters	nr	finky -3 letters	Finky plus 3 letters	nr	nr	nr
S8	✓	K + 2 = 7 therefore k=5	p –2 = 4 therefore p=6	✓	n – 3 = 16 therefore n = 19	N + 3 =19 therefore n = 16	✓	Double q	Halve q
S9	✓	nr	nr	✓	nr	nr	nr	nr	nr
S10	✓	Finky + 2	Finky has to go down by 2	✓	Finky – 3	Frisky + 3	✓	Finky x 2	Finky divided in half

Note: nr – no response

Of the students who were asked to generalise the 'add two rule' using the word 'finky', 2 students were successful in expressing the generalisation. The other students would either talk about the generalisation in regard to adding two letters or express it as 'finky2' without using the mathematical operation involved with the function. S8 required a value for the variable and therefore he used expressions that incorporated single letters.

Nine students' generalisations aligned with Harel's (2001) process generalisation (showing generalisation across a number of steps) or Lannin's (2005) explicit generalisation (linking the dependent variable with the independent variable). The different levels of process/explicit generalisation tended to be related to misunderstandings of the notation system used to represent variables and expressions involving operations. Many of these misunderstanding reflected the categories identified by Küchemann (1981): particularly *Letter as object*, *Letter as specific unknown*, *Letter as generalised number*, and *Letter as variable*. It did not seem that the students were engaging in 'guess and check' either in the initial stages of identifying the rule or in "whole-object" strategies as identified in past research involving growing patterns (e.g., Lannin, 2005; Radford, 2006).

Table 4 presents the levels of expressions for generalisation together with examples of each descriptor for each level. Statements such as 'finky2' were accompanied by utterances such as "You add 2, it is finky2", which aligns with adding two to 50 and obtaining 52. In all there were 60 responses related to describing the generalisation (6 per student).

Table 4. Levels of expression for explicit generalisations together with frequency of student usage.

Level	Descriptor	Example (+2 rule)	Frequency
1	No expression	R: What if I had a made up number like finky and put that into Rosie. What would come out? S: A donkey. R: What do you have to do with it? What does the machine do to it? S: I don't know.	25
2	Letter as object	R: So what do you think would happen to finky? S: Finky add two letters	7
3	Letter as specific unknown	S: It is a K so K plus 2 is & so K is 5.	4
4	Letter as generalised number or variable	R: What do you think would happen if I put in a number called finky? S: 2 Finky you add two.	12
		R: What if I put in a number like finky? What would come out? S: Finky add 2.	12

As indicated in the results, 40% of the students' responses (n=24) incorporated the use of letters as generalised numbers or as variables. This was accompanied with students reiterating that 'finky' meant any number. Most of the responses that were considered as Level 1 responses were proffered by three students, S1, S5, and S9. From the results, stages of expression of justification were hypothesised. These stages relate to the use of numbers and unknowns in the students' general statements, and reflect the stages proffered by past research (e.g., Lannin, 2005). Table 5 presents the three stages with the associated exit points of each student.

Table 5. Stages of expression of justification.

Stage		Descriptor	Exit point
1	Numeric evidence (countable numbers)	Used small countable numbers to justify the rule	S5, S7, S9
2	Quasi - generic evidence (uncountable numbers)	Used quasi-variables to justify the rule	S1, S6, S8
3	Generic evidence (algebraic expression)	Used letter notation to justify the rule	S2, S3, S4, S8, S10

This research makes the distinction between using large numbers to justify generalisations and using algebraic notation. This reflects the distinction that Fujii and Stephens (2001) make with regard to the use of variables and the quasi-generalisation of

Cooper and Warren (2008). We conjecture that for young students, moving from familiar numeric situations to using large uncountable numbers represents a leap in understanding. It shows that students are moving from a simple computational situation to evidencing an understanding of the applicability of that computation across the number system.

Conclusion and implications

This research presents three main tentative conclusions. First, young students can engage in activities that require them to express and justify generalisations. This result suggests that there is a need for young students to experience functional thinking activities within the classroom to develop higher levels of mathematical understanding. It would be suggested that kinaesthetic activities that link directly to the learning context of the student would be beneficial. The level of thinking they exhibited mirrors that shown in past research in growing patterns with older students. In this instance though there is one distinctive difference in these students' responses which is the absence of Lannin's (2005) terms of non-explicit generalisations or recursive thinking, building on the previous term or terms in the sequence to determine subsequent terms. We suggest that this is a result of how the activity was constructed where the signs for the input and output were explicit (represented as input and output cards) and the linking of the data sets was accompanied by physical movement. In addition the input numbers were randomly selected thus ensuring that there was no implicit relationship in one data set (e.g., the input or output numbers).

Second, we conjecture that young students' ability to reach generalisations was assisted by the types of activities that were selected and the way they were presented to the students. The crux of problems involving functional situations is the need to coordinate two data sets, the independent and dependent variables and identify the relationship between these sets. The activities for this research were deliberately chosen so that this relationship was transparent. From a semiotic perspective the signs for each were visible and required the students to be actively involved in their creation. Blanton and Kaput (2005) also chose tasks where the variables were explicitly related, for example, the number of eyes and tails on puppy dogs, and hence the students demonstrated success in this task. In addition, the EPGP study, the function or change process was represented kinaesthetically by gesturing with hands across the front of the function machine. This assisted students to focus on the underpinning concept embedded in all of these activities, which is co-variational thinking.

Third, we also conjecture that the context for growing patterns in previous studies is restrictive and abstract. The position of each term as one of the variables is not transparent and we conjecture this contributes to the use of guess and check and recursive strategies. Additionally, in past research students have been asked to engage in the exploration of functional problem solving situations with little prior experience in co-variational thinking. This adds to their difficulties. Our research suggests that young students can deal with co-variational situations as long as both variables are explicitly represented and the rule is clear for students. The tasks presented in this study focus on the relationships within the function, that is, it is not obscured by other aspects as it is in patterning. When using examples such as patterning sequences, students tend to 'run along' the pattern instead of recognising the covariant relationship between pattern terms and their positions. Additionally, cards were displayed to the students in a random sequence forcing students to focus on the relationship between the input and output (horizontal relationship) rather than on the relationship of just the output cards (vertical relationships).

This paper has focused on students' attempts to generalise from function machine contexts, describing the various ways students express generalisation. Furthering the conjectures presented the *Early Years Generalising Project* is continuing to further investigate functional thinking with larger cohorts of students.

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TEACHER CHANGE IN A CHANGING EDUCATIONAL ENVIRONMENT

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This paper considers the change in teachers' confidence, beliefs, and knowledge with respect to mathematics teaching across a 3-year collaborative intervention, which although planned in a reform-based learning environment, took place as the reforms were rolled back and a new view of curriculum introduced. Of 86 middle school teachers involved at some time during the project only 19 completed both the pre- and post-profiles and of these only 11 had been in the project since its beginning. Teacher change appears more likely to have been related to the length of time in the program than to the state-wide curriculum changes.

Introduction

The Tasmanian project upon which this report is based was titled "Mathematics in an Australian Reform-Based Learning Environment" (MARBLE). The "reform-based learning environment" reflected moves of several Australian states to create values-based curricula "designed to meet current educational needs by making legitimate connections between disciplines" (Department of Education Tasmania (DoE), 2002, p. 11). The aims of the project within the context of the Tasmanian Essential Learnings curriculum (D*o*E, 2002) were to provide professional learning (PL) for teachers to assist them in enhancing middle school students' mathematical understanding necessary for the quantitative literacy needs of today's society (Steen, 2001) and for the further study of mathematics in order to contribute to innovation in Australia (Committee for the Review of Teaching and Teacher Education, 2003).

Research elsewhere had suggested that important features of PL programs included:

> (a) ongoing (measured in years) collaboration of teachers for purposes of planning with (b) the explicit goal of improving students' achievement of clear learning goals, (c) anchored by attention to students' thinking, the curriculum, and pedagogy, with (d) access to alternative ideas and methods and opportunities to observe these in action and to reflect on the reasons for their effectiveness. (Hiebert, 1999, p. 15)

Sowder (2007), in her extensive review, similarly advocated the need for ongoing PL. The challenge faced by the MARBLE PL program was fitting all of these aspects into the time and resources available.

Several papers previously reported on some of the outcomes of the MARBLE project. Pertinent to the current work, Watson, Beswick, and Brown (2006) reported on initial data collected on a fraction problem, indicating the strengths and weaknesses of teachers' pedagogical content knowledge (PCK) in relation to the task. Information such as this formed the basis of the interventions that took place during the project. Initial levels of teachers' confidence and beliefs were covered by Beswick, Watson, and Brown (2006). Change in students' attitudes (Beswick, Watson, Brown, Callingham, & Wright, 2011) and performance (Watson, Brown, Beswick, Callingham, & Wright, 2010) over the time of the project have also been reported. Initial analysis of teacher knowledge was provided by Beswick, Callingham, and Watson (2011). The current paper completes the data analysis by reporting on the changes that took place for teachers over the 3 years of the MARBLE project.

Background context for MARBLE

The background to the MARBLE PL program was the Essential Learnings Framework (DoE, 2002). This curriculum framework identified 18 Key Elements within five Essential Learnings (Thinking, Communicating, Social Responsibility, World Futures, and Personal Futures). "Being Numerate" was identified as a key element in the Communicating Essential and was one of the first Key Elements against which teachers reported in 2005. This emphasis recognised "Being Numerate" as an important cross-curricular understanding and coincided with an increased focus on pedagogy and collaborative practice across the curriculum.

Amid controversy over the implementation of the Essential Learnings Framework, in 2006 a new curriculum was announced by the incoming Minister for Education that would "make [the curriculum] easier to understand, and more manageable for teachers and principals" (D*o*E, 2007, para 1). Mathematics/Numeracy became one of eight defined areas of the curriculum against which both primary and secondary teachers are required to report.

Against this backdrop, the research question for this paper is: What changes occurred for various subgroups of teachers in the MARBLE project in relation to the knowledge and confidence for teaching mathematics?

The professional learning program

The initial experiences provided for teachers in the MARBLE project were summarised by Watson, Beswick, Brown, and Callingham (2007) in relation to mathematical content knowledge, PCK, knowledge of students as learners, and curriculum knowledge. PL topics in the earlier years of the project included quantitative literacy in the media, problem solving strategies, and assessment (formative and summative, and involving the use of rubrics). The final coverage of topics in the project is contained in Beswick et al. (2011). Topics included the use and benefits of concrete materials, planning a unit of work, and understanding common misconceptions with fractions.

The schools in the project were situated in rural areas of the south (five) and north (four, including one Catholic) of the state. Four of the DoE schools were district high (K–10), one was a high (7–10), and three were primary (K–6) schools; the Catholic school was K–10. Except for one planning session with representatives of all schools held at the beginning of the second year, all PL sessions were held within the two clusters of schools. There were 3 whole-of-cluster sessions in the first year, 11 in the second, and 10 in the third in each region. The sessions were largely the same in each cluster but on occasion the specific needs of teachers meant that modifications of content occurred or specific topics were included. Feedback, in addition to that reported here, was sought from teachers at the end of each session, and through meetings with school coordinators, face-to-face interviews with 19 teachers at the end of the project, and surveys of teachers who left the project (and school) during the project.

Methodology

Design and sample

The overall research design was a longitudinal study of teacher and student change with respect to the interventions as part of the project. As noted elsewhere (e.g., Watson et al., 2010) students' attitudes and performance were measured each year. Teachers completed a profile adapted from the work of Watson (2001) when they entered, and at the end of, the project.

It was envisaged that most teachers would be in the project for 3 years but as seen in Table 1, this was not the case. The table contains information on the teachers who took part in the MARBLE project. Some teachers did not participate for long enough to complete either the initial or the final teacher profile.

Table 1. Teacher participation in the MARBLE project.

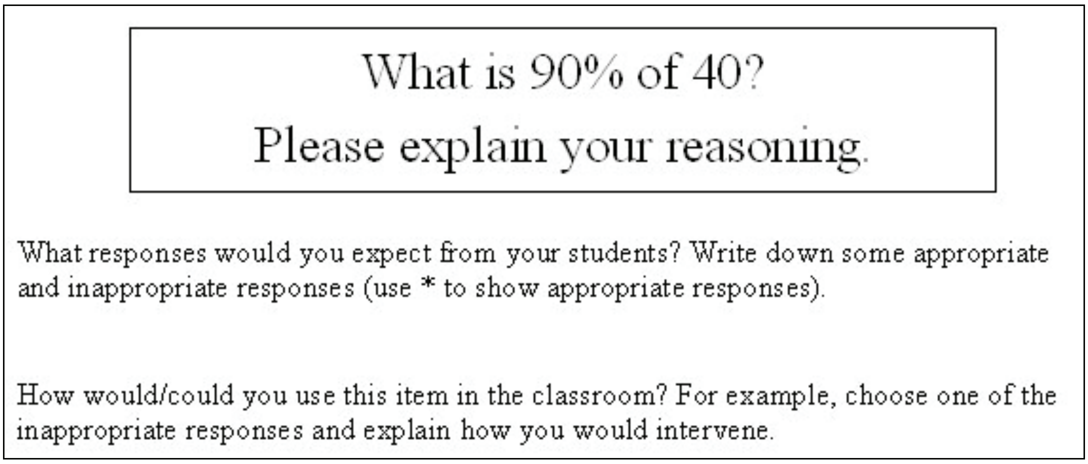
	Year 1	Year 2	Year 3	Total
Number of Teachers	42	47	54	
New Teachers	-	24	20	86
Completed Initial Profile	42	12	9	63
Completed Final Profile	11	3	11	25*

* Of the 25 teachers who completed the final profile, only 19 had completed the initial profile.

Instruments

The initial profile questions provided a data set comprising five sub-scales relating to teaching mathematics: Confidence, Everyday Life, Numeracy in the Classroom, General Pedagogical Knowledge, and Pedagogical Content Knowledge. Coded scores for items in the Confidence, Everyday Life, and Numeracy in the Classroom subscales ranged from 1 to 5, with higher scores representing more confidence to teach the concept (such as fractions) or a higher level of agreement with the given statement (e.g., "I need to be numerate to be an intelligent consumer").

General Pedagogy items were coded hierarchically, with higher scores representing higher levels of pedagogical knowledge. The highest level response (code 3) for the item, "How would you go about improving students' numeracy and mathematical understandings?", for example, indicated that teachers provided an integrated, high-level rationale for their written responses. The PCK items were also scored hierarchically and asked teachers to think about the range of responses their students would give to each of the numeracy items, and then consider how to use the items in the classroom. An example of a PCK item is presented in Figure 1.


What is 90% of 40?
Please explain your reasoning.

What responses would you expect from your students? Write down some appropriate and inappropriate responses (use * to show appropriate responses).

How would/could you use this item in the classroom? For example, choose one of the inappropriate responses and explain how you would intervene.

Figure 1. An example of a PCK item used in both profile administrations.

From the sub-scales a Combined Scale was constructed that was used by Beswick et al. (2011) to suggest a four-level hierarchy for teacher knowledge for teaching mathematics. These levels were labelled Personal Numeracy, Pedagogical Awareness, PCK Emergence, and PCK Consolidation, based on the outcomes of Rasch (1960) analysis, to reflect increasing ability of teachers to express confidence in their capacity to teach topics, to cope with numeracy in everyday life, to agree with student-centred statements about numeracy in the classroom, and to display sophisticated general pedagogical knowledge and PCK for mathematics.

Analysis

The original data set used by Beswick et al. (2011) was augmented by one teacher; the software Winsteps (Linacre, 2006) and the Rasch Partial Credit Model (Masters, 1982) were used for the analysis reported here. Of the 59 individual profile items, 49 were common to both initial and final profiles and were used to link the two profiles for analysis. The 49 link items provided an anchor set that established the difficulties of the items at each test administration relevant to each other and estimates of person ability were identified for each teacher in the original and follow-up profile, anchored to the same set of link item difficulties so that genuine comparisons could be made. These ability measures were used as a basis for subsequent analysis. *T*-tests were used to compare the mean ability levels of all teachers who completed either the initial or final profiles and paired *t*-tests were used to compare those of teachers who completed the profile on both occasions. Effect sizes were calculated as described by Burns (2000), looking at the profile items as a whole and separated into the five sub-scales.

Results

The results for the overall profile and the five sub-scales are presented in four stages, comparing the initial and final profiles completed by the following groups of teachers: all at the start (n = 63) with all at the end (n = 25); those who completed both initial and final profiles (n = 19); those who began in Year 1 and completed both profiles (n = 11); and those who began in Years 2 or 3 and completed both profiles (n = 8).

Table 2 shows that in comparing all teachers who completed the initial profile (n = 63) and/or the final profile (n = 25) there was little change in the overall Combined

Scale, Confidence, Numeracy in the Classroom, or PCK. The change in teachers' reaction to numeracy in Everyday Life was significant and negative. The mean ability score for this subscale for teacher ID17, for example, fell from 4.6 to 1.4, a difference of 3.2 (raw score range of 1 to 5). The only significant positive change for this group occurred in relation to general classroom pedagogical knowledge, which, from the effect size, should have been observable in the classroom.

Table 2. Change for all teachers completing initial and/or final profiles.

	Original (*n = 63*)		Follow-Up (*n = 25*)		*t*	*p*-value	Effect size
	mean	SD	mean	SD			
Combined scales	0.57	0.45	0.61	0.41	0.391	0.697	0.09
General Pedagogy	-0.24	0.82	0.45	0.87	3.530	0.001**	**0.83**
Confidence	0.84	1.21	0.87	0.94	0.120	0.904	0.03
Everyday Life	1.70	1.30	1.03	0.75	2.397	0.019*	**-0.56**
Numeracy in the Classroom	0.52	0.41	0.54	0.41	0.231	0.818	0.06
PCK	-0.03	1.36	0.16	1.34	0.596	0.553	0.14

* Significance <.05. ** Significance <.01.

Table 3 contains parallel results for the 19 teachers who completed both profiles, regardless of when they began with the MARBLE project. The results were in the same direction and were similar to those in Table 2.

Table 3. Change for teachers who completed both initial and final profiles (paired t-tests).

	Original (*n = 19*)		Follow-Up (*n = 19*)		*t*	*p*-value	Effect size
	mean	SD	mean	SD			
Combined scales	0.61	0.45	0.64	0.47	0.165	0.870	0.05
General Pedagogy	-0.08	1.00	0.53	0.98	1.889	0.067	**0.6**
Confidence	0.91	1.39	0.93	1.04	0.050	0.957	0.02
Everyday Life	2.02	1.49	1.17	0.78	2.180	0.036*	**-0.69**
Numeracy in the Classroom	0.55	0.33	0.56	0.38	0.023	0.982	0.01
PCK	-0.30	1.64	0.06	1.48	0.704	0.486	0.23

* Significance <.05.

Table 4 summarises the results for the 11 teachers who were involved in the MARBLE project for all 3 years and completed both profiles. The *t*-values are not significant, except for Everyday Life, due to the small sample size, but the effect sizes are larger than for the other groups of teachers. For the 11 teachers, only Numeracy in the Classroom showed no change, whereas PCK showed a meaningful increase reflected in the effect size. Results for General Pedagogy and Everyday Life were similar to those for the large data sets of which they were a part. Using Burns' (2000) classification of ± 0.4 as a significant effect size for this type of data, the combined scale of all items for these teachers shows an almost significant effect size at 0.38. This differs considerably to the effect size seen in Table 2, showing almost no difference. Using the four-level hierarchy described by Beswick et al. (2011), three of the 11 teachers achieved a higher level in the follow-up profile administration; one moving from Level 3 (PCK Emergence) to Level 4 (PCK Consolidation), and the other two from Level 2 (Pedagogical Awareness) to Level 3. Two teachers shifted in a negative direction, moving from Level 3 to Level 2, however the degree of movement was very small. Other teachers remained within the same level. Overall, the mean ability score from the first profile administration to the second went up for 7 teachers and down for 4 teachers.

Table 4. Change for teachers who participated for 3 years and completed both initial and final profiles (paired t-tests).

	Original (n = *11*)		Follow-Up (n = *11*)		*t*	*p*-value	Effect size
	mean	SD	mean	SD			
Combined scales	0.65	0.37	0.79	0.35	0.926	0.366	0.38
General Pedagogy	-0.20	1.19	0.48	0.97	1.486	0.153	**0.61**
Confidence	1.20	1.18	1.37	0.79	0.627	0.538	0.26
Everyday Life	2.36	1.45	1.30	0.70	2.177	0.042*	**-0.89**
Numeracy in the Classroom	0.55	0.36	0.56	0.39	0.074	0.942	0.03
PCK	-0.85	1.87	0.26	1.38	1.590	0.127	**0.65**

* Significance <.05.

Table 5 summarises the results for the 8 teachers who were involved in the project for 1 or 2 years only and completed both profiles. The results are similar to those of the other participants in relation to an improved general pedagogy, a decrease in relation to use of numeracy in Everyday Life, and no change in relation to Numeracy in the Classroom. The big changes, however, were with respect to Confidence and PCK, which were negative and brought about a negative change in the Combined Scale. These teachers appeared to have experienced an "implementation dip" in terms of the PCK aims of the project.

Table 5. Change for teachers who participated for 1 or 2 years of the project only and completed both initial and final surveys (paired t-tests).

	Original (n = *8*)		Follow-Up (n = *8*)		*t*	*p*-value	Effect size
	mean	SD	mean	SD			
Combined scales	0.57	0.57	0.43	0.55	0.492	0.630	-0.23
General Pedagogy	0.10	0.71	0.59	1.06	1.103	0.289	**0.52**
Confidence	0.65	1.68	0.33	1.09	0.449	0.661	-0.21
Everyday Life	1.54	1.50	1.00	0.90	0.884	0.392	**-0.42**
Numeracy in the Classroom	0.57	0.30	0.56	0.39	0.057	0.955	-0.03
PCK	0.47	0.88	-0.22	1.65	1.039	0.317	**-0.49**

Discussion and conclusions

In answering the research question about change in teacher knowledge and confidence over the 3 years of the MARBLE project, two aspects of the results are considered. The first is the overall disappointing outcome for teachers generally. The second is the better performance of the 11 teachers in the project for 3 years.

The numbers in Table 1 support Hiebert's (1999) view that, regardless of the focus on explicit goals, students' thinking, and alternative ideas, little impact can be expected if the time of exposure is not measured in years (plural). The reasons for the turnover of teachers were not related to the content of the PL as only one teacher of the 86 expressed disagreement with the aims of the project and actively withdrew. The other teachers left the project because of changed roles or schools. Many of the exiting teachers, surveyed informally, expressed thanks for what they had achieved from the program, and some indicated that they regretted leaving.

Although the numbers are small, the more positive outcomes for the teachers who were in the project for the 3 years are encouraging, particularly with respect to PCK. As reported by Watson et al. (2006), the teachers initially struggled with PCK tasks. The improvement suggests that at least some of the requirements set out by Hiebert (1999) and Sowder (2007) were met during the program. Perhaps it is possible to speculate that difference in the PCK outcomes for the 11 teachers in the program for 3 years and the 8 in it for 2 years or less reflect the difficulty in taking up new ideas associated with teaching numeracy and having the confidence to trial them purposefully in the classroom. It may be that the teachers who were in the project for 3 years had similar experiences but persevered and hence came out with more proficiency in their PCK and Confidence. It would appear that at least 3 years are needed to overcome the "implementation dip" that the somewhat radical change in numeracy practice brought about. That the eleven teachers also displayed the same negative change in relation to numeracy in Everyday Life as did the other teachers, suggests that generally all of the teachers became more realistic in their assessment of their ability to handle numeracy in everyday settings.

The authors would suggest, somewhat facetiously, that others should choose for their interventions, schools with little staff movement and systems that do not change their curriculum during a 3-year period. Unfortunately this is not the real world. The Linkage Partner in this project purposely chose two rural clusters of schools where it felt help with numeracy was needed; however, little was done outside of MARBLE to alleviate the problem of teacher retention and issues of rurality. As to system change, although unfortunate and creating an observable underlying tension for teachers, it was not felt by the authors to be a major factor in the outcomes of the research.

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SECONDARY STUDENT PERCEPTIONS OF WHAT TEACHING AND LEARNING APPROACHES ARE USEFUL FOR THEM IN LEARNING MATHEMATICS

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Students' perceptions of what teachers do and what students themselves do that helps them learn gives an insight into what might be effective in a mathematics classroom. This paper looks at student perceptions in general but also specifically in relation to the teaching and learning of mathematics. Data were collected from students at two South Australian schools via an online survey conducted each year for three years. The students were asked questions relating to what teachers did that helped them learn and what they did that helped them learn Mathematics. The results of the survey will be presented and will highlight areas that students think are most important in learning mathematics.

Introduction

The importance of teachers to student learning has been well established worldwide (Darling-Hammond, 2007) and in Australia, a government report *Teachers for the 21st Century: Making the Difference* (Department of Education Science and Training [DEST], 2000) highlighted that not only are teachers central to student learning, but that student needs are changing and therefore the skills teachers need to be effective are also changing. The related area of teacher effectiveness has in recent times been the subject of scrutiny in Australia (DEST, 2000, 2003) and overseas (Darling-Hammond, 2000, 2007; Wang, Haertel, & Walberg, 1993) and much has been written on the qualities of a good teacher (Center for Teaching Quality, 2006; Department of Education and Children's Services [DECS], 2005).

Darling-Hammond (2007) proposed a number of qualities for effective teachers,

- strong general intelligence and verbal ability that help teachers organize and explain ideas, as well as to observe and think diagnostically;
- strong content knowledge—up to a threshold level that relates to what is to be taught;
- knowledge of how to teach others in that area (content pedagogy), in particular how to use hands-on learning techniques (e.g., lab work in science and manipulatives in mathematics) and how to develop higher-order thinking skills;
- an understanding of learners and their learning and development—including how to assess and scaffold learning, how to support students who have learning differences or difficulties, and how to support the learning of language and content for those who are not already proficient in the language of instruction; and

- adaptive expertise that allow teachers to make judgments about what is likely to work in a given context in response to students' needs. (p. 3)

Wilson, Cooney and Stinson, (2005) looked specifically at mathematics teachers and highlighted four main areas that make for good teaching in mathematics; Prerequisite teacher knowledge, promoting mathematical understanding, engaging students, and effectively managing the classroom environment. They went on to examine these areas in more detail and highlighted some more specific practices such as *Connecting mathematics*, *Visualizing mathematics*, *Assessing students' understanding*, and *Refrain from telling*. Although they did indicate that while the “teachers were emphatic that good teaching was not telling, they believed that they should guide the students and even tell in instances where guiding was unsuccessful” (p. 97). Perry (2007) also highlighted that mathematics teachers needed to have “a passion and enthusiasm for both the subject and its teaching” and “to know their children well and to make sure that their lessons were fun and relevant, both for the children and for the teachers” (p. 282). The Australian Association of Mathematics Teachers [AAMT] (2006) has developed research-based *Standards for Excellence in Teaching Mathematics in Australian Schools*, which are organised around the three domains of *Professional knowledge*, *Professional attributes*, and *Professional practice*.

While much of the research into what makes a good teacher is based on data that has been collected from teachers (Perry, 2007; Wilson, Cooney, & Stinson, 2005), students' perceptions of what teachers do that helps them learn (Wang, Haertel, & Walberg, 1993) and what students do to help themselves learn (Hattie, Biggs, & Purdie, 1996, Lawson & Askell-Williams, 2002) has also been the subject of ongoing research. Wang et al. (1993) synthesised research into students perceptions of what helps them learn and identified 28 categories which were grouped into six broad types of influences. One type was *Classroom instruction and climate*, which included eight categories—*Classroom management*, *Student and teacher social interactions*, *Quality of instruction*, *Classroom climate*, *Student and teacher academic interactions*, *Classroom assessment*, *Classroom instruction*, and *Classroom implementation and support*. Kiewra (2002) looked into what teachers can do that develops the learning strategies of the students.

This paper adds to the research around teacher effectiveness from a student perspective and also what strategies they use when learning mathematics.

Method

The results presented here are a part of a larger 3-year study that examined “contemporary learning environments” (with a focus on the use of learning technologies) involving four South Australian schools. This paper reports on students' perceptions of what teachers do that helps them learn and what they, as students, do that helps them learn. The results are from two of the four schools involved in the larger study, one a large metropolitan high school with students from Years 8 to 12 and the other a small R–12 area school.

A three-stage process was used to gather the data for the research around student perceptions of what teachers do that helps them learn. In Stage 1, student focus groups were asked open ended questions. In Stage 2, the notes from the focus group responses were collated by the researchers and common themes extracted. These common themes were then turned into statements, where the words used by the students were incorporated into the statements in order to make the student voice more evident. In Stage 3, the statements were added to an online survey and the students were asked to indicate what they thought were the five most important aspects and also to indicate how often they saw these actions on a five-point Likert scale.

The first time that the survey was run in 2008, an open response question "What advice would you give teachers to help them better support your learning?" was incorporated to allow students to add any other things that teachers did that helped them learn. This was done in order to capture anything not evident from the focus groups. The survey was then run in 2009 and 2010. The same statements were used in all three surveys, as the open response question had not yielded anything new.

The students were asked to nominate what they considered to be their best subject and then in relation to that subject to respond to a series of questions related to what they do that helps them learn. These questions were only part of the 2009 and 2010 surveys.

Each year, classes of students were taken to the computer room and given time to complete the questionnaire. A total of 918 students completed the online survey in 2008, 1105 in 2009, and 624 in 2010. The survey data was imported into SPSS for analysis and the open ended responses were examined by the researchers to identify any additional aspects of teaching that the students considered to be important.

Results and discussion

The student ranking of what they believe to be the most important practice or process that helps them learn was very consistent across the three years, particularly in the top five (see Table 1, next page). Teachers explaining things well was the most important for all three years, with almost half the students rating it either most or second most important in 2010. Given that the data were collected over a three year period, only the year 8, 9, and 10 students would have completed all three surveys. This meant that there were between 100 and 200 new students doing the survey each year. This level of consistency across two quite different schools across three years does strengthen this result.

The top five practices are quite consistent with aspects of the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2006) as well as aspects identified by Wilson, Cooney and Stinson (2005) and Perry (2007). Interestingly, classroom management was rated quite low and not seen as being a significant issue in these two schools, and so it would be interesting to see if this was more highly rated in other more challenging schools. Being extended in class was also rated quite low by the students, as was looking at ways students learn. These are both about teachers' processes/practices that would be difficult for students to observe and would have long-term effects, as opposed to the more immediate effects from the more highly ranked processes/practices.

The students were also asked to rate how frequently they observed the teacher practice or process, on a Likert scale from 1–5 (Never, Some of the time, About half the time, Most of the time, and All the time). The mean of the ratings was calculated and used as a measure to compare across the three years.

Table 1. Student ranking of what they believed was the most important practice or process that a teacher used that helped them to learn.

No.	Aspect	Rank 2008	Rank 2009	Rank 2010
3	Teachers explained things well	1	1	1
1	In general teachers got me interested in the lesson material	2	2	2
17	My teachers were approachable	3	3	3
2	My teachers encouraged me to achieve	4	4	4
9	My teachers provided useful feedback	5	5	5
5	My teachers would check on our understanding of lesson material	6	8	7
15	My teachers were passionate and energetic about teaching	7	7	8
18	My teachers talked to me as an individual	8	9	9
7	My teachers' lessons were well organised	9	6	6
11	My teachers arranged for student to have some choice in class activity	10	13	10
16	My teachers used a variety of ways of explaining things	11	11	6
8	My teachers would generally try to provide for different student's learning needs	12	12	15
4	My teachers told me about ways to remember what we were learning	13	10	11
12	Generally classes were well managed	14	15	14
13	Generally the class environment encouraged me to achieve excellent results	15	14	13
14	Generally I was able to have input to the things I am learning	16	17	17
10	My teachers would extend me during classes	17	16	19
6	My teachers would closely look at the ways in which we were learning	18	18	16
19	My teachers implemented learning experiences with ICT that helped me learn	19	19	18
20	My teachers implemented learning experiences that used ICT to specifically cater for different needs of students	20	21	21
21	My teachers supported students to learn for themselves what ICT to use and when to use it	21	20	20

The top five most important practices, as rated by the students, were experienced by the students quite frequently, and in most cases there was an increase in the mean across the three years of the surveys which would indicate that the practice was becoming more frequently experienced by the students. It is interesting to note that there are some practices that are frequently experienced, such as ―Generally classes were well managed” which was the second highest mean value, that were not considered to be important by the students, who rated it 14th or 15th. Teachers offering choice in a lesson was rated as the 10th most important but was one of the least frequently experienced practices.

The schools involved used the data from the first survey to look at their practices and identify areas that needed to be further developed. The schools used the data in different ways, with one school getting the student leadership team to report the data back to teachers during a staff meeting and talk about what they saw as being important. The other school had the student leadership work with the rest of the students to unpack what they meant by the top ranked practices and what they would like to see improved.

The student data provided a very useful stimulus for discussion and from the data in Table 2 there can be seen an overall trend of increased frequency of important practices experienced by the students.

Table 2. Mean rating of frequency of observation of teacher practice or process.

No.	Statement	2008	2009	2010
1	In general teachers got me interested in the lesson material.	2.95	2.99	3.18
2	My teachers encouraged me to achieve.	3.49	3.36	3.50
3	Teachers explained things well.	3.32	3.30	3.45
4	My teachers told me about ways to remember what we were learning.	2.76	2.76	2.87
5	My teachers would check on our understanding of lesson material.	3.08	2.99	3.07
6	My teachers would closely look at the ways in which we were learning.	2.84	2.76	2.81
7	My teachers' lessons were well organised.	3.66	3.55	3.58
8	My teachers would generally try to provide for different student's learning needs.	3.06	2.97	3.03
9	My teachers provided useful feedback.	3.22	3.15	3.36
10	My teachers would extend me during classes.	2.78	2.72	2.89
11	My teachers arranged for student to have some choice in class activity.	2.65	2.64	2.87
12	Generally classes were well managed.	3.54	3.45	3.61
13	Generally the class environment encouraged me to achieve excellent results.	3.05	3.10	3.30
14	Generally I was able to have input to the things I am learning.	3.10	2.97	3.26
15	My teachers were passionate and energetic about teaching.	3.13	3.09	3.31
16	My teachers used a variety of ways of explaining things.	3.16	3.06	3.22
17	My teachers were approachable.	3.60	3.50	3.75
18	My teachers talked to me as an individual.	3.10	3.09	3.38
19	My teachers implemented learning experiences with ICT that helped me learn.	2.82	2.86	3.21
20	My teachers implemented learning experiences that used ICT to specifically cater for different needs of students.	2.64	2.59	2.91
21	My teachers supported students to learn for themselves what ICT to use and when to use it.	2.85	2.77	3.18

The students nominated the subject that they did best at and were asked to rate a series of statements related to what they do that helps them learn that subject on a 5 point Likert scale (Strongly disagree, Disagree, Neutral, Agree, and Strongly Agree). The data from students who indicated that Mathematics was their best subject has been presented in Table 3 below, showing the means for each of the statements.

Mathematics was their best subject

In 2009, 16.9% of the students surveyed nominated Mathematics as their best subject, second only to English at 17%, this would seem to be a very positive and possibly a surprising result. However, the students who rated Mathematics as their best subject were, not surprisingly, very sure that they could succeed. The students also indicated that they could do better if they made a greater effort and that they really wanted to understand what they were learning.

The data below, although presented in two sections, need to be read together. For example, the data in Table 3 indicate that the students are not very likely to use the World Wide Web for Mathematics, while Table 4 indicates they are less likely to use the web than for most other subjects.

Table 3. Student rating of agreement for statements relating to what they do that helps them learn Mathematics.

Statement	2009 Mean	2010 Mean
I am sure that I can do well in this subject.	4.40	4.42
I practise things over and over until I know them well in this subject.	3.68	3.97
I make a note of things that I don't understand very well in this subject, so that I can follow them up.	3.64	3.90
I make plans for how to do the activities in this subject.	3.22	3.43
I make up questions that I try to answer about this subject.	2.82	3.38
I try to put ideas into my own words when I'm learning something new in this subject.	3.51	3.78
I am deeply interested in this subject	3.64	3.72
I think about my thinking, to check if I understand the ideas in this subject.	3.59	3.78
I draw pictures or diagrams to help me understand this subject.	3.54	3.68
I can get better at this subject if I put in the effort.	4.17	4.22
When I have finished an activity in this subject I look back to see how well I did.	3.86	3.99
I want to really understand what I am learning in this subject.	4.17	4.26
I use the world wide web (e.g., Google, Wikipedia) to help me understand this subject.	2.80	3.08

It is notable that the students used practice as a way of learning mathematics, and the students also highlighted review as being very important—and that effort and understanding were valued.

The means in all of the statements increased from 2009–2010, indicating that the students were more positive about the range of the strategies listed to help themselves learn. There had been a greater emphasis on contemporary learning within the schools and this may indicate that the students were more aware of their own learning strategies.

Differences between learning areas

The section on what students do that helps them learn was structured so that each student nominated one area and answered the questions in relation to that subject only; as such different students responded to each of the subjects and so care must be taken when looking at the results comparing subjects.

Table 4 highlights differences in column means for the different learning areas for the 2009 data. The comparison of column means is presented in Table 4, below, where a letter in a column indicates that there is a significant difference between the means of that subject and the column's subject for the question in that row. These results are based on two-sided tests assuming equal variances with significance level 0.1, and, for each significant pair, the letter of the subject with the smaller mean appears under the subject with larger mean. The letters in the table represent the subject that the students nominated as their best subject: A—no subject nominated (Blank), B—Arts , C—Design and Technology (D&T), D—English, E—English as a Second Language (ESL), F—Languages other than English (LOTE), G—Mathematics, H—Physical Education (PE), I—Science, J—Studies of Society and the Environment (SOSE), and K—Vocational and Employment Training (VET). When reading the table, any letter in the Mathematics column represents a subject that has a mean less than Mathematics, while any subject column that has a G in it will have a mean greater than Mathematics for that question.

Table 4. Differences in column means.

	Subject										
	Blank (A)	Arts (B)	D&T (C)	Engl. (D)	ESL (E)	LOTE (F)	Math (G)	PE (H)	Sci. (I)	SOSE (J)	VET (K)
H_1. I am sure that I can do well in this subject.		A	A				A	A	A		
H_2. I practise things over and over until I know them well in this subject.		D J						D			
H_3. I make a note of things that I don't understand very well in this subject, so that I can follow them up.							D		D H		
H_4. I make plans for how to do the activities in this subject.		G	F G								
H_5. I make up questions that I try to answer about this subject.									B		
H_6. I try to put ideas into my own words when I'm learning something new in this subject.				G					G		
H_7. I am deeply interested in this subject		A D E F G	A G					A G	A D E F G	G	
H_8. I think about my thinking, to check if I understand the ideas in this subject.											
H_9. I draw pictures or diagrams to help me understand this subject.	D	D F H	D F H				D F H		D F H	D	
H_10. I can get better at this subject if I put in the effort.		A							A D F		
H_11. When I have finished an activity in this subject I look back to see how well I did.		A F									
H_12. I want to really understand what I am learning in this subject.									A D F H		
H_13. I use the world wide web(eg. Google, Wikipedia) to help me understand this subject.	G	G	B F G H	G H	G	G			B F G H	B F G H	

Results are based on two-sided tests assuming equal variances with significance level 0.1. For each significant pair, the key of the smaller category appears in the column of the category with larger mean.

Tests are adjusted for all pairwise comparisons within a row of each innermost sub-table using the Bonferroni correction.

Table 4 provides some insight into the different approaches that students use to learn different subjects. Students indicated that they are more likely to use diagrams in Mathematics than in English, LOTE, and PE, while they are less likely to put ideas into their own words than in English and Science. Students also used notes to remind themselves of things that they did not understand more when learning Mathematics than English. Students planned more in the Arts and Design and Technology, which may be a reflection of the types of problems that are set in mathematics classrooms in general.

It would also seem that the students are not as deeply interested in mathematics than many of the other subjects, which may be cause for concern as these are the students who nominated Mathematics as their best subject. Has mathematics been presented as a tool to be used rather than as a discipline with its own knowledge that can be studied in depth?

Conclusion

The results of this study add to the body of knowledge around teacher effectiveness, and in particular it gives a student perspective on what teachers do that helps them learn. The consistency of the results across two schools and three years does indicate that the students are quite sure about what teachers' practices help them learn. While the statements provide some insight to what students value, by their nature they are open to interpretation and so more work is needed to unpack these statements. The student-generated statements support much of the previous work done in the area, and in particular the AAMT standards (AAMT, 2006) include many practices that students identified as being important.

The data on what the students do that helps them learn, while not as well developed as the teacher statements, do provide some useful insights to areas for further investigation. Why is it that students for whom mathematics is their best subject believe that they can do well and really want to understand what they are learning, but do not have the deep interest that other students have for their best subject?

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TEACHERS' USE OF NATIONAL TEST DATA TO FOCUS NUMERACY INSTRUCTION

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With increased accountability attached to students' results on national testing in Australia, teachers feel under pressure to prepare students for the tests. One approach is to use evidence from school and student results to identify areas for targeted teaching strategies to improve students' understanding. Using NAPLAN results lower secondary mathematics teachers in one school implemented mental computation and estimation approaches as well as a strategy to address the literacy demands of typical test items to support student learning before and after the NAPLAN test. An analysis of the professional learning identified approaches to enhance both students' learning as well as teaching practice.

Introduction

Prior to 2008, each state and territory in Australia used state-developed tests to collect student achievement data for the Federal Government. To better standardise the monitoring of student achievement the *National Assessment Program in Literacy and Numeracy* (NAPLAN) was introduced in 2008 (DETYA, 2000). The same tests in literacy and numeracy are now administered to all students in Years 3, 5, 7, and 9. Testing early in the school year potentially provides diagnostic information to teachers about their students' performance in mathematics topics common to all states and territories (Curriculum Corporation, 2006).

Whether we approve of a national testing regime or not, this level of accountability is in place for the foreseeable future with pressure on school principals and teachers to improve results. While the information may be useful after the results are released, teachers of Years 3, 5, 7, and 9 are experiencing increased pressure early in the school year to prepare students for the test. Principals, school systems personnel, and parents are scrutinising the results to determine whether schools and their teachers are 'measuring up'. Public comparisons between 'statistically similar' schools are now possible with the recent release of the *My School* website by the Federal Government which presents statistical and contextual information about schools.

The results from the assessments are reported in individual student reports to parents, as well as school and aggregate reports with substantial information including results for each item and for each student. The school reports enable teachers to analyse the results for each year group to determine which items appear to be understood and which are problematic. In addition, school data can be compared to the Australian student data.

The information is useful to address common errors and misconceptions as well as to aid planning and programming of future learning (Perso, 2009). Rather than abandon good pedagogical practices and have students individually practise test items, NAPLAN items can be used as one source to address key issues in students' mathematical understanding and develop appropriate quality-teaching approaches (Anderson, 2009). The purpose of the project reported here was to engage teachers in using evidence from their own NAPLAN results to identify their students' needs and collaboratively develop pedagogical practices which research has shown to be beneficial in building understanding. In particular, this paper describes and analyses the outcomes of a program conducted in one school by addressing the following research questions.

1. What strategies did teachers choose to use to support student preparation for NAPLAN and how was this different to previous practice?
2. Did the professional learning support have an impact on student learning and on teaching practice?

Literature review

Teaching to the test

High-stakes testing has been criticised for encouraging teachers to limit the curriculum to what is assessed (Abrams, Pedulla & Madaus, 2003) and resulting in the "corruption of indicators and educators" (Nichols & Berliner, 2005, p. 1). While the types of testing being conducted in some states in the United States of America in recent years could be considered higher stakes than the NAPLAN testing in Australia, systems, principals and teachers feel under pressure to prepare students for the tests and achieve good results, particularly given the publishing of the *My School* website . The pressure to raise scores has the potential to distort teaching and learning but there are ways teachers can support students' preparation for high-stakes tests without detracting from real learning (Gulek, 2003). Miyasaka (2000) identified five types of test preparation practices that support student learning and improve achievement— teaching the mathematics content, using a variety of assessment approaches, teaching time management skills with practise in test-taking, reviewing and assessing content throughout the year, as well as fostering student motivation and reducing test anxiety. In addition, Marzano, Kendall and Gaddy (1999) found knowledge of test vocabulary and terminology improves student performance.

Compulsory testing of students in Years 3, 5, 7 and 9 in Australia has the potential to focus teachers' efforts on preparing students for the test by using past papers for practise and limiting learning to technical support such as how to fill in answers (Nisbet, 2004). However, balancing this is the potential benefit of identifying students' strengths and weaknesses with data informing planning and teaching. In a survey of 56 primary schools, Nisbet (2004) reported about two thirds of the schools used the data to identify topics causing difficulties but only 40% of teachers used the results to identify individual students who were having difficulty, and only 22% used the results to plan their teaching. The low proportion of primary school teachers using the data to inform teaching and learning represents a missed opportunity and there is little evidence that secondary mathematics teachers are analysing NAPLAN data in meaningful ways.

An alternative approach

There is an alternative approach to ‘teaching to the test’ but the evidence above suggests teachers require support to analyse and interpret the data and consider alternative practices, to address common student misconceptions and difficulties (Anderson, 2009). Gulek (2003, p. 42) refers to the need for “school practitioners to become assessment literate in order to make the maximum use of test results” and Thomson and Buckley (2009) describe the potential of test item analysis to inform pedagogy. It should be noted the test preparation practices that we are advocating are aimed at improving students’ knowledge, skills and understanding of mathematics and **not** at artificially increasing students’ test scores.

Research has advocated several teaching practices that have the potential to target particular aspects of students’ difficulties in mathematics and numeracy. While many strategies could be considered, in this project, to be based on students’ errors, the following strategies were chosen from research which has shown them to be helpful in increasing mathematical understanding: mental computation, estimation and number sense, and the literacy demands of context-based mathematics questions.

Sources of students’ errors

Common student misconceptions have been identified as a major source of errors. For example, Ryan and Williams (2007, p. 23) use the term “intelligent overgeneralization” to refer to students’ predisposition to create inappropriate rules based on experiences. Some common generalisations include: multiplication makes bigger; division makes smaller; division is necessarily of a bigger number by a smaller number; and longer numbers are always greater in value. The following is an example of a NAPLAN Numeracy item where this type of over-generalisation occurs with few students selecting the correct answer of 22.

What is the answer to $6.6 \div 0.3$?

A) 0.022 B) 0.22 C) 2.2 D) 22

A common fraction misconception occurs when area is not the feature students identify in regional models of fractions (Gould, Outhred, & Mitchelmore, 2006). The “number of pieces” interpretation is a common response. This research explains the responses to the 2008 Year 7 NAPLAN item shown in Figure 1 where only 28% correctly selected the last option.

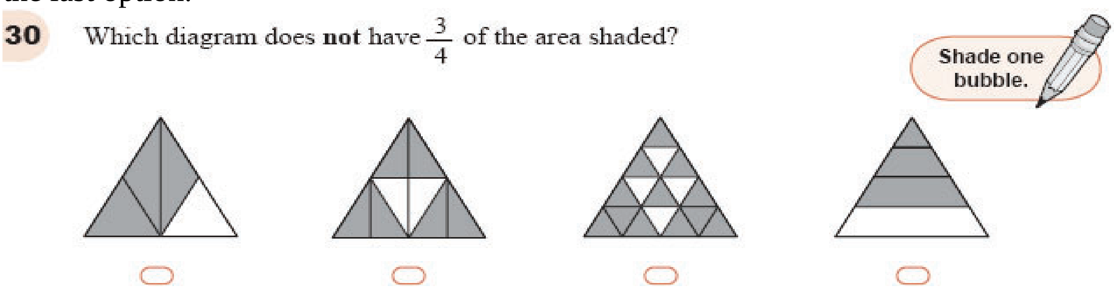

Figure 1. A fraction item from the 2008 Year 7 non-calculator numeracy NAPLAN test.

1. Mental computation, estimation and number sense

In dealing with misconceptions like these, Anderson (2009) points out those encouraging students to apply reasoning about numbers to evaluate answers can be a challenge. She argues that one way to support the development of students' thinking strategies is to use test items that focus on mental computation, estimation and number sense (McIntosh, Reys & Reys, 1997). Options in multiple-choice items may often be eliminated after considering whether the solutions are reasonable. Anderson proposes that after students have estimated the answer, teachers can pose questions such as:

- What strategies could you use to check the solution?
- What would the question need to be to obtain each of the alternative answers?

An estimation focus allows test items to be a source of meaningful mathematical discussion.

2. Literacy demands of context-based mathematics questions

The contextual nature of many NAPLAN items and the associated language implications often leads to claims that these tests are more comprehension than mathematics. However, interpreting mathematical situations in context is what numeracy is all about. Hence, we claim the contextual nature of the items is at the heart of numeracy and deserving of special attention. It seems pointless to pursue repetitive symbolic manipulation exercises to address poor responses to contextual items.

Newman (1983) developed an error analysis protocol to analyse student responses to contextual items. She identified five levels of difficulty (Table 1). Most errors occurred in the second and third levels of 'comprehending' and 'transforming' the text into an appropriate mathematical strategy, not applying the symbolic procedure. By translating each of the levels from Table 1 into a question for students, teachers are able to determine their first level of difficulty (White, 2005).

Table 1. Levels in Newman's error analysis.

Reading the question	Reading
Comprehending what is read	Comprehending
Transforming the words into an appropriate mathematical strategy	Transforming
Applying the mathematical process skills	Processing
Encoding the answer into an acceptable form	Encoding

Engaging teachers in professional learning

Planning professional learning opportunities for teachers in relation to promoting a change in practice requires consideration of several factors such as teachers' knowledge, beliefs and attitudes (Wilson & Cooney, 2002). Rather than change in beliefs and attitudes preceding change in practice, Guskey's (2002) model proposes professional learning precedes the implementation of new ideas in classrooms, which when implemented can lead to a positive change in student learning outcomes, and subsequently, a change in teachers' beliefs and attitudes. This model suggests that teachers need to try new ideas and witness positive student outcomes before they fully embrace such approaches.

Building on Guskey's model, this project aimed to change secondary mathematics teachers' attitudes towards NAPLAN and its usefulness. The approach taken with the teachers encouraged them to use evidence from the previous NAPLAN Numeracy test for their students, to identify topic areas and mathematical concepts of concern, and to develop strategies addressing the particular learning needs of their students

Methodology

One school which had a high NESB enrolment and low NAPLAN results volunteered to participate in the project. Ten teachers of Years 7 and 9 (12 classes in total) were involved. In May each year, Years 7 and 9 students complete two 32-item test papers for Numeracy, one with and one without the use of a calculator. The authors used the school's 2008 NAPLAN numeracy test results to identify specific areas of the curriculum requiring consolidation. Items from NAPLAN 2008 in these areas were used by the authors to compile a short diagnostic pre-test for each of Years 7 and 9 consisting of 5 non-calculator and 5 calculator items. Though the results from 2008 were those of the current Year 8 and 10, not the cohorts involved in the project, they were still considered reflective of teaching approaches in the school because the teachers were the same. Teachers administered the tests in early March, slightly more than two months before the NAPLAN tests in May, 2009. Each teacher corrected their class responses. In the six Year 7 classes, only one class had more than 50% of total responses correct in the calculator and non-calculator pre-tests (same class). In the six Year 9 classes, two had more than 50% of total responses correct in the non-calculator pre-test and no class had more than 50% of total responses correct in the calculator pre-test. These data support the items chosen as being areas of difficulty for the students.

A one day meeting two months before the NAPLAN tests was held between the teachers and the authors. The day consisted of reviewing the students' pre-test responses, considering the key mathematical ideas and misconceptions in the tasks, and exploring a range of possible teaching approaches identified by the authors. Teachers also contributed suggestions about the mathematical issues they saw as relevant and strategies they believed could be used to address student difficulties. As a result, a list of possible strategies was jointly constructed. Each teacher then nominated one or more to implement in their general teaching as well as with targeted NAPLAN items.

Data collected from teachers included teacher questionnaires and interviews plus eight teachers were observed for one lesson by a trained research assistant who was a qualified mathematics teacher. Pre-tests were collected from students in each of the eleven classes. In addition, comparative NAPLAN results for the Year 7 and 9 students in 2009 with their Year 5 and Year 7 results respectively in 2007 aligned with the corresponding New South Wales data have been used.

Results and discussion

The results mainly report the preferred teaching strategies identified and used by the teachers. These data inform on pedagogical practices and potential teacher change during the project. A second section reports on student learning. Given there was only two months of teacher implementation before the NAPLAN test and the length restrictions of the paper, these data are only briefly reported. They are seen as some indicator of the success of the professional learning but not in any way conclusive on their own.

Teaching strategies

During the one day meeting, the teachers reported giving their students practise on NAPLAN type items before the tests. However, there was no use of school data to inform their planning and practice, or approaches to build desired understanding in their general teaching. When each pre-test item was discussed, teachers were asked to estimate the proportion of the school cohort correctly answering each item. They tended to overestimate and were frequently surprised by the low number of correct responses.

From looking at the mathematics involved in the identified areas and the incorrect answers chosen by students, the teachers and authors chose eight strategies as potentially useful for improving students' mathematics proficiency. These strategies contained a mix of general teaching strategies and some for class discussions based around NAPLAN style items. The teachers indicated that they intended to focus on the areas of concern and use strategies from the day not only in their general teaching, but also with NAPLAN items as stimuli for constructive class discussion.

After implementation, teachers completed a short questionnaire where they ranked the strategies in their preferred order of usefulness. Table 2 shows the results from the eight teachers who responded to the questionnaire. Scores were calculated by assigning 1 to the first choice, 2 to the second choice and so on, hence the lowest score indicates the most preferred strategy and the highest score indicates the least preferred (scores could range from 8 to 64).

Table 2. Preferred strategies as reported by the teachers to address students' difficulties.

	Strategy	Score
1.	Promoting interpretation of context-based mathematics questions using Newman's error analysis questions	20
2.	Developing efficient mental computation strategies	29
3.	Using estimation strategies with all calculations	36
4.	Eliminating possibilities in multiple choice questions	41
5.	Checking reasonableness of answers	43
6.	Developing visualisation strategies in geometry (2D to 3D and 3D to 2D representations)	47
7.	Identifying irrelevant information in mathematics questions	52
8.	Developing strategies for answering open-ended questions	58

Their ranking must be interpreted realising they may not have tried some at all and only chose from the specific ones they did implement. None the less, the attractiveness of the ones they did choose to try is a factor in determining effective strategies that promote good pedagogy and are seen as comfortable for use by teachers.

Newman's questions and mental computation emerged as the most popular choices with 7 teachers ranking Newman's in the top 3. Some teachers' comments revealed some believed they were already using such strategies. For example:

> The majority of the strategies I already used prior to the PD except for the Newman's method.

Others found the opportunity to consider new approaches was beneficial to both their teaching and student learning as shown by the comments below from three different teachers.

> Identified their need for mental computation and to read all of the question.
> I found the Newman's questions are very useful. I went through that with all my classes.
> Newman's strategies— worked— ensuring read all of question.

Three teachers' comments suggest their knowledge and understanding of the potential of NAPLAN items and data have improved:

> It gives me an idea of which kind of questions students found hard so I would focus more on those areas.
> Next year I intend to show students a variety of strategies for approaching the numeracy tests. I will also target some specific areas of knowledge that students in the past have had difficulties with.
> The pre-test identified common areas of weakness in my class. Common misconceptions were easily identified by the alternate choice students made when choosing the answer.

Professional dialogue between teachers and the researchers enabled the identification of a range of strategies for implementation in classrooms, an approach acknowledged as successful by the following three teachers' comments:

> It was good to gather with colleagues and to discuss alternate teaching strategies.
> It was especially good to get the chance to do practical maths questions and be the "student" ourselves.
> Focusing on mental computation, visualisation, Newman's as part of each unit, from beginning of the year—encouraging this as a normal part of doing Maths.

Even though teachers indicated they already used some of the teaching strategies in regular lessons, their awareness of the strategies and ability to identify when they were using them increased. Further, they had not used them as a focus for supporting NAPLAN preparation nor in taking items and through these strategies making them a source of constructive class discussion rather than right/ wrong drill and practice. The data here show they were still using some of the learning three months after the NAPLAN tests.

Table 3 shows the strategies which were planned for and actually used by the teachers in the observed lesson. Some teachers used more than one strategy.

Table 3. Observed strategies.

	Strategy	Planned	Observed
1.	Promoting interpretation of context-based mathematics questions using Newman's error analysis questions	3	3
2.	Developing efficient mental computation strategies	2	2
3.	Using estimation strategies with all calculations	0	0
4.	Eliminating possibilities in multiple choice questions	3	1
5.	Checking reasonableness of answers	1	2
6.	Developing visualisation strategies in geometry (2D to 3D and 3D to 2D representations)	3	3
7.	Identifying irrelevant information in mathematics questions	1	0
8.	Developing strategies for answering open-ended questions	0	0

The data set here is not big but still allows for some inference about the classroom practices of the participating teachers.

The top two (Newman's analysis and mental computation) figured prominently but a specific focus on estimation did not. All three who used Newman's analysis actually went through the steps with the class. Visualisation, though not an original popular choice, was used as the basis for three of the lessons. The specific test strategy of eliminating possibilities in multiple choice questions was planned but not widely used indicating lessons became more involved with the mathematics and appropriate procedures rather than test based strategies. As one teacher said to her class, "Does the answer actually fit the question? Have confidence in your ability."

Four of the lessons involved NAPLAN items as a source of class discussion and group work. In all these lessons, teaching went beyond right/wrong answers and looked at procedures. Three involved group work, while one was more teacher centred. The visualisation lessons were three of the four that did not use NAPLAN items. The teachers chose other activities that involved students in groups building objects given specific properties (for example, can you build the shape which looks like this from the front and has the most cubes). The level of student engagement was commented on positively in six of the eight lessons.

Student learning

Student data from 2007 to 2009 for each student were compared to the total NSW data. The groups used in the comparisons were the same in both 2007 and 2009. The mean gain for each group was calculated by averaging the individual gains. The results comparing the mean gains using a one tailed *t*-test showed that the gains by the sample school compared with the state are significant at the 1% level for Year 7 and at the 2% level for Year 9. These comparative data are encouraging and do support a positive impact of the project on student learning but, especially given that only two months of intervention occurred and all the other influences on the students and teachers, the approaches implemented can only to be viewed as one factor impacting on the gains.

Conclusions

There is evidence that engagement in the project by teachers and students coincided with some positive student learning outcomes and new teaching practices. The use of Newman's analysis in particular seems to have provided a better way of dealing with contextual mathematics. Thus the project was seen as successful by the school**.** The mix of using clearly identified strategies in general class teaching with NAPLAN items as a stimulus for discussion appear to be an effective pedagogical combination. The results here are consistent with Martin's (2003) observation that showing students test items and discussing strategies for thinking about questions and responses promotes student confidence and resilience, and enables a greater sense of student control over their learning. In addition, the assessment literacy (Gulek, 2003) of teachers by using data to inform teaching certainly became apparent as part of teaching practice where no indication of doing so previously was evident. However, there is no conclusive evidence about the way the data were used.

The results presented here are not advocating 'teaching to the test', rather they support the notion that there is much to learn from using a school's NAPLAN data to develop pedagogical content knowledge about important mathematical concepts. Nor is national testing being promoted as the most desirable approach to assessing students' knowledge, skills, and understanding. Teachers best carry out assessment as they talk to and observe their students (AAMT, 2008). However, given the reality we face and the fact that many teachers do feel pressure to actively prepare their students for the tests, the approach presented here offers some ideas for a constructive way to do so. Future iterations therefore are supported and, in particular, the results suggest looking for ways to increase long term positive beliefs and ownership by teachers.

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CONCERNED ABOUT THEIR LEARNING: WHAT MATTERS TO MATHEMATICS STUDENTS SEEKING TO STUDY DESPITE ABSENCE FROM SCHOOL OWING TO CHRONIC ILLNESS

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Increasing numbers of young people experience disruption to their education owing to chronic illness. Many seek to continue their learning despite absence from school for prolonged or accumulative periods of time. The need to consider ways to support them arose in the context of a project called *Link 'n Learn* funded by the Australian Research Council (2008-2010). This paper reports on one aspect of a collective case study of students absent from school with diverse types of chronic illness and their mathematics teachers. It highlights that students focussed on their desire for interaction to continue study whereas their teachers were concerned about issues of illness.

What do students worry about when they miss mathematics lessons at school owing to a chronic illness? When they want to continue their studies nonetheless, what matters to them? What concerns do their teachers have about these students' learning during absence? This paper explores these issues for senior secondary students with chronic illness, who want to continue their mathematics studies, and their teachers at school. Understanding more about their concerns has implications for the educational support of increasing numbers of young people who experience disruption to their education while managing a chronic illness.

For most young people a big part of normal life is attending school. For those with chronic illness, being absent and losing contact with teachers and peers may create apprehension about disrupted friendships and falling behind academically (Charlton, Pearson, & Morris-Jones, 1986). Research has found that keeping things as normal as possible decreases their anxiety, increases their sense of control and helps them cope better with treatment (Bessell, 2001; Brown & Madan-Swain, 1993). Opportunities to connect to school and continue their learning may provide welcome distraction. Disconnection from school over time may lead to students becoming reluctant to return to full-time attendance (Bessell, 2001; Haas & Fosse, 2008). Addressing students' social and academic needs also improves their quality of life and employment prospects (Charlton, et al., 1986; Lightfoot, Wright, & Sloper, 1999).

Although on-site hospital schools traditionally oversee the educational needs of inpatients, medical advances and de-centralised healthcare have resulted in shorter stays and therefore reduced access for young people to the available learning support. Those who are not hospitalised, nor well enough to return to school, often spend lengthy periods of time at home: “out of sight” of hospital schools and their own schools. Yet federal and state legislation mandates schools to provide educationally for all of their enrolled students. How might schools avoid their students with chronic illness being “out of mind” during absence? Government-funded home-tutoring programs typically provide one hour of educational support per week for students absent from school for prolonged periods but eligibility and availability vary across educational sectors (Shaw & McCabe, 2008) and these programs are deemed inadequate on their own.

The ARC-funded *Link 'n Learn* project, of which the collective case study is a part, involved a partnership between the Royal Children's Hospital (RCH) Education Institute and the Melbourne Graduate School of Education at the University of Melbourne. It explored the possibilities of school-based educational support utilising communications technologies to connect students with chronic illness to their schools (Wilkie & Jones, 2010). The collective case study focussed on the interaction between senior secondary mathematics students and their teachers to achieve academic continuity, defined in this study as *students' access to, and utilisation of, opportunities to learn effectively so that academic progress is made despite disruption to full-time schooling*. The following section provides details on the context for the study by discussing why academic continuity in mathematics was the focus.

Context for the research

Chronic illness often goes hand-in-hand with absence from school and young people miss out on learning opportunities at school for lengthy periods or accumulatively over time, which often leads to significant gaps in their education. This is of particular concern in the domains of literacy and numeracy (Chekryn, Deegan, & Reid, 1987; Shiu, 2001). Secondary students absent from school undergoing cancer treatment have expressed particular anxiety about mathematics: they tend to be well aware that many university courses require mathematics study at Years 11 and 12 as a pre-requisite. Yet researchers believe that students' independent attempts to keep up with their studies by trying to accumulate factual knowledge are an ineffective way to learn in this domain (Charlton, et al., 1986; Fottland, 2000). There is overwhelming consensus that interaction between teachers and students is fundamental to effective education. Sociocultural perspectives consider mathematical learning as “an inherently social activity” (Schoenfeld, 1994, p. 62) that involves both “individual and collective learning processes” (Van den Heuvel-Panhuizen, 2003, p. 10). Clarke (2001) suggests that locating “learning solely within social practice or solely within the cognising individual” is a mistake (p. 297).

Independent study of mathematics is problematic for young people absent from school. This is corroborated by education advisors at the RCH Education Institute who commented that students bring their textbooks with them to hospital but struggle to learn from them. They also highlighted that mathematics was the hardest subject for them to support, particularly in the secondary years, because of the specialist knowledge required. Research has found that some students had trouble accessing any support at all for their mathematics learning during absence from school; others received some tuition in hospital but this “bore little or no resemblance to what they would have done in school" (Charlton et al., 1986, p. 1345). Visiting teachers who tutor students at home state similar concerns about their lack of mathematics expertise (Searle, Askins, & Bleyer, 2003).

Providing effective mathematics education support for students whether they are in hospital or at home has proved a difficult challenge. Previous research conducted at the RCH Education Institute explored online communication between students with chronic illness and their schools, and recommended the exploration of strategies specifically to support learning in mathematics (Campbell & St Leger, 2006). The positive feedback from student participants about using communication technologies to keep in touch with school, the advent of increasingly flexible and affordable options, and concern for the inadequacy of current educational support, led to ongoing research efforts including the *Link 'n Learn* project. The following section describes the design of a collective case study to address the need for further research into support approaches and to also consider the perspectives of young people who are looking ahead to university and want to continue mathematics study despite chronic illness.

Research design

There are multi-faceted issues and concerns when a student is unwell and absent from school: a complex context at the intersection of medical and educational domains. Students are in and out of hospital, often stuck at home, too unwell to attend school but seeking to continue their studies nevertheless; teachers at school are busy with their classes and teaching, and have a student they no longer see every lesson. The student‘s goal is academic continuity in mathematics, their hope is for support from school, and the possibility is for interaction mediated by technologies. In designing a collective case study, I sought to focus on this interaction between a student and their mathematics teacher—between student–and–teacher pairs—keenly interested in their viewpoints and perceptions, their experiences and issues. A pair best represented my understanding of what constituted ―a case" (Adelman, Kemmis, & Jenkins, 1980).

An important aspect of the study was gathering data over time rather than just at one moment as a snapshot. Young people by definition experience chronic illness over extended periods of time. I chose to construct a collective case study around a small number of students and their teachers (22 participants), methodologically so that I could develop in-depth understanding of their activities, experiences and perceptions over time and explore multiple viewpoints (Stake, 2006). An interpretive reflexive approach focusing on detail also required a manageable number of subjects.

I sought to utilise as many sources of data as possible (Creswell, 2007) while remaining sensitive to the dignity of students during a potentially traumatic period. Initial data were gathered about each student‘s and teacher‘s concerns, interaction preferences and perceived support needs. Informal conversations, observations (hospital and school visits), emails and text messages provided ongoing data about the nature and frequency of communication between a student and teacher. Once students returned to school full-time or the end of the academic year was reached, interviews were conducted individually with the student and their teacher. These provided opportunity for the students and teachers to reflect on their interactions with each other: on what was important to them; their teaching/learning experiences; particular issues they faced;

outcomes of their experiences; and advice they would give to others in a similar situation.

The students in this study had different levels of self-perceived ability in mathematics, various types of chronic illness, and diverse patterns of absence from school (see Table 1).

Table 1. Students' mathematical ability, type of illness, treatment, and absence from school.

CASES	Self-perceived mathematics ability (1-10)	Type of chronic illness	Medical treatment	Absence from school
Adam & Mr Alston	7	nasopharyngeal carcinoma (solid cancerous tumour)	surgery, then cycles of hospitalisation (a week or more), recuperation at home (a few weeks), and outpatient appointments	prolonged (9 - 10 months)
Belinda & Mr Bluett	8	osteosarcoma (solid cancerous tumour)	surgery, then cycles of hospitalisation, recuperation at home, and outpatient appointments	prolonged (7 - 8 months)
Cate & Ms Curtin	5	anorexia nervosa	repeated hospitalisation (up to 6 weeks at a time) and outpatient appointments	prolonged and recurrent (6 weeks at a time repeatedly over 3 years)
Debbi & Mr Davis	8	conversion disorder, chronic fatigue syndrome	hospitalisation for testing (6 weeks), then recuperation at home with outpatient appointments	prolonged (2 months), intermittent (absent weeks at a time; missed approx. half year)
Elijah & Mr Everest	5	osteosarcoma (became terminal at end of year)	cycles of hospitalisation and recuperation at unit near hospital or at home, then palliative care at home	prolonged (whole year)
Faraji & Mr Fabiano	5	reflux nephropathy (causes renal failure)	haemodialysis for several hours in hospital ambulatory ward Mondays & Fridays every week	recurrent (Mondays & Fridays every week)
Gareth & Mr Grady	6	acute lymphoblastic leukaemia (became terminal part-way through year)	cycles of hospitalisation and recuperation at home, then extended hospitalisation	prolonged (previous year then 7 months as participant)
Harry & Ms Heath	8	osteosarcoma	surgery, then cycles of hospitalisation, recuperation at unit near hospital or at home, and outpatient appointments	prolonged (9 months) then intermittent
Irene & Ms Ingleton	7	multiple sclerosis	hospitalisation for testing, then outpatient appointments	intermittent
Joelle & Ms Joskin	8	cochlear implant complications	intermittent hospitalisation for surgery or treatment (up to two weeks each time), recuperation at home	intermittent (previous year), prolonged (3 months), intermittent (1 month)
Kody & Ms Kiselow	7	acute myeloid leukaemia	cycles of hospitalisation, recuperation at home, and outpatient appointments	prolonged (6 months)

The following discussion focuses on one particular aspect of the collective case study: the concerns of the students—what mattered to them when trying to learn mathematics during absence from school—and the concerns of their teachers. Other themes relating to the learning and teaching of mathematics through online interaction and to academic continuity are discussed elsewhere (Wilkie, 2010).

Discussion of findings

Previous research has highlighted the anxiety young people with chronic illness may experience about falling behind academically (Hedström, Ljungman, & von Essen, 2005). I sought to explore this issue further specifically in the domain of mathematics and as it related to students‘ attempts at independent study and their being absent from lessons at school. I also examined teachers‘ perspectives on their concerns for their student‘s ongoing study. These themes are discussed in turn in the following three subsections.

Students' concerns: Continuing mathematics study away from school

All students indicated that *not being able to ask questions while studying* concerned them. A majority considered that *having to figure things out for themselves* was ―most hard". Yet overall there was no correlation between the level of concern about these two issues and students' perceived ability in mathematics, for example:

> The textbook work looks really complicated when no-one's there to explain. (A, 9/5/2008, *high ability*)
> On my own, like if I needed help, I couldn't just ask the teacher. (B, Q6.4, 27/11/2008, *high ability*)
> It was hard doing stuff and not having a teacher there to show me how to do it. (C, Q3.3, 24/11/2008, *lower ability*)

One student's indication that this was *not* a concern for him was surprising since he had given up trying to study on his own quite quickly: ―a month into being sick" (H, Q2.2, 13/11/2009). Perhaps he indicated this to signal to me his high ability; later in the year he admitted that he had struggled with independent study: ―I used to try, I did a bit of commerce work and a bit of maths work myself but I found it too hard so I just stopped" (Q2.1, 13/11/2009). Davis, Hersh, and Marchisotto (1995) found that in mathematics ―better students tend to demand instant understanding" but when learning becomes difficult, this may be ―debilitating" and cause resistance to further study (p. 315).

A majority of students expressed concern about the *motivation to keep going with study*. One said that ―you get behind with it and you just can't be bothered doing it" (C, Q9, 24/11/2008). Another student indicated no concern about his motivation, and indeed his determination to continue study was apparent until terminal cancer intervened. One student's indicating no concern was initially surprising because he expressed a low level of self-perceived ability, yet demonstrated self-motivation throughout the year and completed all the work his teacher gave him. He said ―I'm not good getting ‗A pluses' but I just want it to be a Pass. That's all I want it to be" (E, Q6, 5/11/2009).

Concern about *not having enough energy to keep up with the work* elicited surprising responses: I had expected more students to consider ill health an issue. One student with chronic fatigue syndrome indicated she only found it ―somewhat hard" to manage. Another underwent haemodialysis twice a week (fatigue is a major side-effect), yet he indicated that he did *not* find lack of energy an issue. Of the six students with cancer, only one reported lack of energy as being ―most hard". I had observed first-hand the intense treatment each of these students underwent; and two students with aggressive osteosarcomas maintained positive attitudes even when they were noticeably unwell. Do some students genuinely not experience ill health as affecting their learning or do they intend to show their determination that they are getting on with life anyway, despite their illness? Does ongoing study during illness symbolise this resolve? Whether or not feeling unwell is an issue, students seemed to want to portray that their study and motivation struggles are about the lack of assistance, not ―sickness".

Students' concerns: Absence from mathematics lessons

The two major concerns of students regarding their absence from mathematics lessons were missing out *on hearing the teacher's explanations* and *on being able to copy down notes in class*. Unsurprisingly, all teachers reported high use of these activities in their lessons. *Not being able to ask questions or to seek individual help from the teacher* were also of concern to a majority of students (and their teachers). One student said, ‗I like to get everything done in person with [teacher name]" (E, Q7.1, 5/11/2009). Four students indicated more concern than their teachers about individual help; four pairs expressed a similar high level of concern. In comparing students‘ with teachers‘ concern about taking notes in class, students generally expressed more anxiety than teachers (Figure 1).

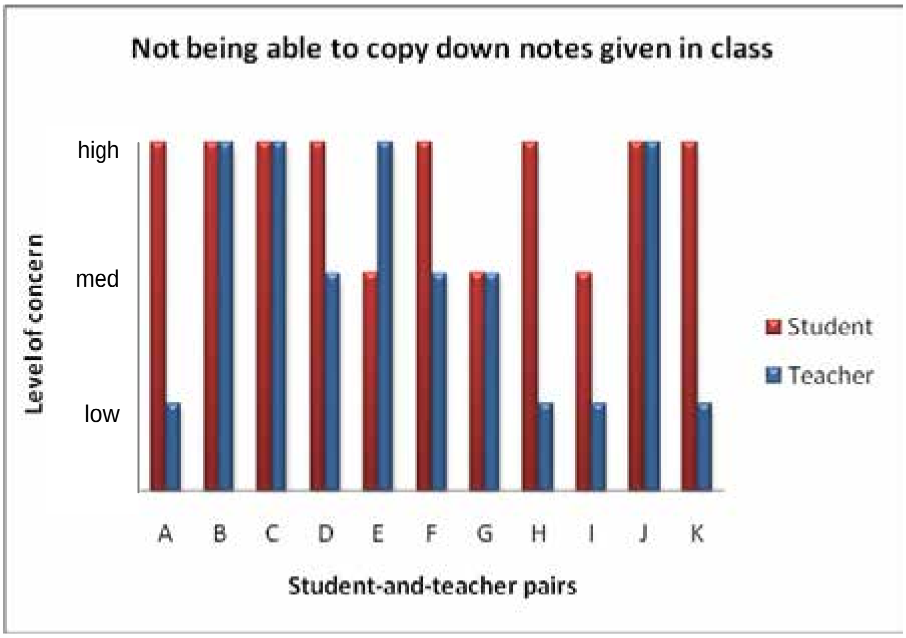

Figure 1. Comparing concerns: not being able to copy down notes.

When initially asked what she needed for mathematics study, one student replied promptly ‗notes from the board" (B, 11/9/2008). Another student preferred the possibility of having lessons videoed, saying ‗I learn better in class" and when ‗I copy down notes" (C, 4/9/2008). One student photocopied notes from her friends but said that ‗because they didn‘t write everything out, it was like a different language" (D, Q4.1, 19/11/2008). It seemed that having another student‘s notes at least provided some information but being able to write one‘s own notes (via videoconferencing or a videorecorded lesson) was preferable.

Being able to work with friends on problems elicited a wide range of initial responses but none of the students with self-perceived high ability in mathematics categorised working with friends as an activity they thought they would miss most. Do students with high ability rely less on the involvement of peers in their learning or would they like to be perceived as not needing the help of peers? Yet one student with high ability later reflected, ‗I was studying on my own and not with the class – and it was different because I wasn‘t learning it in school – but on my own" (J, Q2, 28/7/2009).

There was a range of responses to *missing out on finding out about class work being set* with five students indicating it as a high concern. One student said that if teachers ‗give out what exercises [are] to be done" to a student who is unwell, it ‗gives them a fair idea of what needs to be expected, so when it comes to exams or when you go back to school, again it‘s not a big shock and you don‘t feel out of it" (D, Q10.1, 10.3, 19/11/2008). Those students whose teachers had already been keeping their students up-to-date with what was happening in class expressed no concern. It seems that knowing what is happening in lessons is sought by students, even those who have given up on independent study. It seems to provide at least some sense of connection to school even if it might exacerbate their anxiety about falling behind.

In considering students' concerns overall, there was no correlation between a student's perceived level of ability in mathematics and the nature of their concerns about being absent from lessons. Rather, students' favourite learning activities generally correlated with their issues of highest concern about being absent from lessons. There is the sense that the key concern for students is *missing interaction with their teacher*.

Teachers' concerns: Their student studying during absence

Teachers all indicated that they made high use of *explaining* and *stepping through solutions on the board* in lessons, so it is perhaps unsurprising that a majority expressed high concern about students missing out on those activities. Only one teacher was the exception, indicating little concern, yet his student indicated a high level of concern and had even tried to attend some lessons when he was quite unwell. When I met the teacher he initially said that his student "can teach himself" and that "[he] only needs to be told the chapters being covered in class" (Mr A, 15/5/2008). Later in the year he explained that he had wanted to do more for his student but suggested that he could not have, rather than that it would have been unnecessary.

In comparing students' and teachers' levels of concern about student motivation, there was a noticeable mismatch between several pairs (see Figure 2).

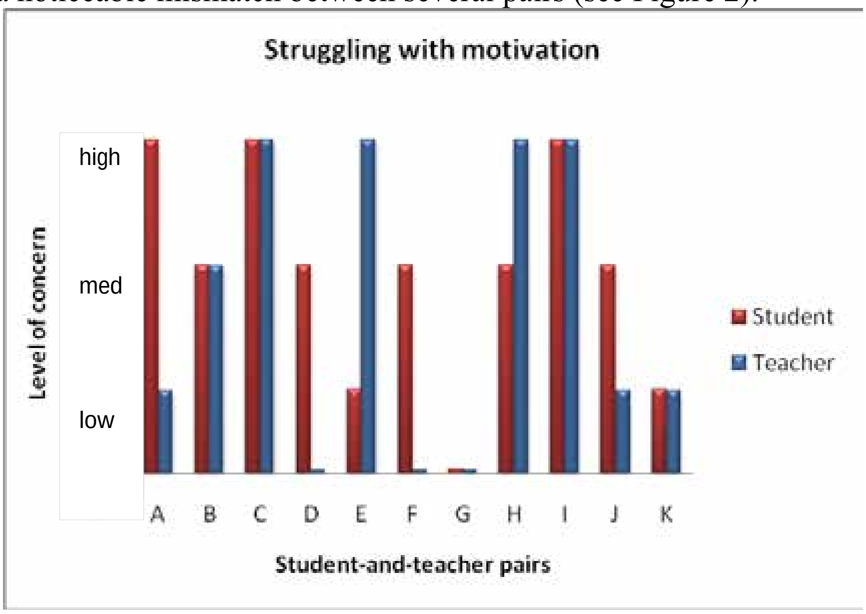

Figure 2. Comparing concerns: struggling with motivation.

In four pairs, the teachers were less concerned than their students; over time the students showed that they were indeed self-motivated but communicated that they struggled. It seemed that staying motivated was an issue that students related more to trying to learn mathematics on their own than to their ill health.

A majority of teachers expressed high concern about their students *struggling to keep up-to-date with work* and seemed to relate this to students' ill health rather than to absence from lessons. Interestingly, three teachers had significant reservations about supporting their students during absence from school and their students each had cancer. The students expressed little or no concern about ill health affecting study but teachers expressed high concern. In later interviews these teachers reflected on their uncertainty about students with serious illness continuing study:

> I thought, ‗Why are you doing this stupid maths when you‘re so ill?‘ to be quite honest. (Mr A, Q6, 30/10/08)
> My first thoughts were… ‗Why would they want to be doing this? Like, who cares about maths–in that situation? (Mr G, Q4.2, 16/10/2009)

Once one of these teachers understood that his student *wanted* to continue study, he was ―happy to go along with what [his student] want[ed]" (Mr G, 10/3/2009). The other two teachers sent a list of topics to their students but remained unconvinced about the appropriateness of study during cancer treatment.

Implications and conclusion

Students in this study seemed to portray that what they need to continue mathematics learning successfully during absence from school are interaction with their teacher and involvement somehow in what is happening in lessons. They seemed less concerned about managing their illness than about their struggles to learn independently–to figure things out for themselves. They seemed to want to convey that being unwell was *not* what they wanted to focus on when talking about their schoolwork; the main issue for them was *involvement* in learning opportunities, not coping with *illness*. One student summed it up poignantly, ―I was more worried about school than I was about being sick". Her suggestion was for teachers to provide ―just that reassurance of how much you need to do" (D, Q11, 19/11/2008). Teachers, however, focussed more on issues related to ill health and expressed concern about their students‘ ability to even cope with study.

In considering these differing perspectives, suggestions addressing the educational support of students absent from school with chronic illness include:

- Giving students the opportunity to specify what types of academic support matters to them, for example, photocopied notes, lesson handouts, email updates, telephone calls, videoed lessons, online interaction with their teacher, videoconferencing during lessons, and modified work requirements;
- Encouraging students to tell teachers that they want to interact with them, and teachers to respond with direction and advice about ongoing study; and
- Reassuring teachers that being as normal as possible in their relationship with students and focussing on learning rather than illness are likely to benefit students.

These could be communicated through videos and brochures accessible online from educational authorities and distributed by school support staff to teachers informing them of: why students might benefit from contact with them during chronic illness; advice from students and teachers who have experienced similar situations; ways to develop modified learning programs; suggested wording for emails; and interaction strategies. Similar resources could be developed for students (and their families) to inform them of: why teachers may worry about study during chronic illness; how to communicate with teachers; and strategies for managing study during absence.

There remain many educational issues to address in supporting young people with chronic illness, unsurprising perhaps given the infrastructure, communication and coordination required: the involvement of families, schools and hospitals in complex contexts. But the value placed on academic continuity through connection by young people themselves, the importance of minimising their educational disadvantage, and the desire to help them participate fully in life, provides motivation to continue.

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QUERIES WITHOUT HINTS, AFFIRMING, OR 'TELLING', THAT SUSTAINED SPONTANEOUS PROBLEM SOLVING ACTIVITY

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Microanalysis of excerpts of video-stimulated post-lesson student interviews illuminated the nature and role of queries that sustained creative problem solving activity. These queries led to student realisation that they 'did not yet know', and the impetus to explore further. In this study, using Seligman's (1995) construct of 'optimism', 'not knowing' is considered 'failure' and 'finding out', 'success'. By responding optimistically to identified 'failure', these students achieved success—developed new mathematical understandings. The intertwined nature of problem solving, and optimistic activity subsequent to such queries is elaborated. This study contributes to the body of knowledge on sustaining autonomous problem solving activity.

Introduction

Mathematics learning associated with developing 'deep understanding' ('relational understanding', Skemp, 1976) differs from predominant teaching practices where:

> … *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks the question, and *mathematical truth is determined* when the answer is ratified by the teacher (Lampert, 1990, p. 29).

The pedagogical approach employed in this study promotes student activity consistent with the title to Lampert's article: "When the problem is not the question and the solution is not the answer". The questions that students in this study focused upon were not those in the teacher's problem, and the results of student explorations were not explicit answers to the problem questions. Students focused their own questions, and developed their own pathways. During such activity, students are "not only choosing the cues and concepts—and often unexpected cues and concepts—but even the very question" (Chick, 1998, p. 17), and are making "not so much direct attempts at solving the problem … [but] thoroughly investigating it, with auxiliary information being extracted from each trial" (Krutetskii, 1976, p. 292). Relational understanding (Skemp, 1976) develops through such exploration—a connected form of understanding where students know why mathematics is relevant and can select and use it in unfamiliar situations. The social element query (Schwarz, Dreyfus, & Hershkowitz, 2009) became a focus when it preceded shifts in thinking about mathematics generated. Seligman's (1995) 'optimism' was employed as a lens because it has been linked with learning gains in mathematics (Yates, 2002), and problem solving (e.g., Williams, 2005).

Theoretically framing this study

Optimism (Seligman, 1995) is an explanatory style associated with how people respond to successes and failures. Sawyer (2007) described problem-solving activity of innovative design teams in industry as: "mak[ing] more mistakes, … [with] as many misses as hits" (p. 16). Similarly, mathematical problem solving leading to insights generally includes 'failures' 'on the way to' attaining 'successes' (Williams, 2006a, b). Thus, for the purposes of this study, mathematical problem solving is considered a situation of adversity where 'failure' is 'not knowing' and 'success' is 'finding out'. An optimistic child perceives successes as 'permanent', 'pervasive', and 'personal' and failures as 'temporary', 'specific', and 'external'. These terms are elaborated later through illustrations. Indicators of optimism were displayed in interviews with students who creatively solved self-set problems (Williams, 2005). For example, Dean struggled to attain a passing grade in mathematics (See Williams, 2005, p. 284), but showed he considered 'not knowing' as *temporary* and able to be overcome: "it always takes me a long time to understand when we first start a new topic". And, he employed *personal effort* to help overcome this temporary state: "I go over and over it until it makes sense". He perceived his successes (finding out) as a characteristic of self (someone overcoming 'not knowing' through personal effort): "and then *I get it*" [Success as Pervasive]. Kerri (Williams, 2005, p. 321), a high achieving girl in the same study, spontaneously constructed new knowledge on several occasions during the research period (see Williams, 2007). Her comment: "last year I did not do as well in maths; the teacher took too big a leaps" showed she limited her 'failure' in mathematics to a particular time frame [Failure as Specific], and identified an external factor that contributed [Failure as External]. Despite their differences in mathematical performances, Dean and Kerri both displayed indicators of optimism, and no indicators of lack of optimism, and each was willingly 'stepped into unfamiliar territory' to explore to develop new mathematical understandings. They did not refer to 'not yet knowing' as 'failure'. The term 'failure' is used in this paper to link to optimism.

In common language usage, 'optimism' is taken to mean feelings of hopefulness and confidence. Seligman's construct includes the perception that personal effort is required to attain successes (as well), not just a hope that it will happen. Martin (2003) and Williams (2003) identified similar constructs associated with student capacity to overcome adversities during learning: 'academic resilience', and 'optimistic exploratory style' respectively. Martin's construct relates to learning in general, and Williams' construct (Seligman, 1995; Williams, 2006a, b) relates to student inclination to explore: enter 'flow' situations.

Flow (Csikszentmihalyi, 1992), a state of high positive affect during creative activity, occurs when a person/group spontaneously develops new skills in response to self-set challenges. Such activity is 'signalled' by intense engagement and loss of all sense of time, self, and the world around as all energies are focused on the task at hand (e.g., Williams, 2006a). During mathematical problem solving, flow conditions occur when a group, or individual student, spontaneously and idiosyncratically identifies an unfamiliar mathematical complexity that was not apparent at the commencement of the task, and decides to explore it (e.g., Williams, 2007). The term *spontaneous* refers to student learning not caused by the teacher (or another 'who knows'):

> We do not use spontaneous in the context of learning to indicate the absence of elements with which the student interacts. Rather we use the term to refer to the non-causality of teaching actions, to the self regulation of the students when interacting ... we regard learning as a spontaneous process in the student's frame of reference. (Steffe & Thompson, 2000, p. 291)

Steffe and Thompson's expression "in the student's frame of reference" is crucial to spontaneity. Social elements (Schwarz et al., 2009)—control, elaboration, explanation, query, affirmation, and attention—have been linked to spontaneity (Williams, 2005). Where activity is spontaneous, control, elaboration, explanation, and affirmation are internal to the student because the student controls the pathway taken, explains and elaborates the mathematical ideas involved, and works out, for themselves, whether the mathematics they generate is reasonable. The role of attention and query in relation to spontaneity still need further investigation.

'Observable cognitive elements' during the process of critical inquiry (Schwarz et al., 2009), were integrated with Krutetski's (1976) 'mental activities' to analyse knowledge construction (Williams, 2005). *Recognizing* involves identifying a context in which a previously constructed mathematical entity applies, or identifying mathematics relevant to a context (Schwarz et al., 2009). *Building-with* involves using a mathematical procedure that has been recognized, in a context in which it has previously been used (non-spontaneous) or in a new context (spontaneous). Krutetskii's 'mental activities' form subcategories of building-with associated with spontaneity: element-analysis (examining a problem element by element), synthetic-analysis (simultaneous analysis of several elements), and evaluative-analysis (synthetic-analysis for the purpose of judgement). *Constructing* involves integrating previously constructed knowledge to develop new insight (Schwarz et al., 2009; Krutetskii, 1976), checking internal and external consistency, and recognizing its usefulness in other situations (Krutetskii, 1976).

The research question for this study was "What are the nature and role of queries that sustained spontaneous problem solving activity in this study?"

Research design

This section describes the context (schools and students), data sources, and excerpts selected, and the pedagogical approach employed (including the tasks, composition of groups, and the types of interactions intended to support spontaneous student thinking).

Context, data sources, excerpts selected

Two Grade 5 students were the focus of excerpts in this study: Tom [Excerpt 1] and Lenny [Excerpt 2, 3]. They attended either a Northern Suburbs Government Primary School, or a Southern Suburbs Catholic Primary School in Melbourne. The broader study from which this data was selected captured problem-solving activity in upper elementary school classes with three tasks undertaken each year over a two-year period. Six 80-minute sessions were undertaken each year with the researcher as teacher implementing the task, and the classroom teacher participating. Four cameras in the classroom captured the activity of each group of 3-4 students in the class as they worked with the task, and briefly reported group findings to the class every 10-15 minutes. Work generated by groups during these sessions was collected and used to support student discussion during interviews. Video-stimulated post-lesson student interviews were undertaken individually with four students after each lesson. Students were asked questions including whether they learnt anything new, how they learnt it, and to find parts of the lesson that were important to them (including, if possible, anything that influenced their process of learning something new). Tom and Lenny were each interviewed after the lessons from which these excerpts were taken. Three excerpts of video data and associated student interviews were selected for microanalysis. Each excerpt included at least one query (self-query or external), and the student continued to control the pathways they explored, and the questions they focused upon after these queries, and simultaneously displayed high positive affect. Tom, in Excerpt 1, was working with the final task in the first year: “The Fours Task”. Lenny, in Excerpts 2 and 3, was working with the sixth task (at the end of the second year): “Marketing Through Blue Smarties”. Queries from different sources were the focus of each excerpt.

Engaged to Learn pedagogy

This approach was developed (see Williams, 2009) to provide opportunities for flow situations. The class undertook three to four cycles of group work followed by reports to the class. Questions asked by other students, the teacher, or the teacher-researcher were intended to be non-confrontational (no contradicting). Rather, they were expected to be requests for elaboration or explanation that were not focused beyond the content presented. For the purpose of sustaining spontaneous activity, the teacher-researcher and classroom teacher did not provide mathematical input or hints, or agree with or dispute pathways taken during these cycles. Instead, they tried to ask questions to elicit further thinking. Such questions are illustrated herein by the type of interviewer queries in Excerpt 3.

Group composition

Groups (3-4 students) were composed by the researcher-teacher informed by video data from group interactions during previous tasks, and teacher background knowledge. Students with similar paces of thinking (differs to student performance) were grouped together so they were more likely to develop new ideas at the same rate. Such composition was intended to reduce possibility of some members ‘falling out of flow’ because the challenge became too great, or not entering flow because the challenge was insufficient. A group member likely to buffer negative influences was included in each group where possible. I had developed these grouping strategies as a teacher before I knew about flow, and before I realised that the ‘positive group member’ was optimistic.

Tasks

Each task was accessible through a variety of representations and levels of mathematical sophistication, and included concrete materials to support student experimentation. The two tasks undertaken during the excerpts selected were:

The Fours task (Tom)

Make each of the whole numbers from one to twenty inclusive using four of the digit four and as many of the following operations and symbols as required:

$$+ \ + \ - \ - \ \times \ / \ \div \ \sqrt{} \ . \ (\,)\ ^{2}$$

Develop strategies to generate these integers faster than other groups. Groups spent three minutes with individuals generating possibilities alone. Then they shared their findings, and ideas, in their group. During reporting sessions, groups could focus on any of the following:

- Two numbers they had generated;
- Something they had found;
- Something that was not working that other groups might be able to help with;
- A ‘big picture idea’ that helped generate numbers faster; or
- Anything else they thought could be useful to other groups.

Reports later in the task tended to focus more on the later dot points.

Marketing through Blue Smarties (Lenny)

Design an advertising slogan by constructing a Blue Smartie Promise to attract lovers of blue Smarties to buy. Remember broken promises are not good for the company. Each group starts with a small-unopened box of Smarties (coloured candy), predicts the number of blue Smarties in their box (giving reasons for their predictions), opens the box, counts, and discusses their findings with their group. Groups then reports to the class, and add a tally to the board (See Figure 1). Each class member then predicts, opens, and counts blue Smarties from a new box, and adds their data to Figure 1. Groups analyse the data and report on their analysis. Groups then try to develop a Blue Smartie Promise. The feasibility of keeping each promise is then discussed.

Results and discussion

Excerpt 1: Queries from member of Tom’s Group

Tom participated in the cycles of group work and reporting on ‘The Fours Task’. Alf, another group’s reporter explained how his group made 17. In doing so he expressed four divided by four as a mathematical object:

> … we did four times four to get sixteen but we needed one more … we had two extra fours … then we did four times four plus … four over four … so it would be like saying four times four plus one … four over four is one whole, so that is just like saying one, and four times four plus one you get seventeen[1]

Tom excitedly realised he could apply this idea more generally: “when he [Alf] said four over four … is the same as one just that sentence just flung me like quickly in my mind *ahhh I could use that*”. When group work recommenced Tom stated:

> … we need … a strategy to figure out every single one … like what Alf and Ken’s group did because four over four… one could come in handy for everything that is a not multiple of four- so … from sixteen you need one to get to seventeen ... umm- something minus four over four to get to fifteen

Tom undertook element-analysis in identifying the structure of Alf’s calculation (stem plus four over four). He could see he could vary the stem and use either plus, or minus, four over four for different integers. Tom elaborated his idea further to his group:

> For four you can get three and five using four over four and for eight you can get six and seven and for twelve you can get eleven and 13 and for 16 you can get 15 and 17 and for 20 you can get 19.

[1] Key to Transcripts: ‘…’ text omitted that does not alter meaning; ‘-’ changed direction to comment; ‘[text]’ explanatory text added by researcher. ‘/’ cut across another’s statement.

In doing so, he continued to focus on the part after the stem (plus or minus one) and did not elaborate on making stems. Gabrielle (another group member) requested further explanation and Tom focused again on the part after the stem. Eventually, Gabrielle took the pencil from Tom, shifted the paper towards him, and tapped on the sheet: "*How how how*?" Tom elaborated the end part again. Gabrielle did not give up though; she queried in a different way: "… so if somebody asked you to … give answers to every single number …?" This query did not draw attention to the stem, nor contradict Tom's idea that the stem could be any multiple of four. In responding to Gabrielle's query, Tom realised his idea was only partially correct: "… if you are going to do like 12- you can't do 12- you won't be able to do it because four plus four plus four is the only way to get 12". Gabrielle's queries captured Tom's attention with her repeated emphasis on the word '*why*' and her tapping. Her queries did not contain mathematical input, and did not draw specific attention to what was problematic. Yet Tom identified what he did not yet know [failure] as he responded, and spontaneously tried to find out more as a result.

Tom's subsequent cognitive activity was simultaneously optimistic [synthetic-analysis; Failure as Temporary]. In simultaneously considering the structure and its usefulness for generating different integers, he perceived not knowing as temporary: "… so the only one that you can do it for is 16 … oh unless you do eight- so four plus four- yeah so four eight- so four plus four minus four over four …". By continuing to vary parts of the calculation as he tried to see what was possible, and considering the outcome each time, Tom made decisions about whether he needed to explore further (evaluative-analysis: synthetic-analysis for the purpose of judgement). He simultaneously demonstrated that he perceived not knowing could be overcome by the personal effort of looking into the situation to see what could be changed to find out more [Failure as Specific, Success as Personal]. Thus, in response to Gabrielle's queries, Tom's initial 'not fully correct construct' became 'more correct'.

Excerpt 2: Lenny focused his own query

During the "Marketing Through Blue Smarties Task", students added tally marks to record how many blue Smarties were in their box of Smarties (See Figure 1). For example, the five tallies beside the top box in the second column of boxes represented five boxes containing six blue Smarties.

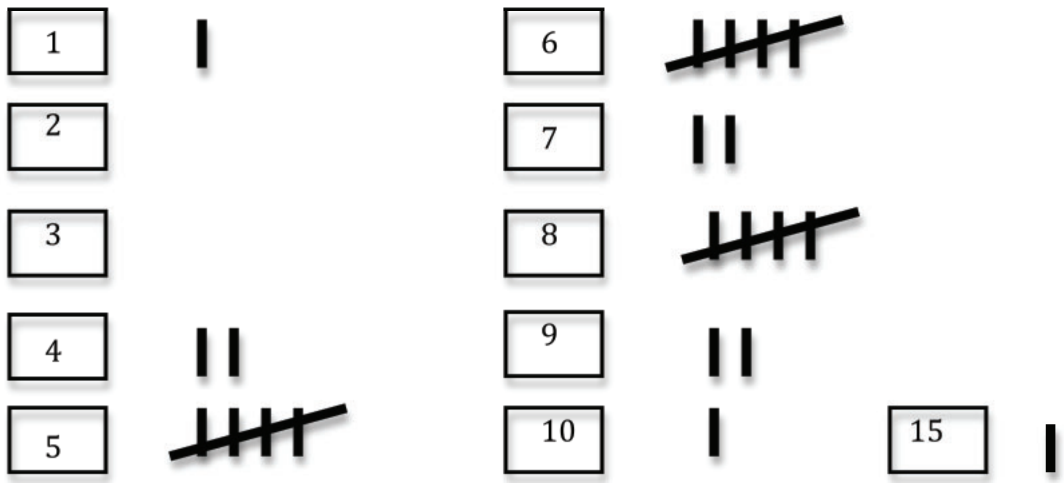

Figure 1. Diagram on board: tallies of numbers of blue Smarties in boxes.

Lenny silently wondered why some groups had so few blue Smarties in their boxes: "I found that really *really* surprising … even the *four* (pause) because that is *half* (pause) what I thought it would be". This was the start of Lenny's thinking about chance in this situation. He recognized 'average' as relevant for examining this: "I was trying to think (pause) what the (pause) *average* was" but he was not sure how to find it. He explained one way he tried to find the average and why he did not think it was right:

> I think I did it wrong but … I added it all up and I think it added to twenty four and then … I forget what we we – *I* – was supposed to do then so I just counted all the ones that had (pause) … the ah numbers next to them and then I think there was nine and then I divided it by nine.

Lenny could not recall the procedure for finding an average "I can't remember how (pause) to do it properly" so tried a possibility using what he did remember (an add, and a divide). He knew the result was not reasonable (evaluative-analysis, identified 'failure'): "I counted the fifteen as one". When asked what he did, he elaborated: "*I added all of them* [the tally marks] *up like* (pause) so I added this one- three sss- eight" and considered the size of the answer was not reasonable: "It was two and a bit … so I did it wrong [confident voice]". When asked how he knew it was wrong, Lenny showed some conceptual understanding of the term 'average': "I knew the (pause) it's- there's- if there was eight (pause) six and five each (pause) *mmore* of them are over five so how is it under [around?] two?" He also knew a low number of blue Smarties in a box would bring down the average: "And I knew probably it would be around five six because the *one* would bring it down a fair bit …". Lenny's evaluative-analysis undertaken from more than one perspective helped him identify his 'failure' to work out how to find an average. He continued to puzzle over this [Failure as Temporary]. Lenny's personal effort expended on trying to work this out was reflected in his lack of awareness of group interactions around him: "*Yeah* I didn't really put that much into our ... promise because I was [soft laugh] *trying to figure out the average*". Across the time of the interview, Lenny became more articulate in expressing why he considered he was not correct. It is unclear whether this happened because he had worked out ways to express himself more clearly, or because he had extended his thinking: "I just went *one* (pause) *two* and stuff but I didn't count like (pause) … yeah I didn't count all *of them as 15* (pause) I just counted them as *one* each".

In this excerpt, Lenny queried what he saw on the board based initially on the prediction he had made. As a result, he spontaneously posed a question to help him consider this further: "What is the average?" As he did not remember how to work this out, that became his focus. He used what he did remember to develop possible calculations (synthetic-analysis) and made judgements about the reasonableness of the answers he generated (evaluative-analysis). In doing so, he simultaneously displayed optimistic enactment [Failure as Temporary, Failure as Specific, Success as Personal].

Excerpt 3: Interviewer queries support Lenny's puzzling about table

Lenny knew there was something the matter with the procedure he was using to find the average, because he was counting each box as containing one Smartie. He had not found a way to overcome his problem though. This could have been because he did not understanding the table in Figure 1. Lenny continued to display intense interest when he had decided his answer was wrong. As interviewer, I was asking questions in a '*wondering*' rather than confrontational way, and softly as though I was not expecting an answer: "so I wonder how many blue Smarties are there altogether?" Lenny responded with intensity and immediately began again to try to find this number (by counting tallies]:

> Lenny: *I don't know*- its two (pause) three (pause) oh eight (pause) mmm thirteen …? [Failure Identified, exploration continued].
> Interviewer: [Using language used previously by Lenny]: So are you counting the number of *Smarties* there, or are you counting the number of boxes?
> Lenny: I am counting the number of (pause) how many *lines*.
> Interviewer: [softly] I wonder what those lines stand for (pause) whether they stand fo/?"
> Lenny: [Cut off query with excited reply] /They stan- that st- that one stands for *one* and that one [one of the tallies beside the four] stands for (pause) *four*".
> Interviewer: [soft wondering] "Four what?"
> Lenny: Four [long pause then confidence answer] *blue Smarties*.

Lenny was excited at the result of his synthetic-analysis. He realised he could use the numbers in the boxes together with the tallies to find the number of Smarties. He demonstrated failure was temporary and success personal as he began to interpret the table to answer the question he focused on (how many blue smarties?): "I'd have to count *one* … add *four* [correcting himself to]- and then I would have to add *eight* which would be nine and then I'd have to add five fives …". The queries from myself as interviewer used language Lenny had introduced to encourage Lenny to elaborate his thinking. They were generally soft questions that were not necessarily intended for Lenny but could have just been me wondering about ideas that were developing. The more specific question about whether Lenny knew how to find the number of blue Smarties now, was based on what Lenny had been talking about in the interview. Lenny's intense interest in this question was demonstrated by the emphasis in his response. He focused on this identified lack of knowing and continued to think about it. Lenny's use of the term 'lines' and the interviewer's attention to this in the subsequent query was almost immediately followed by Lenny's excited realisation of how to interpret the table.

Conclusions

These excerpts illustrate queries from different sources (group member, self, expert other). Their role in each case was the same. They led to the student identifying a failure (not yet knowing) and intently continuing their spontaneous exploration. These queries did not contain mathematical input or hints or affirmations or contradictions, so did not eliminate spontaneity. They drew attention (in some way) to something that required further elaboration. In each case, evaluative-analysis occurred as a result as the student began to construct new understandings (mathematical structure, concept of average, meaning of table), and the activity was simultaneously optimistic. This study begins to make transparent some of the links between optimistic activity and learning gains:

- Exploring further when finding something is 'not yet known' [Failure as Temporary]
- Persevering by experimenting to find a way to find out [Success as Personal, Failure as Specific].

This research adds to the body of knowledge about increasing relational understanding through problem solving. It illustrates the nature of queries that sustained exploration.

Further study of problem solving activity in other contexts would be useful to see whether queries of the same nature perform the same role elsewhere, and to find other types of questions that achieve this. This study should be useful to teachers and teacher educators interested in developing questions that promote problem solving activity.

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"MY SELF-ESTEEM HAS RISEN DRAMATICALLY": A CASE-STUDY OF PRE-SERVICE TEACHER ACTION RESEARCH USING BIBLIOTHERAPY TO ADDRESS MATHEMATICS ANXIETY

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We read books to find out who we are. What other people, real or imaginary, do and think and feel is an essential guide to our understanding of what we ourselves are and may become. Ursula Le Guin

Pre-service primary teachers‟ mathematics anxiety affects their future teaching of mathematics. This makes them less likely to engage with mathematics, impacting on the attitudes and performance of their future students. Hence, teacher education is a crucial site for further research. Bibliotherapy, incorporated into a fourth-year pre-service teacher‟s action research during her final practicum, helped identify the impact of previous experiences on her mathematical identity. With each cycle of her action research, supported by the bibliotherapy process, the pre-service teacher was able to develop greater insight, leading to a more positive projection into her future as a teacher of primary mathematics.

Introduction

Recent mathematics curriculum documents, for example, the Australian Curriculum (Australian Curriculum and Assessment Authority [ACARA], 2010), are based on the premise that all students are capable of learning mathematics. This counters the traditional view, in which only a few students were expected to succeed. Mathematics has been perceived as a critical filter (Sells, 1978), associated with elitism and social stratification (Tate, 1995). Beliefs that success in mathematics relates to participants‟ inherent worth still dominate thinking (Gates & Jorgensen, 2009). Failure in mathematics can have a powerful emotional impact that may extend far beyond the mathematics classroom (Boaler, 1997). The impacts of mathematics instruction produce for many an enduring state of mathematics anxiety. This anxiety has been associated with inappropriate teaching practices, and a prevalent belief that success in mathematics is determined by ability rather than effort (Stigler & Hiebert, 1992).

This paper, part of a larger project investigating the use of bibliotherapy, is written within a framework of action research, as a tool for addressing primary pre-service teachers‟ mathematics anxiety. This will add to existing frameworks for the study of affect in mathematics education (see, for example, Hannula, Evans, Philippou, & Zan, 2004).

Theoretical framework and literature review

This research is located at the intersection of the literature on the impacts of mathematics anxiety on primary teacher mathematics education, and bibliotherapy.

Mathematics anxiety

Mathematics anxiety is a learned emotional response, characterised by a feeling that mathematics cannot make sense, of helplessness, tension, and lack of control over one"s learning. Mathematics anxiety has been associated with inappropriate teaching practices, and a prevalent belief that success in mathematics is determined by ability rather than effort. Ma"s (1999) meta-analysis of studies of elementary and secondary students found significant relationship between anxiety towards mathematics and achievement in mathematics.

Theoretical models of mathematics anxiety have multidimensional forms that incorporate attitudinal (dislike), cognitive (worry) and emotional (fear) aspects, (Hart, 1989; Wigfield & Meece, 1988). Baroody and Costlick (1998) suggested that children who develop mathematics anxiety tend to fall into a self-defeating, self- perpetuating cycle, and described a mathematics anxiety model that illustrates how beliefs can lead to anxiety, which reinforces unreasonable beliefs.

The impact of teachers" beliefs about mathematics can be far-reaching in promoting positive outcomes for students, and remains an important focus for educational research (Leder, 2007). Many primary or early childhood pre-service teachers (PSTs) have a fear of mathematics, and see themselves as unable to learn effectively. A great deal of research has been done in this area, but is outside the scope of this paper. Thompson (1992), in an extensive review of research into affective elements of mathematics education, noted that the difficulties in promoting teacher change were intimately connected with both what teachers know and believe. Ambrose (2004) reports that mechanisms that have the potential to change beliefs are those providing emotion-packed, vivid experiences, becoming immersed in a community, and promoting reflection on beliefs.

Bandura"s theory of self-efficacy indicates the significance of teachers" beliefs in their own capabilities on student learning and achievement. Bandura (1994) defines self-efficacy as "people"s beliefs about their capabilities to produce designated levels of performance that exercise influence over events that affect their lives . . . Self-efficacy beliefs determine how people feel, think, motivate themselves and behave" (p. 71). People need a strong sense of efficacy before they try to apply what they have learnt or try to learn new things. Teachers" beliefs about their own ability are a significant factor in their approach to teaching mathematics and even militate against their willingness to teach upper primary classes (Wilson, 2009). High teacher efficacy leads to improved student performance learning and achievement (Allinder, 1995; Ashton, 1984; Gibson & Dembo, 1984; Madison, 1997).

Many students come to tertiary teacher education with limited mathematics understandings, and a pattern of avoidance and anxiety. Researchers of primary (elementary) PSTs report high levels of mathematics anxiety, low confidence levels to teach mathematics and low mathematics teacher efficacy. For a more detailed discussion of these issues see Wilson (2009).

Researchers have concluded that high levels of teacher mathematics anxiety can be perpetuated in classrooms (for example, Furner & Berman, 2005). When students are marginalised and do not identify themselves as confident learners of mathematics, they are unlikely to map mathematics into their future identities in a positive way (Boaler, 1997). The way individuals perceive themselves as learners of mathematics is integral to their subsequent identity as teachers. In previous research (Wilson, 2007; Wilson & Thornton, 2008) many PSTs described an interaction during their schooling that led to them identifying themselves as persons who couldn"t learn mathematics, and said that this still impacted on their self-images as future teachers of mathematics.

Identity brings together affective qualities and cognitive dimensions. Ricoeur (1994) suggests that people make sense of their own personal identities in a similar way to their understanding of the identity of characters in stories. Identities are mobile, and remain open to revision. Walshaw, (2004, p. 557), argues that "teacher education must engage the identities of pre-service students", and describes the journey of a pre-service secondary teacher, Helen, who, "through a process of formation and transformation, finally at the end of the year, understood who she might become" (p. 563).

In summary, PSTs with mathematics anxiety are less likely to engage with mathematics and have low confidence and low self-efficacy, impacting on their identity as teachers of mathematics. It is for these reasons that teacher education has become a crucial site for further research.

Bibliotherapy

Bibliotherapy is a technique that was developed in psychology and library science. It involves guided reading of written materials used in gaining understanding or solving problems. The procedure is based on reading about the dilemmas of, and identifying with, the protagonist, followed by individual or group discussion in a non-threatening environment. The reader is an active participant in the process and interprets through their own psychological perspective and perceptual lenses, but feels safe because they are not experiencing the crisis.

Advocates for bibliotherapy identify both cognitive behavioural and psychodynamic benefits. Shrodes (1950), a pioneer in bibliotherapy, attempted to explain how literature could aid therapeutic work. Her psychodynamic model focused on the processes of identification (or universalization), catharsis (or abreaction) and insight (and integration) as the key steps for therapeutic benefit to occur. Many writers since then, for example, Morawski (1997), have used similar constructs.

The stages of bibliotherapy used in previous research, (Wilson & Thornton, 2008) are that the reader:

- identifies with and relates to the protagonist (identification)
- is emotionally involved and releases pent-up tension (catharsis)
- learns through the experiences of the character and becomes aware that their problems might also be addressed or solved (insight)
- recognises that we are not alone in having these problems (universalisation)
- can envisage a different future identity (projection).

In bibliotherapy, whether used in groups or individually, it is the additional work that goes on in the group or between the therapist and client that leverages the potential benefits, not just exposure to the literature. Researchers comment on the therapeutic dynamics of the group, such as getting feedback from others, and hearing other perspectives. Hynes and Hynes-Berry (1986) describe an important feature of interactive bibliotherapy as "the triad of participant-literature-facilitator" (p. 11).

Bibliotherapy has been used in preparing PSTs to teach students with emotional and behavioural disorders, and students with special needs (Morawski, 1997). Previous research used bibliotherapy during mathematics units for PST to examine their attitudes towards themselves as learners and teachers of mathematics (Wilson, 2009; Wilson & Thornton, 2008;). The significance of the changes in response to the bibliotherapy process was that they contributed to the understanding of aspects that drive the development of their mathematical identity. Themes identified through the analysis of previous research strongly suggest the importance of insight as a major factor in bringing about a positive projective identity. The potential of bibliotherapy is that it is a stimulus for this revision, and the planning cycle of action research.

Methods

This paper reports an action research project by a primary PST, which aimed to investigate and develop her professional knowledge and practice. Shannon was in her fourth year of a primary Bachelor of Education degree. She examined how she might address the impact of her mathematics anxiety on her mathematics teaching practices during the final practicum of her course.

Bibliotherapy was used within the framework of action research as a tool for addressing her affective responses to mathematics. Action research has been identified as a powerful process for reconstructing and transforming practice (Somekh, 2005). However, Salzman, Snodgrass, and Mastrobuone (2002, p. 2), state, "in spite of action research‟s ability to help teachers gain unexpected and valuable insight into the realities of their own classrooms, there appears to be limited innovation at both the pre-service and in-service levels to help teachers develop action research skills." The goal was to understand what had influenced Shannon in the development of her teaching practice, by examining the relationship between her beliefs and her classroom practice, and their impact on her professional identity.

Three instruments were used in the action research cycle:

1. Initially, working with the researcher, Shannon completed a short questionnaire about her self-perceptions as a learner and teacher of mathematics, and past experiences that contributed to these. This was repeated at the end of the project.
2. The second procedure was a cycle of pre- and post- self-assessments of each mathematics lesson. These comprised a survey and short questions completed before and after each lesson, (including feelings, preparedness and teaching success, rated on scale of 1- 10; and notes on level of confidence, what went well, what would be changed in future).
3. The third instrument was a journal of written reflections. Previous papers about mathematics anxiety were provided as part of the process of action research. The readings formed the stimulus for the written reflections, as one of the means of incorporating bibliotherapy into the action research process. The reflections were shared with the researcher and fellow PSTs undertaking action research projects, during and in a presentation and discussion after the practicum.

The reflections were triangulated with the answers to the questions and lesson assessments, and the conclusions were then reviewed in the light of student feedback, and related to the outcomes of previous research using the bibliotherapy framework.

Results and discussion

Shannon identified a critical incident around a packaging problem in a mathematics class "The teacher decided she didn‟t like mine and held it up to the class (with me standing near her) and started berating my design and my ability and saying this is the sort of thing that she would expect from someone much younger. I felt humiliated … I think that after this time, I really started to withdraw from maths."

Using readings to clarify understanding of learning is central to the bibliotherapy technique. The reflections on readings showed strong identification: "I think this perfectly describes how I feel about maths – especially the tension". Identification is one of the stages of the bibliotherapy process. Shannon also commented that the findings of the readings were interesting and relevant, for example she identified with the reports of mathematics anxiety starting in primary school and related this to her school experiences.

An important part of the initial reflections on the readings revolved around the view of herself as a learner of mathematics that Shannon had developed during her schooling: "I‟ve always been able to „keep up‟ but not necessarily understand what I was doing", and "I found that if I understood a formula I was happy about my ability, but if I felt overwhelmed it would be because I wasn‟t good at maths".

Her preliminary comments gave voice to the concern of researchers to ensure that negative learning experiences will not reinforce negative beliefs and feelings about mathematics in the future students she will teach, and echo the concerns of teacher educators who identify this as an issue. A major concern was that she would "inadvertently pass on my fear and anxiety of maths to my students. I don‟t want them having the same negative experiences that I have had". During her presentation and discussion with peers, she again emphasised the strength of the concern she felt at the start of her practicum. "I was concerned that I would instill [sic] in students the same feelings about maths as what I have". This echoes previous research findings (Wilson, 2009), where teachers‟ comments reflect a concern for their students that negative learning experiences will not reinforce negative beliefs and feelings about mathematics.
The reflections on individual lessons indicated that her assessment of her feelings before the lessons stayed in the range from 6 to 8 ½, but that the level after the lessons had a much broader range, from 2 after the first lesson, rising to 8 ½, plummeting to 4 and then rising back to 8. Shannon commented that when she became flustered in lessons, the "lesson focus would change dramatically" and this lowered her assessment of her feelings after the lesson. She related this to her attitude. "If I felt confident before the lesson started, I most often felt good about it afterwards, however if I went into the lesson with a negative attitude, then I most often had negative feelings about that lesson afterwards". Her positive experiences increased her confidence that she would be able to decrease her level of anxiety. "I may even be able to change my negative attitude of this subject over time".

As the practicum progressed, the focus of Shannon‟s comments moved from reflections about her own inadequacy to the reassurance she felt when students responded positively to the lessons. With each cycle of her action research, supported by the bibliotherapy process, the PST was able to develop greater insight, eventually leading to a more robust projection into her future as a teacher. “I found that acting confident in maths actually made me feel more confident and I was then able to more clearly convey the concepts”.

It might take more time for some students to go through the stages of bibliotherapy, although it is important to realise that everyone is unique and there is no schedule for the process. Shannon reflected, “I know my anxiety about teaching maths has not disappeared”. The positive impact of the experience is shown by her motivation to continue with more readings and reflections as she completes her course and begins teaching.

The final answers when the initial questions were repeated provided evidence that the she had shown an emotional response to the readings, had reflected on her own experiences and had engaged in some stages of the bibliotherapy process. Reflection on each lesson during the action research cycles resulted in considerable development of her ability to analyse and critique her own practice, and to improvements in her interactions with students.

Her reflection on her experiences was followed by a consideration of what it could mean for the future and the implications of her insights for her teaching. Her assessment of her increased confidence was authenticated for her by feedback from the class, which corresponded to her feelings about the lessons. In lessons that did not go well, she felt the class “was struggling to understand what I was talking about”. However, as the action research cycle progressed and she was able to demonstrate more confidence, “the majority of the class said they felt better about the maths when I felt better about teaching it”. Shannon projected herself more confidently into the classroom teaching situation and wrote about the importance of positive attitudes. As Carnellor (2004) writes, “Positive attitudes not only enhance the quality of learning, but also the degree to which learning and understanding become embedded in the real-life experiences of the individual” (p. 5).

The final answers and reflections demonstrate the potential of bibliotherapy to change the way PST feel, as the she summed up her experience by saying: “I believe my self-esteem has risen dramatically”.

Conclusion

This research connecting bibliotherapy to cycles of action research is innovative as it brings together analysis of reflections of a pre-service teacher with a study of the beliefs, attitudes and insights that shape her mathematical identities. The juxtaposition of bibliotherapy with action research is potentially a powerful strategy in addressing mathematics anxiety in PSTs. Bibliotherapy, used as part of the process of action research, is able to address Ambrose‟s (2004) criteria for changing beliefs, as it can provide emotion-packed experiences, encourage PST to become immersed in a reflective community and connect beliefs and emotions, and teacher practice.

Supporting PSTs to develop reflective and metacognitive skills empowers them to take these skills into the classroom, and monitor and critique their practice. The special feature of the bibliotherapy approach of eliciting PST reflections stems from its ability to call forth cognitive responses paralleled by emotional responses.

The power of bibliotherapy, as exemplified in Shannon"s action research, lies in the way that her cognitive responses were allied with emotional responses. It changed her understanding of her difficulties and anxieties in the mathematics classroom. Through this research she put her own experiences into perspective, developed enhanced self-images as a learner of mathematics, and changed her assessment of her capacity to learn and teach mathematics. As Wolodko, Willson and Johnson (2003) write "our challenge is to help pre-service teachers confront their past experiences and anxieties about teaching and learning of mathematics. If these are openly dealt with during their university education, fewer teachers may be content to teach just as they have been taught" (p. 224).

Bibliotherapy, allied with action research, provides a new framework that has much to offer. This offers another way that the bibliotherapy process could be incorporated into teacher education courses. It provides teacher educators with a shared language to talk about cognitive and emotional responses in terms of the processes of identification, catharsis, insight, universalisation and projection. Hence, it provides teacher educators and researchers with a framework and language for communicating research outcomes.

Finally, negotiating this issue has the potential to transform learning and teaching beyond that of the PST to the future students. Bibliotherapy allows PSTs to reconstruct their own experiences, and re-evaluate their identities as learners and teachers of mathematics, potentially affecting not only their current study but also their future teaching of mathematics and hence the attitudes of their future students. The potential exists for teachers who have gained insights through this process and, an understanding of the process during their training, to use their experience to help their students address and overcome their own mathematics anxiety.

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DEVELOPING ALGEBRAIC THINKING: USING A PROBLEM SOLVING APPROACH IN A PRIMARY SCHOOL CONTEXT

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This paper reports on an ongoing research project investigating how problem solving can prepare students to think algebraically. The student examples presented highlight how investigating and solving mathematical problems from a structural and generalised perspective can develop the thinking associated with algebraic reasoning.

Introduction

For more than 50 years there has been a call by many experts in mathematics education research, curriculum design and policy development that students in primary school should learn and understand a level of mathematics beyond computational procedures. Directly related to this request has been a response to include algebra within the primary school curriculum. One reason why many western democracies have undertaken the challenge of reforming the primary school curriculum has been the steady decline in the participation rates of students undertaking advanced mathematics courses at a secondary school level (MacGregor, 2004; Stacey & Chick, 2004). Consequently, the declining participation rates and limited engagement with mathematics has slowly impinged on the availability of competent individuals who wish to, or are able to, pursue careers in the mathematical rich vocations offered at a tertiary level (Brown, 2009 p. 5). The inclusion of algebra in primary and middle school mathematics curricula reflects the belief that not only is algebra needed to participate in the modern world; it also provides ―an academic passport for passage into virtually every avenue of the job market and every street of schooling" (Schoenfeld, 1995).

Currently, most primary and middle years mathematics curricula do not solely emphasise the teaching and learning of formal algebra. Instead, the emphasis in these formative years is about developing a conceptual understanding of algebra and in particular the thinking associated with ―doing" algebra, often referred to as algebraic thinking. Algebraic thinking is the activity of doing, thinking and talking about mathematics from a generalised and relational perspective (Kaput, 2008; Mason 1996). Ultimately, algebraic thinking is founded on the ideas and concepts of elementary mathematics and in turn these ideas are used to solve increasingly sophisticated problems. It encompasses all mathematics strands and is built on a conceptual understanding of number and computational fluency, the reasoning of geometry and the processes associated with measurement and statistics.

The potential value for using problem solving is that it may broaden and develop students' mathematical thinking beyond the routine acquisition of isolated techniques and procedures (Booker 2007; Booker & Windsor 2010; Carraher & Schliemann, 2007; Kaput, 2008; Lins, Rojano, Bell & Sutherland, 2001). The thinking required to solve problems can be extended from methods tied to concrete situations—the backbone of primary school mathematics—to experiences that develop an ability to problem solve using abstractions. To consider problems from an algebraic thinking perspective acknowledges that students can adapt their ways of thinking, they can express mathematical generalisations and it can provide an entry to algebraic symbolism that is meaningful (Carraher & Schliemann, 2007).

Research project

The aim of the research project reported here is to explore and gain insights into the effectiveness of using a problem solving approach that facilitates and promotes certain aspects of algebraic thinking. The research aims to provide an improved and deeper theoretical understanding of algebraic thinking and how it can be developed within a primary school context. The intention of the investigation is to develop and implement lessons that actively facilitate algebraic thinking by building on students' problem solving experiences. Furthermore, the research project will seek answers to the following questions:

1. Can problem solving be used to develop algebraic thinking in the primary school context?
2. To what extent are primary school students equipped to use algebraic thinking strategies when solving mathematical problems?
3. What is the effect on students' ability to move from arithmetic to algebra, once a broad problem solving approach that explicitly develops algebraic thinking has been implemented?

Methodology

Part of this study is set in a Year 7 class in a State Primary School that draws from a pre-dominantly lower socio-economic background. Within the cohort of 27 students there is a wide variation in their understanding of mathematics and this position is supported by their 2010 National Assessment Program – Mathematics results. Furthermore, results from Booker Screening Tests (Booker, 2011) re-confirm the diversity and wide ranging mathematical abilities within the group. It would be reasonable to suggest that this class reflects many of the difficulties, challenges and rewards those classes and schools in similar socio-economic areas deal with on a daily basis.

This qualitative research project uses the method of design research and is greatly informed by the research methodology developed and used by Cobb (Cobb & Bauersfeld, 1995; Cobb, 2007). A key aspect of Cobb's interpretation of design research is the importance of collecting primary sources of data by observing and registering mathematical activity by the participant observer/researcher. In addition to this, Cobb also argues that by constantly reflecting on participant actions and synthesising the data a cycle of enactment, analysis and further refinement can allow for generalisations about learning based on all the different elements found within classrooms. Within the context of this study, student work samples and digital video recordings of individual students, small group interactions and whole class presentations and discussion, form the basis of the observations to be analysed.

Findings

This is an ongoing study and it must be noted that the analysis of the data is in its infancy however, the view that has emerged in the early analysis and based on the types of problems the students can solve, the approaches they have adopted and the way they have discussed and presented their results, indicate a growing ability to consider mathematical problems from an algebraic thinking perspective.

At the beginning of the study, it was hypothesised that students would need to develop ways of thinking that moved them from the computational thinking that dominates much of their enacted primary curriculum. A major hurdle to overcome within the cohort was an assumption and behaviour that to solve mathematical problems simply requires numbers to be manipulated. To reduce the influence of this perception, one of the foci of the study was for students to share with their classmates the reasons why and how they developed their solutions. The emphasis to share their mathematical reasoning was a powerful way to motivate the students. By encouraging them to develop a variety of different solutions they began to see the interconnectedness of the mathematics, which in turn influenced their ability to generalise their solutions. The discourse and argumentation that took place assisted individual students to reflect on, modify and delve into all of their mathematical knowledge in order to solve the problems. The opportunities to discuss and exchange mathematical ideas allowed many of the students to overcome the behaviour of calculating using the numbers from within a problem. One particular student's explanation for solving an assortment of structurally related problems was indicative of the way many of the students began to think about the problems. No longer did students immediately try a guess and check method but they attempted to find a generalised approach to the related problems.

> *Nikki:* It's something you can just do for everything ... I've done the problems before but I have never really thought about them. I can do all these problems now because I know a way that works for all of them.

Setting the stage – An overview of the lessons

There is a degree of consistency with regard to the implementation of the lessons throughout the research project, with each 45–60 minute lesson following a similar cycle. Each lesson was introduced with a whole class question where each group, usually made up of 4 students of varying mathematical abilities, were given the same question. After each group had completed the question they had to explain their solution to the researchers, classroom teacher or peers. In preparing their explanation the group had to consider ―Why do you think you are right?" which directed them to address their thinking and mathematical ideas, rather than ―How did you do it?" which emphasises the procedural steps to solve the problems. The next part of the lesson cycle involved giving each group a contextually different yet structurally similar problem. With each new question the mathematics became increasingly more complex Depending on the difficulty of the problems most groups would complete between two to four problems per lesson. The lesson would conclude with a whole class discussion in which students would present their solution.

It cannot be over-emphasised how important the group and class discussions were in igniting and developing different mathematical ideas. During each part of the lesson cycle the collaborative manner in which many of the students conducted themselves was highly productive. Clearly, they engaged with the mathematical discourse of their peers and this had a profound effect on their mathematical thinking. Within the context of this class, the students valued and developed a greater understanding of sophisticated mathematical ideas and this was highlighted by the motivating and knowledgeable applause that followed a mathematically significant event within the group.

Learning and interaction

The following two sessions described are from Weeks 11 and 12 and are towards the end of the teaching sessions. They demonstrate how students can build an algebraic perspective of problem solving. The focus of the prior lesson was to develop a broader understanding of equivalence, in particular the thinking required to manipulate both sides of an equation. At this stage some of the students were using shortened forms of recording and in some respects their own symbolic representations mirrored the formal algebraic symbolism encountered in secondary school. Keiran (2007) describes this as a *generational* activity where students actively create representation of situations, properties, patterns and relations and many of the symbolic meanings children assigned to their thinking can be viewed as algebraic.

Whole class problem

The following problem was given to the whole class.

> *You are given a balance scale, a lump of clay, a 50 gram weight and a 20 gram weight. Describe how you would use these materials to produce a 15 gram lump of clay.*

The thinking described by Thomas is indicative of many students in the class. He demonstrates an understanding of working on both sides of an equation and understood the relationship between the weights and the clay.

> *Thomas:* Here's what I am thinking. If you've got a 50 gram weight and a 20 gram weight, this side is 30 less than the other. Okay, so you get a lump of clay and put it on there and if it balances out then that is thirty and then you half and you get your 15 grams.

This introductory question built a particular way of thinking that emphasised an interpretation of equivalence based on a balance scale metaphor. The idea that for every mathematical action there is a reaction provides a powerful basis for solving problems using an algebraic perspective. This understanding was then carried through to the next series of problems where the relationship could be expressed as two equations.

Group questions and class discussion

Once each group had presented their explanation they were given a choice of problems to solve. Each group could decide which problem they wished to solve and were encouraged to use their own solution process. While many of the children still used counters and diagrams, a number were now using their own shortened symbolic forms. Sarah's group decided to solve the problem:

One block of weight A and one block of weight B weigh 90 kilograms. Two blocks of weight A and one block of weight B weigh 115 kilograms. How much do three blocks of weight A and one block of weight B weigh?

Liam's group, however, chose the following problem:

At the Flourish and Botts bookstore the first Harry Potter book and the second Harry Potter book together cost $45. Two copies of the first Harry Potter book and three copies of the second Harry Potter book costs a total of $125. At this bookstore how much is the first Harry Potter book?

Both Sara (Figure 1) and Liam's (Figure 2) explanations highlight the significant value of identifying the mathematical relationships between the two unknowns. Both of them were able to use a system of equations to solve the problems. They were able to write this symbolically and their explanation confirms this understanding. Furthermore, after Liam had completed his explanation, Sarah, referring to Liam's example, commented that her problem ―is exactly the same as the one we did before". Sarah's statement showed how she acknowledged the problems to be structurally similar even though the content and context were different. Her mathematical focus was not the specific answer to the problem but how both problems could be interpreted in structural terms. An important aspect of algebraic thinking is the ability to consider the interrelationships and generalisation of problem situations and if these generalisations are understood students' mathematical abilities can flourish.

Sarah: Because weight A and B are 90 kilograms, there's two A's and B together there and they weigh 115 kilograms. So you take away the 90 away from 115. It equals 25 kilos. So 1A is 25 kilos and 1B is 65 kilograms.

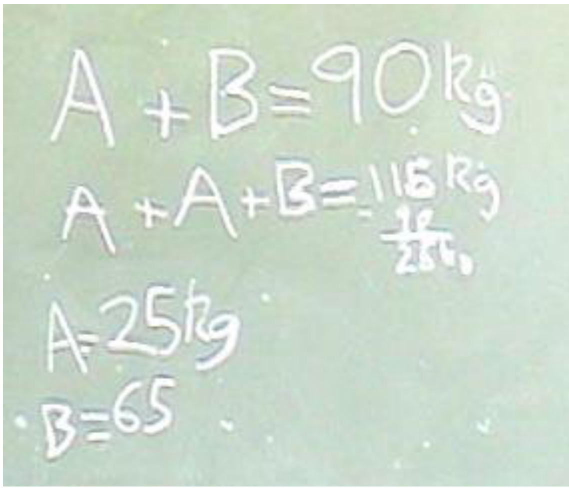

Figure 1. Sarah's explanation to the class.

Liam's explanation follows.

Liam: What we did was 45 double equals 90 so that means that those two together equal 90 (circles 1 and 2). That one is 45 and that one is 45 which is 90 and the one left over is 35 (writes 2 = 35) and that means 45 take-away 35 which means 1 equals 10.

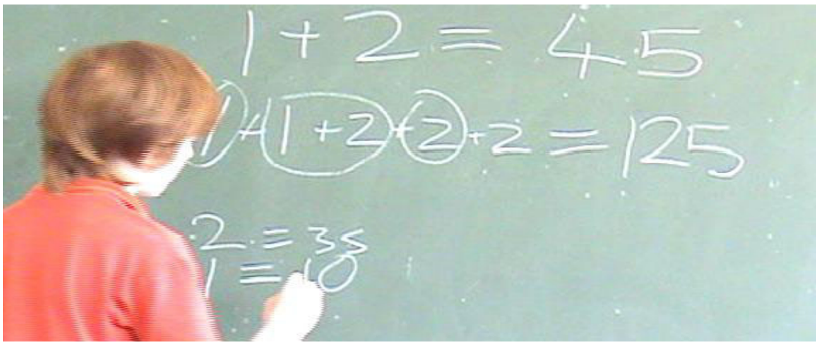

Figure 2. Liam's explanation using 1 and 2 as his symbols for the books.

Holly and Amelia were having difficulty with this problem and they were asked to reflect on how they solved addition and subtraction problems involving unlike common fractions.

> *At the local sports store, all tennis balls are sold at one price and netballs are sold at another price. If three netballs and two tennis balls are sold for $47.00, while two netballs and three tennis balls are sold for $38.00, what is the cost of a single tennis ball?*

Holly explained how she used a factorisation method when both common fractions were unlike and showed an example to Amelia, who through-out the prior lessons had demonstrated an increased awareness and recognition of the mathematical relationships within the problems. The two students then set about solving the problem (Figure 3) and referring to the two netballs and three tennis balls Amelia explained to Holly that the relationship would be maintained if the balls were ―increased by a factor of three".

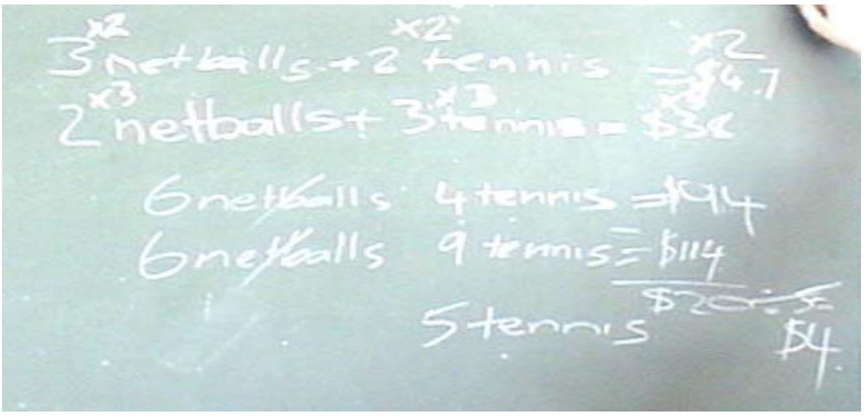

Figure 3. Holly and Amelia's explanation to the class.

In the following lesson and building on from Sarah's, Liam's and Holly's explanation, Dougal's group (Figures 4 and 5) developed a solution using counters and two calculators, whereas Emma's group (Figure 6) showed their thinking using a very detailed diagram for the following problem:

> *At an art store, brushes have one price and pencils have another. Eight brushes and three pens cost $7.10. But six brushes and three pens cost $5.70. How much does one pen cost?*

At this point in time both Dougal and Emma's groups did not understand the factorisation process outlined by Holly and Amelia. However, both had developed an understanding of how to subtract like terms in order to isolate one of the variables. In analysing their interpretation of the problems, the use of the digits 1 and 2 by Liam, Dougal's counters and Emma's diagram of the brushes and pens replaces the conventions associated with using x and y to represent the two variables yet the thinking and to a certain degree the mathematics mirrors a more formal symbolic representation. In developing algebraic thinking these students were capable of developing their own solutions. It must be emphasised that the students' symbolism was not forced upon them, but reflected their own thinking. While it is tempting to move as soon as possible to a formal, symbolic approach as the basis of school algebra, this move may lessen the significance and power of algebra to many learners. The opportunity to be grasped is one that develops a general way of solving problems that allows students the freedom to internalise their thinking and builds an understanding of this symbolism.

Figure 4. Dougal using two calculator and counters to complete the problem.

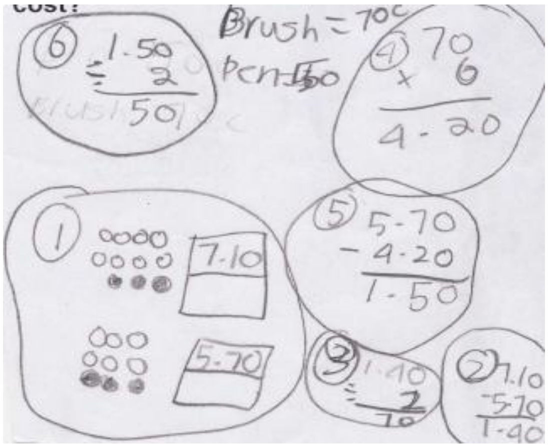

Figure 5. Dougal's written explanation of the same problem.

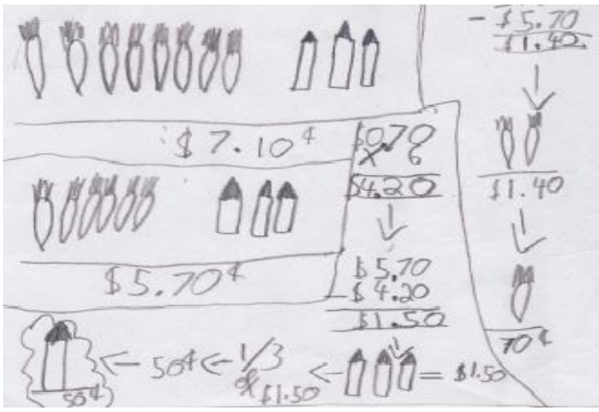

Figure 6. Emma's diagram to solve the problem.

Discussion and conclusions

The current outcomes of this research project indicate that a problem solving approach that develops algebraic thinking and provides students with the foundations in which to reason algebraically. The foundations of the approach are based on facilitating and encouraging students to represent and solve structurally related problems in a variety of ways and giving them opportunities to articulate and generalise their solutions. As a student's generalised and relational thinking develops their initial verbal descriptions give way to more mathematically based explanations, preparing them for the more concise, symbolic arguments that will eventually develop into the formal algebra used in further mathematics. In particular, students can be helped to construct algebraic notation in a meaningful way through their representations using materials, diagrams, models, tables and graphs in their search for patterns and generalisations. This approach empowers a way of thinking about mathematics that can offer students a more meaningful conceptualisation of algebra. By developing algebraic thinking using a problem solving approach, students develop a way of thinking that builds from their own mathematical understanding and provides an entry point into more sophisticated mathematics.

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ADAPTING ASSESSMENT INSTRUMENTS FOR AN ALASKAN CONTEXT

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The latest curriculum development effort of the Math in a Cultural Context, a long-term Alaskan project, includes Indigenous knowledge (IK). Collaborating with Yup'ik elders, MCC has identified a powerful set of mathematical processes used in constructing everyday artefacts. The knowledge of elders provides a unique way to teach Rational Number Reasoning. Measuring the efficacy of curriculum developed from IK requires a reliable and valid assessment instrument, which captures the mathematical content and learning trajectory established by Indigenous knowledge. An appropriate assessment instrument was unavailable; hence adapting questions from other instruments was undertaken. This paper describes the process of adapting an Australian fraction assessment for use in this Alaskan context.

Context

The underperformance of schools serving American Indian (AI) and Alaskan Native (AN) students and communities has been one of the most vexing and enduring issues in education. Federal reports have, for almost a century, advocated approaches that recommend educational programs connect school and community (Meriam, Brown, Cloud, & Dale, 1928; Executive Order No. 13,336, 2004), as a way to redress the continuing lower academic performance of AI/AN students, particularly in the mathematics domain. To address this problem, the Math in a Cultural Context (MCC) project has developed a long-term curriculum and professional development project in collaboration with Yup'ik Eskimo elders from southwest Alaska. The project has expanded to both urban and rural school districts and has been implemented across Alaska's diverse geographical and cultural regions: Athabaskan, Inupiaq, Tlingit and Yup'ik. The current project takes place within five diverse Alaskan school districts.

MCC curriculum development is underpinned by the use of everyday Indigenous activities that are mathematically rich, with the potential to engage students and improve their understanding. Subsistence activities such as gathering berries and constructing a fish rack have become the foundation for a supplementary elementary curriculum and associated professional development materials. Positive impacts on AN students' mathematics performance have been observed when using MCC's supplemental curriculum (Lipka & Adams, 2004; Lipka, Yanez, Andrew-Ihrke, & Adam, 2009; Sternberg, Lipka, Newman, Wildfeuer, & Grigorenko, 2006).

Repeatedly, elders have demonstrated how they use body proportional measuring and symmetry/splitting in tailoring clothing, constructing buildings, and star navigating. MCC's approach is to work with Yup'ik Eskimo elders and Yup'ik teachers, mathematicians and math educators, educators and Alaskan school districts with an aim of integrating Yup'ik and Western knowledge for the purpose of improving students' mathematics knowledge and performance (Lipka, et al., 2009). As this two-decade-old project has matured, we have increasingly recognised the mathematically laden ways that Yup'ik elders use their knowledge to solve everyday problems. Our most recent mathematics curriculum development and learning trajectory begins from Indigenous knowledge (IK) and the Indigenous worldview. Constituting mathematics curriculum from IK, that is both an authentic representation of Yup'ik cultural practice and school mathematics is a turnaround from the not so distant colonial past (Lipka & Andrew-Ihrke, 2009).

MCC is currently developing Rational Number Reasoning (RNR) and geometry curriculum materials that intertwine Yup'ik constructions with fractions, ratios, and proportional reasoning. Historically, elders did not and could not rely on exogenous tools to construct items so they employed body symmetry and body-part relationships as a precise form for measuring proportionally so their end-products (e.g., clothing, boots and kayaks) fit the user. Central to both Yup'ik everyday practices and the development of a RNR and geometry curriculum for elementary school students lies a set of generative concepts gleamed from elders' practice. The dynamic way in which body proportional measuring and symmetry interact presents an integrated perspective on teaching measuring, geometry, patterns, numbers, and early algebraic thinking.

An important and common Yup'ik measure is the "knuckle", which forms the basis for constructing a square, which can be transformed into geometrically pleasing patterns that adorn squirrel parkas or become the basis of circles used for ceremonial headdresses, as shown in Figure 1. In both cases, the knuckle measure is ½ the length of the constructed square and ½ the length of the diameter of a circle, thus establishing a 2:1 or 1:2 relationship. The square then becomes the base from which a circle is made—both are shown in Figure 2. Other Yup'ik body proportional measures are also used for constructing a variety of projects. For example, a kayak measure is approximately 3:1-Yagneq (arm span) to the length of a kayak.

(a) Pattern on a squirrel parka

(b) Ceremonial headdress

Figure 1. Yup'ik artefacts.

(a) The knuckle measure *(b) Square to make a circle template*

Figure 2. The knuckle measure and square to make a circle.

Like existing MCC modules, the RNR curriculum incorporates tasks to engage students in creating their own representational models. Using the knuckle measure, students' first construction is creating a square from uneven material. Through the process, students and teachers observe how symmetrical splits create congruent areas (see Figure 3), learn basic Euclidean geometry—2 points create a line, parallel and perpendicular lines. Rather than verifying the square using an Euclidean proof that the four sides are equal length and all angles are right angles; from a Yup'ik perspective, the square is verified using transformational geometry, "It is about what you do to the shape that stays the same … that is a reflection … the two sides of the mirror—the image and the original match" (Lipka & Andrew-Ihrke, 2009, p. 9). Students learn that one-fold creates ½, a second or recursive fold creates ¼, and a third-fold creates ⅛, which forms a foundation for multiplicative thinking.

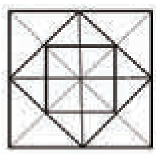


Figure 3. Constructing and folding a square demonstrates multiplicative thinking and geometry.

RNR curriculum and accompanying professional development materials are being developed for grades 2–6. It is expected that students taught using the new materials will improve their conceptual understanding for the targeted mathematical content. Thus an assessment instrument coupled with the appropriate statistical analysis must answer the research question: "To what extent do the new materials support students' mathematical understanding of fractions, ratios, and proportional thinking, overall and in each grade?" Hence this paper describes the adaptation of an Australian fraction instrument to create valid and reliable instruments for use by MCC.

Assessment of efficacy

The RNR project will adopt a similar research design to previous MCC curriculum development projects. Those projects employed a quasi-experimental pre/post-test design, in which intact classes were assigned as a control group or an experimental group (Lipka & Adams, 2004; Lipka, et al., 2009). All students were tested prior to the commencement of the teaching of the unit and at its completion. The experimental group were taught using MCC's supplemental curriculum, while the control group used their usual curriculum materials, typically the district adopted mathematics text.

On previous occasions MCC have used or adapted assessment instruments available from other research projects. When no suitable instrument met their needs, pre-test and post-test instruments were constructed by selecting appropriate questions from the National Assessment of Educational Progress (NAEP) and *Trends in*

International Mathematics and Science Study (TIMSS). Items were also created to reflect the major mathematical components of the module. Instruments were also piloted to assess and compare the difficulty between pre-test and post-test instruments, and determine their reliability (Lipka & Adams, 2004).

Assessment of fraction understanding instrument

The *Assessment of Fraction Understanding* (Wong, 2009) was identified by the MCC team as an instrument that could be used or adapted for the RNR project. The instrument was intended for use to establish students' level of knowledge and understanding of fraction equivalence. A learning pathway developed from empirical evidence enabled the three aspects of learning to be identified for students: (a) knowledge that has been mastered; (b) likely misconceptions that will be exhibited; and (c) knowledge required to further conceptual understanding (Wong, 2009, 2010).

The *Assessment of Fraction Understanding* (AFU) comprised two forms, one for one for grades 3 and 4, and another for grades 5 and 6. *Form A* comprised 25 constructed-response items, while *Form B* comprised 25 constructed-response items. Eighteen items were common across the two forms, which enabled students to be compared across grades without the need for all students to be administered all items (Wright & Stone, 1979). Items incorporated area models (i.e., circular, rectangular and square), number-line models, unit recognition, partitioning, equivalence and fraction language. A full description of the instrument, its development, and its testing is found in Wong (2009).

Assessment Adaptation

Modification of the *AFU* to meet MCC specifications required the addition of ratio and proportional reasoning items, and parallel forms for grades 2 to 6. MCC's long-standing partnership with Yup'ik elders and teachers and the development of assessment instruments in previous projects, provided a process from which instrument development/modification was undertaken. The process used to adapt the *AFU* comprised six main steps:

1. Yup'ik elders demonstrate the cultural activity to be incorporated in the RNR curriculum, aligning the instrument to indigenous knowledge.
2. Explore the mathematics embodied within the cultural activity.
3. Present research on student learning of fractions, proportional reasoning and ratios.
4. Develop an item bank.
5. Develop/modify assessment instrument.
6. Pilot assessment instrument.

The first four steps of the process were undertaken at a weekend Teacher Leadership Workshop conducted by MCC, with Yup'ik elders, teachers and educators. Following introductions, Dora Andrew-Ihrke, a long-term MCC adjunct faculty and Yup'ik cultural expert, described and demonstrated, in English, cultural activities considered suitable for the RNR curriculum. Evelyn Yanez, also a long-term Yup'ik consultant to MCC, interjected occasionally with relevant Yup'ik words, explanations, and how Yup'ik stories can support RNR. They described how they visualise the process of tailoring, creating patterns, and showed appropriate cultural artefacts. Both Dora and Evelyn responded to questions generated from their demonstrations. They then guided the workshop group in completing a number of activities, which enabled the participants to become familiar with mathematically embedded processes of body proportional measuring and splitting/symmetry.

The second step of the process was to identify and clarify the mathematics embodied in the cultural activity and re-contextualise the knowledge of elders to fit modern schooling (Lipka & Andrew-Ihrke, 2009). Discussion of the mathematics, such as constructing a square, how it could be incorporated in the classroom was undertaken. Teachers also explored how they could use the approach to develop fraction sets based on body proportional measuring and symmetry/splitting.

The next stage of the process was to present to the attendees, the learning trajectory or pathway identified by Confrey, Maloney, Nguyen, Mojica, and Myers (2009) for developing rational number understanding, and the pathway of learning linked to the *AFU* (Wong, 2009; Wong, 2010). Also, discussed were how learning pathways can inform teaching and learning fractions, ratios and proportional reasoning, and how the pathway would be recalibrated for indigenous knowledge.

Prior to creating items suitable for inclusion in the MCC assessment, a discussion of assessment design considerations was conducted; bias, common errors, types of problems (e.g., symbolic, pictorial, routine/non-routine, procedural, conceptual), item difficulty, clarity of instruction, and duration of assessment were discussed. Teachers and educators then worked in grade level groups to examine the applicability of *AFU* assessment items from the item bank and create items suitable for their grade, to assess the mathematical thinking embedded within the cultural activities demonstrated. Items created were catalogued and added to the item bank.

After the weekend workshop, pencil and paper assessments for grade 2, grade 3–4 and grade 5–6 were created and emailed to the teachers for review. From the comments received, the assessments were revised and two versions, A and B created. Both versions comprised the same number of items, however for three grade 2 items, five grade 3–4 items and one grade 5–6 item, however one version incorporated a diagram that was absent from the other. For example, item 6 from grade 3–4 version B is shown in Figure 4; the diagram was omitted in the version A.

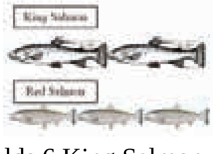
Grade 3- 4 (version B)	Grade 5- 6 (version A)
6. For every 2 King Salmon there are 3 Reds.  If a fish rack holds 6 King Salmon, how many Red Salmon would the fish rack hold?	9. Mark and John have identical candy bars. Mark ate $\frac{4}{5}$ his candy bar and John ate $\frac{2}{3}$ his candy bar. Who ate more? **Grade 5-6 (version B)** 9. Mark and John have identical candy bars. Mark ate $\frac{1}{4}$ his candy bar and John ate $\frac{1}{5}$ his candy bar. Who ate more?

Figure 4. Sample items from the grade 3- 4 and grade 5- 6 assessments.

For the grade 5–6 assessment, the fraction quantity in four items varied across versions. An example is shown in Figure 4. Common items, such as the king salmon item, were used to link the instruments across grades and versions. Items were also retained from the *AFU* unaltered to enable comparison of learning pathways from different samples.

Pilot testing of the MCC assessments

The development of the assessment and its piloting was presented to teachers as a recursive process which continued until the assessments were tested and verified statistically to be both reliable and valid. Therefore, teachers and educators who attended the weekend workshop agreed to pilot the new assessments at their schools. Three iterations of piloting and modification of the instrument were undertaken and discussed as follows.

First Iteration: Versions A and B

The first round of pilot testing was conducted by a MCC staff member and the first author at the Alaska Native Cultural School, a school with a majority of AN students, in Anchorage, Alaska's largest city. The assessments were administered grade 2 to 6 students, with the number of participants and the version they completed shown in Table 1.

Table 1. Sample of students tested by version and grade.

Grade	Version A & B	Version C	Version D
2	18	26	92
3	20	20	89
4	17	12	70
5	24	16	99
6	23	18 (version A)	122

During test administration, students were asked to work alone; they could ask for clarification of questions and were offered paper for folding. Grade 2 students were administered each assessment item verbally, with the administrator reading each question aloud to the whole class. An overhead projector was also used to guide students through the assessment and to ensure answers were written in the correct location. The time taken for students to complete the assessment, their composure and actions were observed during assessment administration for all grades.

It was observed by both test administrators that students in grades 2, 3, 4, and 5 had difficulties completing their assessment. Hence, marking the completed assessments at the point of data collection was undertaken. Rather than determine a total score for each completed assessment, responses for individual questions were examined, along with a comparison of responses for items with pictures and no-picture, and grade 5–6 items with different fraction quantities. This type of review also provided an indication of item difficulty (Bond & Fox, 2001).

Responses to the king salmon question are listed in Table 2, stratified by picture/no-picture, grade and response. The correct answer 9 appears in bold type. The number of responses for answers 3, 6, and 18 are included, along with an "other" category, which includes whole number answers not listed, answers with fractions, and "non-attempts". Of the grade 5 students who completed the picture question, 27%

(*n* = *26*) answered it correctly, compared to 50% (*n* = 8) who completed the non-picture version. Of the grade 6 students who completed the picture question, 36% (*n* = *28*) answered it correctly, compared to 60% (*n* = 5) who completed the non-picture version. Determining whether pictures provided an advantage was not possible due to the small sample sizes. Review of all items within all assessments confirmed the assessments were too difficult for grades 2 to 5.

Table 2. Responses to the King Salmon Question

For every 2 King Salmon there are 3 Reds. *If a fish rack holds 6 King Salmon, how many Red Salmon would the fish rack hold?*											
Picture						No-picture					
	Responses						Responses				
Grade	3	6	**9**	18	other	Grade	3	6	**9**	18	other
2 (n =9)	2	1	**0**	1	4	2 (n =10)	0	5	**1**	0	4
3 (n = 8)	0	0	**1**	1	6	3 (n = 12)	0	1	**2**	1	8
4 (n = 6)	0	0	**3**	0	3	4 (n = 11)	0	1	**3**	1	6
5 (n =26)	1	1	**7**	1	6	5 (n = 8)	1	1	**4**	1	1
6 (n =28)	0	0	**10**	3	5	5 (n = 5)	0	1	**3**	1	0

Discussions of the difficulty of the assessments were undertaken with the classroom teachers and MCC principal investigator, and it was decided that adjustments to the assessments were needed prior to visiting the second school the following day. From the results of the review and observations during assessment administration, no adjustments were made for grade 6. Major revisions as listed, were undertaken resulting in the creation of version C:

- Grade 2 – Reduce the number of items and incorporate items with diagrams.
- Grade 3-4 – Use the grade 2 versions as the basis for creating a new instrument and add some difficult items.
- Grade 5 – Use the grade 3-4 versions for creating a grade 5 instrument.
- Ensure adequate link items across all grades.

Some items were also reworded or reorganised. For example, the fish in the king salmon item (see Figure 4) were repositioned vertically as they would appear in real life. An item aimed at addressing the paper folding process was also reviewed for clarity of instruction, as the pictorial representation of the process was ambiguous. With the assistance of classroom teachers, a number of attempts at rewording the item highlighted the difficulty in creating pencil and paper items which reflect the underlying mathematical concepts revealed by Dora's cultural activities. Hence a companion performance-based, hands-on assessment (one-to-one interview) was created for administration to a subset of students who also completed the pencil and paper assessment.

2nd Iteration: Version C

The second round of testing was conducted at Dillingham City School, a rural school with a majority of AN students, within the Bristol Bay region. Version C was administered to students from grades 2 to 5, while versions A and B of the grade 5-6 assessment, were administered to grade 6. The number of students tested is shown in

Table 1. The process of reviewing the response to individual items as undertaken in the first iteration was also undertaken for all grades, which confirmed the assessments comprised items with a range of difficulties. No items had either a 100% or zero percent success rate.

One-to-one interviews with the first author were conducted with four of the students: (a) to ensure questions were interpreted as intended; (b) to gauge the difficulty of the items and assessment overall, (c) to identify the mathematical thinking exposed by the question; and (d) to uncover likely strategies to be employed. Those students did not undertake the assessment with their class. It was found that item wording did not pose a problem to answering the items and different strategies were employed by students to answer items.

Iteration 3: Version D

Final changes to the assessments were undertaken to ensure consistent representation of items across and within forms. For example some fractions were in-text (e.g., 1/4, 5/8), while others were in vertical format (e.g., $\frac{1}{4}$); all were changed to the vertical format. Both authors administered Version D at five elementary schools and one middle school in Juneau, the capital of Alaska. Not all grades or classes within grades were tested at each school.

The analysis of grade 3-4 data was undertaken first as this instrument contained the greatest number of common items between the grade 2, grade 5 and grade 6 instruments. The assessment was shown to be reliable using Cronbach’s alpha = .88, $n = 159$, calculated using SPSS v16. Using RUMM2020, all but three of the items in assessment fit the dichotomous Rasch model (RUMM Laboratory, 2004a). These items will be reviewed to determine any necessary changes.

Data coding and preliminary analysis for grade 6 assessments is underway. Although Cronbach’s alpha = .90, $n = 122$, review of the instrument and responses to items showed that further rewording of items is necessary. Two items were found to violate the assumption of local independence during Rasch modelling, hence were omitted from a second analysis. Further results showed that four items violated the assumptions of item fit and three polytomous items exhibited disordered thresholds (RUMM Laboratory, 2004b); these items require further analysis and review with changes to the grade 6 assessment expected.

Once analysis of all grade level assessments is complete, the data will be aggregated and Rasch modelling conducted on the entire data set. This will provide a preliminary learning trajectory commensurate with learning fractions, ratios, and proportional reasoning from indigenous knowledge.

Conclusion

The RNR pencil and paper assessments were designed to reflect IK knowledge. To do so, it was imperative that the assessment developers and teachers understood the cultural activities and mathematics embedded within those activities. One difficulty encountered in developing a culturally valid instrument was preserving the cultural knowledge in an authentic form. Three iterations of development and testing were undertaken. Preliminary analysis shows that further item development is needed to improve instrument reliability and validity. The adaptation of the *AFU* instrument to another cultural context presented enormous challenges. However, the possibility of establishing an assessment instrument that reflects IK and calibrates a learning trajectory that follows the cultural activities and learning process gleamed from Yup'ik elders' knowledge represents " a first." The refinement process is expected to continue during the next few years.

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MATHEMATICS AND GIFTEDNESS: INSIGHTS FROM EDUCATIONAL NEUROSCIENCE

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Exceptional performance, or giftedness, in mathematics is complicated by the variety of conceptual approaches to studies of giftedness as well as the broad and diverse nature of mathematics as taught in modern educational institutions. This paper outlines approaches to giftedness in mathematics that are based in studies of cognition within the discipline of educational neuroscience, approaches that conceptualise giftedness within a context that is sensitive to modern biology and, at the same time, inclusive of modern research in the social and behavioural sciences. Based on such approaches, exceptional performance in mathematics is discussed in relation to cognition and performance as a product of internal processing and environmental connectivity of the human organism. Such approaches have facilitated the development of an overarching framework for learning and memory that may enable a view, within the constraints of empirical science, of educational concepts related to exceptional performance. This framework may provide useful insights into the identification and education of students who may be gifted in mathematics.

Introduction

Exceptional performance, or giftedness, in mathematics appears to be a topic of great interest to researchers and teachers worldwide and there appears to be no lack of studies of the mathematically gifted. There appears to be, however, little in the way of common ground between many such studies, or studies of giftedness and cognition more generally, with the differing approaches used seemingly based in concepts and assumptions that appear to bear little relation to each other. There appears to be also no overall conceptual framework within which to compare such studies (e.g., Samuels, 2009) and, perhaps as a result, no overarching conceptualisation of giftedness as an aspect of cognition and behaviour (e.g., Kaufman & Sternberg, 2008). Studies of giftedness in mathematics appear, additionally, to lack cohesion due to the broad and diverse nature of the subject of mathematics as taught in modern educational institutions (e.g. Davis, 2003; Organization for Economic Cooperation and Development (OECD), 2003). There appears to be also a lack of cohesion apparent in disagreement about empirical, or even descriptive, comparisons of performance across cohorts in the many categories of the subject of mathematics. This complex situation is given an added dimension of arguments about whether educational institutions can function effectively in the educational development of the gifted (e.g., Diezmann & Watters, 2002; Ericsson, Nandagopal & Roring, 2009; Freeman, 2006) and by the view that studies of gifted performance in mathematics may be directed only at the aspects of mathematics that are determined as valuable in a particular society, depending on who is making such determinations and on their rationale for any such determination (Hertzog, 2009; Kaufman & Sternberg, 2008).

There have been, however, attempts to investigate overarching conceptualisations of cognition, and to investigate gifted performance within such conceptualisations. Modern educational neuroscience, for example, has attempted to incorporate an evolutionary perspective into studies of human cognition in order to place such studies in a context of human interaction with environment, a context that includes social interaction and other aspects of behaviour (e.g., Cotterill, 2001; Edelman, 1987; Margoliash & Nusbaum, 2009). Some such research has merged concepts derived from evolutionary biology and studies of cognitive function with concepts derived from education and the information sciences (e.g., Buss, 1999; Geary, 2005; Sweller, 2007, 2010) and some research has, in turn, merged such concepts with those related to connectivity of processes and pathways in organismal and non-organismal structures and systems (e.g., Barabasi, 2002; Buchanan, 2002; Sporns, Tononi & Kötter, 2005).

The results from such interdisciplinary and combination studies have been used to erect a broader, more flexible framework that describes learning and memory processes in terms of information processing systems more generally (e.g., Woolcott, 2009a, 2009b, 2010a, 2010b). Mathematics education, within this framework, can be viewed in many ways as similar to education in any subject category at any level of a broad spectrum of performance. This framework suggests further that, in treating a human individual as a information processing system, there may be differing, but sometimes overlapping, component information systems that may process information in different ways and over different time frames, but which may contribute to an assessable performance in any culturally-valued subject, not just mathematics. In considering exceptional performance in mathematics, therefore, it may be useful to consider aspects of an individual's performance that give an individual a degree of expertise, both within and across a number of culturally-valued knowledge domains.

Mathematics, performance, and educational neuroscience

In the modern age, mathematics learning is an important part of the societal accumulation of culture (knowledge and skills) and this learning is assessed, as is all learning, through observation of performances based in muscular contractions that indicate any resultant memory storage (Cotterill, 2001; Llinás, 2001). The types of performance vary from simple eye blinks to complex sequences of movement seen in sports performances, and include talking, reading and writing. Learning and memory processes and their relationship to performances in motor tasks have been the subject of considerable recent research both in the natural sciences and the social and behavioural sciences, and some of this research has been directed at examining individuals who demonstrate above-normal performances that are valued in particular societies. This includes performances that exceed the normal in pen and paper tests, such as in the Mathematics Olympiads, but also those performances that demonstrate other types of above-normal expertise, such as seen on the concert platform, in the chess arena, and on the sporting field; at various levels from local and national through to international (e.g., Ericsson, 2005; Zhu, 2007). Such assessments of expertise may be largely norm-referenced with standardised intelligence tests, competitions, or other types of performance assessments conducted with this in mind (e.g., Vialle & Rogers, 2009). Some such assessments may be used to grade individuals for various reasons, for example, in order to assign monetary or other incentive awards in competitions. Although such performance assessments are not always used in any directly formative way, they may be used to indicate progress towards a goal of increased expertise or expert knowledge—for example, through guided practice (Ericsson et al., 2009). In institutional education, such assessments may serve as a guide to the quality and content of education that is provided to some students within subjects or within year groups and, recently at least, have been used to determine the allocation of resources, including an improved teacher to student ratio, to individual students or groups of students identified as gifted, and this includes those students gifted in mathematics (Moon, 2007; Vialle & Rogers, 2009).

As well as research into examining comparative performance, there has been also research into the determination of potential future performance, with support obtained for the effectiveness of some such determinations—for example, in assessments used to assess potential ability in mathematics and to assist in development of training regimes (e.g., O'Boyle, 2005). Although results from some assessments used to determine potential academic ability, such as intelligence quotient (IQ), spatial intelligence, or crystallised intelligence assessments have been correlated with academic performance in mathematics, there are limitations in applying such results to programs designed to increase expertise (e.g., Haier, 2009). Haier and associates (see, for example, Colom et al., 2009; Haier, 2009; Haier & Jung, 2008) have, however, developed a neural model, the parieto-frontal integration theory (P-FIT) that correlates the amount of grey matter (neuronal cell bodies) activated across a number of different brain regions with test scores from several such assessments, and this model may be useful in determining general intelligence, at least, based on the brain's measurable characteristics. There may be, however, many other factors that may play a role in both performance and ability (Samuels, 2009), with quick processing time—which is linked to white matter (neuronal connections)—also likely to play a key role in any assessment of potential intelligence (e.g., Haier, 2009).

Giftedness, including giftedness in mathematics, has been related to gender and age differences (e.g., Haier, Jung, Yeo, Head & Alkire, 2005; Halpern et al., 2007; Shaw et al., 2006) and exceptional performance in mathematics, specifically, has been linked with hemispheric bias and interhemispheric connectivity (O'Boyle, 2005) as well as developmental variation *in utero* (Baron-Cohen, 2003) in human males. It has been difficult, however, to relate giftedness to specific genetic attributes and Plomin and associates (e.g., Davis et al., 2007) have suggested that this is because the genes that contribute to superior learning and memory and related performances, may be generalist genes that contribute to development of many parts of the human organism. Further, modern research in learning and memory has also indicated that some types of giftedness may not be subject-specific, being related to general attributes of a human cortical advantage, such as a superior ability to generalise, superior attentional or working memory processes, or superior ability in problem solving. Some researchers, for example, have related superior working memory and attention to high scores in assessments of the general factor of intelligence (*g* factor) or fluid intelligence (Colom et al., 2007). Such superior functionality has been considered a neuropsychological characteristic of gifted people (Geake, 2009a). Although executive function, including working memory (short-term memory) and related inhibitory processes, has been implicated specifically in mathematics performances (e.g., Bull, 2008), this may be largely because such processes relate to generalised skills that are concerned with the utilisation of strategies. Such neuronal processes appear to be related also to creativity, adding support to the suggested relationships between intelligence, giftedness, and creativity (e.g., Cotterill, 2001; Geake, 2009a; Jung et al., 2009).

Some recent studies have attempted to describe fully the neuronally-based pattern analysis carried out during mathematics by comparing brain function in individuals with savant syndrome, including individuals with autism spectrum disorder, and neurotypical individuals, where both are considered as gifted in mathematics (e.g. Casanova & Trippe, 2009; Treffert, 2009). Some such studies (e.g., Happé & Vital, 2009; Mottron, Dawson & Soulières, 2009) have indicated that the detection, integration and completion of patterns, and the requisite grouping processes, function in the negotiation of the phenomenological world, a tacit support for the arguments that any study of human cognition must be sensitive to the consideration of evolutionary processes (e.g., Calvin, 2004; Dehaene, 2004, 2009). In association with this pattern analysis is the ability to produce new material within the constraints of the integrated structure, a process which Mottron et al. (2009) refer to as creativity. In gifted individuals who are neurotypical, this integrated structure may be determined by automatic hierarchies that govern generalisation and memory processing through information loss and the limitation of the role of perception. Grandin (2009), a noted researcher who has autism and savant syndrome, has argued that the orientation towards pattern analysis that may be recognised as mathematics, as well as resulting from environmental interaction, may be due to differences in connectivity within individuals.

A better understanding of pattern analysis as a component of mathematics is, obviously, an important issue in understanding exceptional performance in mathematics. Snyder and associates (e.g., Snyder & Mitchell, 1999) have suggested, however, that the algebraic and algorithmic patterns and processes taught in mathematics may not correspond to the patterns and processes that they are designed to activate, and this is supported by Baars (1995) in proposing that humans use heuristic processes and analogies, rather than algorithmic processes, in dealing with patterns of environmental input. Although several capacities have been described for the brain—for example, problem-solving, decision-making and action control—Baars considers that one of the strengths of the brain, and the entire nervous system, may be in remembering and cross-analysing patterns observed from the real world, which is arguably an intrinsic mathematics capacity.

A flexible framework for cognition and giftedness

Although there is little in the way of consensus on how to accommodate information from differing studies of giftedness in mathematics, and giftedness in general, some of the parallels drawn between concepts within modern educational neuroscience and other disciplines have been used to erect a broader, more flexible framework within which to examine giftedness specifically and cognition more generally, (e.g., Woolcott, 2009a, 2009b, 2010a, 2010b). This flexible framework describes learning and memory processes in a broad sense in terms of information processing systems, and this is similar to the descriptions of human cognition and evolution in terms of natural information processing systems that have been used in some educational studies, such as those concerned with cognitive load theory (e.g., Sweller, 2007, 2010). This framework was developed from a consideration that learning and memory can be said to involve three temporally connected, but separable, stages in information flow:

1. environmental information input to or output from an individual;
2. processing of resultant information changes within the individual (information processing); and
3. changes in the observed state of the individual resulting from any such information processing.

In this flexible framework, the concepts of learning and memory have been generalised across both organismal and non-organismal structures, and all matter and energy described as information. All discrete organisations of matter and energy within the universe (in the sense of Gribbin, 1994) are described as information processing systems, with changes in information within such discrete organisations described as processing (Woolcott, 2010b, 2011). Learning and memory are described in terms of the overarching range of possibilities or potentialities of any change of matter and energy within such information processing systems where such change results from information input or output.

Within this framework, a human can be considered as a discrete matter and energy entity and human connectivity can be considered in terms of interactions with environment of the human information processing system and, as well, any designated structure within the human system can be considered also as a similarly discrete entity. On this basis human learning and memory can be described as a function of human connectivity with environment, as well as a function of connectivity within the central nervous system and, in particular, of neuronal connectivity within the brain. This framework supports the consideration separately of the differing aspects of human cognition within a dynamic system, and allows also a formalisation of the partitioning of cognitive structures, which is, in practice, a common method in dealing with learning and memory in cognitive psychology and the natural sciences (Woolcott, 2010b, 2011). For example, such dynamism operates, not only during storage of discrete information in long-term memory, but also in spatiotemporal sequencing of memories (Calvin, 2004; Postle, 2006) and in the linkage of emotions and chemical reward with learning and memory (Damasio, 1994; Le Doux, 2000; Panksepp, 1998). Neuronal patterns that develop with such intrinsic and dedicated flexibility act to adapt each human to a range of environmental inputs, including input classified as mathematics.

Since this framework supports explanations of cognition couched in terms of the interaction of component systems within the human organism, it supports the view that learned concepts are not necessarily uniquely subject-dependent. It is well known, that, even though some regions of brain activation may correspond to concepts described as, say, mathematics or reading, many common brain regions may be activated during processing of information in any subject (Dehaene, 2004; Geake, 2009a, 2009b). Lakoff and others (e.g., Lakoff & Núñez, 2000) have referred to such commonality of learning processes in terms of conceptual metaphors, as well as cross-domain mappings that preserve inferential structure and which are essential for linking conceptualisations generally, but which serve also for the linking of concepts in subject categories.

In considering a human individual as a type of universal information processing system, there may be differing component systems that may process information in different ways and over different time frames, but which may contribute to an assessable human performance, even if these systems sometimes overlap. The consideration of the human cognitive system as separable components suggests that it may be more useful to consider only those aspects of an individual's performance that may be viewed as superior, where those aspects result from components of that individual as an information processing system, rather than to consider that a student who has a superior performance in any one aspect is gifted in other ways as well. In this way giftedness may be conceptualised as the degree of expertise that an individual may have obtained in a culturally-valued knowledge domain, or the potential expertise in such a domain for which the individual may have an assessed performance, so long as it is recognised also that various components of the student's cognitive and related systems may contribute differentially to that expressed expertise. The consideration of separable information processing components may be useful also in examining aspects of giftedness such as motivation and emotion (e.g., Cotterill, 2001; Geake, 2009b).

An additional advantage of a flexible framework that supports a view of separable cognitive systems is that such a framework accommodates the concept of giftedness as the acquisition of knowledge in specialised domains in individuals that may otherwise have differences in cognitive connectivity, such as may occur in higher functioning in individuals within the autism spectrum (e.g., Casanova & Trippe, 2009; Grandin, 2006, 2009). Differences in connectivity between component systems, such as seen in neuronal hyper-connectivity and hyper-plasticity, may lead to the development of expertise, or giftedness, or may result in lack of expertise depending on what is being assessed (e.g., Casanova, 2010; Markram, Rinaldi & Markram, 2006). The flexible framework also accommodates the differences in abilities as explained by Haier and associates in their P-FIT model (e.g., Haier & Jung, 2008; Colom et al., 2009; Haier, 2009), since each component of the cognitive system, as described in the P-FIT model, can be treated effectively as a separate system in describing information transfer, storage, and recall.

Conclusion

Identification of giftedness, and the development of expertise based on that identification, may benefit from a broad approach that views human performance in a framework of interacting information processing systems, some of which have components in the environment external to the human organism. The framework outlined here indicates that education may have the potential to develop, through selective teaching to the system at large, any interacting systems that give rise to particular performances or abilities that are considered culturally valuable, whether these lie within, across or outside of the subject of mathematics or which link mathematics with other subjects. It may be necessary to re-evaluate our cultural mathematisation to more fully incorporate knowledge of brain processing that acts naturally across subject areas, particularly as it relates to the high level of expertise that is an expected result of gifted education.

This framework appears to offer reconciliation also of some of the disparate approaches that have been taken in studies of giftedness (see, for example, Perleth & Wilde, 2009) since the framework allows some comparison of such differing approaches through consideration of parallels that may be present between differing analogies and assumptions (e.g., Woolcott, 2009b, 2010b, 2011). Comparison and evaluation of such differing approaches may be useful in elucidating learning and memory processes to be used in education and teaching, including teaching of the gifted (Woolcott, 2009a, , 2010b). For example, the consideration that problem solving is the main function of learning and memory in the human interaction with environment (e.g., Grillner, 2003; Tonegawa et al., 2004) may be central to any educational strategy and, therefore, an important aspect of giftedness in mathematics. Gifted education, as is the case with education more generally, therefore, should develop such problem-solving ability through learning, in order that each individual maximise the potential for such interaction and the subsequent growth of contextually-linked information connectivity in long-term memory (for example, see Edelman in Sylwester, 1995).

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THE BIG IDEAS IN TWO LARGE FIRST LEVEL COURSES OF UNDERGRADUATE MATHEMATICS

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What is important to teach students within the mathematics discipline? Identifying the fundamental concepts (or big ideas) of mathematics is being looked at in the development of the Australian National Mathematics curriculum. But what do lecturers at university consider to be the "big ideas" in the mathematics courses they teach? Seventeen lecturers were interviewed about thirteen mathematics courses to establish what they considered were the key areas of learning within these courses. This paper reports on the interviews conducted with lecturers from two large first year classes. Their responses indicate that teaching mathematics involves a lot more than mathematics alone.

Introduction

Currently Australian schools are preparing to adopt a National Curriculum for four subjects, including mathematics. The *Australian Curriculum—Mathematics* is intended to focus on the fundamental concepts in mathematics that should be taught. Establishing what the "big ideas" or mathematical concepts that students need to grasp in mathematics is also relevant in a university setting.

How well a student grasps a new concept can be described in terms of *Concept usage* which matters in mathematics, as students are not only required to understand mathematics concepts, but also to use them in processes that require other abilities such as the use of logic and critical understanding (Moore, 1994). *Concept image* is the term used to describe how a concept is understood and seen by the student (Tall & Vinner, 1981). The *concept image* is developed over many years and is influenced by the student's experience from within and beyond their education. The *concept image* develops either consciously or subconsciously with more experience. Thus the student may have a very different understanding of the concept from that held by their discipline. The discipline's understanding of the concept is described as the *concept definition*. For example, when it comes to developing formal mathematical proofs, understanding of *concept usage* becomes important (Moore, 1994). Without this understanding, the ability to use proof techniques diminishes. For any form of mathematics, *concept definition*, *concept image* and *concept usage* are all important factors.

Some concepts have a greater impact on a student's learning and can be described as threshold concepts. A threshold concept is a term that has emerged within the literature on higher education learning and teaching as a way of thinking about how students come to understand the key ideas of their discipline. The theory of threshold concepts was initially developed by Meyer and Land during a national research project in the UK in the economics discipline (Cousin, 2006). It was believed that understanding some crucial concepts, which they called threshold concepts, was essential to becoming an economist (Cousin, 2006). The theory of threshold concepts has now been taken up and developed by many other disciplines including mathematics.

Each mathematical concept can be described by the *concept definition,* applied according to the *concept usage* and seen by the student through their *concept image.* However, these concepts can be categorised into two levels of importance: core concepts that define important stages in learning and threshold concepts that, once grasped, will change the way the student thinks (Meyer & Land, 2003). Threshold concepts are most likely to be transformative, irreversible, integrative, bounded and troublesome (Meyer & Land, 2006).

Once a threshold concept is grasped, the student is said to gain new insight into what they are studying so that the material they are working on becomes clear and obvious. They are therefore changing from the *concept image* they possess to an understanding of the *concept definition* held by the discipline. In mathematics, complex numbers, limits, proofs and calculus have all been described as examples of threshold concepts (Easdown, 2007; Meyer & Land, 2003; Pettersson & Scheja, 2008).

The purpose of this study was to identify the "big ideas" that mathematics lecturers want their students to learn. For ease of communication, the term "areas of learning" was adopted in interviews to describe these "big ideas". These areas of learning will be presented and discussed in terms of the core versus threshold classification, as well as in terms of the students' progression through concept image to concept definition and concept usage. The areas of learning also include skills that cannot be classified as mathematical concepts.

Research design

The interviews discussed here are part of a larger integrated study being conducted at The University of Queensland that will use interviews, student surveys and analysis of course assessments to investigate how students' behaviour and attitude affect their ability to understand key concepts in mathematics.

The lecturers were sent the following list of questions before each interview:

1. Please list four to five main areas of learning in the course. These can be concepts, skills, or topics that you consider of key importance for the students to attain by the end of the course.
2. Are any of these key concepts similar to those in other courses?
3. Do you feel that students are made aware of the key concepts they are expected to understand?
4. What do you do when teaching the course to aid the students in gaining understanding of these concepts?
5. Does the assessment in the course encourage students to gain understanding of these key concepts?
6. Does the course assessment determine whether a student has understood these key concepts?

7. How do you think the students' grades reflect their understanding of the course material (and whether they have understood the key concepts)?

Initially, responses were recorded by taking hand-written notes that were sent back to the lecturer for verification. Later interviews were audio recorded, which allowed for greater accuracy and depth of information collected.

The majority of lecturers interviewed have had extensive teaching experience in a variety of universities and had taught the courses for several semesters.

Results

Interviews were conducted with 17 lecturers. Two courses were selected for analysis in this paper. The lecturer who taught the first course, *Mathematical Foundations*, identified areas of learning similar to those in other courses, but gave very different reasons for his choices. The second course, *Multivariate Calculus & Ordinary Differential Equations*, is typical of courses taught by two lecturers.

Mathematical foundations

Mathematical Foundations is a course that caters mainly for engineering students who have not completed advanced mathematics at secondary school. The pace of mathematics covered is quite fast for some students as the content usually covered in two years at school is taught in one semester (13 weeks) at university. In the first semester of each year enrolments are around 700 students.

The lecturer, Tom, identified the following four major areas of learning:

1. Limits;
2. Recognition of different number systems;
3. Proof by induction; and
4. Describing physical problems in the language of mathematics.

He pointed out that his intention was not to teach students detailed procedures in each area, unlike many other interviewees, but to introduce them to mathematical thinking; for example, he wanted students to understand how and why different number systems were introduced. This was to show them the relevance of the different types of numbers in the course, such as complex numbers.

Tom wanted students to be able to describe physical problems in the language of mathematics. He explained that this skill showed the usefulness of mathematics in solving "real life problems". It also brings together the areas of mathematics in the course. Showing students the relevance of what they are learning has been shown to encourage students to adopt a deep learning approach (Entwistle & Tait, 1990). These four areas of learning are all used extensively in later courses, where they are classed as assumed knowledge.

Tom felt that all the areas of learning were tested for understanding in the course assessment. He generally found that, although most students understood limits, their algebra skills were poor. In the final exam, which was regarded by Tom as quite difficult, number systems (in the form of complex numbers), proof by induction and applications were always tested. Tom said that a student would need to know well all the areas of learning to achieve a high distinction. However some students could pass the course by only understanding one of the areas of learning. A credit could be achieved if that area were applications. It is not possible to gather which key areas of learning a student had understood only by looking at their final grade.

Tom's descriptions show that the course was not just about learning concepts but more about using the concepts to teach the students how to think and act like mathematicians.

Multivariate calculus and ordinary differential equations

This is a large first level course that usually has around 1000 students enrolled in the second semester. As with the previous course, the majority are engineering students, but there are also some students majoring in mathematics. The course complements previous courses by introducing students to more advanced aspects of calculus. They solve a variety of problems involving functions of several variables, partial derivatives and parameterisation of curves and line intervals.

Due to the large number of students enrolled, there are often several lecturers teaching the course. The two lecturers interviewed, Bob and Alice, have each been teaching this course for several semesters. Bob's responses relate only to the part of the course that he taught, whereas Alice's responses relate to the entire course. Alice teaches the later part of the course with much of the material related to the early part.

Table 1. Areas of learning: the analysis of Bob's and Alice's responses.

Analysis: Areas of learning	**Bob's responses**	**Alice's responses**
Critical thinking	Critical thinking	
Applications	Applications: to motivate and show how calculus can be applied	Using ordinary differential equations in modelling applications.
Graphical interpretations of plane interception	Linear algebra: looking at how planes intercept	
Vector calculus		Basic understanding of line integrals and the use of parametric curves in their evaluation; parametric representation of curves in 2 and 3 dimensions.
Computational aids		The use of *Matlab* or other computational aids to assist in visualisation/understanding of concepts.
Understanding functions of more than one variable	Differentiation of functions of more than one variable Graphing functions of more than one variable by visualising graphs from functions, using geometric interpretation to make predictions	Linear and quadratic approximations to functions of more than one variable Using partial derivatives to analyse key features of functions of more than one variable. (e.g. tangent planes, max/min problems, Lagrange multipliers) Rates of change of functions of more than one variable, interpreting this in graphical terms
Ordinary differential equations		Solving and interpreting solutions of certain ordinary differential equations

Critical thinking, applications, and computational aids are the three areas of learning that relate to the entire course. Bob stated that one of the most important skills he wanted to teach students was to critically think. As he stated, “No one should graduate from a respectable university without being able to critically think”. Though critical thinking is not exclusive to mathematics, Bob felt that many students in his course lacked this skill, which in turn affected their ability to engage in learning mathematics. Although Bob saw critical thinking as a major issue that needed to be addressed with his students, it was not mentioned by Alice.

Alice often uses *Matlab* herself to demonstrate problems in lectures. She would therefore see and demonstrate the value of *Matlab* to her students. Bob did not see the necessary benefit of Matlab in the course. During the interview he mentioned that, though Matlab was meant to be an additional tool to aid understanding, he doubted that this was achieved.

Both lecturers valued applications but saw their use very differently. Bob saw applications as a means to motivate students, whereas Alice saw applications as learning tools that will teach students more about the course content.

The other responses from both lecturers relate to the course content. Both placed emphasis on understanding and interpreting mathematics. Alice seemed to place great importance on understanding course content, whereas Bob appeared more interested in developing students’ mathematical thinking. The students taking the course would find a very different emphasis placed on learning from one half of the course to the other.

All these areas of learning are developed and used in more advanced courses. Both lecturers emphasise the areas of learning when lecturing; however, Alice also mentioned that all hers are in the course profile.

To aid students to grasp the areas of learning, Bob encourages student involvement and discourages students from sitting in the back two rows. He said his students are well behaved and happy to ask lots of questions. To emphasise critical thinking, he often tells his students that they should “Think, think, and think again”.

The weekly assessment for the course contained challenging questions which Bob hoped extended students and encouraged understanding. However, he was not convinced that this was achieved as many students struggled with these questions. Alice was more confident than Bob that students did achieve understanding.

The final examination was considered by both lecturers to determine students’ understanding of the areas of learning they had identified. Bob explained that the final examination had some challenging questions different from those seen in lectures or assignments. He expected that a high distinction would indicate that a student had grasped all the areas of learning, but conceded it was possible that students could pass without the ability to critically reason and with only understanding about half of the areas of learning. He considered the students who failed to be the ones who struggled with most of the areas of learning and made basic mathematical errors. Although both lecturers were confident that student grades correlated with the level of knowledge acquired, they were unsure whether their students had grasped any particular important area of learning they had identified.

Discussion

Analysis of the full set of interviews indicated that lecturers not only have very different ideas of what is important for students to learn within their courses, but also different ways of justifying common choices. So two lecturers may identify areas of learning as important, but for different reasons. For example, proof by induction was identified as a key mathematical concept in two courses, the mathematical foundations course described above and the first level discrete mathematics course. The same area of learning has very different purposes in each course. In the first course, it was to show students that mathematics is established on well-founded reasoning, and in the discrete course it was so that students would understand that proof by induction is an alternate method of proof that is not necessarily intuitive, but extremely useful. Sometimes the importance of the concept is what the concept demonstrates rather than the *concept definition* or *concept usage*. The way lecturers teach mathematical induction would create very different *concept image*s by the students.

The courses included in this paper also place different emphases on core versus threshold concepts. For example, it was not surprising to find that in the mathematical foundations course, three of the areas of learning Tom described are considered threshold concepts (Easdown, 2007; Meyer & Land, 2003; Pettersson & Scheja, 2008). Many of areas of learning from both courses can be seen as core concepts, threshold concepts or possibly as processes with more importance placed on their *concept usage*. This interpretation will depend strongly on how they are assessed. For example, within the topic of understanding functions of more than one variable, a student can be assessed on the process, or on their understanding, or on the definition. Some areas of learning involve abstract ideals not necessarily related to a particular concept. These are more like graduate attributes such as critical reasoning or being able to convert real life problems into mathematics.

Even within courses taught by more than one lecturer there are differences in emphases. In the multivariate calculus course, Bob's goal is to raise students' mathematical thinking ability and not just for them to grasp the course content. He felt strongly that students' lack of critical thinking skills affected their engagement in mathematics. However, taking mathematics courses has not been shown to increase students' critical thinking skills (Terenzini, Springer, Pascarella, & Nora, 1995), even though critical thinking skills can be increased by student involvement in other courses and outside their formative study (Terenzini, et al., 1995). Alice has a different goal, since she seems to place more emphasis on students mastering essential content and skills. These differences raise interesting questions about the relative emphasis in the two halves of the course on developing *concept usage*, *concept images,* and *concept definitions*.

These lecturers and many others interviewed indicated that the assessment processes they used did not allow them to know with confidence whether students – apart from those achieving high distinctions – had grasped the areas of learning they considered most important. This finding suggests that it is not enough to identify the "big ideas" for inclusion in a mathematics course; assessment tasks must be capable of eliciting these ideas from students in a way that lecturers can recognise.

This study is currently being expanded to identify the "big ideas" of second and third level courses. This will establish a more detailed picture of how mathematics is taught, developed and connected for the undergraduate student majoring in mathematics, which in turn can inform teaching.

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META-RULES OF DISCURSIVE PRACTICE IN MATHEMATICS CLASSROOMS FROM SEOUL, SHANGHAI AND TOKYO

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This study extends a previous study on spoken mathematics (Clarke & Xu, 2008) and seeks to compare the discursive practices in classrooms from Seoul, Shanghai, and Tokyo, with a particular focus on meta-discursive rules (Sfard, 2001) that regulate exchanges between the teacher and students. The analysis centres on the events when the topic of linear equations was introduced. The similarities and differences of the three classrooms suggest that while the shared macrocultural values and beliefs frame the social activity of the classrooms in similar ways, the meta-discursive rules of classroom microculture determine the opportunities for student learning in mathematics.

Introduction

The benefits of engaging students in mathematics classroom dialogues have been highlighted in a number of recent publications (e.g. Alexander, 2008; Walshaw & Anthony, 2008) and the intensity and quality of classroom discourse have been the focus of many studies (e.g., Mercer, 1996). While there seems to be an universal assumption about the significance of student mathematical talk in learning mathematics, our studies on spoken mathematics in 22 well-taught classrooms internationally (Clarke & Xu, 2008; Clarke, Xu, & Wan, 2010) revealed significant differences among those classrooms characterised as “Asian” in the opportunities that each classroom afforded for the students to employ relatively sophisticated mathematical terms in both public discussion and private student interactions.

Extending the previous study, the study reported in this paper attempts to compare the discursive practice in classrooms from Shanghai, Seoul, and Tokyo and to examine the role played by culture in the constitution of that practice. In this study, I want to go beyond simply considering culture as a set of values and beliefs that are brought in by the participants or as external influences that are imposed on them, but to see culture as an integral part of how the work in the classroom was carried out and sustained. For the clarity of the paper, I define “culture” to be “any aspect of the ideas, communications, or behaviours of a group of people which give them a distinctive identity and which is used to organize their internal sense of cohesion and membership” (Scollon & Scollon, 1995, p. 127). In this paper, I distinguish microculture from macroculture. I use the word macroculture to refer to a set of ideas, communications, or behaviours embraced by the majority of people in a particular society (e.g. Chinese culture), whereas microculture defines regularities and patterns of interactions specific to mathematics classrooms, usually from the perspective of the researcher. The main purpose of this paper is to examine the microculture of mathematics classrooms with a particular focus on the meta-discursive rules that regulate patterns of classroom exchanges between the teacher and the students.

Meta-discursive rules in mathematics classrooms

Studies of mathematics classroom microculture have been focused on the “normative” aspects of teaching and learning. These include studies of patterns of social interactions and those rules and norms specific to a content area. For example, the work by Cobb, Yackel, and others studied classroom social norms, sociomathematical norms, and classroom mathematical practice (Yackel & Cobb, 1996). Examples of socio-mathematical norms include what counts as a mathematically different, efficient, sophisticated, or acceptable solution. Sfard (2001) addressed more general aspects of the normative aspects of classrooms, and coined the phrase “meta-discursive rules” to encompass those rules that regulate or govern discourse rather than those object-level rules concerning the relationships between mathematical objects. According to Sfard (2001), the meta-rules in mathematical discourse include those that underlie the uniquely mathematical ways of defining and proving; rules that regulate and guide interpersonal exchange and self-communication; the way symbolic tools should be used in the given type of communication, and so on. These meta-rules are the observer’s construct and mostly act “from behind the scenes”.

The importance of meta-rules of classroom discourse has been acknowledged by several studies. For example, van Oers (2001) argued that “participation in a mathematical discourse presupposes the observation of a set of meta-rules that regulate the discourse and the practice in general” and these rules are “culturally bound, intersubjective entities” (p. 79) that are developed as a result of participating with others in a community of practice. In addition, in a study of classrooms in Korea and the US, Pang (2000) provided evidence to show that sociomathematical norms rather than those general social norms determine the opportunities for student learning of mathematics.

Methodology

Based on the work by Yackel and Cobb (1996) and Sfard (2001), a particular focus of this paper is on the meta-rules underlying the discursive practice in classrooms. The analysis of the lessons centres on the events in which a new mathematical topic was introduced. I selected three classrooms located in Shanghai, Seoul, and Tokyo respectively, from the dataset of the *Learner’s Perspective Study* (LPS) because of a shared focus of content on “linear equations” or “linear function”. The LPS research design was detailed elsewhere (Clarke, 2006). In brief, three teachers who were considered as competent by local standards, from three different schools, were selected in each city. A sequence of lessons was videotaped for each teacher using three cameras (teacher camera, whole class camera and focus student camera) and video-stimulated post-lesson interviews were conducted with both the teacher and the students. Other materials collected include student written work, instructional materials, and so on.

This paper reports the analyses of the first three lessons from each of the classrooms studied and the teacher interviews. The guiding question of the analysis is “What are the similarities and differences of meta-rules that regulate the discursive practice in the three classrooms?” To address this question, the data analysis was conducted in two phases. In the first phase, three lessons from each of the three classrooms were analysed to reveal the forms and functions of activities involved in introducing the new content. In the second phase, classroom dialogues and interview accounts were examined in detail to uncover the meta-discursive rules governing those exchanges. The paper discusses the meta-discursive rules related to:

- *The nature of mathematics:* What is mathematics and who defines the rules and principles?
- *Ways of learning mathematics:* How is mathematics learned in the classroom?
- *Mathematical language:* What is considered to be the appropriate use of mathematical language?
- *Mathematical explanation:* What counts as a valid and acceptable explanation?
- *Mathematical solution method:* What is regarded as an acceptable solution method?

I will discuss these meta-rules in relation to the beliefs, values and expectations from a broader macroculture and the traditions of a particular education system. Based on the comparison of the meta-discursive rules in the three classrooms, I conclude the paper by examining the affordances of these rules on student mathematics learning and drawing some implications for studies of mathematics classrooms.

It should be emphasized that the selection of these three classrooms is not intended to signify any form of national typification. Instead, I want to illustrate the distinctive pedagogy that each classroom employs and to show how the meta-discursive rules shaped the forms of knowledge allowable in each classroom.

Introducing linear equations in the three classrooms

Despite a common focus on linear equations, observation of the lessons showed different tasks and activities employed in each classroom. In the Shanghai classroom (SH1), the topic of the first lesson was on linear equations in two unknowns and solutions. Particular attention was paid to clarifying the meaning of linear equations in two unknowns and the concepts of a solution and a solution set. The second and the third lessons introduced the rectangular coordinate axes and coordinates as ―a graphical method” for solving linear equations in two unknowns.

In the Seoul classroom (KR1), the emphasis of the first lesson was on the difference in the graphs of a linear equation in two unknowns, when the condition for variable X is a natural number as compared to the graph when X is a real number. Lesson 2 focused on the notion of the intersection of the two straight lines as the solution of the simultaneous equations, and Lesson 3 continued this focus and introduced the method of elimination by addition and subtraction.

The three lessons in the Tokyo classroom (JP1) were conducted around the same task: a staircase problem, which served as a context to introduce general forms of linear function. In the first lesson, the teacher invited the students to brainstorm about the variables that can be examined in the stair problem, and the class explored the relationship between the number of steps and the perimeter of the stairs in three forms of representation: a table, a formula, and a series of figures. In the second lesson, the class was asked to relate the mathematical relationship between the number of steps and the perimeter to the changes displayed in the figures. The students were also asked to formulate relationships between two variables of their choice. The definition of a linear function was introduced in Lesson 3.

Meta-discursive rules in the three classrooms

Doing mathematics as a collective activity

The early analysis of spoken mathematics in LPS classrooms revealed both similarities and differences in the way classroom dialogue was orchestrated in each classroom. Figure 1 shows the number of teacher utterances, student utterances, and choral utterances in each lesson analysed in this paper. The figure demonstrates that while teacher talk was the most dominant form of talk in all three classrooms, there are significant differences in the way in which choral utterances and individual student utterances were valued. While very few choral utterances were found in the Tokyo classroom, this form of utterance was the most important means through which the students were given voice in the classroom in the Seoul and Shanghai classrooms.

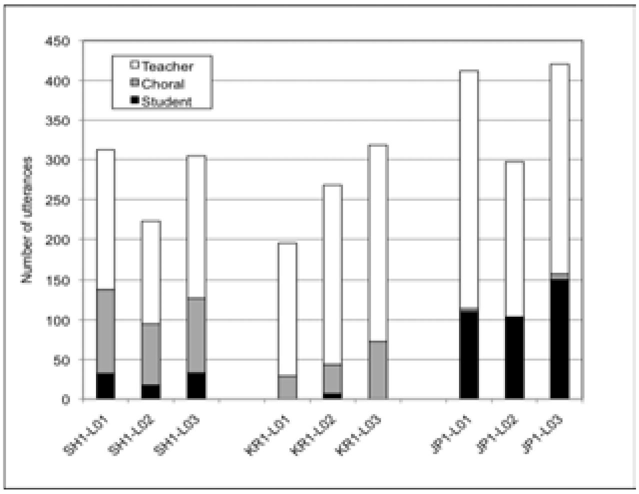

Figure 1. Number of public utterances in each lesson.

Further analyses of the classroom data revealed differences in the role of the students and the value attached to student contribution in public classroom discourse. In the Shanghai classroom, despite the classroom discussion being regarded as heavily guided by the teacher, the students were given many opportunities to contribute to the public classroom discourse, usually through teacher invitation. The activities, such as drawing a coordinate plane or defining the quadrants, were conducted in a way that the conclusion could be seen as the result of the collective contribution of the whole class. And this was crystallized in the form of board notes. Such an approach of building on student contribution was expressed in this Shanghai teacher's interview:

> One characteristic (of a typical lesson) is that the teacher is the facilitator of learning. This lesson shows that students are the active agent in learning, from the beginning till the end. That is…(I raised) questions that let them to answer, and towards the end, students generate their conclusions. Even when we talk about the sample problems, the teacher does not tell them the conclusion directly. <u>It is the students who have to think and talk about the problems by themselves. The role of the teacher is only to guide them. In other words, students are the active agent.</u> (SH1-IntT2)

While the Shanghai teacher weaved student input into a coherent ongoing classroom discourse, student contribution to the public discussion in the Seoul classroom was minimal, with most student responses consisting of a simple mental calculation or agreement with a statement made by the teacher. The teacher's reluctance to the “new” way of teaching was clearly expressed in his interview:

> These days there are many open classes in which students actively discuss in the class, I think the way of teaching is changing. But I think the teacher should teach. I think it is better. In the beginning, I teach and in the last part of the class I make students discuss what they learned. It is a good way to teach math. I don't oppose to the open class. But I think teacher's explanation is more important in teaching math. (KR1-IntT2)

Compared with the emphasis on collective action in both the Shanghai and the Seoul classrooms, the students in the Tokyo classroom were given autonomy to generate their own formulation and come up with their own method of solving the problem. In the interviews, the teacher stated the importance of students having their own opinions and raising these opinions in the public discussion. For example, in one interview, she said:

> Um, it went totally different from what I have planned, so I wouldn't be able to evaluate this class. But I had another thing I wanted to do in class if it had gone as I planned. That plan was to begin talking about a graph of a linear equation in general. So I had two plans for this lesson. But it was not important to do as planned. Students discuss with each other, and have their own opinions—that is the most important. And I think it is what was good about this lesson. (JP1-IntT2)

This Japanese teacher valued the opportunity for the students to share their opinions with their peers, which was considered more important than teaching the lesson as planned. The observation of the Japanese lessons also showed that student expression of lack of understanding was acceptable in the classroom and adequately resolved by the teacher. Arguably, this classroom is a different place from the one in which students are rarely given the chance to voice their own opinions.

While all the three classrooms can be regarded as belonging to a collectivist culture associated with Confucius Heritage, the form of collectivism was differently performed in each classroom. While in the Shanghai and Seoul classrooms, the students were given opportunities to verbally participate in the classroom discourse as a collective, the teacher in the Tokyo classroom respected the different opinions of individual students, and orchestrated the classroom discussion so that these student opinions were voiced and shared within the classroom as a community.

The use of mathematical language in the classroom

The significance attached to the use of standard mathematical language also differs. In comparison with the other two classrooms, the Shanghai classroom showed a distinctive emphasis on the accuracy of mathematical language (see Figure 2). Through the classroom discursive interactions, the students were assimilated and institutionalised into a discourse of school mathematics that encourages the accurate use of standard mathematical terms. The modelling of mathematical language by the teacher was a deliberate strategy, and the students were expected to follow such a model.

The value attached to the use of accurate mathematical language and the completeness of student response was clearly conveyed in the teacher interviews.

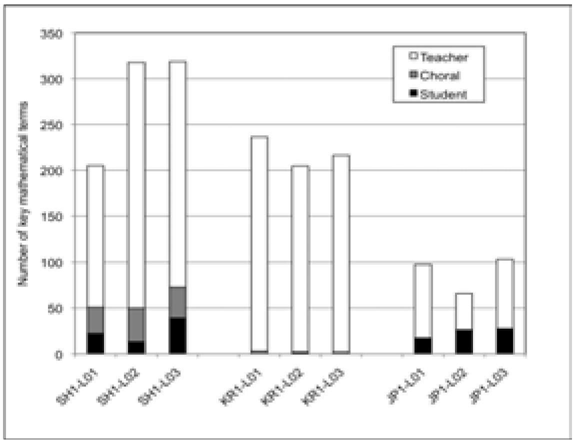

Figure 2. Frequency of key mathematical terms employed in each lesson.

For example, in the second interview, the teacher said:

> I asked one student to answer me. He could tell me what was the first step, what was the second step. The answer was quite complete, especially he said the first step is to transform an equation to an algebraic expression with unknown to represent another unknown. What he said is very good. He said the second step was…put this algebraic expression into another equation to substitute the unknown in that equation. That is to make the system of linear equations in two unknowns into an equation in one unknown. Then … after that … what to do after finding out this unknown. Find another unknown by substituting the value of the other unknown got. This language, that is this mathematics language, is good. (SH1-IntT2)

The use of standard mathematical language can be regarded as a normative aspect of this particular classroom. This finding is consistent with Leung's (1995) study of Beijing classrooms, in which he reported that 15 out of the 36 lessons observed demonstrate the stress placed on the use of accurate and rigorous mathematical language. Compared with the Shanghai classroom, the accurate rehearsal of mathematical language was much less prominent in the other two classrooms. This suggests that the emphasis on the verbalization of mathematics language may represent a distinct feature of Chinese classrooms.

Mathematical explanations

In many mathematics classrooms, it is not sufficient for students to simply provide an answer to a problem. Providing explanations is considered to be an essential component of mathematics discourse (Lampert, 1990). In the Shanghai classroom, the students were frequently asked by the teacher to provide explanations for their answer to a particular mathematical problem. Many of these explanations required the students to employ mathematical concepts or rules to justify their responses. This systematic way of defining and applying mathematical concepts (mediated by specifically designed tasks) could be seen as a reflection of beliefs about the nature of mathematics and beliefs about what students should be able to do in mathematics. This is well grounded in a tradition of school mathematics in China that emphasizes basic knowledge and basic skills. As Li (2006) observes, under this tradition, the teaching process is usually deliberately organized to ensure that teachers and students concentrate on concepts, theories, rules, skills and techniques.

Compared with the Shanghai classroom, in the Seoul classroom the rules or principles of solving a linear equation or simultaneous equations were given with little explanation from the teacher nor requested from the students in terms of the underlying meaning of the mathematical operation. The focus of the lessons was to help the students understand the procedures of solving particular groups of equations rather than an explicit focus on the meanings of concepts or the relationship between different representations. The emphasis on procedures in Korean classrooms was also reported in the study by Park and Leung (2006). Such an approach can be regarded as reflecting a view that mathematics is composed of a given body of knowledge and truth, and the task of teaching is to impart this body of knowledge to the students. In addition, such a –transmissive” way of teaching might be influenced by the male dominant culture in Korea in that this is a class in a girls’ school with a male teacher.

In the Tokyo classroom, students’ contributions were accepted and acknowledged no matter whether or not they were –mathematical” in a strict sense. In this classroom, mathematics was about formulating relationships and expressing them in different representations such as a table, a formula or figures. The students were interrogated to explain their understanding of the underlying relationships between variables and between representations of different form. For example, in the second lesson, the students were probed about their understanding of the proportional relationship between the number of steps and the perimeter of the stairs displayed in different representations.

It can be argued that the rules governing the legitimacy of mathematical explanations in classrooms reflect the different priorities that each teacher had in developing their students’ mathematical understanding. In the Shanghai classroom, the intention is to get the students to understand the meaning of mathematical concepts, such as a solution. The Japanese teacher, on the other hand, tried to get the students to understand the mathematical relationship between two variables by using a range of representations. In comparison, the Korean teacher tried to get the students to understand the procedures for solving different types of equations. While one may argue that these differences are constrained by the different mathematical tasks presented in each of the classrooms, the meta-rules for the acceptance of certain student explanations and the rejection of others reveal more fundamental differences in the teachers’ pedagogy and their beliefs about the nature of mathematics and mathematics learning.

Diversity and simplicity of solution methods

Rather than restricting the class to a particular way of solving mathematical problems as demonstrated by the teacher in the Seoul classroom, different methods or solutions were encouraged by the Tokyo teacher. The encouragement of diverse ideas was demonstrated in two interrelated aspects: firstly, the students in this Tokyo classroom were given autonomy to generate their own formula about the variables of their choice; secondly, the students were encouraged to consider the relationships displayed in different representational forms from various perspectives. The teacher’s respect for diversity of solution methods was conveyed in her interview:

> I think it is important to make them raise their hands when we had some opinions opposing to each other. It is not for deciding by majority. I do this to see what each student has in their mind. (JP1-IntT2)

The Shanghai classroom also provided the students with opportunities to display various solution methods, but the purpose of displaying different solution methods was to examine which method was better and simpler in solving particular types of problems.

> In this way, we list students' different ways of solution, and compare them. We can analyze which method is better and students can get the correct way in the process of solving the problems ... This problem, students can do it themselves. But after solving the problem, most of them do not think whether there is a simpler method. … Some students do the problem correctly, but in a very complicated way. But a few of students do it correctly, and use a simpler method. We encourage students to make it simpler when solving a problem. (SH1-IntT3)

Arguably, the emphasis on diversity and on simplicity represent two different meta-discursive rules, each having consequences for student learning. The respect for diversity of solution methods without evaluation of their superiority in the Tokyo classroom could foster student creativity, but it might overlook the consideration of the relative validity of those methods. On the other hand, the public evaluation of different solution methods may help students to see the merits of certain methods in terms of their simplicity and efficiency, but it might encourage rigid approaches to problem solving by fostering a belief in one single “best method”. Indeed, as Sekiguchi (2006) argued, maintaining the productivity of mathematical activity requires a delicate balance between the three components of a value system: validity, efficiency, and creativity.

Conclusion

From the outset, there are similarities among the three classrooms studied, such as teacher-dominated whole-class teaching as the predominant mode of instruction in all three classrooms. However, this superficial similarity masks the different functions of whole class discussion and the distinctive characteristics of such discussion displayed in each setting. As I have demonstrated in the above comparisons, the balance between uniformity and individualization was differently maintained in each classroom. While the Shanghai teacher expected the conclusions to be built upon student inputs, the Seoul teacher conceived that the role of the students was to follow the examples set by the teacher. Moreover, both the Shanghai and the Seoul classrooms encouraged uniform and collective action by the students. In comparison, the students in the Tokyo classroom had opportunities to raise their individual opinions.

The comparison of meta-discursive rules also reveals some fundamental differences in the criteria that each teacher used to make judgement about what is “mathematical” and what constitutes “student capability in mathematics”. In the Shanghai classroom, the students were required to use standard mathematical language as modelled by the teacher. In addition, to be considered as mathematically capable, the students should not only be able to articulate their understanding of the mathematical concepts or principles in standard mathematical language, but also be able to apply their understanding in solving mathematical problems. In the Seoul classroom, to be regarded as mathematically capable, the students were required to understand the conditions of X and the consequence of these conditions on the solutions and the graphs of an equation. In this classroom, understanding means to know and to be able to apply those “established” mathematical routines and principles in solving various. In contrast, the students in the Tokyo classroom were interrogated by the teacher regarding their understanding of the relationships between different representations. Understanding, here, meant being able to see the underlying relationships between the variables as expressed in different representational forms and the connections between these. Since each meta-discursive rule affords different opportunities for student learning in mathematics, it can be argued that students in the three classrooms were in fact learning different ―mathematics" in spite of a common focus on the topic of linear equations.

The similarities and differences between the three classrooms have implications for cross-culture comparative studies and studies of teacher competence. The findings suggests that while an examination of shared macrocultural values and beliefs (e.g. respect for authority) may help us to understand the similarities in the social organization (teacher-dominated whole class instruction), we need to look for meta-discursive rules of the classroom microculture in order to understand what determines opportunities for student learning in mathematics. More importantly, the diversity of discursive practices demonstrated in the three classrooms that are usually characterised as ―East Asian" in several major international studies (e.g. TIMSS) suggests that teacher competence should indeed be conceived as a cultural construct reflective of local cultural norms, national aspirations, and traditions of particular educational systems.

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PROFESSIONAL DEVELOPMENT OF MATHEMATICS AND SCIENCE TEACHERS IN COMMUNITIES OF PRACTICE: PERCEPTIONS OF “WHO IS MY COMMUNITY”

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This paper investigates views of mathematics/science teachers and higher education faculty interacting in professional development projects adding a community of practice component. Knowledge acquisition in a community of practice relates to ongoing interactions among members as they perform their roles and responsibilities. In particular, the paper reports each group’s perceptions of community and discusses implications for state-level programs funding professional development projects.

Introduction

As paradigms for teacher professional development shift from a “training-and-coaching model” whereby university-generated research is disseminated to teachers through workshops and university courses (Corcoran, 1995) to a learning community model that promotes educators learning together about professional matters (Darling-Hammond, 1996), professional development endeavours must reflect a community orientation. Projects to “promote improved instruction in mathematics and science for Texas school children by providing professional development for their teachers” (http://www.thecb.state.tx.us/os/TQ/), such as those funded by the *Texas Teacher Quality Grants Program* (TQGP), rely on the experiences and expertise of higher education faculty. However, higher education content faculty in the United States seldom interact with education faculty and classroom teachers outside these programs; therefore, including a community of practice component may produce challenges for state-level programs.

Guiding framework

The National Council of Teachers of Mathematics (NCTM) describes professional development for mathematics teachers in terms of community by describing roles for various stakeholders in mathematics education, including higher education (NCTM, 2000). More specifically, the National Staff Development Council (NSDC) proposes a community model stating that educators should organize “into learning communities whose goals are aligned with those of the school and district” (http://www.nsdc.org/standards/index.cfm). Descriptions of effective professional development of teachers also suggest a community design in which teachers learn in teams, reflect together on their learning, and connect their learning to the classroom (Lee, 2001; Little, 2003). Learning in a community is a theme that is interwoven throughout Timperley's (2008) ten general principles for effective teacher professional development, in terms of student outcomes, that are based on her synthesis of ninety-seven studies from around the world. Knowledge of content and instructional practices are hallmarks of most effective professional development programs, but this knowledge does not solve the problem of enactment (Darling-Hammond, Bransford, LePage, Hammerness, & Duffy, 2007). Teachers must also adapt their practice based on this knowledge. Providing opportunities for teachers to practice and reflect on instructional approaches is crucial to them moving from knowledge to action, and communities of practice provide a forum for this sustained, long-term professional learning.

As state-level programs transition to funding community-oriented professional development projects, many adopt the paradigm of a community of practice, that is, a ―group[s] of people who share a concern, a set of problems, or a passion about a topic, and deepen their knowledge and expertise in this area by interacting on an ongoing basis" (Wenger, McDermott, & Snyder, 2002, p. 4). A basic model for a community of practice entails three components: domain, community, and practice. Community involves the social feature of the group that develops trust and contributes to learning in a safe environment as well as the roles and responsibilities of members (Wenger, et al., 2002). However, models for the design of communities are difficult to describe (Barab, Barnett, & Squire, 2002; Barab, MaKinster, & Scheckler, 2003; Barab, Schatz, & Scheckler, 2004; Hung, Chee & Hedberg, 2005; and McConnell, 2005). In addition, aspects of communities of practice relate to members interacting in group settings (Glazer & Hannafin, 2006), and group interaction among teachers offers strong affective and supporting components to acquisition of knowledge (Rovai, 2002). As Little (2006) summarizes in speaking about the potential of professional communities, ―For more than two decades, research has shown that teachers who experience frequent, rich learning opportunities have in turn been helped to teach in more ambitious and effective ways. Yet few teachers gain access to such intensive professional learning opportunities" (p. 1).

Geography also poses challenges for educational endeavors. For example, the state of Texas is the second largest in land area in the United States and is slightly larger than France (https://www.cia.gov/library/publications/the-world-factbook/index.html). The state's population is over 24 million people compared to Australia with slightly over 21 million (http://www.census.gov/). Texas educators teach a common set of standards to about five million students from diverse ethnic and economic backgrounds (African American 14%, Hispanic 48%, White 34% Others 4%, Economically Disadvantaged 56.7%) (http://ritter.tea.state.tx.us/perfreport/snapshot/2009/state.html). Since research indicates that geography affects human activities such as art and culture (Hassani, 2009), economic status (Gittell, 2009), health care (Arcury, Gesler, Preisser, Sherman, Spencer, & Perin, 2005), entrepreneurship (Gupta & York, 2008), per capita income as well as university education (Basher & Lagerlof, 2006), it seems reasonable that geography could also affect educators' concept of community.

Thus, an issue for state-level programs is the structure of professional development in a community of practice designed by higher education faculty. If teachers are to gain knowledge and to change their classroom practices based on that knowledge by interacting with and developing trust among community members, then the question arises ―who is my community?" In particular, how do higher education faculty and teachers who comprise a community of practice perceive community in terms of membership? How do they perceive the roles and responsibilities of members?

Method

Exploration of the concept of community as perceived by directors and participating mathematics and science teachers occurred through a case-study design. Qualitative research through structured interviews, observations, field notes, and other ―rich" data sources offers researchers avenues to answer questions such as ―What is going on here? What does this mean? Why do the participants behave this way?" Nine projects funded by the *Texas Teacher Quality Grants Program* comprised a case for this study that served as a pilot for the program's state-wide evaluation. To account for Texas' geographical influences, the projects, chosen by TQGP staff, represented six geographical regions of Texas. Interviews with nine project directors and eight sets of teachers took place face-to-face on the campuses of the higher education institutions that received the funding. One set of teachers answered questions during one of their project's online sessions. Table 1 depicts a summary of the projects.

Table 1. Participating Teacher Quality Grant Program *projects.*

Geographical Region	*Number of Teachers*
East Texas	3
Coastal Region	3
South Texas	4
Central Texas	2
North Texas	5
West Texas	3

Digital voice recordings and field notes recorded the data collected during each site visit. As part of the interview process, the researcher stated that TQGP views their projects to be communities of practice and gave each interviewee a sheet of paper with Wenger et al.'s (2002) definition of a community of practice recorded on it. Then the researcher read the definition out loud to the interviewees. Following this reading, project directors and teachers answered questions that asked them to state who were the members of their community and to describe the role and responsibilities of the members. Next observations of professional development activities by the researcher occurred and recordings of interactions between teachers and projects directors took place. Analysis of the data transpired through a triangulation process that compared project directors' responses, teacher-participants' responses, and observations during project activities as directors and teachers interacted.

Findings

A disconnect existed between project directors' and teachers' perceptions as to the members of a TQGP community of practice (see Table 2). Seventy percent (70%) of project directors stated that both teachers and project staff are part of the community; however, only 32% of teachers reflected this same view. Some teachers (37%) described their community as one that excludes higher education faculty and consists only of participating teachers in the project. Other teachers (32%) ignored the TQGP project and described their community as teachers, administrators, and students in their schools. Other perceptions of community held by teachers extended the concept of community to include the school and the community at large; whereas another limited community to TQGP participants who were going through a Master's degree program together.

Table 2: Members of community.

Perception of community membership	*Percent of project directors*	*Percent of teachers*
Teachers, principal, administrations, students, parents, business leaders, university staff & faculty	10%	16%
Teachers in TQGP projects pursuing Masters degree	0%	5%
Teachers only in TQGP project	0%	37%
Teachers in TQGP project & project staff	70%	32%
1) Teachers & staff in current TQGP project 2) Teachers in past/current TQGP projects pursing Masters degree	10%	0%
Teachers, students, & administrators at school	10%	10%

Both project directors and teachers described community membership in visual terms (see Figure 1). One project director described her TQGP community of practice in terms of two TQGP communities with overlapping members, but possessing different goals.

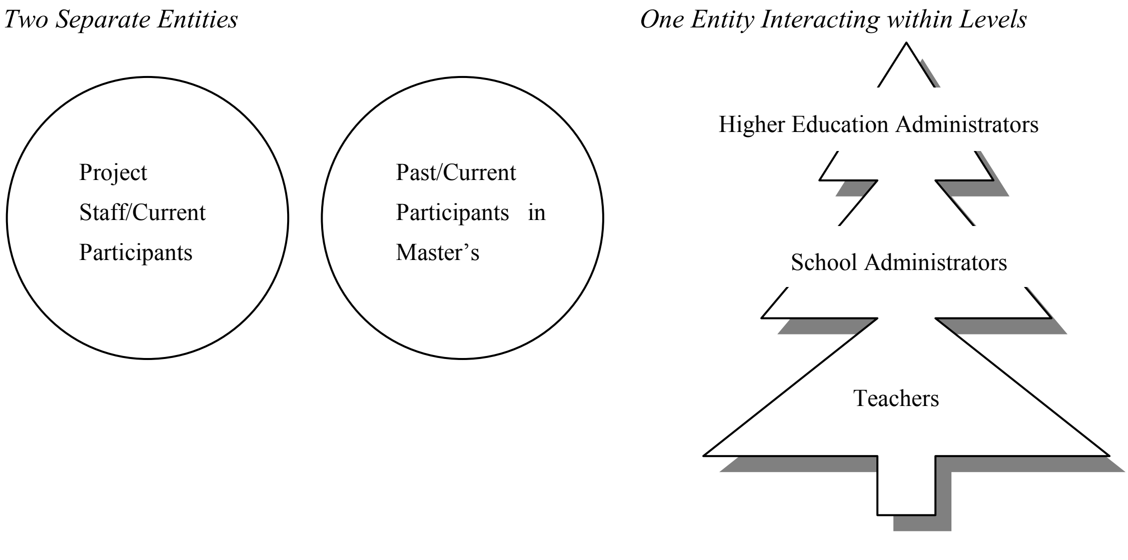
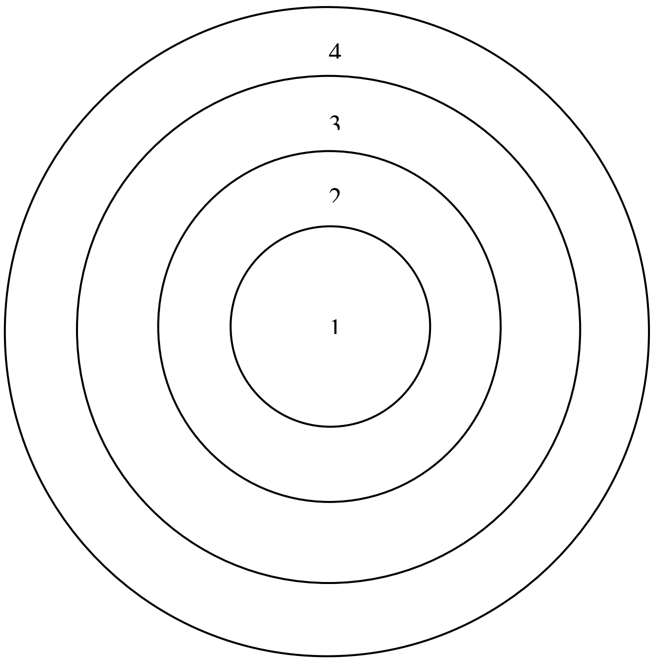


Figure 1: Visualisation of community by two project directors

Interestingly, the interviewed teacher from that project who enrolled in the Master's program expressed only one view—those past/current participants in the master's degree program—and completely discounted other teachers who did not pursue the degree. Another project director described the TQGP community as a tree whereby members at a particular level of branches communicate with each other but have limited communication with other branches. This description resembled, to some degree, NCTM's community as teachers can undergo professional development with various stakeholders in mathematics education but not in one activity.

In contrast, a teacher from a different project that was conducting lesson study, a site-based professional development model originating in Japan (Fernandez, 2003; Fernandez & Yoshida, 2004; Isoda, Stephens, Ohara, & Miyakawa, 2007; Lewis, 2002; Takahashi, 2000) described her TQGP community in terms of concentric circles, as shown in Figure 2. She referred first to a very small nucleus of teachers in her immediate lesson study group and expanded outward to include all teachers and staff in the project. She then extended community membership to others outside the project who provided expertise and support to those in the TQGP project. Similarly, another teacher in a project located in a different geographical region of the state that was beginning to implement lesson study expressed her TQGP community completely in terms of their lesson study effort, identifying teachers in the project, project staff, and consultants who assisted them with the lesson study process. These descriptions reflected more of NSDC's learning community concept.



1 Teachers in immediate lesson study group.
2 Higher education and other teachers who are knowledgeable about lesson study.
3 Teachers in TQ Project in other lesson study groups.
4 Others who help and support teachers.

Figure 2: Visualisation of TQGP community by participant.

Perceptions about the roles and responsibilities of members in the majority of TQGP projects held higher education faculty as dispensers of knowledge and providers of classroom activities. Members who are teachers were recipients of knowledge who discussed the activities in the TQGP community setting, took them back to their schools, and worked them with their own students or with other teachers. One project director added that he believes these roles and responsibilities result from teachers' perceptions of what constitutes professional development and not from the design intended by the project director who wanted teachers to take a more active role.

Members in another TQGP project perceived roles and responsibilities changing as project activities continued throughout the year. In this project, teachers and project directors were researchers in an outdoor learning environment during the summer. However, in the fall roles and responsibilities changed to participants being students in a class taught by project staff. These types of roles and responsibilities for community members followed more of a ―training-coaching” model for professional development.

In projects that included an outreach component requiring site-based interactions, both teachers and project directors tended to view roles and responsibilities of members in terms of each one possessing some type of expertise that is of value to the community. In projects implementing Lesson Study, both groups cited learning together with each member being a different resource for the group, providing knowledge of content, pedagogy, curriculum, and student misconceptions. Their perceptions of the roles and responsibilities in their community is a feature of the Lesson Study model, a model that moves teachers from recipients of knowledge disseminated by others to practitioner-researchers of student learning (Takahashi & Yoshida, 2004). This view of roles and responsibilities as group learners with access to resources aligned to professional development described by Lee (2001) and Timperley (2008).

Discussion and conclusion

Since TQGP project directors' and teachers' perceptions of community do not align, these projects offering professional development using a paradigm of a community of practice are not well defined. Project directors consider themselves part of the community; however, the majority of teachers do not. Teachers generally perceive their communities in terms of other teachers. Those teachers who do include project staff view them as outside resources and supporters. This latter view is prevalent among members in projects with site-based components that require directors to interact with teachers in their classrooms. Although most projects do reflect community in terms of higher education as a stakeholder in mathematics and science education, teachers do not work side by side with higher education faculty to plan and contribute to their own professional development. Since a community of practice is about people learning together, this factor may contribute to teachers' exclusion of project staff from their concepts of community.

In most projects, roles and responsibilities of members follow traditional forms of professional development with higher education faculty being givers of knowledge, designers of activities, and modellers of pedagogy and teachers being recipients of knowledge, takers of activities, and implementers in their classrooms. However, in projects where interaction among members occur in the schools, especially in those implementing lesson study, descriptions of the roles and responsibilities view each member as an expert. For example, higher education faculty offer support and provide knowledge of content and pedagogy; whereas, teachers provide experiences about students thinking, curriculum, etc. Connecting learning in a community to student learning is a feature of effective professional development of teachers.

This analysis of TQGP projects as communities of practice reveals that more thought needs to be put into the design of professional development by higher education faculty. Since most project directors who structure these projects, especially those in content departments, have little experience with learning communities, state-level programs need to consider how to provide this experience for them. In addition, these experiences need to include working in schools with teachers, especially in their classrooms, in outreach efforts to shift project directors' thinking from teachers as students to teachers who have students. Ultimately, the perceptions of project directors and teachers about members in their professional communities will play a major role if site-based professional development, which is indicative of improved teaching in mathematics and science for school children, materializes.

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YOUNG CHILDREN'S UNDERSTANDINGS ABOUT "SQUARE" IN 3D VIRTUAL REALITY MICROWORLDS

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This paper reports an investigation of primary school children's understandings about "square". 12 students participated in a small group teaching experiment session, where they were interviewed and guided to construct a square in a 3D virtual reality learning environment (VRLE). Main findings include mixed levels of "quasi" geometrical understandings, misconceptions about length and angles, and ambiguous uses of geometrical language for location, direction, and movement. These have implications for future teaching and learning about 2D shapes with particular reference to VRLE.

Introduction

When asked "What is a square?" or "What do you know about squares?", children would give a variety of responses such as:

> It has four equal sides.
> They're a quadrilateral, a regular quadrilateral. They have four corners, four right angled corners.
> Squares can go onto 3D shapes, like on the bottom of a pyramid.
> Square based pyramid.
> And a cube and rectangular based prism. (Year 4s)
>
> They're 2D shapes.
> They have four lines of symmetry.
> It's an enclosed shape.
> Every side is the same.
> There's four angles.
> Four right angles. (Year 5s)
>
> Is a 2D shape. It has four edges, four vertexes.
> Four sides are the same.
> Four corners of the square are 90 degrees. (Year 6s)
>
> It times a number by itself.
> It's a regular quadrilateral means it has 4 equal sides and 4 equal angles.
> The four angles are all 90 degrees, if not 90 degrees it can't be a square.
> All sides are even.
> It has 2 sets of parallels. (Year 7s)

Upon reviewing their responses, it seems that the critical properties of a square (i.e., four equal sides and four equal angles) are all understood by year 4-7 children. However, examining their understanding simply by spoken language is not necessarily sufficient. In order to probe more deeply into children's thinking and understanding, the researchers have developed a 3D virtual reality learning environment (VRLE), allowing young children to express their thinking and construct their understanding about shapes and geometry via a variety of semiotic resources (Yeh & Nason, 2004a).

This study originated from the *Spatial Thinking And Reasoning* [STAR] project, in which the VRLE named VRMath 2.0 is the vehicle for investigating and developing young children's spatial abilities. VRMath 2.0, puts simply, is a combination of 3D LOGO and Web 2.0 environments. Traditionally, LOGO turtle graphics has been a powerful tool for learning geometry. However, its significance has been limited by its 2D graphics. A 2D square drawn in a traditional LOGO environment is as a square drawn on paper—a legitimate bird's-eye view of a square. What we taught young children traditionally was also based on this mindset and communication in either symbolic language or visual concrete materials has focused on the top view of the 2D square. The geometrical understanding based on this is what we would call ―quasi" understanding that needs to be challenged and further qualified. This study is informed by ―new paradigms for computing, new paradigms for thinking" (Resnick, 1996, p. 255) and ―empowering kids to create and share programmable media" (Monroy-Hernandez & Resnick, 2008, p. 50), and has introduced new practices to traditional turtle graphics. The 3D LOGO graphics (and Web 2.0[1]) are providing new opportunities for young children to develop a more holistic learning and geometrical understanding, and new opportunities for researchers to reveal how this new practice enables young children to think and do things differently. This paper reports the first trial of the STAR project about how young children develop their ideas of squares in VRLE.

Literature review

Semiotics as the epistemological stance

This research has taken a semiotic view about meaning-making as its epistemological stance. Semiotics is the study of signs, where a sign (representamen), is something that stands to somebody (interpretant) for something (object) in some respect or capacity (system of signs) (Peirce & Buchler, 1955). Human cognition or ―meaning-making" is irreducible to any one element of this triadic relation among the sign, object, and interpretant. Signs are incomplete representations of the objects and thus meaning making must be an on-going process, and meaning must be constantly qualified and challenged.

The multiple semiotic resources for learning mathematics proposed by Lemke (2001) are of particular relevance to this research project, is. Lemke outlined three semiotic resources—typological, topological, and social-actional—which have informed the design of VRMath 2.0 for mathematical meaning-making. According to Lemke, typological semiotics represents meanings by types or categories such as spoken words, written words, mathematical symbols, and chemical species. They are discrete, point-

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[1] Due to the complexity and scope of this paper, the sharing aspect of Web 2.0 is not included in this report.

like, and distinctive signs. In contrast to this, topological semiotics makes meaning by continuous variations in such as size, shape, position, colour spectrum, visual intensity, pitch, loudness, and quantitative representation in mathematics. Social-actional semiotics provides a context such as building a bridge that constantly reinforces the meaningfulness of mathematics in the real world situation. The epistemological assumption for this study thus is that better learning occurs when multiple semiotic resources (i.e., typological, topological, and social-actional) are provided for meaning making.

Learning and understanding about 2D shapes

Recognition of basic 2D shapes including circles, triangles, rectangles, and squares is usually developed quite early in lower primary years, or even before school. Children at this stage are able to name the above basic shapes when see them but are not noticing properties of those shapes such as the number of sides and angles. In terms of Van Hiele's (1986) level of geometric understanding, most children in years 4 and 5 have understanding of Level 1 (visualisation, recognise figures by appearance) and Level 2 (analysis, recognise and name properties of geometrical figures), but not Level 3 (abstraction, perceive relationships between properties and between figures).

Concrete materials and computers have been the main resources for teaching and learning about 2D shapes and regular polygons. In particular, the LOGO programming language and its 2D turtle graphics has been widely used for learning and creating 2D shapes during the 1980s and 1990s (e.g., Clements, 1999; Noss, 1987). In the LOGO environment for example, a square has been transformed or represented typologically as:

FD 50 RT 90 FD 50 RT 90 FD 50 RT 90 FD 50 RT 90

or

REPEAT 4 [FD 50 RT 90]

This typological transformation of a square represented a new paradigm of thinking and doing, as well as a different level of geometric understanding.

However, as stated earlier, the traditional LOGO environment was limited by its 2D graphics. Furthermore, the traditional LOGO environment lacked the topological representations of a square. The 3D LOGO environment of VRMath 2.0 provides continuous viewpoints of a square in 3D virtual space, where learners' fixed mindsets about a bird's-eye view of a square can be challenged.

Revisiting Microworlds

The term ―microworld" arose in the context of introducing the LOGO programming language (Papert, 1980). However, the idea of a microworld is not necessarily limited to LOGO programming. Edwards (1995) found that microworlds were analogously used in a variety of environments such as simulations, intrinsic models, interactive illustrations, and discovery-based learning environments. He then concluded that microworlds should be seen as being the embodiments of mathematics. He argued that the value of microworlds went beyond their reifying link between the representation and the mathematical entity to providing the opportunity for learners to kinaesthetically and intellectually interact with a system of mathematical entities, as mediated through the symbol system of a computer program.

Hoyles, Noss, and Adamson (2002) noted that the microworlds environments led directly to the idea of constructionism, arguing that effective learning will not come from finding better ways for the teacher to instruct but from giving the learners better opportunities to construct. This idea has been central to the development of VRMath 2.0, aiming to provide better opportunities for learners to construct and engage learners in interlinked mathematical entities and representations (semiotic resources).

The instrument: VRMath 2.0

VRMath 2.0 is a new implementation of its predecessor VRMath (Yeh & Nason, 2004b) enhanced by technological changes. Figure 1 is a snapshot of a square drawn in the prototype used in this study.

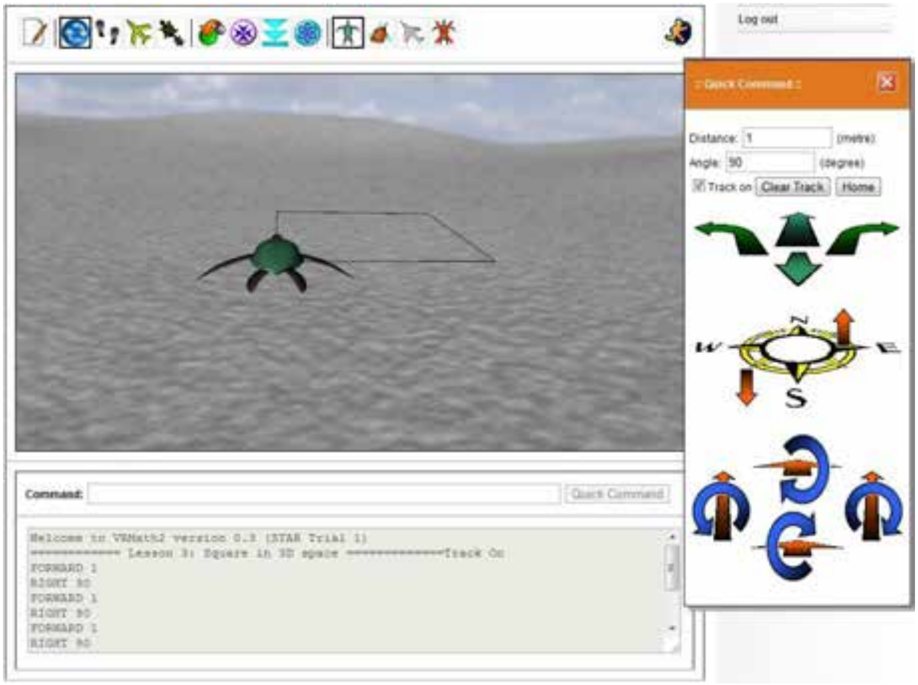

Figure 1. VRMath 2.0.

The VRMath 2.0 environment is rich in typological and topological resources. Typological resources include the Tool bar with icons, the Quick Command window, the Command field and the Message box. These resources have certain discrete meanings imposed on them. For example, the Quick Command has sets of icons that produce moving (change location) and turning (change direction) commands such as *forward 1* and *right 90*. These language commands, when clicked, will produce a topological change of the turtle in the 3D virtual space. Another main topological resource is the 3D navigation. When learners navigate in the 3D virtual space, for example, they perceive continuous views of the square in 3D space.

Because it employs virtual reality (VR) technology, the 3D space is measured in metres instead of pixels. The 3D space also enables full 3D rotations on three axes with six turns, and six fixed movements (i.e., up, down, east, west, north and south). These moving and turning commands can be classified into two groups as egocentric (i.e., forward, back, and the six turns) and fixed frame of references.

Method

There were 12 participants, three from each of grades 4 to 7. Each of the three students from the same grade were administered a lesson of 45 minutes as a small group teaching experiment (Steffe & Thompson, 2000). The lesson administered for this study was the third lesson titled “Square in 3D space”, which involved (1) discussion about squares; (2) drawing a square on paper; (3) interpreting square procedures, and (4) drawing a square in VRMath 2.0. Prior to this lesson, all 12 participants had been introduced to the interface and environment of VRMath2 in lesson 1 and 2.

During the lesson, the discussions were audio-recorded and later transcribed. Participants' drawings were collected and their interactions (e.g., using Quick Command and navigation) with VRMath 2.0 were automatically logged into an online database. Field notes were also taken if the researchers observed any developments.

Results

Discussions about squares

Participants' responses to “What is a square?” or “What do you know about squares?” were presented in the introductory session. The critical properties of a square were all mentioned, with some additional information. For example: “2D squares can be found on 3D shapes” and “a regular quadrilateral” by year 4; “four lines of symmetry” by year 5; “four edges and four vertexes” by year 6; and “times a number by itself”, “two sets of parallel lines” by year 7. This initial assessment showed that from year 4 to 7, students were able to articulate some properties of a square.

Another question that the researchers asked was “Is a square a rectangle?” All participants answered “No”, except one year 7 student who had previously said that a square has two sets of parallel sides. The prominent reason for arguing that a square is not a rectangle was that a rectangle must have two long sides and two short sides. The year 4s' discussions were typical:

Researcher: Is a square a rectangle?
J: It can be. Stretched out a bit.
N: If the sides aren't equal.
J: My brother told me this and our teacher was telling us about this too. She used to be a year 6 teacher. They had a argue about this [sic]. She said a square can be a rectangle, but a rectangle can't be a squ ... I mean, a rectangle can be a square, but a square can't be a rectangle. But now I don't believe that anymore because the squares have four equal sides and four corners and the rectangle has two long sides and two short sides. So, I believe that it's partially a square but it can never be a full square.
Researcher: So a rectangle can't be a square…?
J: Fully.
Researcher: But can the square be called a rectangle?
R: Not really.
J: No, they're partially, they're bits and pieces that are the same.
R: A rectangle is basically a square stretched out.
J: Yes, and a square has four equal sides.
Researcher: So a rectangle can't have four equal sides?

J: No, because then it would be a square.

Draw a square on paper

When asked to draw a square on paper, all participants again drew quite consistent views of squares, as shown in Figure 2.

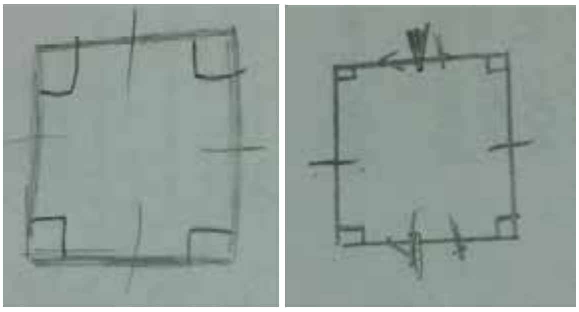

Figure 2. Drawings of squares.

Participants from year 4 to 7 all used the same notation. They used right-angle brackets to denote right angles, and a small dash on every side to denote the same length. One error found on the right hand side square in Figure 2 was the year 7 child who used one dash and two dashes to express his idea again about ―two sets of parallel lines”. However, this could actually denote that the four sides are not equal.

Interpreting square procedures

After drawing squares on papers, the participants were given two sets of procedures and asked to interpret them to see if they created a square:

Which sequence will make a square?

FORWARD 1	NORTH 1
RIGHT 90	EAST 1
FORWARD 1	SOUTH 1
RIGHT 90	WEST 1
FORWARD 1	
RIGHT 90	
FORWARD 1	
RIGHT 90	

Because these commands had been introduced in a previous lesson, the researchers were testing the participants' understanding about these commands. To our surprise, participants had quite different interpretations, particularly to the first procedure. A year 4 girl drew on paper as she interpreted the first procedure (see Figure 3).

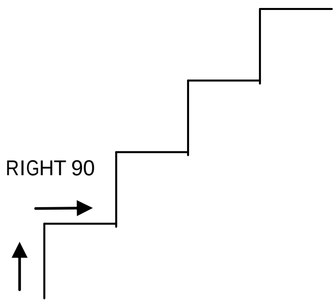

Figure 3. Year 4's interpretation (zig-zag).

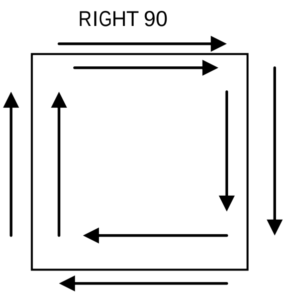

Figure 4. Year 6's interpretation (2 squares).

A year 6 boy also misinterpreted RIGHT 90 and thus predicted the first procedure to be two squares (see Figure 4). Year 5 children were also puzzled by RIGHT 90 but soon realised it was a turn when they acted it out physically. Year 7 participants did not make any mistake on the first procedure. All participants were able to recognise that the second procedure produces a square. And from that, all agreed that using compass movements is easier to create a square.

Draw a square in VRMath 2.0

Participants first tested the two square procedures in VRMath 2.0 to see if they really produced squares. As they followed the procedures, they created a square in virtual space. But due to the perspective in the 3D environment, the square does not look like a square. The researchers then asked, ―Is that a square?” and ―How do you know that that's a square?” Despite having just interpreted the procedures as squares, and being aware of the distance set to 1 metre and degrees set to 90, the participants started to give surprising responses.

Year 4s seemed to be quite confident. They thought it should be a square although it did not look like one. A year 4 girl said ―It looks like one. If you go up like …”, then navigated above the square to get a top view. Two year 5s also navigated to confirm that it was a square. Another year 5 boy was not so sure. He did not navigate (claiming that he ―didn't know how to move it up”): instead he said ―I don't know because I can't see because … I don't think it is”. Later, this boy actually commented that ―That's not a square, it's a trapezium.” One year 6 was very much in doubt about the square. He did navigate to try to make it a square but could not get a perfect viewpoint. When the researcher questioned how he could be certain whether or not it was a square, this year 6 participant tried to use a ruler and a protractor to measure the square on the computer screen. The year 7s were very sure that the two procedures would produce squares. One year 7 did not even navigate to see but simply recognised the square. A noticeable behaviour for another year 7 was that he navigated to a good viewpoint (top view) whenever he drew a line.

The participants were then encouraged to create different procedures for a square. The results were very creative. They created squares on different planes in the 3D virtual space. One year 7 created a very simple procedure for a square using both egocentric and fixed frame of reference (FOR) commands.

FORWARD 1 EAST 1 BACK 1 WEST 1

While the above procedure did create a square, the researchers challenged ―what if the turtle starts at RIGHT 45 direction? Would your procedure still produce a square?” He then predicted yes but found it to be a diamond (see Figure 5).

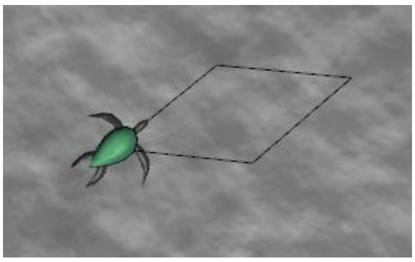

Figure 5. Diamond shape produced by mixing two FORs.

He quickly realised and explained to the researcher that because –east is always that way" and –west is always that way" (pointing to west)—a step closer to the big idea that the fixed FOR does not change the turtle's direction.

Discussion and conclusion

In this paper, we have presented our new practice that builds on the traditional LOGO. The traditional LOGO microworld is still a powerful field. It links mathematical concepts/entities with symbolic and visual representations. However, when examined under a semiotic framework, this tradition lacks the continuous representations of topological resources. In light of this, our new practice employs the 3D LOGO microworld with Web 2.0 technology to form VRMath 2.0. VRMath 2.0 is a vehicle for investigating and developing young children's spatial abilities. After its first trial, we have identified the following points of discussion.

1. Children may only develop –quasi" geometric understandings within traditional teaching and learning. For this paper, we probed 12 year 4–7 children's understandings about –square". Their seeming understanding about 2D squares became fragile when challenged in a 3D environment. We found that the traditional and legitimate bird's-eye view to be rigid. When viewing a 2D square from a 3D perspective, or when creating a 2D square in 3D space using LOGO programming language, young children's understandings about squares could be changed easily, even when they were fully aware of a square's critical properties.
2. It was also noticed that in 3D movements, some children were confused about the moving (change location) and turning (change direction). In real world situations, we are often changing the location and direction together. However, in mathematics, moving and turning have to be separated. If the turtle is changing direction, then it will not change its location and vice versa. We would like to term this as –component movement", which is essential for mathematical reasoning.
3. In terms of language use, this study found that the word RIGHT (as a turning command), can be interpreted as a moving command. When this happens, the children have ignored the number of degrees following the RIGHT command. No doubt this could be a diagnostic indicator of children's understanding, but it also presents a semantic, rhetorical, and communication problem of the learning environment.
4. Children seemed to have developed a new definition of rectangle, which unfortunately is not in line with that of the wider mathematics community. The source of this misconception is unknown. Educators should be informed about this misconception so they can communicate with children using the same mathematical language and discuss alternate definitions with them.

To conclude, we would like to return to our epistemological stance on semiotics. VRMath 2.0 can be seen as a complex sign system. It provides multiple semiotic resources including typological, topological, and social-actional representations for mathematical meaning-making. Due to the scope of this paper, we have not yet reported the sharing and social aspect of VRMath 2.0. We believe that VRMath 2.0 is a pertinent vehicle for investigating and developing spatial abilities and geometric understanding, which, including VRMath 2.0 itself, will need to be constantly challenged and qualified.

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TEACHERS' INTERACTIONS WITH STUDENTS LEARNING THE "EQUAL ADDITIONS" STRATEGY: DISCOURSE PATTERNS

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The study investigated interactions between nine teachers and their Year 5–6 students during a lesson on the ―equal additions" strategy for subtraction problems involving difference. Two quantities were compared (e.g., $445 vs. $398), a quantity was added to both (rounding up the subtrahend), and students asked about the two differences ($447–$400 and $445–$398). Teachers‘ use of so-called ―indicator words" was analysed. Those using words such as ―difference" and ―how much more" frequently had more students who chose the equal additions method to solve post-test problems. The findings reflect the challenges of bringing about deep and lasting change in teaching (and learning) mathematics.

Introduction

Mathematics education reform in western countries has resulted in a shift in emphasis away from training students in the use of rote-learned skills and procedures, towards helping students to develop deep conceptual understanding (Fraivillig, Murphy, & Fuson, 1999; Goya, 2006; Skemp, 2006). Problem solving processes, including thinking, reasoning, and communicating mathematically, have received far greater attention than in the past (see Ministry of Education, 1992, 2007). Sfard (2008) links these together, defining thinking as self-communication.

Researchers have become increasingly interested in the nature of the learning that takes place during classroom mathematics lessons, and there has been a sharpened focus on the interactions between teachers and their students (Rye, 2011). Discourse analysis, or conversation analysis, has become a popular means of gaining insights into the teaching and learning processes within classrooms (Cohen, Manion, & Morrison, 2007; Mercer, 1995; Mercer & Littleton, 2007; Perakyla, 2005). Such an analysis looks at the ―organisation of ordinary talk and everyday explanations and the social action performed in them" (Cohen et al., 2007, p. 389). It has been characterised as a kind of psychological ‗natural history‘ of the phenomena that has interested researchers. Cohen et al. suggest that researchers need to be highly sensitive to the ―nuances of language". According to Hodgkinson and Mercer (2008) classroom talk, the means by which children make sense of the ideas of their teachers and peers, ―is the most important education tool for guiding the development of understanding and for jointly constructing knowledge” (p. xi). Consequently more attention needs to be given to improving the quality of classroom talk.

Several writers have noted that teachers exert a high degree of control over the ways children engage in conversation in the context of classroom learning, and sometimes children are prevented from engaging productively by the actions of their teachers (Hodgkinson & Mercer, 2008), and in particular teachers who put a high priority on the management of behaviour, and who control who gets to talk, when they talk, and about what.

Classroom talk and thinking has been categorised in many different ways. For example, Mercer (1995) has distinguished ―Exploratory talk” (where speakers engage critically but constructively with each others‘ ideas) from ―Cumulative talk” (where speakers build positively but uncritically on what others have said), and ―Disputational talk” (which is characterised by disagreement and individualised decision-making). Talk can also be examined using a *linguistic* lens (talk as spoken text) or a *psychological* lens (talk as thought and action) (Mercer, 1995). Barnes (2008) contrasts ―Exploratory talk” (new ideas being tried out that are often hesitant and incomplete) with ―Presentational talk” (well-shaped talk, adjusted to the needs of the audience).

Several researchers have examined the nature of classroom talk in the context of mathematics lessons (e.g., Mercer & Dawes, 2008; Mercer & Littleton, 2007; Mercer & Sams, 2006). Solomon and Black (2008) noted that children‘s opportunities to contribute, and the type of talk directed towards them by the teacher, varies. Their work focuses on the way that some children readily develop an identity of engagement with mathematics, while others adopt an identity of exclusion from mathematics—a process that may begin from quite early in a child‘s school career. Further, teacher questioning can narrow the range of possible responses when teachers continue to ask questions in order to get a pre-determined answer (i.e., ―cued elicitation”). Mercer and Littleton examined the incidence of ―indicator words” assumed to reflect the thinking that occurred during the exploratory talk of students engaged in joint problem solving.

The aims and focus of the research

The present study set out to explore the use of language by teachers while teaching the ―equal additions” strategy for solving subtraction problems with a Compare structure. In contrast to the more common Separate structure that involve taking away an amount from a *single* quantity, Compare problems involve comparing *two* different quantities to find the difference (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Fuson, 1992)—see Table 1. Teachers‘ language is examined, alongside the use of the equal additions strategy by their students in solving post-test problems.

Table 1. Problem structures for “Separate” and “Compare” problems from Carpenter et al. (1999).

Separate (Change: Take from)	<u>Result Unknown</u> Ana had 13 plums. She gave 5 to Sam. How many plums did Ana have left? 13 – 5 = ☐	<u>Change Unknown</u> Ana had 13 plums. She gave some to Sam. Now she has 8 plums left. How many plums did Ana give Sam? 13 – ☐ = 8	<u>Start Unknown</u> Ana had some plums. She gave 5 to Sam. Now she has 8 plums left? How many plums did Ana start with? ☐ – 5 = 8

Compare (Difference)	Difference Unknown Ana has 13 plums. Sam has 5 plums. How many more plums does Ana have than Sam? 13 – 5 = □	Compare Quantity Unknown Sam has 5 plums. Ana has 8 more plums than Sam. How many plums does Ana have? 5 + 8 = □	Referent Unknown Ana has 13 plums. She has 5 more plums than Sam. How many plums does Sam have? 13 – 5 = □

Method

Nine teachers (7 female and 2 male) of Year 5-6 (nine- to eleven-year-old) students from four schools (serving communities ranging from low to high socioeconomic status), each with one instructional group (a total of 64 students) participated in the study (see Table 2). One teacher from each school had previously worked with the researchers, and that teacher agreed to ask the other teacher/s working at the same level also to be involved in the study. Teachers‘ classroom experience ranged from two to approximately 25 years. Experience in working with the Numeracy Project approach ranged from two to about eight years.

Students were given some written assessment tasks prior to the first lesson, then a similar assessment after the third lesson. This study focuses on the second lesson, which was designed to teach the equal additions strategy for subtraction (Ministry of Education, 2008, pp. 38-39). During the lesson, the teacher wore a portable digital audio-recorder attached to a flexible belt, with a lapel microphone to pick up his/her language to the children (and some responses from children who were close to the teacher). The researchers observed the lesson and noted non-verbal (contextual) information that could assist with the interpretation of the transcripts of audio-recordings. Actions with materials, written recording in the group workbook, and in students‘ individual mathematics books were photographed to capture some of this nonverbal information.

In the Equal Additions lesson, the first scenario used in Book 5 is as follows:

> Problem: ―Debbie has $445 in her bank account, and her younger sister Christine has $398. How much more money does Debbie have?"
> Make piles of $445 and $398. ―Now suppose that Grandma gives Christine $2 to give her a ‗tidy‘ amount of money. To be fair, Grandma gives Debbie $2 also." Discuss why 445 – 398 has the same answer as 447 – 400 and then record 445 – 398 = 47 on the board or modelling book.

The book then provides other examples of equations that can be turned into word problems and solved using materials (e.g., paper money).

Results

Data from the transcripts of the teachers‘ language while teaching the Equal Addition lesson were analysed to check the use of particular terminology during the lesson. Students‘ responses on the written assessment tasks given after the third lesson was analysed to see which students chose to use equal additions to solve the Compare problem and other subtraction problems. Table 2 shows the frequencies for teachers‘ use of particular terminology and the identities of particular students in their groups who used equal additions for the compare problem (those who used it for another subtraction problem are shown in brackets). Gail referred to ―difference" far more often than the other teachers (n = 30). She was also the second highest user of ―how much more” (*n* = 6). She chose to illustrate the idea of difference using small numbers (4 vs. 2), showing what happens when one is added to both numbers (5 vs. 3), that the difference remained the same. Several times she referred to the way “the distance between [the two numbers] stays the same”. At the very beginning of the lesson, she referred to a number line activity the students had done prior to the lesson, then part way through the lesson she asked students to:

> Think of a number line… and you‘re looking for the difference between six and two, the difference there is a space of one, two, three, four, right, now if you add two to both of those, one, two. Has the difference between both of them changed? [A student says “No”] It hasn‘t, has it, but if we went like this and you added two to one and not the other, okay, it‘s bigger isn‘t it. The difference between it has changed, so it becomes bigger.

It was interesting to note that three of the five children in Gail‘s group used equal additions on a post-test problem, the greatest proportion of any group. Cara referred to “difference” 12 times and was the most frequent user of “how much more” (*n* = 7). She also referred to the “distance between the two [numbers].” Four of her ten children used equal additions on the post-test. Ben started the lesson by using the Separate (“take away”) structure rather than Compare, but later referred to “difference” nine times. He only used “how much more” three times. Two of his students used equal addition on a post-test problem, and one student (B6) used equal subtraction for one problem. Three teachers (Ann, Dot, and Iris) did not refer to “difference” at all.

Table 2. Number of times particular words or expressions were used by teachers and children who chose to use Equal Addition to solve a Compare problem (or another subtraction problem) on the post-test.

Teacher	“Difference”	“How much more”	“Why”	“How”	Group size	Children using Equal Addition
Ann	0	1	6	20	8	
Ben	9	3	8	39	8	B4 (B5)
Cara	12	7	7	22	10	C3, C4, C7 (C9)
Dot	0	0	32	29	8	D8
Ed	4	4	7	31	5	E4
Fay	1	3	3	47	6	F3
Gail	32	6	23	31	5	G1, G2 (G5)
Hana	2	4	14	48	7	H1, H4
Iris	0	2	6	15	7	I5

Several other key issues that emerged were the importance of teachers using consistent language and their awareness of problem structure. Although it was not clear whether or not any of the teachers understood about different problem structures, Cara and Gail were very careful in their use of mathematical language and special terminology with the children.

Several teachers, including Ann, Dot, and Iris, took the first example from the book (see description above), which was structured as a *Compare* (difference) problem and turned it into a *Separate* problem. Dot said:

> Right, I had $445 right, K had, K asked me if she could have a loan of $398 and being the giving, caring person that I am, I said sure. How much money did I have left over? Right, I want you to think about the tidy numbers, using tidy numbers.

One student (D1) was concerned that if two was added to one number, it needed to be taken off later. Dot explained to D1:

> [D1], what I think you‘ve been confused with is if we did it to one of these numbers, if we added two the one number, then yes, we do have to take it away but we did it to both numbers. If we just added two to 398 and 445 the same, then yes, we would have to take that two away, but because we do the same treatment to both numbers the gap remains the same.

Several students commented at this point that they were lost, so Dot then decided to bring the lesson to a close as they had run out of time for further explanations. When Dot was asked in the post-lesson interview if she planned to follow up anything particular from the lesson in the future, she did not have a plan to address the confusion described above. It was interesting to observe later that on the post-test, D1 continued to subtract from the difference the amount she had added to the subtrahend initially, making her answers consistently incorrect.

Ben introduced his lesson by sharing with the group how, in preparing for this lesson, his own mathematics had been extended.

> This is one of these really cool exercises, now I mentioned before that since doing this, my understanding of maths has really improved. What this next lesson is, is actually a really cool lesson for an area that I think we‘ve got a bit of a weakness in as a class, looking at one particular type of operation. Now, so what we are going to do is, we‘re going to look at, looking at [Reading the learning intention for the lesson] how to solve subtraction problems by Equal Addition that turns one of the numbers into a tidy number.

Ben then asked the students ―What sort of problem are we looking at?" One student (B1) suggested a missing addend structure: ―398 plus what equals 445?" Ben would not accept this missing addend structure because the learning intention in the resource book focused on subtraction. He said:

> Oh okay, so [B1], you‘ve gone for that first one, reversing strategy, so you‘ve gone for 398 plus what equals 445, yeah. If we just look at the learning intention, which is to solve subtraction problems by using Equal Addition, are we using a subtraction problem here?

One child answered ―No." Ben continued:

> Is this still a good strategy? Yep, but we‘re going to look at just using subtraction, so what problem am I going to write down here to show subtraction?

Another child suggested ―445 take away 398". Ben affirmed that response:

> Nice, so we are going to use 445 take away 398 equals, we think it might be 47 [suggested earlier by one of the other students].

Ben then asked whether adding two to both numbers would change the answer. Some students thought it would increase the answer by four, but others believed that the answer was still the same. Ben tried to get those students to explain why:

> What are we actually looking at, we‘re looking at, what? ... So when I give the answer, what‘s the answer? Okay, if we take the answer we say let‘s say that‘s 47, we‘re happy that it‘s 47. What does the 47 actually mean?

One student suggested ―Numbers?" to which Ben responded ―Excellent, nice, okay."

There was further discussion but it did not appear to produce what Ben was wanting so he explained:

> Okay with subtraction, we‘re really looking at the difference between these two numbers, so the difference between 445 and 398 is 47, so we‘re just looking at difference, so the numbers here, you can change the numbers either way and it‘s not going to affect the outcome. Does that make sense?

At least one child agreed, but another was concerned about what happens if different amounts are added to different numbers. Ben responded:

> Ah now, good question. Will that affect the answer, if you‘re not adding the same amount to each side—because you‘re looking at difference? But that‘s a good question, that‘s a very good question.

He then gave them another problem, but did not stick consistently to either a Compare or a Separate structure.

> Okay, let‘s have a look. I‘m going to give you another problem. Here you go. This time [B1]‘s got 367 apples and [B5] would like to have some—he thought he could probably eat 299 apples ‗cause he‘s sort of feeling a little bit hungry, he hasn‘t eaten for a while. So [B1] started off with 367 and he is going to give [B5] 299 of those ‗cause he‘s quite generous. Now can you predict, now thinking about using that Equal Addition, will that help us solve the problem?

At least one child responded ―Yes.”

> Keeping in mind that we‘re looking at the difference between these two numbers, not necessarily the numbers themselves.

One student (B4) suggested that the answer was 68. When asked by Ben, how he did it, he responded:

> I gave each of them one more.

Ben then pressed for understanding.

> So just while we are doing this, but with [B4] adding on one more, have we changed the difference between the numbers?

The students responded with both ―Yes” and ―No.”

> We‘ve changed the numbers, but have we changed the difference between the two numbers?

This time the students knew they were expected to answer ―No.” However, it was not clear whether or not they really understood why they had answered ―No.”

Discussion

The analysis of indicator words showed a consistent pattern in terms of the relationship between the frequency of teachers using the term ―difference” and the number of students from their instructional group who chose to use equal additions to solve a problem on the post-test at least one week later. Mercer and Littleton (2007) used the relative incidence of indicator words to examine improvements in children‘s talk from *before* to *after* a programme designed to increase the quality of their talk during group-based learning. However, this tool has also proved useful in the present study for analysing differences among the teachers in their awareness of the Compare structure for subtraction. Finding that the highest incidence of using equal additions on post-test problems (60% of students in instructional group G) was associated with the teacher who had the highest incidence of referring to ―difference" (Gail) suggests that the content of teachers' language may be important in revealing critical differences in the effectiveness of their teaching of mathematics. Teachers who did not refer to ―difference" had no more than one student who chose to use the equal additions strategy on the post-test.

The findings of this study suggest that teachers' understanding of problem structure may be an important component of their content and pedagogical content knowledge (PCK) in mathematics. This is consistent with the work of several writers who stress the importance for primary teachers of having a deep and connected understanding of mathematics in order to teach it effectively (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). However, bringing about reform in mathematics education is challenging and time consuming (Anthony & Hunter, 2005).

Analysis of the lesson transcripts showed that teachers stuck very closely to the lesson description in the resource book (Ministry of Education, 2008, p. 38), mostly using the IRE (Initiation, Response, Evaluation) pattern in their interactions with the students. Although the teachers appeared committed to teaching for understanding, many of the lessons were taught in a fairly procedural manner. An alternative to following the instructions in the resource book for the scenario described in the procedure could have been to begin the lesson by letting the students solve the problem in their own preferred ways. If no student spontaneously used equal additions, then the teacher could suggest trying out this strategy to check its effectiveness. When this approach was used with Bachelor of Teaching (Honours) students, they seemed to be particularly impressed with the elegance and efficiency of the equal additions strategy after having initially tried a less efficient strategy of their own choosing. Alternatively, multiple ten-frames with beans, including some grouped in canisters of ten, seem to show far more clearly than paper money the number of beans that need to be added to the subtrahend to make it into a tidy number. It would have been good to see the Yr 5–6 teachers encouraging the students in their groups to discuss their ideas with peers, justify their solution strategies, and resolve differences in viewpoints.

Conclusions

Observing the equal additions lesson highlighted for us just how complex a process like subtraction can be. Teachers need to have a deep and connected understanding of mathematics, including knowledge of problem structure and number properties. Although resource books such as the one used for this lesson include some useful activities, it is vital that the underlying purpose and structures are clearly articulated, and teachers realise that they need to study the lesson until they fully understand it. Otherwise teachers may pick up such a book and follow a lesson prescriptively, and because they missed the point of the lesson, cause further confusion for their students. The findings of this study highlight the fact that mathematics education reform is difficult, and it takes considerable time to shift classroom discourse patterns.

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PROFESSIONAL PAPERS

IMPLEMENTING PROBLEM SOLVING IN AUSTRALIAN CLASSROOMS: ADDRESSING STUDENTS' AND TEACHERS' BELIEFS

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Most teachers believe learning how to solve problems is an important goal, and report teaching problem solving in mathematics lessons. Some students have different views about what occurs in mathematics lessons. These inconsistencies may be a consequence of different understandings about the purpose of school mathematics and what constitutes problem-solving activity. *The Australian Curriculum: Mathematics F to 10* emphasizes the important role of problem solving in learning mathematics and has supported teachers' implementation by embedding problem solving and reasoning into the content. However, effective implementation may only be realized when students' and teachers' beliefs are addressed.

Introduction

Teachers generally endorse a focus on problem solving in school curriculum and agree that problem solving is an important life skill for students to develop. Given the amount of policy advice and resource development, there are concerns about the limited opportunities for Australian grade 8 students to engage with problems other than those of low procedural complexity (Hollingsworth, Lokan, & McCrae, 2003). This suggests that teachers' beliefs about the importance of problem solving are not being supported by actions in their classrooms. There may be good reasons why problem solving seems to have a less prominent place in mathematics classrooms than may be intended. Frequently teachers' plans are thwarted by a range of contextual factors that include interruptions, and the urgent daily requirements that tend to take up so much of teachers' time. Another constraint may be students' views about problem solving. If teachers and students have different perspectives about problem solving in learning mathematics, it is possible that teachers' best efforts may be thwarted by students' actions.

Implementing problem solving in mathematics classrooms

Developing successful problem solvers is a complex task. According to Stacey (2005), there are many factors involved (see Figure 1). Students need deep mathematical knowledge and general reasoning ability as well as heuristic strategies for solving non-routine problems. It is also necessary to have helpful beliefs and personal attributes for organizing and directing one's efforts. Coupled with this, students need good communication skills and the ability to work in cooperative groups.

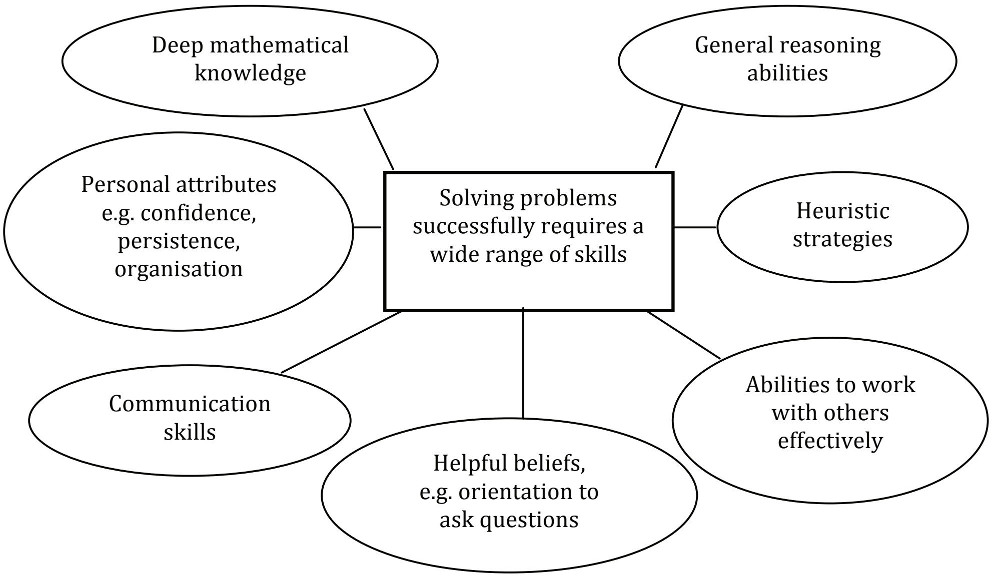

Figure 1. Factors contributing to successful problem solving (Stacey, 2005, p. 342).

Since the introduction of problem solving in school mathematics in Australia in the 1980s, teachers have had many opportunities to build knowledge about teaching problem solving and using problems as a focus of learning in mathematics classrooms (Clarke, Goos, & Morony, 2007). Advice for teachers has been provided in a range of publications including books and professional journals, in national curriculum statements (Australian Education Council, 1991) as well as in state and territory curriculum documents. Such advice has been accompanied by pre-service and in-service programs to change teaching practices from more traditional approaches to contemporary or reform methods where teachers use non-routine problems and problem-solving tasks as a focus for learning (Anderson & Bobis, 2005).

Problem-solving tasks can be used in classrooms in different ways. Wright, (1992) described two distinct approaches teachers adopt according to their beliefs about student learning. The 'ends' approach to problem solving presents students with problems at the end of the topic after skills and procedures have been rehearsed. This is based on a belief that students need mathematical content and procedures before they can solve unfamiliar problems; an approach also referred to as teaching *for* problem solving. The 'means' approach to problem solving uses problems as the focus of learning with problems used to provide a stimulus for student thinking. This approach has also been referred to as teaching *through* problem solving. A third possibility is for teachers to teach *about* problem solving so that students learn a range of heuristic strategies. All approaches have merit since they involve the students solving problems at some stage during mathematics lessons.

Teachers' views about problem-solving tasks and teaching approaches

The development of successful problem solvers requires regular experiences with a range of question types. To explore primary school teachers' problem-solving beliefs and practises including their preferences for particular questions types or tasks , the author collected data using a questionnaire (Anderson, 2003). If a problem is defined as “a task for which the students have no prescribed rules or memorized procedures” (Van de Walle, 2003, p. 67), then mathematical problem-solving tasks do not need to be elaborate or complex. However, whether a task is a problem or not will depend on the level of understanding of the particular students the teacher is working with. To focus teachers’ responses to the questionnaire, a classification of mathematics question types was developed for each of: exercise, application problem, unfamiliar problem and open-ended problem (Table 1).

Table 1. Background information categorizing question types for two-digit addition.

Background Information:
For the purpose of this questionnaire, the following definitions are given to assist understanding of the terms used. The examples are questions used by teachers when teaching two-digit addition.

Mathematics Question Type:	Example:
Exercise (a routine question for practising skills)	35 + 27
Application problem	If there are 32 oranges in one box, 37 in a second box, and 35 in a third box, how many oranges are there altogether?
Unfamiliar problem (a problem type students haven’t seen before)	There are pigs and chickens in a farmyard and altogether there are 23 heads and 68 legs. How many pigs and how many chickens are there?
Open-ended problem	□ □ What might the missing numbers be? + □ □ 1 3 4

Teachers were asked to indicate how frequently they used each type of question (Table 2). Overall, the majority of teachers used application problems and exercises more frequently than open-ended problems or unfamiliar problems. This may be a result of a reliance on textbooks and worksheets, or it may be that teachers are more confident with these particular student question types.

Table 2. Frequency of use of student question types (%), n=162.

Types of Questions	Rarely	Sometimes	Often
Exercises	5	27	68
Application Problems	4	26	70
Unfamiliar Problems	37	52	11
Open-ended Problems	22	58	20

Judgements about teachers’ problem-solving beliefs were made on the basis of their level of agreement with questionnaire items. The responses from 162 primary-school teachers to forced choice and open-ended comments were used to place each teacher in a category of having mostly traditional beliefs (n=23), mostly reform-oriented beliefs (n=20) or mixed beliefs about the use of problem solving in mathematics lessons. To compare teachers’ reported beliefs with their reported practices in classrooms, further items on the questionnaire asked them to report frequency of use of a range of teaching practices (see Anderson, White, & Sullivan, 2005).

Generally it seems that the reported beliefs and the reported practices are linked. The teachers with more traditional beliefs reported using strategies that were compatible with a transmissive style of teaching in that they frequently had students working alone, they provided detailed explanations about how to do problems, and they frequently set exercises for skills practice. The teachers with more reform beliefs reported using practices that gave responsibility to the students by encouraging group work, providing less initial explanation, encouraging individual recording, and allowing students to explore mathematical ideas. Both the traditional and reform teachers reported frequently modelling problem-solving processes and discussing problem-solving strategies with their students. Perhaps the difference between the teachers is not so much on the value they attribute to problem solving but on how students learn to solve problems and how students respond to their problem-solving lessons.

Further analyses of questionnaire responses revealed key beliefs which impact on teachers' choice of question types (or problems) and how they were presented to the students. In summary, for many teachers, the main beliefs impacting on, or constraining, problem-solving implementation were:

- students need to learn the 'basics' first before they can do problem solving;
- students give up if they cannot do a maths problem quickly;
- open-ended problems are more suitable for gifted and talented students;
- students with language difficulties have trouble doing application problems;
- problem solving is more appropriate for students in the upper grades of primary school; and
- problems take up too much time in the over-crowded mathematics curriculum.

Many of these beliefs reflect the contextual nature of beliefs with teachers adopting different problem-solving practices based on the characteristics of the students in their class. For at least some primary school teachers, confidence and experience also impacted on whether they implemented problem solving in their classrooms.

Research into secondary mathematics classrooms has revealed similar beliefs with many teachers reporting not enough time and lack of resources as reasons constraining problem-solving implementation (Wilson, Fernandez, & Hadaway, 1993). Beswick (2005) used a survey to gather data about secondary mathematics teachers' beliefs and a classroom environment survey to collect data from their students. The relationship between beliefs about the nature of mathematics, beliefs about mathematics teaching, and beliefs about mathematics learning are summarised in Table 3 (Beswick, 2005, p. 40).

Table 3. Relationships between beliefs.

Beliefs about the nature of mathematics (Ernest, 1989)	Beliefs about mathematics teaching (Van Zoest et al., 1994)	Beliefs about mathematics learning (Ernest, 1989)
Instrumentalist	Content-focused with an emphasis on performance	Skill mastery, passive reception of knowledge
Platonist	Content-focused with an emphasis on understanding	Active construction of understanding
Problem-solving	Learner-focused	Autonomous exploration of own interests

Beswick’s (2004) investigation of ‘Andrew’s’ beliefs revealed he held a problem solving view of mathematics and a constructivist view of mathematics learning. However, his beliefs and practices varied between grades providing evidence of the impact of context. In addition, there were differences between Andrew’s reported student-centred classroom teaching approaches compared to his students’ views about the classroom environment, particularly for the grade 10 students. As noted by Beswick “the data suggest that in his grade ten class Andrew was more likely to set the tasks and to be the arbiter of correct solutions” (p. 115). However his students reported the classroom was less student-centred than Andrew reported. These differences need to be explored if problem solving is to be valued by both teachers and students and if real problem solving is to form a key component of classroom activity.

Students’ views about problem solving

Stacey’s framework in Figure 1 indicates that developing successful problem solvers requires more than choosing appropriate problems, and deciding when and how to introduce them to students. Teachers also need to consider students’ personal attributes and their beliefs about mathematics and the role of problem solving in learning mathematics. Students’ beliefs about problem solving may influence their reactions to teachers’ attempts to use problem-solving approaches in classrooms. Because of previous experiences, students may develop narrow beliefs about mathematics with problem solving viewed as not legitimate mathematical activity, particularly if it is not associated with working from the textbook. Schoenfeld (1992) identified student beliefs which do not support a problem-solving approach in classrooms. These included:

- mathematics problems have one and only one right answer;
- there is only one correct way to solve any mathematics problem;
- mathematics is a solitary activity, done by individuals in isolation; and
- students who understand the mathematics will be able to solve any problem in five minutes or less.

For problem solving to be a focus of mathematics classrooms, students’ beliefs about the nature of mathematics may need to be challenged yet this can be frustrating for teachers. Identifying differences in teachers’ and students’ views about problem solving helps to develop teachers’ understandings about the challenges they may face.

More recently, Anderson (2008) used the same question types from Table 1, to collect data from seven teachers and the students across grades 3 to 6 to identify and explain which question types were problems and how frequently they were used, as well as the frequency of a range of teaching strategies. For the open-ended problem, few students in grades 4 to 6 indicated it was a problem compared to application and unfamiliar problems. The main reasons given suggested it only required the application of an algorithm with no recognition there were multiple solutions. Students’ experiences of these types of questions provided some indication of the source of their beliefs as teachers indicated they hardly ever use unfamiliar or open-ended problems, with students frequently doing exercises to practise skills and procedures.

Teachers and students were asked to report the frequency of use of particular teaching strategies in mathematics lessons. Several items revealed similar views about what was occurring. For example there was agreement between teachers and students across all grades that students hardly ever posed their own problems or were able to choose which problems they could solve.

Other items revealed differences between teachers and their students. Teachers reported rarely having students work alone or getting students to record answers to problems in their own way. However, the majority of students in every grade reported they frequently worked alone and rarely developed individual recording methods. Teachers reported they more frequently showed students exactly how to solve mathematics problems, allowed students to use calculators and concrete materials, and used real-life problems with students. Students indicated these strategies were used less frequently. These differences were not extreme but the teachers were surprised by what the students suggested. Students' responses provided an opportunity for fruitful discussions about problem-solving strategies and how teachers' efforts could be further developed to improve students' attitudes to problem solving in mathematics lessons.

Some teachers did have more traditional beliefs and used practices in their classrooms which reinforced a view of problem solving as doing routine exercises rather than open-ended or non-routine problems. While other teachers reported more reform-oriented beliefs, their practices did not necessarily reflect their beliefs. If problem solving is to become a regular component of mathematics lessons, new approaches will be required by all teachers of mathematics across all of the grades of schooling.

Problem solving in the *Australian Curriculum: Mathematics*

Problem solving is one of four proficiency strands in the new *Australian Curriculum: Mathematics F to 10* (Australian Curriculum, Assessment and Reporting Authority, 2010, p. 3) and is described as follows.

> Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable.

This description for problem solving suggests students need to actively engage with a range of important processes during mathematics lessons. For this to occur teachers will need to select tasks which allow for student choice about the mathematics and the problem-solving strategies they use to model and investigate situations. Importantly, students also need to be able to communicate their solutions in their own ways. Problem solving involves investigating new and somewhat challenging situations that require time and effort. For many students, problem solving will need to be more than just doing questions which are applications of the mathematics they are learning right now.

It is an ongoing challenge for teachers to develop successful problem solvers given the constraints acting against teachers' intentions and best efforts. The challenges for teachers are to:

- ask fewer questions in each lesson but use questions or problems which require reasoning;
- use rich problem-solving tasks including investigations and open-ended questions;

- discuss with students the role and purpose of problem solving in learning mathematics; and
- allocate time for students to grapple with the underlying mathematical ideas.

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OVERCOMING LANGUAGE BARRIERS IN WORD PROBLEM SOLVING: A FRAMEWORK FOR INTERVENTION

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We describe a pedagogical approach aimed to assist second-grade Filipino children to solve additive word problems in English, a language primarily encountered only in school. The impact of the intervention is exemplified through a case study of one child with sufficient understanding of additive structures but with poor English skills. The intervention provided linguistic and mathematical scaffolds focussed on linking the text, situation, and problem representations. Conveying additive part-whole concepts in Filipino using a range of representations strengthened knowledge of additive situations and facilitated success for English word problems. However, correct solutions did not necessarily imply coherent mappings between the strategy and the text.

Aerwen you 8-ge yingbi. Ranhou Jun zai gei-le ta yixie. Xianzai Aerwen you 14-ge yingbi. Jun gei-le ta ji-ge yingbi?

If you do not speak Chinese, how would you approach the word problem above? When translated to English, the word problem reads, "Alvin had 8 coins. Then Jun gave him some more coins. Now Alvin has 14 coins. How many coins did Jun give him?" This is an example of a Missing Addend problem which can be solved by many first-graders (Carpenter & Moser, 1984). However, with insufficient proficiency in the language of the problem, even a very simple problem becomes difficult, if not impossible, to solve with semantic understanding. Unfortunately, many children face this challenge because they learn mathematics in a language not widely spoken in the community. Such is the case in several African nations (Obondo, 2007), in remote Indigenous communities in Australia (Simpson, Caffery, & McConvell, 2009), and in the Philippines (Young, 2002) where this study was conducted.

As part of a two-year project, we developed a series of assessments and interventions (Bautista, Mitchelmore, & Mulligan, 2009; Bautista, Mulligan, & Mitchelmore, 2009) aimed to help Filipino children solve additive[1] word problems in English, a language they generally encounter only in school. Our intervention was based on Kintsch's (1986) model of word problem solving. While the outcomes of the intervention are discussed elsewhere (Verzosa, in press), this paper exemplifies the instructional scaffolding for one child who was mathematically competent but unable to solve word

[1] Additive word problems refer to word problems that may be solved by either addition or subtraction. The reader may refer to Carpenter and Moser (1984) for a taxonomy of additive word problems.

problems in English. We describe the outcomes of our intervention so that others may (1) learn the strengths and limitations of our pedagogical approach, and (2) adapt Kintsch's framework for designing an instructional sequence and assess children's performance in solving word problems.

The Kintsch model

Kintsch's (1986) framework (elaborated by Nathan, Kintsch, and Young (1992)) was informed by theories on how readers process text. Within this framework, word problem solving consists of three inter-related mental representations—the *textbase*, *situation model*, and *problem model.* The textbase depicts the meaning of the words in the text and how these relate to each other. It also represents a superficial type of comprehension. For Kintsch (1994), true comprehension occurs when one constructs a representation not only of the text itself, but also of the *situation* described by the text. Called the situation model, it may contain elements not explicitly mentioned in the text. Finally, the problem model for additive problems is a part-whole representation of the given and unknown quantities in the problem.

An example

The textbase representation of the sentence "Alvin had 8 coins" contains representations of the concepts of a person (Alvin), of coins, and of possession. The situation model may be an image of a little boy with eight coins in his pocket. It depends on a reader's prior knowledge and goals for reading text, and can differ from one person to another (Grabe, 2009).

The problem model for the Missing Addend problem given at the beginning of this paper specifies a mathematical part-whole relation between Alvin's initial (part) and final (whole) sets of marbles. This mathematical structure elicits a corresponding subtraction strategy (14 – 8) to determine the unknown part.

The situation model

Meaningful problem solving entails carrying out a strategy based on a situation model, rather than simply on an incoherent textbase consisting of keywords from the text (Mayer, 2003; Thevenot, 2010). Additionally, an appropriate situation model promotes correct solutions, as children often solve problems by directly modelling the situation described by the text (Carpenter & Moser, 1984). These direct strategies are what Brissiaud and Sander (2010) call *situation-based*, and not necessarily connected to problem models. Situation model construction hinges on *linguistic* and *mathematical* knowledge, elaborated as follows.

- **The solver should know most of the words in the text.** In a clever experiment, Hseuh-Chao and Nation (2000) asked 66 adults to read one of four versions of a 673-word simple text. The versions differed in the proportion of words (20%, 10%, 5%, 0%) replaced by nonsense words. While most of those reading the intact text could answer questions about the text, none of those reading the 20%-nonsense version, and very few of those reading even the 5%-nonsense version could. Their analysis further suggests that at least 98% of the words in a text should be known in order for adequate comprehension to take place.

- **The solver should possess an understanding of the additive part-whole structure of sets and a flexible understanding of number meanings.** Text comprehension research shows how domain knowledge is important for comprehension (Hirsch, 2003). A reader unfamiliar with cricket, for example, could not be expected to comprehend a newspaper article about cricket. The same applies to word problems. We found that some children could not solve Missing Addend problems even when problems were narrated to them in their first language (Filipino) because they were limited to conceptualising disjoint sets with known quantities (Bautista & Mulligan, 2010). Further, some could only conceptualise number as the final number-word in the counting process. They did not realise that numbers may exist outside the counting activity, as when these represent an unknown transformation in the Missing Addend problem (Nunes & Bryant, 1996).

Using Kintsch's model for intervention design

Guided by the three components of problem solving in Kintsch's (1986) model, we designed a pedagogical approach aimed to strengthen each component as well as appropriate mappings between them (Figure 1). Our basic conjecture was that we could develop the three components of word problem solving *simultaneously,* rather than sequentially. In practical terms, this meant not having to wait for children to acquire skills necessary to construct a coherent textbase before providing them with opportunities to develop their mathematical knowledge.

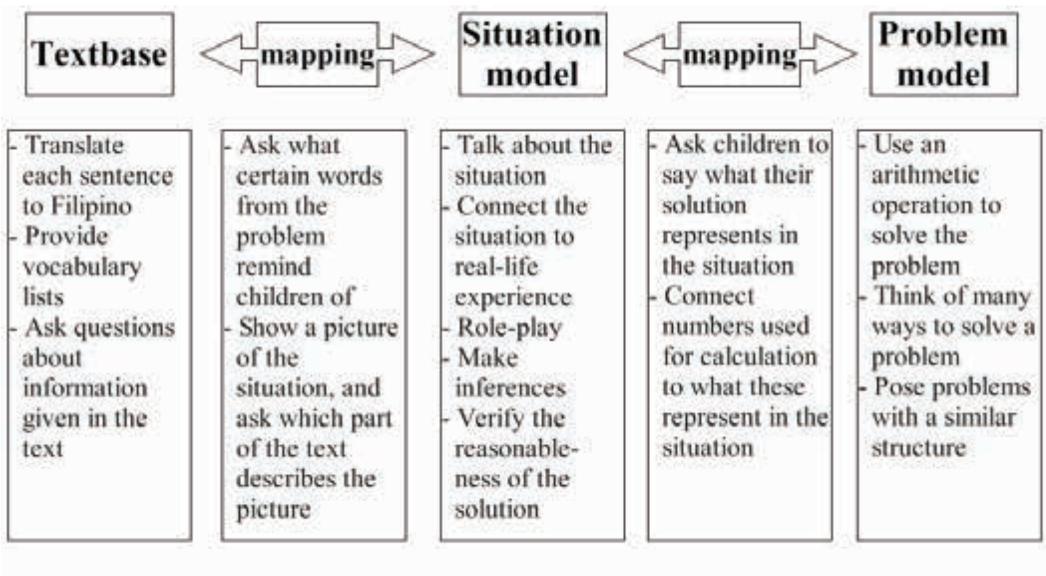

Figure 1. Pedagogical. approach based on Kintsch's model.

Because the children in our study had not yet acquired communicative English skills, we conveyed mathematical concepts in Filipino, using a range of representations. For example, the tasks in Table 1 show the various ways we represented the Missing Addend problem at the start of this paper. In this way, it became possible to communicate mathematical structures while circumventing the English language difficulty. At the same time, linguistic support was also provided (see strategies listed under Textbase in Figure 1).

Table 1. Various representations for the Missing Addend task 8 + = 14

Mode of representation	Typical Tasks or Activities
Concrete	Screening task (Wright, Martland, & Stafford, 2000): Briefly display 8 blocks. “I will join some blocks to the 8, but I will not tell you how many.” Join 6 blocks to the original 8, without showing the child the number of additional blocks. “Now, there are 14 blocks altogether. How many blocks are in the bag?” [presented in Filipino]
Pictorial	
Verbal-pictorial	“*Wish ko lang* [I wish I had]” task (Kolson, Mole, & Silva, 2006): Show 8 dots. “I have 8 dots. I wish I had 14. How many dots do I need?” [presented in Filipino]
Textual	Gina had 8 bags. Ramon gave her some more bags. Now, Gina has 14 bags. How many bags did Ramon give Gina? [presented in English or Filipino]
Symbolic	8 + = 14

Note: These representations were provided to the children in the order prescribed above, which correspond to increasing levels of abstraction.

The study

The intervention described in this paper was carried out over seven two-hour sessions spread over three weeks. Ninety Filipino children who had just finished second grade took part. This paper presents an analysis of the outcomes of the intervention through a case study of Vilma[2], who was typical of children with sufficient understanding of additive structures but with poor English skills. She was nine years old and had just finished Grade 2 when she participated in the intervention. She completed four assessments. The first was a written test pre-intervention. Individual interviews were also administered over three stages—before, immediately after, and four months after the intervention.

The interview consisted of the following steps:

- Step 1. Present a problem written in English.
- Step 2. If a correct solution was not reached, present the same problem written in Filipino (linguistic scaffold #1).
- Step 3. If a correct solution was not reached, narrate the same problem as if it were a story (linguistic scaffold #2).

Problems used for post- and delayed post-assessments shared similar structural features with those given in the pre-test, and differed only in their surface characteristics (e.g. using pencils instead of coins), and number triples. The number triples were all in the range 1-20, and were based on Carpenter and Moser’s (1984) study.

To determine how the intervention influenced word problem solving performance, data from each interview were analysed with respect to the linguistic scaffold required

[2] pseudonym

by the child to solve the problems and the mathematical strategy used to calculate the correct numerical solution.

Intervention outcomes

Pre-intervention written test results

In the written test, Vilma answered two out of ten additive word problems (five in English, five in Filipino). She also used idiosyncratic strategies to solve word problems. For example, she solved the English problem *Jimmy has 5 cards. Tony has 7 cards. How many cards do they have altogether?* by multiplying 5 and 7. Moreover, she used the same strategy (addition) for all five Filipino word problems, even when three are solved by subtraction.

Interview results: Which linguistic scaffold helped?

Table 2 displays the stage (i.e., English, Filipino, narrated), if any, at which a correct solution was reached. Before the intervention, Vilma could solve problems only when these were narrated to her. Her understanding of the statement *Rica has 12 books* was “Notebook*”*, and she did not know what *has* meant. Filipino translations also did not help. Even when reading Filipino text, she could not identify the giver (Alma) in the Missing Addend problem in Table 2.

Table 2. Stage during the interview where a correct solution was reached.

Problem type	Sample problem	Pre-test	Post-test	Delayed post-test
Join	Alvin had 3 coins. Then Jun gave him 8 more coins. How many coins does Alvin have now?	NA	F	F
Separate	Alex had 14 dogs. Then Alex gave 6 dogs to Carla. How many dogs does Alex have now?	NA	N	E
Missing Addend	Jolina had 7 pencils. Then Alma gave her some more pencils. Now Jolina has 12 pencils. How many pencils did Alma give her?	N	E	E
Part Unknown	There are 11 marbles. Four of these belong to Jimmy. The rest belong to Mia. How many marbles does Mia have?	N	N	E
Start Unknown	Mark had some *pan de sal*[3]. Then he gave 6 *pan de sal* to Chris. Now Mark has 8 *pan de sal* left. How many *pan de sal* did Mark have in the beginning?	N	N	N
Compare	Rica has 12 books. Luis has 7 books. How many more books does Rica have than Luis?	X	E	F

Note: E – problem solved in English; F – problem solved in Filipino; N – problem solved when narrated; X – problem incorrectly solved at each stage; NA – not asked.

The Compare problem was not correctly solved at any stage of the interview. It was difficult to convey the Compare structure verbally because she did not know the words “*higit* [more]” or “*lamang* [extra].”

There were some improvements after the intervention. She could already solve a Compare problem, but she continued to rely on narration strategies for most problems.

[3] A type of bread common in the Philippines.

She also solved two problems in English. While she remembered the meaning of some words (such as *gave*) that were taught during the intervention, she could not identify the giver from the Join problem text, *Bing gave 6 bottles to Ted.* She also still could not understand simple statements such as *Alex had 14 dogs.*

The delayed post-assessments show that she relied less on narration strategies. She solved three problems without assistance, but these did not include the Compare problem which was solved in English in the post-test. Similar to her post-test results, Vilma still required a Filipino translation for the Join problem. She interpreted the Join problem as a Missing Addend problem—she stated that the question was, "*Ilan yung binigay sa kanya* [How many were given to her]?," and stated her answer as, "*Tatlo yung binigay sa kanya* [Three were given to her]."

Interview results: Which mathematical strategies were used?

Vilma used a range of strategies when solving problems. She did not always use an arithmetic operation, and she did not always need to directly model the action in the problem. For example, she solved the Part Unknown (pre-intervention) problem by performing a "*bawas* [take-away]" calculation, but cited an addition fact [7+5=12] to solve the Part Unknown problem (post-intervention) involving 12 – 5 = ? As a second example, she did not need to represent and concretely compare two sets when solving the Compare problem after the intervention—she solved the problem by performing a take-away strategy.

Discussion

We analyse Vilma's performance by relating it to the three components of problem solving. Before the intervention, Vilma was a typical example of a student who already possessed the mathematical knowledge required to solve all problems that were presented to her, with the exception of the Compare problem. Moreover, she was not limited to solving problems that could be directly modelled. Rather, she recognised how the structure of the problem situation linked to her problem model representation of sets and operations.

The main difficulty was that Vilma could only access her mathematical knowledge when problems were narrated (in Filipino) to her. This initial profile suggested that Vilma could not construct a situation model from the text because she was unfamiliar with common English words. Additionally, her undeveloped reading comprehension skills prevented her from retrieving information from the text.

The intervention exposed Vilma to a range of additive situations. Thus, she developed mathematical knowledge (a problem model) related to comparisons, allowing her to solve a Compare problem, post-intervention. Moreover, she solved two problems in English. These findings strengthen the argument we presented at the outset, about how domain knowledge (in this case, knowledge about various additive situations) contributes to enhanced situation models, and thus, correct solutions for English problems.

However, *our findings also suggest that an appropriate situation model may not necessarily be based on a coherent textbase.* For example, Vilma solved two problems in English, and yet she still could not understand the statement *Alex had 14 dogs.* A plausible explanation could be that, upon reading the text, Vilma recalled some of the additive situations discussed during the intervention, and based her situation model on her recollection. This assertion is supported by Vilma's performance in the post- and delayed post-assessments. *While she could correctly solve a Missing Addend problem in English, she required a linguistic scaffold to solve the Join problem which was definitely easier*—of the 88 children in Carpenter and Moser's (1984) study, none could solve Missing Addend, but not Join, problems. Thus, a glaring limitation in Vilma's progress was in the mapping between the textbase and situation model. The minor intervention of providing Filipino translations of English words commonly found in word problems did not seem to help.

Unless the language difficulty is addressed, Vilma would have no other option but to base her situation model on a few words and the given numbers in the text. Returning to the Alvin problem at the start of this article, a non-Chinese speaker may be forced to impose a situation on the text, and perform a calculation based on this situation.

Implications

This case study cannot be generalised to all children learning mathematics in an imported language. However, we consider the results to be a good description of how linguistic difficulties impede performance. Results also show how it is possible to help children conceptualise a wider range of additive situations in spite of difficulties in language and reading comprehension. The findings point to several implications, as follows:

- It should be recognised that learning in an imported language is challenging for both teachers and students. Recall that we used Filipino to convey mathematical situations and concepts. If we had to carry out our intervention in English, the children's unfamiliarity with the language would have forced us to remain at a textbase level of discourse, focusing on key words as cues for a strategy.
- Because language policies may take time to change, other avenues for providing language support such as code-switching or targeted professional development programs need to be explored and prioritised. Otherwise, children may have to cope with learning mathematics in an imported language in their own ways.
- Language difficulties should not be an excuse to delay mathematics teaching of developmentally appropriate content knowledge. Mathematical concepts should still be conveyed, using various representations as a way to circumvent the language barrier.
- Interviews or informal conversations that scaffold children towards correct solutions are necessary to help teachers identify mathematical strengths or weaknesses that may be initially masked by language difficulties.
- As we have done, teachers may adapt our pedagogical approach, based on Kintsch's (1986) framework, to design an instructional sequence and identify both children's progress and pervasive difficulties with respect to the three components of problem solving.

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BEGINNING TEACHERS' MATHEMATICAL KNOWLEDGE: WHAT IS NEEDED?

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Over the past decade there has been growing interest in describing and measuring the kinds of mathematical knowledge needed by teachers. Such efforts are in parallel with the development of national standards for teachers, indicating levels of expectation across the years of teachers' careers. This presentation provides an opportunity for teacher educators and teachers to consider the nature of mathematical knowledge needed by beginning teachers at all levels of schooling. Discussion will be informed by data from an ALTC funded national project that aims to improve the quality of pre-service teachers' outcomes in mathematics and by the AAMT Standards framework.

Introduction

Interest in beginning teachers' mathematical knowledge is not new. At the first MERGA conference, Brown (1977) described growing concerns about the mathematics knowledge of pre-service teachers and what he described as "anti-mathematical" backgrounds. In response to these concerns a remediation program was described "which is almost identical to that necessary in the lower secondary or upper primary schools" (p. 45).

In 1987, Shulman's seminal work identified three domains of teacher knowledge: subject-matter knowledge, pedagogical content knowledge, and curricular knowledge. Subject-matter knowledge includes all of those ideas fundamental to the domain, pedagogical content knowledge extends to such matters as useful forms of representation, explanations and examples of the domain, and curricular knowledge includes understanding of how the subject-matter is organised over the years of schooling (Shulman, 1987).

A number of studies have deepened understanding of the kind of knowledge that teachers need for teaching. Mewborn (2001) showed that crude measures of teacher knowledge, such as the number of mathematics courses taken, were insufficient to characterise teachers' mathematical knowledge for teaching. Hill, Schilling and Ball (2004) developed measures of teachers' mathematical knowledge for teaching (MKT) using multiple choice items that could be described broadly as mathematics content knowledge set in a classroom context. Watson (2001) used a profiling approach with a range of questions that addressed all of Shulman's (1987) knowledge types. Using a similar instrument, Beswick, Callingham and Watson (2011) demonstrated that the different knowledge types could be considered as a single domain, providing an holistic conception of teacher knowledge for mathematics teaching that included beliefs about and attitudes towards mathematics as well as classroom focussed mathematics understanding. Further, they showed that the domain had a hierarchical structure in which general pedagogical knowledge and pedagogical content knowledge (PCK) related specifically to teaching mathematics were at the upper end of the scale and everyday numeracy was at the lower end.

Callingham and Watson (in press) focussed on pedagogical content knowledge restricted to the area of statistics. They used items of two main types – those in which teachers were asked to identify likely responses from their students to a particular question, and then to suggest appropriate interventions to one of these responses, and secondly, those in which they chose their "next steps" in response to questions showing students' actual answers. These items attempted to capture both the diagnostic element of teachers' knowledge and their understanding of students' learning in the domain of statistics. A four-level hierarchy of teachers' PCK was identified which could be used to both identify teachers' understanding and also measure teacher change.

The Australian Association of Mathematics Teachers (AAMT) developed a rich description of the characteristics of exemplary mathematics teachers (AAMT, 2002/2006) through a project that brought together teacher expertise and research findings. This description of *Standards for excellence in teaching mathematics for Australian Schools* has three domains: Professional Knowledge, Professional Attributes and Professional Practice. These domains address the various knowledge types described by Shulman (1987) and aim to provide a basis for identifying exemplary teachers of mathematics. More recently, the Australian Institute for Teaching and School Leadership (AITSL) (2011) published a set of generic teaching standards that described seven standards across three domains: Professional Knowledge, Professional Practice and Professional Engagement. Of particular interest is that the AITSL document included four levels to describe different career stages, including graduate standards. The graduate standards are particularly relevant to the project reported here, which has a focus on improving pre-service teachers' mathematical outcomes.

These various recent developments describe a rich context in which the collaborative project described here takes place. An increased attention to the forms of knowledge required for teaching mathematics, along with explicit descriptions of teaching standards at various levels, and a new mechanism for the accreditation of teacher education courses together require thoughtful responses by those engaged in mathematics teacher education. The systematic use of evidence to support professional opinion in the shaping and refining of mathematics teacher education programs is a critical part of that response, and the major focus of the project.

Background to the study

Building the Culture of Evidence-based Practice in Teacher Preparation for Mathematics Teaching (CEMENT) is a two-year project that aims to produce:

1. Evidence-based changes to mathematics education teaching within participating universities;

2. Recommendations about effective models of teacher education for teaching mathematics;
3. Processes for bringing about change at unit and course levels; and
4. Progress towards a national culture of evidence-based practice in relation to mathematics teacher education.

The project team (authors) represent seven universities across all states and the Northern Territory, which include diverse institutions delivering a wide variety of teacher education courses. The mathematics education taught within the differing programs varies in the amount of time allocated, the nature of the content and delivery and the placement within the overall course structure. In order to meet the aims of the project, data were needed about what pre-service teachers at the end of their course knew and understood about mathematics teaching. There were limitations on the nature and amount of data that could be collected. Because of time and manpower constraints and the national nature of the study, it was decided that an automatically scored web-based survey would be used, which in turn limited the nature of the items. The focus of the survey needed to go beyond content knowledge of mathematics alone, and to include aspects of pedagogical content knowledge. In addition 10 items addressing teacher beliefs about mathematics and its teaching were included. Collaboratively, the team developed items that included all of these domains. A selection of these items was piloted with students at the University of Tasmania who were undertaking mathematics education units over the summer semester. This pilot study is the focus of this report.

Method

Sample

The students in the sample were all undertaking a pre-service course for primary teaching. The majority (*n* = 52, 86.7%) were studying off campus and were split almost equally between part-time (*n* = 29, 48.3%) and full-time (*n* = 31, 51.7%) study. Of the respondents, one-quarter (*n*=15, 25.0%) were aiming to graduate in 2011, with a further 29 students (48.3%) aiming to graduate by 2013.

Students were asked about their previous educational experience. Of the 55 students who responded, 23 (41.8%) had secondary schooling only, and 25 (45.5%) had a certificate level qualification, possibly reflecting some vocational training prior to university entrance. When their mathematics backgrounds were considered, 21 (38.2%) had only studied mathematics to Year 10, 11 (20.0%) had studied a non-pre-tertiary mathematics subject and 17 (30.9%) had studied a pre-tertiary mathematics subject in Year 11/12.

The sample was, therefore,towards the end of pre-service teacher education and had educational and mathematics backgrounds that have been reported elsewhere as typical of pre-service teachers (e.g., Ainley, Kos, & Nicholas, 2008). No information was collected about gender but the enrolment in primary education is predominantly female.

Instruments

A 45-item online test was undertaken by 60 pre-service primary teachers at the University of Tasmania. The instrument consisted of 10 items addressing beliefs about mathematics, 13 items addressing mathematics content and 23 items that addressed pedagogical content knowledge (PCK). Examples of items are shown in Table 1.

Table 1. Examples of items used in the pilot test.

Item category	*Example*
Beliefs	Mathematics is a beautiful and creative human endeavour
Beliefs	Students learn by practicing methods and procedures for performing mathematical tasks
Content knowledge	Which one of the following contains a set of three fractions that are evenly spaced on a number line? **A)** $\frac{3}{6},\frac{3}{5},\frac{3}{4}$ **B)** $\frac{3}{4},\frac{19}{24},\frac{5}{6}$ **C)** $\frac{3}{4},\frac{19}{24},\frac{7}{8}$ **D)** $\frac{4}{5},\frac{5}{6},\frac{7}{8}$
Pedagogical Content Knowledge (PCK)	A Year 5 teacher asked her pupils to determine the value of the following calculation on their calculators: 2 + 3 x 4 = The class was surprised to find that some student calculators gave a result of 14, while others gave a result of 20. Which of the following best matches your likely response to this situation? **A.** Use the difference as a motivation to teach the students how to use the correct order of operations, highlighting an acronym such as BODMAS. **B.** Show the students how to use parentheses or brackets when entering expressions into their calculators. **C.** Check school booklists and supplies to make sure that only one kind of calculator was available to students in the class. **D.** Ask the pupils to explain the different results, and use their explanations to discuss the order of operations as an arbitrary convention.

The Beliefs items used a five-point Likert scale from strongly disagree to strongly agree; content items were scored right or wrong. Following discussion among the project team, the PCK items were mostly scored dichotomously as right/wrong. Some PCK items, however, provoked considerable discussion and scoring was determined on the basis of an agreed hierarchy. The PCK item shown in Table 1, for example, was scored as A = 1, B = 2, C = 0 and D = 2 on the grounds that the responses for B and D both represented good “next steps” for developing understanding, and the response to A was reasonable but not of the same quality as the two scored at 2.

Data analysis

Data were analysed in various ways to provide a range of information. First the scored responses were analysed using Rasch measurement to provide quality control information about the items by a consideration of fit to the Rasch model (Bond & Fox, 2007). Three scales were produced: Beliefs about mathematics (BELF, 10 items), Mathematical Content Knowledge (MCK, 13 items); and Pedagogical Content Knowledge (PCK, 23 items). From each of these scales a measure of performance for each student was obtained in logits, the unit of Rasch measurement. These measures were used as a basis for comparisons between groups based on the background variables. Finally, frequency counts of students’ choices provided some diagnostic information.

Results

All of the three scales showed excellent fit to the Rasch model indicating that within each scale the items worked consistently together to measure a single construct that could be used to make inferences about students' performances. Performance measures were obtained for every student on each one of the three scales and used for further analysis.

Between groups analysis

Comparisons were undertaken between groups based on full-time/part-time enrolment, education background and mathematics background. No comparison was made between distance and face-to-face students because of the low numbers of students studying on-campus. No statistically significant difference was found among any of the groups on any measure. This finding is not surprising given the homogenous nature of the sample.

Performance on different kinds of scale

Boxplots of the distributions of students' performance measures on each of the three scales are shown in Figure 1. The scales show a monotonic decline in median score from BELF, to MCK to PCK indicating that of the three scales students found the pedagogical content knowledge more difficult than straight mathematics content knowledge, which was more difficult than endorsing beliefs about mathematics.

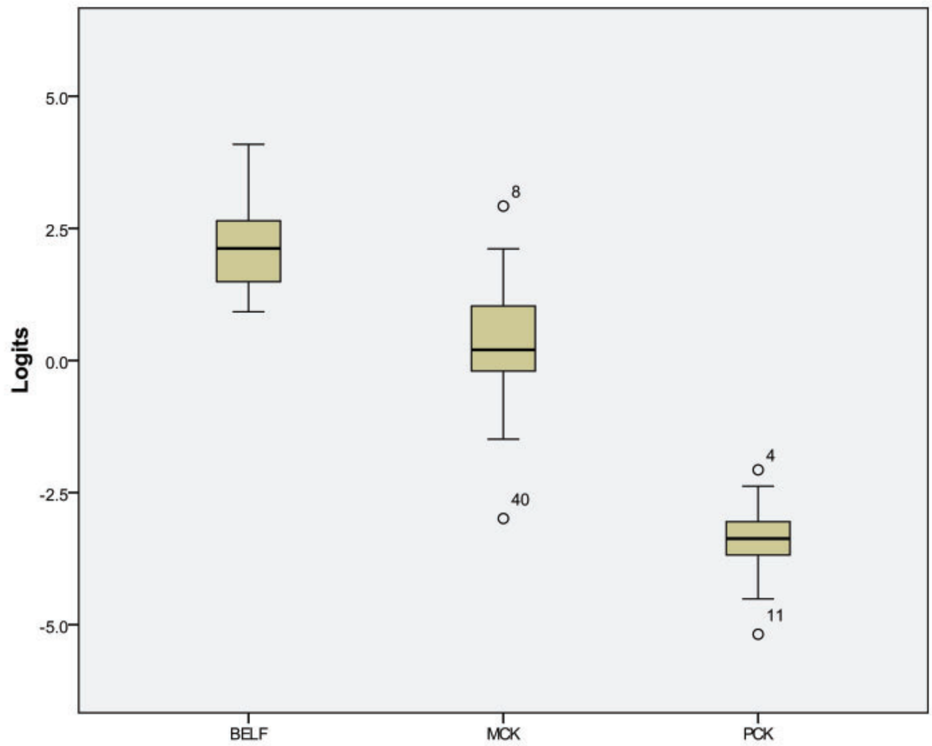

Figure 1. Distributions of students' performance measures (with outliers shown).

To explore this finding further, results from the Rasch analysis output were examined to identify specific items or groups of items that students found difficult. These findings are reported for each of the three scales.

Beliefs about mathematics

The most strongly endorsed items were those indicating a broadly student-centred view of mathematics learning, such as “The teacher must be receptive to the children’s suggestions and ideas” and “Teachers must be able to represent mathematical ideas in a variety of ways”. Students, however, also strongly endorsed “Acknowledging multiple ways of thinking may confuse children”, in apparent contradiction to the other two items. At the other end of the scale, students found it difficult to endorse “The procedures and methods used in mathematics guarantee right answers”, possibly reflecting an emphasis on process rather than product. “Mathematics is a beautiful and creative human endeavour” was also difficult for students to endorse, although it is not clear whether they disagreed with the beauty and creativity or with the human endeavour. “Mathematical ideas exist independently of human ability to discover them”, however, was also fairly difficult to endorse, suggesting that students understood mathematics as a human activity but did not see it as creative or beautiful.

Mathematical content knowledge

Among the MCK items, the most difficult was identifying the prime factors of 30. Unexpectedly, however, a majority of students (n = 30, 54.5%) chose the option listing all factors of 30 rather than the anticipated attractive distracter of “1, 2, 3, 5” suggesting that the students understand the notion of factor but not the idea of prime factor. The next most difficult item was the fraction item shown in Table 1. Only 15 (27.3%) students answered this correctly. Surprisingly, at about the same level of difficulty was “The product of an odd number and an even number is odd”, to which students had to choose from the options “always true”, “sometimes true” and “never true”. Only 16 students (29.1%) responded correctly. Whereas the prime number item was based on knowledge of mathematical language, both of the other two items were more conceptual in nature, raising issues about students’ underlying understanding.

At the other end of the scale, the easiest items were combinations based on a menu, a definition of congruence, identifying an incorrect representation of ¾, and a two-step computation based on reading currency conversions from graphs. The remaining items were all at about the mean difficulty level and consisted of a number of items based on geometry including an angle calculation, and one requiring an algebraic expression to describe a linear pattern. It seems that for this group of respondents, work on geometry and algebra would benefit them in terms of their mathematical development.

Pedagogical content knowledge

The easiest PCK items included the item shown in Table 1 about teaching an algorithm, and an item about choosing an appropriate representation to develop children’s understanding of proportional reasoning that was also scored with multiple codes. It is possible that by rescoring these items to try to allow for all reasonable possibilities that the items have lost their discriminatory power.

The other items all tended to bunch together on the scale which means that they provided a lot of information across a narrow range. It is possible that with a larger and more diverse sample, this difficulty might be overcome.

Of all the items, those addressing teaching aspects of measurement and geometry appeared slightly more difficult. Respondents could not, for example, identify rhombi from a collection of 2D shapes, and suggested incorrect teaching explanations for students. One surprising item addressed materials suitable for developing subitising skills. Students were provided with a description of subitising and a choice of five possible materials: number line, dominoes and dice, number expander, MAB, and a large collection of objects, all represented pictorially. Of the 48 respondents, 17 (35.4%) chose MAB and 18 (37.5%) chose the large collection rather than the dominoes and dice (n = 9, 18.8%).

Discussion

This pilot study is part of a much bigger project that aims to provide useful tools to universities so that they can monitor their pre-service teachers' mathematical development in three domains: beliefs and attitudes, mathematical content knowledge and pedagogical content knowledge. The items trialled produced coherent scales but additional work is needed on the PCK items to ensure that they discriminate more effectively. As a first attempt, however, the project team was relatively satisfied with the instrument.

The finding that PCK was more difficult than MCK and BELF is consistent with other research in the area (Beswick, Callingham & Watson, 2011). This finding raises issues for mathematics education about how best to develop PCK in pre-service teachers. Although MCK and PCK are inextricably linked, it seems that mathematics understanding alone is not sufficient.

The nature of the items that respondents found difficult provides information that can be used to revise courses in the relevant university. More work is needed in areas such as geometry and measurement, which have received little explicit focus compared with fractions and proportional reasoning, for example. Students appear to have difficulty with choosing appropriate representations and materials for teaching, and this could be addressed in workshops and online activities.

The standards and frameworks available at present (e.g., AAMT, 2002/2006; AITSL, 2011) provide useful information about desirable attributes but little support for developing these. The instrument described represents a starting point for providing data about some aspects of these attributes so that pre-service teacher education can develop courses and approaches based on information rather than solely on the opinions and beliefs of teacher educators. Further items have been developed by the project team, as well as similar instrument intended for high school mathematics pre-service teachers, including those who are likely to be teaching outside their specialisation. These instruments will be trialled and modified throughout 2011, and information provided to all participating universities to inform future course development.

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REFINING THE NAPLAN NUMERACY CONSTRUCT

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This paper provides an outline of the work done by ACER Mathematics test development staff for NAPLAN Numeracy in 2009 and 2010. It examines how the NAPLAN Numeracy construct can be further refined and its relation to psychometric and construct validity. The paper discusses some of the test development processes of NAPLAN Numeracy, some of the difficulties in constructing the test and some of the processes used to establish the validity of the measurement.

A brief overview of NAPLAN Numeracy

NAPLAN is the Australian National Assessment Program for Literacy and Numeracy. It has been operating in Australia since 2008 and is currently managed by the Australian Curriculum, Assessment and Reporting Authority (ACARA), who have issued guidelines for implementing the numeracy components. (ACARA, 2010) Prior to ACARA, NAPLAN was managed by various agencies under the joint oversight of the Australian States and territories and the Federal Government. In 2009 and 2010 the Australian Council for Educational Research (ACER) was contracted to conduct test development for NAPLAN 2010 and NAPLAN 2011. The paper describes some of the issues encountered by the team of ACER staff engaged in developing NAPLAN Numeracy, from a test development and test design perspective,. The views expressed are not those of ACARA or necessarily those of ACER.

Numeracy as a test construct

Presenting mathematics problems in something like a real world context possibly dates back nearly as far as the existence of writing (Gerofsky, 2004). Formal testing is at least two-thousand years old (Black, 1998). In recent years large scale testing of numeracy and mathematics at points in schooling has been undertaken in most developed nations. Such testing has included international tests of samples of students such as TIMSS (Trends in International Mathematics and Science Study (TIMSS), 2007) and PISA (Organisation for Economic Co-operation and Development (OECD), 2005). In three large, English speaking countries (the UK, the United States and Australia) large cohort tests have been implemented at a state level to monitor student progress and school performance. However the fact that many developed nations engage in numeracy/mathematics testing does not by itself establish the validity of such tests.

Messick (1989, p. 13) defines validity as: "An integrated evaluative judgement of the degree to which empirical evidence and theoretical rationales support the adequacy and appropriateness of inferences and actions based on test scores." The dominant view of validity within the field of psychological and educational testing is known as construct validity (Kane, 2006).

The full implications of the issues around what inferences will be made and what actions taken in respect to NAPLAN test scores is a complex one that extends into questions of educational policy. Consequently a full discussion of the construct validity of NAPLAN Numeracy is beyond the scope of this paper. More narrow approaches to validity include the concept of content validity. Kane (2006, p. 19) describes the content validity model in these terms:

> The content model has most frequently been applied to measures of academic achievement. A content domain is outlined in the form of a test plan or blueprint, which may involve several dimensions (e.g., content per se, cognitive level, item type), with different numbers of items assigned to each cell in the plan.

This model has been implemented in NAPLAN Numeracy with a set of assessment guidelines (ACARA, 2010) outlining content aspects, cognitive aspects and item format aspects of the assessment.

In addition to the assessment guidelines and as a means to clarify further the assessment guidelines, there are several other features and process that are relevant to the validity of NAPLAN Numeracy:

- clarifying the nature of what is being tested;
- using a psychometric model against which the quality of items can be judged and data analysed after trial;
- expert item writing, item writer training and item writing guidelines;
- The use of expert review to moderate the appropriateness and correctness of content;
- ongoing study of test results to inform future test development.

All NAPLAN items undergo a trialling process. Post-trial items are analysed using the Rasch model (Wright, 1980). The Rasch model is a mathematical model of test scores for tests that satisfy a number of important pre-conditions. Using the Rasch model to analyse tests allows for sensible comparisons of test scores between different year groups, cohorts and test papers. Use of the Rasch model requires that the items in the test are:

- one-dimensional: the items all test the same underlying skill or ability;
- locally independent: individual items should not affect the probability of answering other items correctly;
- uniformly discriminating: the chance of answering an item correctly should increase in a uniform way for students of increasing ability.

These conditions insure that students are assessed against a standard that is both consistent and stable across a series of tests.

NAPLAN and numeracy

NAPLAN tests two major domains: literacy and numeracy. Numeracy has been historically defined in multiple ways (Doig, 2001). Perhaps the most relevant Australian definition is the one produced by The Australian Association of Mathematics Teachers (AAMT):

> In school education, numeracy is a fundamental component of learning, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:
> - underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic);
> - mathematical thinking and strategies;
> - general thinking skills; and
> - grounded appreciation of context. (AAMT, 1998)

This definition has been widely used by many Australian agencies and reports including the National Numeracy Review (Council of Australian Governments (COAG), 2008).

The AAMT definition also provides a useful description for test developers for the variation in styles of items that should be found in a test of numeracy:

- some items that address underpinning mathematical concepts;
- some items that address mathematical thinking and problem solving strategies;
- some items that include general thinking skills—including general reasoning;
- some items that are grounded in a directly meaningful context.

Of course many items will fall in more than one of those categories and it is unlikely that any one item will address all of those categories well. The exact proportion of each of those categories is not pre-defined and clearly variations in the extent to which a test has items that cover these categories will determine the 'flavour' of numeracy test actually produced. The balance for 2010 NAPLAN Numeracy was, to some extent, determined by the existing forms of the test in 2008 and 2009. For purposes of accurate comparison of achievement over time there was a clear need for this later test not to vary too much from the effective construct created by the first two years of testing.

Content structure of NAPLAN Numeracy

The primary content source for NAPLAN are the *Statements of Learning* (MCEETYA, 2006)—the set of nationally agreed curriculum outcomes between the Australian States and Territories. These statements describe content Australian students should have met by the ends of Year 3, 5, 7 and 9. Testing occurs in May, so students have effectively only had one term in those respective years. Ideally NAPLAN content would be determined by content statements for Years 2, 4, 6 and 8 respectively. This issue is most pronounced in Year 3, where no statements of learning for a preceding year level exist. Consequently to sample adequately numeracy content for NAPLAN, other documents need to be considered.

The *Statements of Learning* already describe a common core between state curricula and so provide a way to cross-compare state curricula. In theory by cross-checking individual statements with curriculum documents, content can be categorised as either taught at the target year or before the target year. For example a Year 9 Statement of Learning could describe content in a given curriculum targeted at Year 8 or at Year 9. If targeted to Year 8 then this would suggest that it was suitable content to be tested in Year 9 NAPLAN—at least in that state.

By examining all state curricula, the *Statements of Learning* content common to Years 2, 4, 6 and 8 could, in theory, be deduced. However that approach would rely on curricula organising content overtly on a Year by Year basis. For sound pedagogical reasons many state curricula are not organised that way and instead group content into stages that cover more than one year and that allow for more flexibility in student development. Even so those stages are not necessarily the same two year intervals as the Statements of Learning and some classification along these lines can be done. For example Pythagoras' Theorem described in the Statements of Learning within Year 9 Measurement, Chance & Data (MCEETYA, 2006, p. 15) and in the New South Wales 7-10 Syllabus (NSW Board of Studies, 2003, p. 124) Pythagoras appears in MS4.1, that is at Stage 4. Stage 4 in NSW is a stage the majority of students are expected to have completed by the end of Year 8 (NSW Board of Studies, 2003, p. 5). However in the Queensland Scope and Sequence Year 1–9 document (QSA, 2008) Pythagoras is cited at Year 9 of the Measurement sequence. Both NSW and Queensland documents are consistent with the *Statements of Learning*, but in the case of Pythagoras it would clearly not be content that could be equitably tested in Year 9 NAPLAN.

To turn the *Statements of Learning* into a practical tool for guiding test development and content sampling requires some re-organisation. For each sentence in the Professional Elaborations section, multiple, distinct 'topic points' were identified by the team. A topic point was written so that:

- it reflected the language of the statements;
- it could be potentially mapped to other curricula;
- it was more general than a item descriptor but specific enough that item descriptors could reflect the language of the topic point.

This process was to enable a set of content descriptions to which items could be at least partially mapped and from which in turn, mappings to other curriculum could be made. Topic points were then classified as active or inactive depending on their suitability. Inactive topic points included content that could not be directly tested (for example constructing three-dimensional models) or which fell beyond the content scope (such as Pythagoras' theorem, as explained earlier).

Year level classification of topic points was done firstly on the basis of The Statement of Learning Year (3, 5, 7 or 9) then modified by reference to state curricula and other sources and from feedback from external review of items. Topic points were further moderated by internal review and an on-going process of additions, deletions and modifications.

All NAPLAN Numeracy items are reviewed multiple times both by the contractor (in this case ACER), the managing organisation, and by State, territories and key stakeholders. Feedback on items allowed us to refine the topic points further. Topic points were arranged in a basic database with multiple fields of metadata. They were grouped first by the four main categories used in the *Statements of Learning*: Number; Algebra, Function and Pattern; Measurement, Chance and Data; Space. Each of those categories could be further subdivided to establish categories of more practical use for test balancing, content tracking and curriculum mapping.

Item writers were obliged to classify all items against one or more topic points. More complex items might be classified against multiple topic points. This allowed item writers to concentrate on only one direct source of content, while still allowing for mapping of items to curricula.

Item difficulty

Items developed for NAPLAN are required by the assessment guideline to be classified by "quarters." Quarters are defined by firstly measuring the trial sample population on a Rasch measurement scale. The ability range of that sample is then divided into four intervals of equal size when measured in logits. These intervals (quarters) are numbered by descending order of difficulty, i.e., quarter 1 is the highest difficulty and quarter 4 the lowest. The aim of this process is to create a broad classification of item difficulty that reflects the spread of ability in the population.

The relation between facility (as per cent correct) and a Rasch logit scale is intrinsically non-linear (Wright & Stone, 1979). However when scores at either extreme are ignored the relation between scores and logits can be approximated to a linear relationship. Consequently a more intuitive description of the quarters can be given in Numeracy (not that this necessarily follows for other domains in NAPLAN).

- Quarter 4: Items with greater than 75% correct facility.
- Quarter 3: Items with less than 75% correct but greater than 50%.
- Quarter 2: Items less than 50% but greater than 25%.
- Quarter 1: Items less than 25% correct.

Actual difficulty is determined by trialling but an estimate of difficulty is required to ensure an adequate spread of items. One approach is to examine the past performance of existing items in NAPLAN Numeracy. To this end we developed a simple database of the numeracy items that appeared in NAPLAN 2008 and NAPLAN 2009 and then added data about these items from publically available sources. However these data on past performance provide only a crude way of judging item difficulty. With more novel items difficulty can be hard to estimate prior to trialling.

Although no substantive model exists for predicting item difficulty of NAPLAN Numeracy items prior to trialling, it can be inferred that, as items are classified on the basis of content and cognitive domains, these two aspects (mathematical content and cognitive processes) are the basis of item difficulty. This inference though, assumes that these two domains provide a sufficient description of NAPLAN Numeracy items. Other aspects such as cognitive load (Sweller, 1999) may play a role also and non-mathematical aspects, in particular reading/language demand may affect item difficulty. Although sources of non-mathematical difficulty should be minimised it is inevitable that some will persist as the test has to be communicated via a linguistic medium (in this case written words and symbols). An important priority for future research in NAPLAN should be an examination of the underlying properties of items that affect difficulty.

The cognitive domain of NAPLAN Numeracy

Using the AAMT's definition of numeracy it should be expected that a test of numeracy includes items that address mathematical thinking, problem solving strategies and general reasoning. The assessment guidelines for NAPLAN Numeracy include a cognitive dimension (Knowing, Applying, and Reasoning) and the Statements of

Learning include a substantial section on Working Mathematically. Also the proposed *Australian Curriculum: Mathematics* includes 'proficiency strands': Understanding, Fluency, Problem Solving and Reasoning (Australian Curriculum, Assessment and Reporting Authority (2010b). Of those four strands, Understanding has an overarching role (Kilpatrick, 2001).

The cognitive dimension given in the guidelines uses the same terminology as the TIMSS assessment framework Cognitive Domain (TIMSS, 2007) but similar structures can be found in the PISA assessment framework for Mathematical Literacy (OECD, 2005) which includes three competency clusters: reproduction, connections and reflection. The models used in TIMSS, PISA and the Proficiency Strands are not identical nor are they necessarily commensurate but they each use a three part structure. That structure includes a level at which routine knowledge and skills are included (e.g., fluency, knowing or reproduction); a level at which more problem orientated skills are included (problem solving, applying, connections) and a level at which more complex cognitive skills are included.

The TIMSS assessment framework (2007) provides the closest match to the NAPLAN Numeracy assessment guidelines. The three levels are described thus:

> The first domain, knowing, covers the facts, procedures, and concepts students need to know, while the second, applying, focuses on the ability of students to apply knowledge and conceptual understanding to solve problems or answer questions. The third domain, reasoning, goes beyond the solution of routine problems to encompass unfamiliar situations, complex contexts, and multi-step problems. (p. 33)

The extent to which a test item is knowing, applying or reasoning is clearly relative to the student's experience of the task. A task classified as 'reasoning' may actually be better classed as 'applying' if they have experienced similar problems before. Similarly what may be 'knowing' for some students may require a degree of application of knowledge for students less familiar with the content.

Context

Numeracy is grounded in context (COAG, 2008). Whereas mathematical skills underpin numeracy, numeracy also requires that people be able to apply those skills flexibly and efficiently. Consequently some NAPLAN Numeracy items demand that students apply their skills to non-abstract problems. Context may also be used as a means of engaging students in the tasks. These uses of context are different but complementary and the role a given context may play in any particular item for any particular student may vary. Use of context in problems can add a level of difficulty to a task by requiring students to make sense of the situation, and then model it mathematically. Some tasks may involve a further step where an abstract 'answer' has to be re-interpreted in light of the context (for example the context may determine whether to round up or round down the result of a division).

The influence of calculators on content

Since 2008 NAPLAN Numeracy has followed a policy where the Year 3 and Year 5 papers were to be conducted without the use of calculators. The Year 7 and the Year 9 papers are presented in two sections: a calculator-allowed section and a non-calculator section. The two sections allow students to experience tasks with and without calculators.

Simply dividing papers into calculator and non-calculator sections, however, is not sufficient to specify the exact approach of both sections. The approach taken with the 2008 and 2009 papers can be seen, by inspection, to be two sections that have a similar spread of skills and content but which vary in style only in a small number of items. In this approach students encounter a mix of items in the non-calculator paper some of which require no calculation and some of which require some calculation (using either mental or pencil and paper strategies). The calculator-allowed paper then follows a similar structure, with a mix of items, some requiring no calculations, some requiring calculations that could be attempted without a calculator and some which could not reasonably be solved in the time without a calculator. This approach presents students with a situation where they must decide what arithmetic tools (cognitive or mechanical) they need to employ for any given item.

The calculator status of items was classified using a modified form of a scheme used in the National Assessment of Educational Progress [NAEP] (2008) assessment framework. This scheme categorised items as Inactive, Active or Neutral with respect to calculator use. However given the ambiguous nature of the “Neutral” category we found that the scheme had to be further refined as follows:

- Calculator inactive: items for which calculators are irrelevant, for example an item asking students to identify a square.
- Calculator neutral: items that could be reasonably answered with or without a calculator but for which some students would choose to use one (if permitted).
  - Neutral non-calculator: a neutral item designated for the non-calculator paper.
  - Neutral calculator allowed: a neutral item designated for the calculator allowed paper.
- Calculator active: items which, within reason, require the use of a calculator or would be too time-consuming for students to complete.

Item writing guidelines

For the purpose of training item writers and to create a set of consistent standards to judge items by, a set of item writing guidelines was produced based on published best practice from a number of authoritative sources. The key sources were the following:

- Thomas Haladyna’s (1999) 30-point checklist. The most substantial study of multiple-choice writing rules has been conducted by Thomas Haladyna of Arizona State University.
- National Mathematics Advisory Panel (2008). In 2008 the National Mathematics Advisory Panel produced a report on school mathematics in the USA. The Assessment Task Group studied some of the issues relating to formal system-wide assessment including State-wide assessment and the NAEP national (sample) assessment. The task group’s report identified seven “Non-Mathematical Sources of Difficulty or Confusion in Mathematics Test Items That Could Negatively Affect Performance”.

- Educational Testing Service (ETS) International Principles for Fairness Review: guidelines (2007). This major testing agency in the USA publishes a set of principles for fairness designed to avoid bias at a content level and at item review.

Language

NAPLAN Numeracy is intended as a test of numeracy not literacy. However English is the medium in which the test is set and items will necessarily involve some language demand, particularly items with some context. Our aim was to ensure as far as possible that language is not what makes a difference in item performance. As well as general research on language issues within mathematics and surveys of terminology use across states (see Connolly, 2009), we also examined research specific to language issues in item development—specifically Abedi and Lord (2001). From these sources and from the stipulations of the Assessment Guidelines we derived a set of language guidelines.

Discussion and conclusion

Describing NAPLAN Numeracy as a numeracy test is not sufficient to define the construct nor do the established Assessment Guidelines give a clear definition of the test. Complex choices have been made in the establishment of NAPLAN Numeracy, which have overt and hidden implications on the nature of the test. To establish fully the validity of the test requires not only further research but also clear statements as to the intended purpose of NAPLAN Numeracy as a form of assessment. Established theories of construct validity emphasise that validity needs to be judged not only against content but also against the nature of the inferences made with regard to the test data and the actions taken with regard to those inferences.

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ANGLE TRISECTION: TWO CLASSICAL CONSTRUCTIONS TESTED

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This paper covers material relevant to a number of topics in the Measurement and Geometry content strand of the K–10 *Australian Curriculum: Mathematics*, as well as embracing ideas and skills linked with the proficiency strands of Problem Solving, Understanding, Fluency, and Reasoning. It uses material in some current 11–12 mathematics curriculum documents. It also asks the reader to work through a non-trivial spreadsheet computation and verify that it does compute the required output correctly. It also introduces readers to one significant and useful problem-solving skill that is not mentioned in the above content strand—although Euclidean geometrical ideas, coordinate geometry and trigonometry all appear before Year 10, no suggestion is made regarding the possibility of attacking a problem given in one of these areas by transforming it into an equivalent problem in another area. (In general, changing the representational system in which a problem is presented, in order to re-pose the problem in a different relevant system in case this might facilitate a solution, is a useful problem-solving tool.) This sort of flexibility in thinking is very useful in mathematics and other fields.

Dedicated to the memory of Vic Czernezkyj

In the mid 1980s, Mr Joe Keating, a Sydney accountant, first contacted John Mack to discuss with him the three classical geometrical construction problems. These are to duplicate the cube, to square the circle and to trisect an arbitrary angle, using only a pair of compasses and a ‘straight edge’—i.e., a ruler without distance markings on it, used only to draw an arbitrary line or a line through two given points. During the 19th century, results in the theory of equations were used to show that none of the above is possible, although each can be done with only a slight relaxation of the above rules.

The challenge to produce valid constructions was taken up by many interested people prior to and after the impossibility result was proven, often with extremely accurate approximate constructions being developed.

Keating became interested in the angle trisection problem and has over the years proposed several constructions for it. Sometimes, when asked to analyse such a proposal, it is possible to demonstrate that a claimed geometrical property of it, which if true would give an exact method, is in fact false. In this case and in other cases, it is now possible, thanks to good geometrical and mathematical software packages, to coordinatise exactly the proposed method and, say, compute accurately the values given by it when applied to angles at one degree intervals between 0° and 90°. These values may then be compared with the exact values of one-third of the given angle.

In 2009, Keating provided a new trisection construction for analysis and recently offered a simplified version of it. In this article, we shall coordinatise his methods and then compute the results obtained by applying them to the angle values given above. Some of the lettering of his original drawing has been preserved in the diagrams below.

The first method

Essentially ∠KCX = θ, the angle to be trisected. KOML is a hinged parallelogram. As the angle θ varies, the arm KL swings, but the two short sides of the parallelogram KO and ML remain parallel to the line ACX. LM varies in length, and OK stays equal in length and parallel to LM. The angle ∠OAX = α is the 'trisection' of ∠KCX.

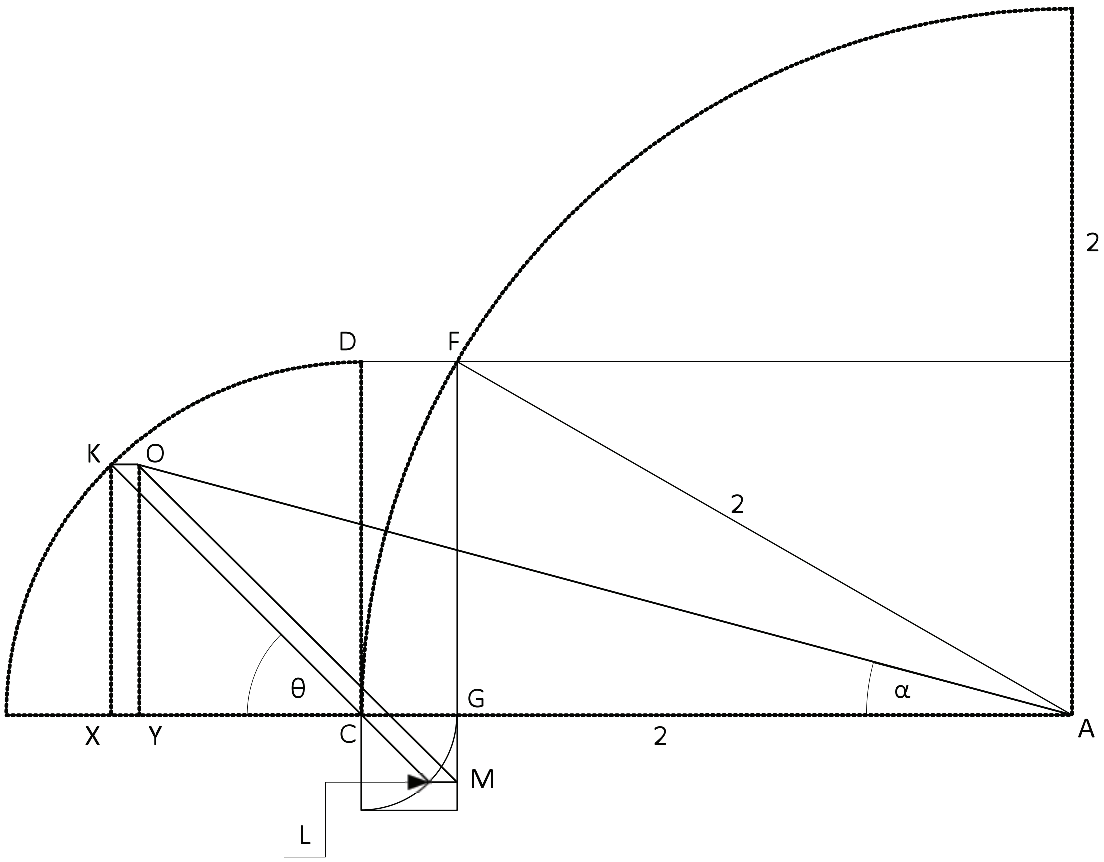

Figure 1. Keating's Method 1 construction of α, an approximate trisector of angle θ

In terms of the diagram (Figure 1), where we have set C as the origin (0,0) and CA as the *x*-axis, we have the relevant data:

C = (0,0), A = (2,0), AF = AC = 2, GF = CD = 1

From these points we can determine by trigonometry that G has coordinates $(2 - \sqrt{3}, 0)$, from which CG is $2 - \sqrt{3}$ units. The coordinates of K are $(-\cos\theta, \sin\theta)$, and the coordinates of L are $(\cos\theta, -\sin\theta) \times (2 - \sqrt{3})$.

There are two circular arcs, (centre C, radius 1), (centre A, radius 2) and a derived circular quadrant centred on C, radius $2 - \sqrt{3}$. ($CG + GA = 2$, and $GA = 2\cos 30° = \sqrt{3}$.)

From Figure 1 and information, we have that the distance

$$LM = (2 - \sqrt{3})(1 - \cos\theta).$$

Now,
$$\tan\alpha = \frac{OY}{AX - XY} = \frac{KX}{AX - ML}.$$

But
$$KX = \sin\theta,$$

and
$$\begin{aligned} AX - ML &= 2 + \cos\theta - (2 - \sqrt{3})(1 - \cos\theta) \\ &= 2 + \cos\theta - (2 - \sqrt{3}) + (2 - \sqrt{3})\cos\theta \\ &= \sqrt{3} + (3 - \sqrt{3})\cos\theta \end{aligned}$$

and so
$$\tan\alpha = \frac{\sin\theta}{\sqrt{3} + \left(3 - \sqrt{3}\right)\cos\theta}.$$

The question then arises: How different is α from $\frac{\theta}{3}$?

A spreadsheet was used to evaluate and graph these two values. The columns of the spreadsheet (available with the electronic copy of these proceedings and as a resource at www.aamt.edu.au/Professional-reading/AAMT-conferences/AAMT-MERGA-2011-files) and a description of each column are:

Theta (θ)	Angle θ varied by rows from 0° to 90°
Tan(theta) Tan θ	
Sin(theta) Sin θ	
Cos(theta) Cos θ	
Tan(theta/3) Tan $\frac{\theta}{3}$	
Tan(alpha) Tan α	
Delta = Tan(theta/3) – Tan(alpha) or, $\delta = \text{Tan}\,\frac{\theta}{3} - \text{Tan}\,\alpha$	Difference between tangents of ‘trisected’ angle α and actual third of θ
Arctan(tan(alpha)) $\alpha°$	Actual angle α in degrees
Delta2 = Theta/3 – alpha (deg) or, $\delta_2 = \text{Tan}\,\frac{\theta}{3} - \alpha°$	Actual degree difference between ‘trisected’ angle and actual third of θ

This is a good spreadsheet exercise in the sense that it involves careful formula construction. Once an initial row is complete, however, it may be replicated downwards. From the spreadsheet, a graph was produced showing the values of Delta and Delta2 as defined above.

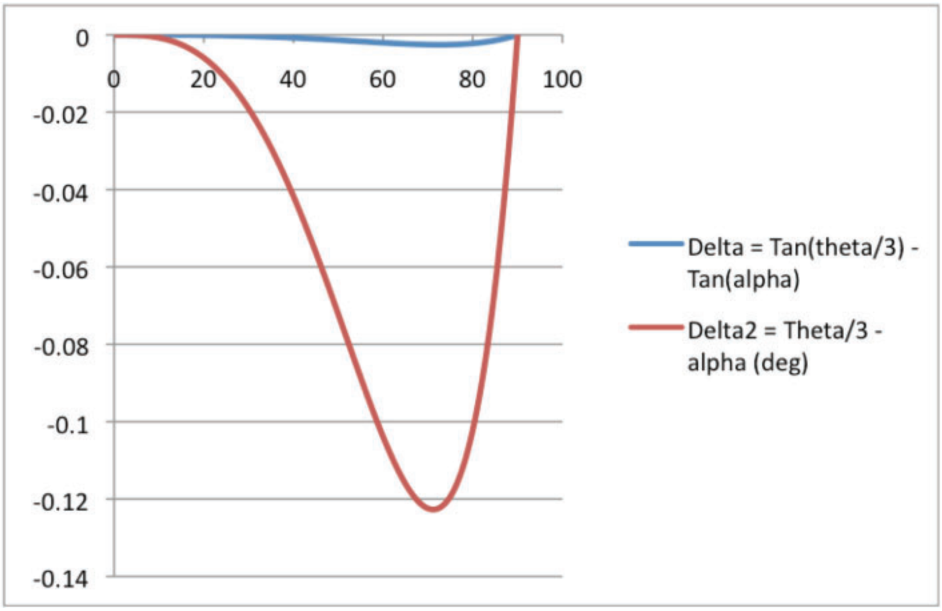

Figure 2. Graphical representation of errors associated with Keating's Method 1.

In summary, the model works really well, giving a maximum error of –0.1227° at 71° for theta (θ). However, like all 'classical constructions' it cannot be perfect.

Second simplified method

Keating's original drawing has been modified in terms of lettering only, to conform to the lettering used on the diagram above, and to keep some similarity in angle names and measures (Figure 3). As before, θ is the given angle and α is the constructed 'trisection'.

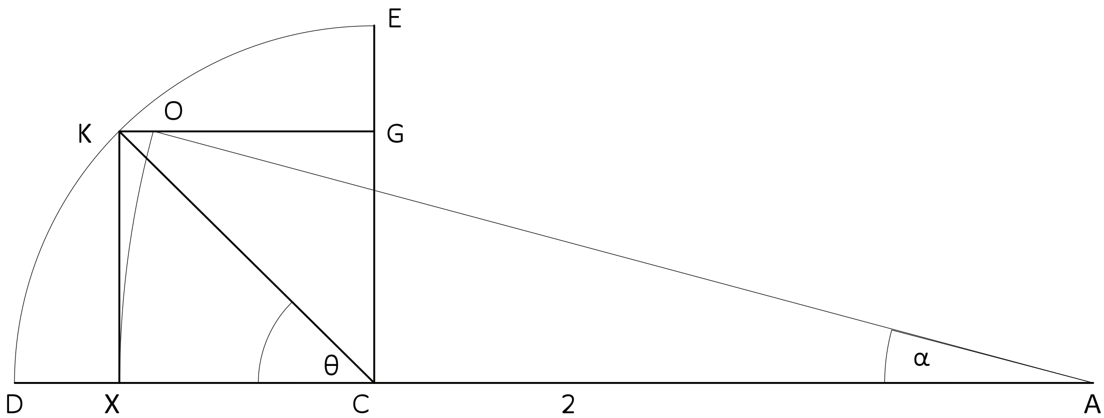

Figure 3. Keating's Method 2 construction of α, an approximate trisector of angle θ

In this diagram, DC = EC = 1, AC = 2, KG is parallel to DCA, KX is parallel to EC which is perpendicular to ACD, OA = AX.

We can calculate that AX = OA = 2 + cos θ and KX = sin θ. From this we have:

$$\sin \alpha = \frac{\sin\theta}{2+\cos\theta}$$

Our spreadsheet analysis is given in the same table, and shows the simplified second construction is less accurate than the first. The greatest difference in this case occurs at about 72° with a difference between constructed and actual angle of approximately –0.3235°.

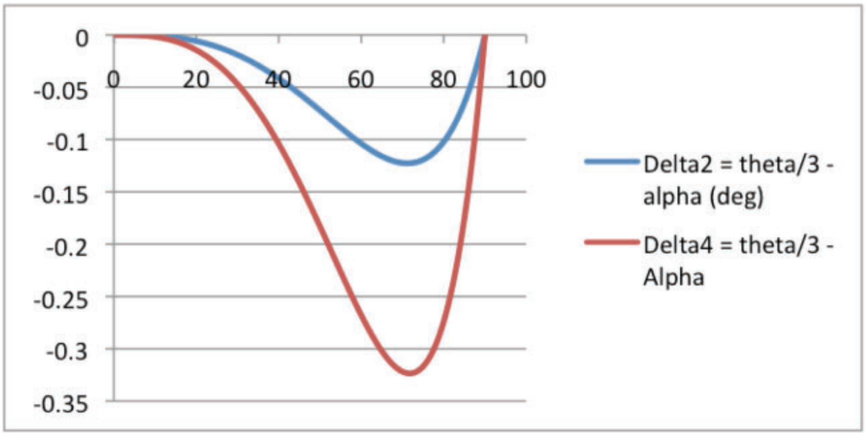

Figure 4. Graphical representation of errors associated with Keating's Method 2

There are available many approximate trisection constructions, as well as exact constructions that break the classical 'rules'. The book by Yates (1971) is a good source for these. But it is an interesting exercise to challenge students to devise their own constructions (using, for example, one of the geometry software packages) and to see if they can explain or calculate the errors produced by them. For example, why is it that the 'obvious construction', namely by trisecting the chord DK shown in the diagram for the second method, does not give a trisection of the angle θ?

John Mack's rolling circle

During discussions about this paper, John suggested that a description of an old, simpler method involving a rolling circle, be included. While going beyond the permitted Euclidean tools, it provides an easier means of trisecting, or in fact manipulating any angle by reducing the circular arc to a straight line. Exact distances, which are multiples or rational fractions of a given distance on a given line can be constructed within the rules of the game.

Assume we have a unit circle with angle θ defined. Also assume we can roll the circle along the line x = 1 without slipping, allowing every angle θ to be mapped to a unique position on the line x = 1. Define the rolling transformation by:

$$\text{Roll }(K(\cos\theta, \sin\theta)) = (1,\theta)$$

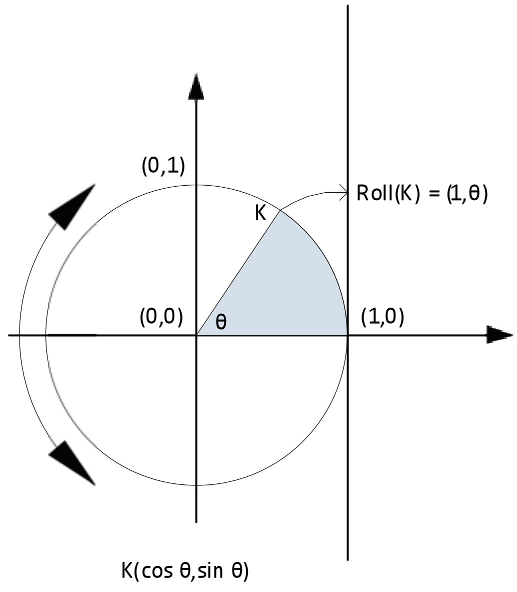

Figure 5. Rolling transformation of unit circle to line.

Every angle on the unit circle can be transformed to a point on the line. To trisect an angle on the circle one can transform it to the line, trisect the line segment and then use the inverse function:

$$\text{Unroll }(1, \alpha) = K^{-1}(\cos \alpha, \sin \alpha)$$

The trisection of a line segment is a classic construction. In this case extend the base line along the *x*-axis by three units. Join the derived point (1,θ) to (4,0). Then construct parallels through (2,0) and (3,0). By similar triangles, this cuts the line segment from (1,0) to (1,θ) into thirds. This construction can be modified from thirds to any integral fraction.

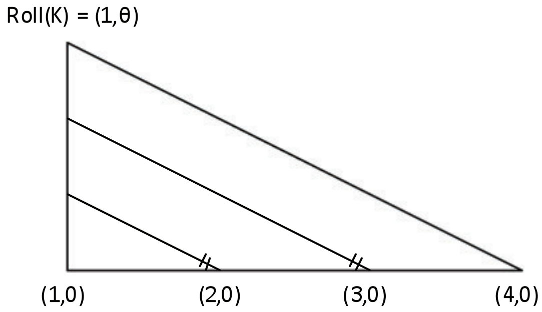

Figure 6. Trisecting a line segment.

This method provides the following, which are asserted and left as exercises for the reader:

- 1 radian can be accurately ‘constructed’;
- 1° can be accurately constructed and in fact any commensurable number of degrees or radians can be accurately constructed;

- π can be accurately constructed, and hence the circle can be squared, as it is possible to construct, using classical methods, a square whose area is equivalent to a rectangle of sides π and 1.

Eves and Dürer

Howard Eves has published many books and other papers on geometry. Two of them (at least) (see Eves (1969) and Eves (1972)) give a number of angle trisection methods, including citations from various sources. Notably Eves quotes Albrecht Dürer's construction. The following is taken from Eves (1969, p. 87).

> An excellent example is the construction given in 1525 by the famous etcher and painter, Albrecht Dürer. Take the given angle AOB as a central angle of a circle (see diagram below). Let C be that trisection point of the chord *AB* which is nearer to *B*. At *C* erect the perpendicular to *AB* to cut the circle in *D*. With *B* as centre and *BD* as radius draw an arc to cut *AB* in *E*. Let *F* be the trisection point of *EC* which is nearer to *E*. Again, with *B* as centre, and *BF* as radius, draw an arc to cut the circle in *G*. Then *OG* is an approximate trisecting line of angle *AOB*. It can be shown that the error in trisection increases with the size of the angle *AOB*, but is only about 1" for angle *AOB* = 60° and about 18" for angle *AOB* = 90°.

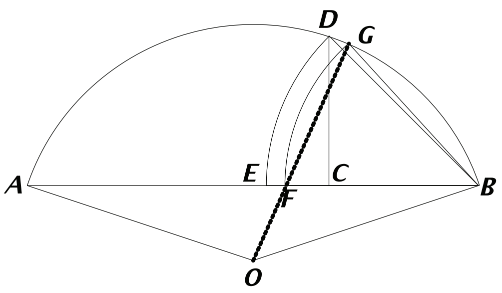

Figure 7. Howard Eves' version of Durer's trisection construction.

We close by asking: Why is it that the following obvious construction does not work? Construct an isosceles triangle with the given angle as vertex. Using classical methods, trisect its base and join the points of trisection to the vertex, thus trisecting the angle there.

Acknowledgment

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X = GIFTED STUDENTS + REGIONAL SCHOOLS + ONLINE COURSES

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Gifted, regional or online, each would surely be challenge enough. But a handful of high schools in regional WA formed a cluster to deal with all three variables! Here is the story of our evolving program to extend more able students, I talk about the strategies we have used to find the students, run an online course and provide enriching activities for a group spread over two or three hundred kilometres in half a dozen schools. Practical and positive, this session looks at where we have come from, how we have met the challenges so far and where we go from here.

Our situation

As they all do, our state education system has been going through more changes. This has included much decentralization, with little to do with curriculum coming out of the central office of the Western Australian Department of Education and Training, and more recently, regional offices being closed or restructured. Schools now need to make their own provisions. Sensibly, our schools within the Great Southern region of WA have recognized that a cluster of schools is more likely to be able to provide for diverse groups of students than the separate schools working individually. By the formation of a cluster and by providing funding for co-ordination, a number of smaller schools have been able to deliver normal Economics and Literature courses using an online mode to a group of students drawn from a number of schools. Almost by accident, the suggestion of a course for gifted students of mathematics was made. The target audience was Years 8-10 from senior high schools (catering for Years 8-12), high schools (Years 8-10) and district high schools (Years K-10). I was asked to provide some mathematics enrichment for gifted students. Unlike the other teachers working in Literature and Economics, who were delivering accredited courses to senior secondary school students, there was no set course and no particular outcomes - the world was my oyster!

This paper describes the nature and development of the resulting mathematics enrichment course to cater to the needs of the students identified as gifted.

Selection of participants

So we wanted to provide a course for gifted maths students in the geographically diverse cluster of schools in our region. Who are the students? It is not a matter of just talking to the top few performers in each year. Some courses I had completed with GERRIC (Gifted Education Research, Resource and Information Centre) some time ago made me aware that many gifted kids are underachieving in our schools. We wanted to make some provision for the bright kids who might not have been inspired or challenged for a while. GERRIC also taught me the value of parents – sometimes they will observe details that a teacher has not been able to observe. So in 2010, we asked each of the regional schools to forward information about the planned opportunity to all of the students in the age group. The following year, in 2011, we have asked schools to pass the information to students in the top third or half of the cohort.

A letter outlines the opportunities that are available, as well as the process through which a student can apply to take part. Generally this involved seeing the Deputy Principal or the school office to collect an application package. The package required input from the student, a parent and two teachers. Table 1 shows the input requested from teachers. It is interesting that no matter how hard one tries, there will always be people whose responses are not very helpful. When responding to the issues below, for example, one teacher could only manage one "Good" on the form; not too helpful, really.

Table 1. Teacher input.

Ability in this subject Briefly describe a situation demonstrating the student's ability in this subject area.
Work ethic Describe the student's work habits, organization and preparation for class.

Parents were asked to say why their child should be in the program, with reasons offered including the need for challenge, the ability of the child or the importance of mathematics.

Students were asked to handwrite responses to these three stimuli.

- Please describe your understanding of the term *mathematics*.
- What do you consider to be the most important part of mathematics? Explain your reasons.
- What part of mathematics do you find most interesting? Please give reasons for your answer.

Imagine my excitement when I kept seeing algebra or problem solving referred to as important and even interesting on the forms of the students applying. These were not typical Year 9s! The fairly narrow view of mathematics as number or measurement set me the challenge – could I lead these students to an understanding of the richness of mathematics? Would they be happy to consider mathematical ideas, and think, talk and write about them?

Activities

Students who applied for mathematics enrichment were offered a couple of additional project opportunities during the year. The Mathematical Association of Western

Australia (MAWA) ran *Mathoquest*, which involved different project topics for teams of students at different year levels. Support was offered for anyone wanting to have a go – but none did. I tried again later in the year with a project based on some material on medicine from the UK, once again with a resounding response of silence. Even the thought of prize money was not enough. Is it just me or are students less prepared to do that bit extra these days? Maybe it's the part-time jobs that so many students have these days —they have no time left for any extra from school.

At the end of the year, a day camp was offered to all applicants. Most, but not all, of the students who attended had been involved in one of the courses. A teacher from each of the participating schools brought the students to our coastal campsite for the day. The program involved a guest speaker who spoke about mathematics in his job as a surveyor. Students were then randomly allocated to teams of four for the remainder of the day. A number of problem solving and mathematical activities and races were conducted before a sausage sizzle lunch. After lunch, teams participated in a strategy game competition, the format of which is based on the games camps that MAWA runs each year. Some limited sponsorship allowed for the provision of some small prizes for the winning teams, with every child going home with a bag of goodies. Students enjoyed the day, relishing the competition and the day out of school doing something they, but often not many others, enjoyed.

Course

Most of my time went into the Mathematics Enrichment Course, which was entitled *Flatland and beyond*. Thankfully, with the resources to run three separate courses during the year, I was able to offer a place to all the Year 9 and Year 10 students that had applied, as well as a couple of the outstanding Year 8 students.

I have been a fan for many years of Edwin A. Abbott, the author of *Flatland*. (Did you know the A also stands for Abbott?) The great animated film from 1965 with Dudley Moore narrating provides a lot of scope for discussion about shapes, people, dimensions, ideas, convincing – it's a pretty rich source. For me, it is the ideal vehicle to get kids talking about mathematical ideas.

So we started the course with viewing the film. The kids found the language hard to manage; they needed repeated viewing to even get some idea of what is going on. I also suggest that they may like to watch a more recent film on *Flatland –The Movie*. This has a different slant from the original film, but at least seems more in touch with the modern world and much more accessible to the students. Table 2 shows the outline program used for the course that was developed.

The second session involves discussion of various concepts raised in the film, including social issues. By the third session, I have asked students to also read some of the original text of *Flatland*. We move the focus more to the idea of dimensions, particularly the assumptions we make when existing within a particular world. The consequences of different sets of assumptions and the importance of making them explicit are stressed.

Table 2. Outline program.

Week	Activities	Follow up activities
1	Face to face meeting, technology, view film	Prompt sheet questions
2	Discussion of *Flatland* issues – social class, different ideas	Read excerpts from *Lineland*, Sphere visit
3	*Lineland* visit – parallels to sphere, higher dimensions	
4	Assumptions – truth - setting boundaries of problem	
5	Assumptions in problem solving Problem solving model Breath problem	Complete solution to problem
6	Discuss breath problem Brick problem using structure	Complete solution to problem
7	Discuss brick problem With assumptions made, move on to the result Justification Language - rectangles	Complete solution to problem
8	Convincing and proving If … then statements Digits problem Squaring problem	Complete solution to problem
9	Convincing Cubing problem	Complete solution to problem Reflection
10	Reflection Assumptions Problem solving model	

With the focus on the assumptions, I introduced students to a model of problem solving used extensively in WA senior secondary school courses for a number of years. The model involves steps of

- Clarify
- Choose
- Use
- Interpret and check.

While the original model also included the aspect of Communicate, I emphasize the need to communicate thinking throughout the process. It is not something that is just done at the end. I find myself using this structure with all of my students, knowing that a student is more likely to have a successful problem solving experience when using a specific process, particularly when the steps are few, clear and manageable.

Quite a bit of time is also spent on justification, the use of “If… then” statements to draw conclusions and the fact that the problem is being solved for an audience. As a consequence, the solution needs to be presented to the audience in a manner such that they are convinced by the argument. I aim to present problem solving as a human endeavour with a purpose more significant than merely passing a maths test.

In the process of the course, students grapple with a number of problems, mostly ones that are simply stated, and also ones that involve accessible concepts. The students all have something they can bring to bear on the problem. The advantage of the online medium is that at this point, the students are able to share their ideas. Hearing what others have to say often triggers more thought. The interactive nature of the session, where I am watching what each student has to offer is also important: I am able to direct students to paths they may have missed, thereby ensuring success with the problem.

While some students are happy to follow up on sessions with thorough attempts at writing up a solution to the problems, all of them become very engaged in the discussions. I am convinced that the social nature of the problem solving leads to student success. The level of response from students to the early problems is quite disappointing. By the end of a course, the responses are much richer. While some of this richness can be attributed to individual thinking, my sense is that the progress is largely a result of the student interaction in a safe environment with other students who are not ashamed to admit to an interest in problem solving.

Online medium

The Cluster of schools provided access to *Centra*, an online meeting tool. Students use a microphone and headphones so they are able to hear and speak to other participants. They are also able to type and write onto their computer so it is shared with the whole group. We were also encouraged to use *OzProjects*, a *Moodle*-based site for Australian educators that allows students to download materials, among other things. With a day for professional development on *Centra* and one for *OzProjects*, I began the construction of a site, figuring how to put up the materials that we would use and how to get kids to access them. With no prior experience of online learning, I decided to find some guinea pigs. Three lovely top Year 9 students in my class and a similar group from Katanning, some 250 km away, volunteered to join me for a couple of trial sessions. We were able to spend a couple of hours looking at some useful problem solving skills while I got used to the new format for lessons.

One of the most significant issues for me was that I did not realize how much I usually rely on visual cues in my normal classroom. Take away the eye contact – you don’t see the spark or the confusion, you are not aware of the day dreamer and you don’t see interactions between students. The use of a response grid like that in Table 3 helped me ensure that all students were accountable and that they were actually engaged most of the time. Each time there is a question or issue, every student writes his own response in his section of the screen. With a signal to indicate that you have finished, I only have to wait for the last signal to be able to move on. This way, I get feedback from every student. I am often able to follow up on all kinds of ideas, not always the ones that I expected.

Table 3. Response grid.

Fred	George	Harry
Anne	Mary	Fran

The use of video conference will alleviate these issues to some extent, hopefully, as we move into a trial of these kinds of media. The ability to see who you are talking to and the availability of visual cues certainly make it easier to have conversations. I have found, though, that the use of the grid, with its inherent accountability and the fact that in a short space of time everyone knows what everyone else is thinking is just too valuable to give up. So my choice, as I move into the era of using the video conference facilities, is to also have students with a laptop connected to meeting software, in our case Ebeam. My slides and presentations are in front of every child, as too are the responses of their peers. We still have the accountability and the scope provided by the grid without having to go around the whole group speaking about the issues.

The students generally responded well to using the *Centra* – it provided them a chance to interact with different students. Its use within school time has thrown up a few interesting issues:

- Support within different schools can be quite variable; for example, one librarian did lots to help students with all sorts of technical issues while another would not hand out headphones and sent students out during sessions in order to shut the library for lunch time.
- Withdrawal from a regular class caused some issues for students having to keep up to date with the class – which made the workload much more of a burden.
- Schools do not see the need to provide supervision or availability of help, so students are on their own, which can be fortunate. For example, in the first stages of one course, two girls were working together; one of the girls could not get *Centra* to play the game. So they shared headphones and workspace, and both participated in the lesson via a single computer.

The learning management system (*OzProjects*) also brought with it some concerns:

- Not all students saw the need to use it! They were asked to access materials from the site for the next session, or upload responses to the site for my perusal. Despite very explicit, repeated instructions, some students did not access it. This was common to other courses – there was a need to have a person physically present to remind them of these expectations. Year 9 students, no matter how bright, are not the mature, responsible beings we might wish them to be.
- There were difficulties with uploading or downloading from the site. Knowing that schools had help available, I have suspicions that this may have been an excuse for not doing things.

Feedback

I believe that it is important to value the efforts made by students and to acknowledge their successes. At the completion of each course, each student receives a two-page certificate. The first is a generic certificate of participation, including dates and basic details of the course. Every student is able to use this in their portfolio. The second page specifies attendance at, for example, 7 out of 10 sessions, and the completion of, for example, 4 out of 6 tasks. A paragraph of specific comments about the participation, learning or achievements of the student is also included. There are some students for whom this page will not be very complimentary, while for others, it is outstanding. A copy of this feedback also serves as valuable information for the school, particularly in the few cases where the student performance was fabulous.

Reactions

Discussion of a film is not what students expect in a maths course. Even when the introductory information is quite explicit, the message doesn't seem to hit home very well. So they are out of their comfort zones—and I am fairly sure they do not see immediately the value of what they are doing. They generally find it interesting, get quite involved and feel as if they learn something about dimensions but underneath it all, I don't think they see their learning as having much to do with maths. Quite differently, the problem solving process is something they see as having direct value to themselves. Almost all of the students quote the problem solving model as one of the positive aspects of the course.

Where do we go from here?

2011 is seeing the fourth course run, this time using video conferencing as the main tool, with meeting software to provide a computer link for each student as well. Each time a course is advertised, a small group takes up the opportunity. While many parents contact me for applications or see teachers in their own schools, very few follow through to the courses. Problem solving sessions for students who had previously done courses have been offered, to no avail. Many able students seem not to want to commit to something that involves work outside class, or extra work from normal classes.

For a small group of students, however, the courses provide an opportunity to work in an area that interests them, to unashamedly revel in mathematical ideas without ridicule, to ask questions that are thoughtful and deep and to have them answered extensively, to receive accolades for their thinking, not just marks for a task; these are the students who provide me with a sense of purpose and the thrill of having watched a child make a mathematical discovery or grapple successfully with a complex idea. Every student takes away something positive from the course, usually some techniques for use with problem solving. Some students, though, make huge intellectual growth and achieve a great sense of joy of learning. So the question remains, how can we provide better opportunities that are more attractive to gifted students, particularly when we have been able to overcome many of the issues associated with the isolation of small schools through the use of ever-improving technology?

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MODELLING AS REAL WORLD PROBLEM SOLVING: TRANSLATING RHETORIC INTO ACTION

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Work carried out under the banner of mathematical modelling, usually deals only with parts of the modelling process, overlooking that aspect most crucial to developing expertise – formulating a mathematical problem from a messy real world context. There is a dual purpose in this enactment of real world modelling: to solve a problem at hand, but over time to enable students to become better and more independent modellers of problems. The presentation will identify or reinforce essential components of modelling activity using illustrative examples, including how some students have approached a modelling task.

Introduction

A stated priority within the forthcoming *Australian Curriculum: Mathematics* indicates that "mathematics aims to ensure that students are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens" (Australian Curriculum, Assessment and Reporting Authority, 2010). Such abilities can only develop if mathematical experiences are drawn genuinely from these same areas of personal, vocational, and civic contexts. Whatever other purposes they serve, text book problems cannot meet this need.

Much work carried out under the banner of mathematical modelling, skates over the aspect most crucial to developing expertise—formulating a mathematical problem from a messy real world context. This essential component is also referenced elsewhere in the modelling process—such as interpreting mathematical results, and evaluating whether an alleged solution fits the needs of the original problem.

The intention here is to highlight components of modelling activity essential to the above curricular goal using illustrative examples, including how some students have approached a modelling task.

Modelling as real world problem solving

This perspective derives from the use of mathematics to model problems in fields outside education. Some, such as Pollak (telecommunications), Burkhardt (physics), and early ICTMA[1] contributors, have taken their insights specifically across into education, (Pollak, 2007; Burkhardt, 2006). Others (e.g. Pedley, 2005) provide external reference criteria for those working within the educational field. Modelling in this vein has two concurrent purposes – to solve a particular problem at hand, and also over time to develop modelling skills that empower individuals to solve problems in their world (personal, vocational, and civic). Characteristic of this approach is a cyclical modelling process – containing elements such as the following (Figure 1) drawn from (Pedley, 2005) in his Presidential address to the Institute of Mathematics and its Applications.

1. Understand the real problem situation.
2. Frame an appropriate mathematical question
3. Formulate a model, using simplifying assumptions etc
4. Analyse the model
5. Compare mathematical outcomes with reality
6. Modify and repeat until an adequate solution has been found.

Figure 1. Modelling process.

The arrows on the right indicate that iterative back tracking may occur between any phases of the modelling cycle. This diagrammatic translation of Pedley's message is a compact version of the modelling chart familiar from many sources (e.g., Galbraith & Stillman, 2006). Such diagrams both describe the modelling process, and act as a scaffolding aid for individuals or groups while developing modelling skills through successive applications. The solution to a problem must take seriously the context outside the mathematics classroom within which it is introduced, and its evaluation involves returning to that context. It cannot live entirely in a classroom.

Sources of problems

Real world problems need to start and finish exactly there – in the real world. Examples include: a primary school class collected data on passing traffic and successfully mounted a mathematical case to the council for lights at the school crossing. A girl provided a convincing case to her parents that she could finance and care for a pony she had set her heart on. A student redesigned the culture that he used for growing tomatoes hydroponically. Senior students investigated the problem of siting speed bumps along a new college drive. Students enacted moves associated with optimising goal shooting opportunities in soccer, and in an unrelated context, Australian Rules football. Many problems can be stimulated by newspaper reports on various topics, and the web is a superb resource on almost any issue of current interest – such as climate change. Whatever the motivation, an essential need is for students to be thoroughly familiar with the context surrounding any problem, and this may involve anything from careful

[1] International Conference on Teaching Mathematical Modelling and Applications

reading of material to physical relocation and enactment outside the classroom, as described above.

Sample modelling problems

Excerpts from two modelling activities are given below. One is an historical case reviewed by a mathematician (Pedley, 2005), the other concerns a problem undertaken by a group of year 10-11 students in 2009.

Example 1

Geoffrey Taylor's analysis of the atomic bomb test in New Mexico in 1945 followed the publication in *Life* magazine in 1947, of photos of the expanding blast wave (Figure 2), taken over a succession of small time intervals.

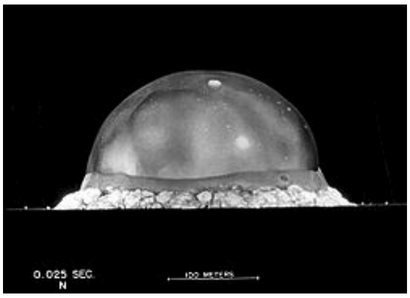

Figure 2. Blast wave photo. [Source: http://en.wikipedia.org/wiki/Nuclear_weapon_yield]

From the photographs Taylor estimated the energy of the blast, using mathematics accessible to senior high school mathematics. The fundamental assumption is that the radius of the blast wave (R) depends on the time elapsed since the explosion (t), the instantaneous energy released (E), and the density of air (ρ).

Thus $R = C\,(t^a\,E^b\,\rho^c\,)$, where C is a dimensionless constant .
Now dim $R = [R] = L$, $[t] = T$, $[E] = ML^2\,T^{-2}$, and $[\rho] = ML^{-3}$
Thus dimensionally we need: $L^1 = M^{(b+c)}\,L^{(2b\text{-}3c)}\,T^{(a\text{-}2b)}$.
Equating dimensions: $b + c = 0$; $2b - 3c = 1$; $a - 2b = 0$; and hence
$a = 2/5$, $b = 1/5$, and $c = -1/5$. So $R = C\,(Et^2\,/\rho)^{1/5}$, where $C \approx 1$ from known data.

From the photographs corresponding values of t and R were known, and used in a plot of log R against log t: $\log R = \log\,(E\,/\rho^{1/5}) + 2\log t$.

The unknown, E, can be calculated from the intercept on the vertical axis – again requiring no more than secondary school mathematics.

Indeed a single estimate for ($E = R^5\rho/t^2$) can be obtained directly from the photograph above, which contains a scale representing 100 m, and a label indicating that it was taken at $t = 0.025$ (s). Expanding the photo from the link, taking measurements, and using the given scale to estimate the radius suggests a value for R of about 132 m. Given that the density of air is 1.2 kg/m^3 we obtain a value for E of about 7.7×10^{13} joule. This converts to an energy equivalent of about 16.7 kilotonnes of TNT. Taylor's subsequent letter .to the Americans that "I see that the atomic bomb you detonated had a power equivalent of about 17 kilotons of TNT", or some similar wording, caused great consternation at such a revelation of classified information!

Reflection on modelling matters

This is a striking example of a modelling problem that was identified and developed from an article in a magazine. It reinforces that the popular press is a fertile source of problems, many of which are suitable for school students to address, sometimes using alternative approaches. We note that the mathematics needed above is no more than secondary level, yet the problem itself was a significant one. Again we focus attention on the formulation stage; for it was there that deep understanding of the context was essential; for example in knowing which three input variables were sufficient, and why other potential candidates such as specific heats could be ignored. And the assumption that $C = 1$ was based on knowledge of a combination of factors that had previously been assessed. Similarly, in any authentic modelling situation, time must be allocated to engage the context thoroughly, for this is necessary if a viable mathematical question is to be identified, appropriate assumptions made, and an appropriate model formulated. It is a far cry from "read the question carefully".

Example 2

This problem was developed by a group of four students (Years 10–11) participating in the modelling challenge at A. B. Paterson College in November 2009. The modelling took place over two days and involved seven scheduled class hours—including requirements to produce a poster and make a presentation. Additional time was at the initiative and discretion of students. Internet access to material on climate change stimulated initial interest in the topic, together with the fact that the students lived on the Gold Coast, and were directly familiar with the object of their study. Edited descriptions from the students' modelling report (that included a 28 page *PowerPoint* presentation) follow.

Real world problem

Climate change is used to describe the changing nature of the world's weather patterns. Increasing temperatures due to climate change have been reported to cause rising sea levels due to the expansion of water around the world.

Mathematical question

At what point in time will the Q1 (or a later building) lobby[2], which is 3 m above sea level, be submerged due to sea level rises, and what will be the mean maximum temperature of that month in Surfers Paradise at that time?

Defining the variables

The independent variable, x, is the time in months, in one month intervals, since January 1938. Therefore: $x = 0$ denotes January 1938; the first month of temperature recorded.

The dependent variable, y, is the mean maximum temperature of Surfers Paradise over one month measured in degrees Celsius. (Using mean temperature or mean minimum temperature, would give a similar pattern of results.)

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[2] Q1 is a recently completed building at Surfers Paradise.

Assumptions

1. The influence of rainfall and evaporation in the ocean is negligible since this would only contribute to the water cycle, in turn feeding back into the oceans.
2. The whole surface of Surfer's Paradise is a flat plane 3 metres above sea level that contains no obstructions to the path of the ocean as it rises.
3. The trend demonstrated in the data set used continues into the future.
4. There is a correlation between temperature and rising sea levels.
5. The melting of the polar ice caps does not contribute to the rise of the sea level which is entirely caused by expansion of water due to heat. If this assumption proves invalid, the predicted time will be too far in the future.

Comment

The assumptions are careful and relevant, containing additional justifications not included here. They foreshadow potential limitations as well as necessary simplifications.

Finding a pattern: A suitable model

Visual analysis of the temperature data would indicate them to be periodic, since the temperature during a year would rise and fall depending on the seasons (Figure 3).

Building a model

1. The general form for a periodic function is $y = a\sin 2\pi/b(x\text{-}c) + d$, where a is the amplitude, b is the period, c is the phase shift, and d is the vertical translation of the function. Thus it would not be possible to predict a change in the mean maximum temperature of Surfers Paradise unless an equation representing the general change in the climate was used in place of the constant d value.
2. To obtain the equation for d, one must determine the average rate at which the temperature increases. A linear regression performed upon the data gave an equation of $y = 0.001x + 24.64$. This equation shows that the equilibrium line of the periodic function for the mean maximum temperature over time graph is sloping upwards.
3. This linear equation can be substituted into the general periodic equation instead of d.
4. The a value in a periodic equation is the amplitude of the wave; essentially half the range of values for the y axis.
5. The b value in a periodic function is the period of the function being the average distance between two crests (or troughs) on a wave. Calculating the period as 12, the b value for the periodic equation is 0.5235. To determine the value of c, the equation can temporarily exclude the c value and be used to predict values for y. Because these values will have been translated incorrectly due to the lack of a c value; they can be matched up with corresponding y values and the x distance between these selected corresponding points can be used to determine the value of c.

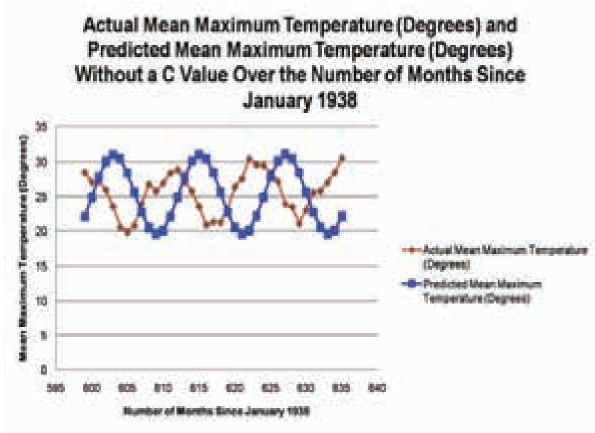

Figure 3. Mean maximum temperature graphs.

One pair of crests and one pair of troughs have been identified for analysis. The adjacent troughs have x-values of 605 and 609—indicative of a horizontal phase shift of 4 months. This indicates that predicted y values have been shifted 4 months forward, and thus the c value is –4. Thus the equation to predict the mean maximum temperature over a month in Surfers Paradise becomes $y \approx 5.75\sin(0.5235(x+4)) + (0.001x + 24.64)$.

Comment

Some innovative thinking is apparent here, particularly in the replacement of d by a linear expression to capture the slowly increasing base temperature. There is some loose description that entangles the period with the coefficient, but the students have the mathematics right. Their incorporation of the translation constant c is clever. Here they inferred a value of 4 from the graph—the actual value is 3 (1/4 of a period).The numerical impact on outcomes is miniscule.

Evaluating the model

A perfect model would have an actual versus predicted equation of $y = 1x + 0$, whilst a linear regression performed upon the data in this graph (below) gave an equation of $y = 1.3x – 7.554$, with an R^2 value of the linear regression to be 0.89 (Figure 4).

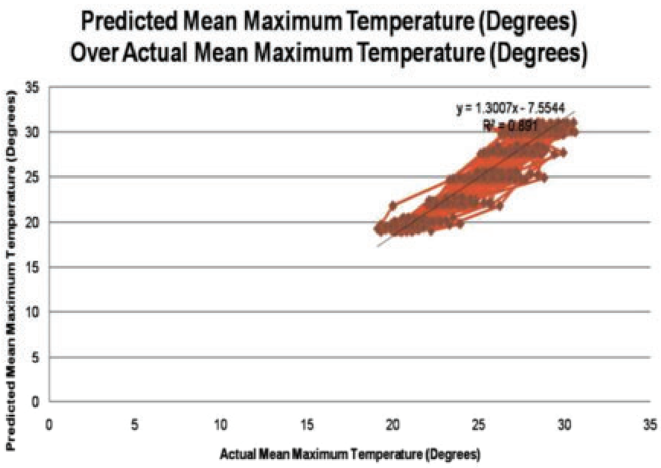

Figure 4. Regression of model values on actual values.

Test: When used to generate the mean maximum temperature of Surfers Paradise over the 80th month from January 1938 the model equation:

$$y \approx 5.75\sin(0.5235(x+4)) + (0.001x + 24.64)$$

gave a value of y ≈ 24.702, within approximately 0.0977 of the actual recorded value of 24.8. Thus it would be reasonable to assume that the model could predict the future trend of the mean maximum temperature of Surfers Paradise.

Comment

Conceptually they should have regressed actual values onto model generated values, which also gives a positive *y*-intercept. The purpose was to show that the model was globally doing the right thing, so the subsequent use of the wave formula was justified. This was also tested, although a range of points rather than a single point should have been used here.

Solving the mathematical problem

Water at 10^0 C increases in volume by 0.0088% when heated by 1 degree. The volume of Earth's water is approximately 1.3 billion km^3, and its surface area is 361 million km^2.

Using: $v = lwh$.

Temp increase $\times$ 0.00088 $\times$ 1300000000 = 361000000000 $\times$ h

Assuming a height increase of 3 metres to reach the lobby of the Q1, the equation gives a required temperature increase of 0.94667° C.

Such a change in temperature must be sustained; it should be the mean of the mean maximum temperatures generated by the model.

The equation for its equilibrium line is $y \approx 0.001x + 24.64$, enabling calculation of the time at which the mean of the mean maximum temperatures will have increased to the point where the sea level has risen by 3 metres.

Adding increase to current temperature we need $26.2217 = 0.001x + 24.64$, which gives $x = 1581.7$. This is 132 years (approx) from 1938, or 61 years time. It is therefore apparent that the sea level calculation above was incorrect. (Recalculation gave a temperature increase of 9.4667 degrees, and $x = 10101.7$ (842 years); i.e., in 771 years time.) At that point the predicted mean maximum temperature from the wave formula is 35.05 degrees.

Conclusion

This involved summarising the findings, comparing them with a prediction of around 400 years by Kurt Wayne, and revisiting the possible additional effects of melting ice not considered in this model. Four limitations of the model were listed, and recommendations made concerning preparations for rising sea levels in residential areas. (Substantial additional material presented by the students cannot be included here.)

Comments

The students picked up an arithmetic error, caused by misreading expansion data, by realising their predicted date was not feasible, and subsequently corrected the error. They revisited their original caveat concerning the possible impact of melting ice. Other predictions can be generated using Pacific Ocean data, expansion rates at warmer temperatures etc. These give wide ranging predictions from about 300 years from the present time upwards. Indeed this serves to illustrate why there is so much debate about the impact of climate change—itself a worthy outcome.

A feature was the way the students continually spun between phases three and six of the modelling process—testing, evaluating, and revisiting was a way of life for them. Apart from the modelling, the students deepened their understanding of mathematics topics they invoked as part of the solution process—sometimes extraordinarily so.

Seeing is believing

Curricular goals for students to use mathematical knowledge productively in vocational, personal, and civic contexts require official commitment and support that has not in the past been seriously provided. At the coal face a crucial element is found in the contrapositive of the adage "seeing is believing" namely "not believing means not seeing". We will not see it happen until belief in what students are capable of achieving means that the right conditions, encouragement, and priorities are provided. Are we up for the challenge?

Acknowledgement

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COMPUTER ALGEBRA SYSTEMS IN THE CLASSROOM: IS THERE A BETTER WAY?

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Ballarat Grammar staff have been using computer algebra systems [CAS] in classrooms since the first Victorian Certificate of Education [VCE] exams permitted the use of CAS in 2002. The question now is whether all staff have taken teaching and learning with CAS as an integral part of their teaching throughout the school, and whether the students are way ahead in the process. And, importantly, where does this process fit into the *Australian Curriculum: Mathematics*? The use of technology in that curriculum will provoke an interesting debate amongst the states.

Introduction

Computer algebra systems, with the full use of a symbolic manipulator along with a grapher and a statistical platform, have been in use in senior mathematics externally set examinations in Victoria since 2002. This makes 2011 the tenth year of using CAS in Year 12 classrooms and examinations. In 2010, the CAS in examinations became the handheld technology of choice, with its use assumed in all technology-active assessments across all three VCE Mathematics subjects: Further Mathematics, Mathematical Methods (CAS), and Specialist Mathematics. To create the situation where such an innovative instrument became mainstream, the experience with this particular technology since 2002 has been somewhat perplexing. From the heady days of ‘I Can do Maths Now’ (Garner, 2002) and the more evenly stated ‘CAS: A Time for all Seasons’ (Garner & Pierce, 2005) the argument about what comes first, the mathematics or the technology, still rages. A comment often heard is: “We are just teaching technology now”. As a strong believer in teaching with CAS, and not just performing tricks and checks with CAS, I write this paper with a different perspective, that of an observer of the use of CAS in the middle years.

The *Australian Curriculum: Mathematics* (Australian Curriculum and Assessment Authority (ACARA), 2011) mandates technology in its various forms as a vital part of any mathematics classroom. In his keynote address to the 2009 biennial conference of the Australian Association of Mathematics Teachers (AAMT), Peter Sullivan spoke of “challenges within the national mathematics curriculum” (National Curriculum Board in Sullivan, 2009, p. 36). An example he presented is an open-ended number exploration used to introduce the idea of equations with unknowns. "Often students can be posed the following task: For the equation 3a + 2b = 70, what might be the values of a and b?" (Sullivan, 2009, p. 39) These tasks have a problem solving dimension allowing the student to decide his or her own path towards guessing and checking. Technology is one way in which students can not only systematically guess and check, but also record their answer. Sullivan summarised by saying the tasks presented can be useful "especially if accompanied by the associated pedagogical actions by teachers" (Sullivan, 2009, p. 41). The link between technology and teacher action appears vital.

Change in pedagogy?

Garner and Leigh-Lancaster (2003, p. 375) explored the thesis that "teaching with CAS necessarily entails significant new pedagogy rather than an extension of existing approaches" and that in using CAS in the classroom "issues of locus of control and ownership of the technology arise … CAS unexpectedly empowers students". Garner (2007, p. 100) reported that, "[I]t is clear that, for some teachers, this movement of control to the learner can be frightening and challenging" (Garner, 2007, p. 100). The pressure teachers feel is reflected in Chick's (2007, p. 14) writing about Pedagogical Content Knowledge (PCK) and the importance of teachers understanding the depth of mathematics of the current topic, so that if the students take them somewhere outside what is planned, then the teachers can be part of that discussion.

While the power is with the student, it is also with the teacher. A range of strategies can be employed by the teacher, deciding which to use when. "Such decisions will be influenced by … the teacher's perceptions of the value of using technology such as CAS for students' learning" (Wander & Pierce, 2009, pp. 168–169).

A search for 'calculators' on the ACARA website (www.acara.edu.au) yielded five entries:

- Finalising Phase One of the Australian Curriculum
- Phase 1 - The Australian Curriculum
- Survey Results
- K–10 Curriculum Directions for Revision
- Framing Paper Consultation Report: Mathematics.

From the following comment (ACARA, 2011), it is clear that the use of CAS has created discrepancies across the states:

> the issues raised have been resolved and work is underway to make the agreed adjustments … in the use of calculators this has involved a formulation that will support the variation in practice that currently exists across the states and territories.

It is suggested that the curriculum "assumes teachers will make use of available digital technology, including calculators, in teaching and learning contexts". The survey of stakeholders reports a need for "a change in thinking and not just a change in the tools used." And it was also noted that "guidance was sought around the appropriate stage and level to introduce calculators. Feedback highlighted the fine line between introducing calculators too early, which may not allow students to develop their own mathematical skills" (ACARA, 2011).

The decision of CAS first, or by-hand skills first, is an ongoing dilemma amongst teachers. Some worry that the teaching of calculator syntax will take precious time away from time-poor mathematics classes. Within the crowded curriculum, one could say that yet another demand on teachers' time is one demand too many. The other end of the spectrum is that calculator syntax can be 'taught' as an ongoing process while students, and importantly also their teachers, use CAS for everyday tasks. Many teachers seem to fear change. Well-honed skills in presenting mathematical arguments are at risk of being superseded. A pre-service teacher initially felt that this new tool, for her, "devalued her own hard won mathematical prowess" (Garner, 2009, p. 86). Pierce, Ball and Stacey (2009) write about CAS use in the middle years:

> teachers with strong backgrounds in both mathematics and teaching were the most difficult to convince to use CAS in their teaching… If a teacher's current practices are already successful, then they may question the need for change. (p. 1163)

Some literature

Susie Groves, in her keynote address to the 1996 annual conference of the Mathematics Education Research Group of Australasia (MERGA), which focused on technology, concluded "Research … has established that technology can alter the nature of classroom mathematics … appear(ing) to show that the use of technology leads to positive learning outcomes" (Groves, 1996, p.17). Jones and McCrae, on assuming graphics calculators in VCE examinations, state "the effect has not been to trivialise the majority of questions, but rather to broaden the methods available to answer many questions" (Jones and McCrae, 1996, p. 307). Pierce, Ball and Stacey (2009) write:

> CAS is valued for calculation and manipulation capabilities, the option of alternative representations, the opportunity for systematic exploration, and for prompting rich discussion. However the technical overhead, initial workload for the teacher and unresolved questions about the perceived relative contribution of machine and by-hand work to learning currently pose obstacles to teaching with CAS in the middle secondary years (p. 1149).

The efficacy of introducing CAS in the middle years as a learning tool, rather than solely or mostly for preparation for VCE study, is discussed. Edwards (2003) comments on "the tendency of CAS to simplify expressions in unexpected ways and to provide an output that is not presented in the fashion that students are used to seeing in standard texts" (Pierce, Ball and Stacey, 2009, p.1157). Some teachers say the varied forms of output are a stimulus for learning. For example, Garner (2009, p. 86) writes, "While some teachers describe working with CAS as 'just button pushing', others see the power of CAS to connect the continuum of the multiple representations of a function and thereby provide a powerful teaching and learning tool."

Use of technology in a 2011 middle school classroom

At Ballarat Grammar in Victoria, the Casio *ClassPad* is adopted at the beginning of Year 9 as the handheld calculator of choice for use in the mathematics classroom. Garner (2009) wrote about two Year 12 students.

> Jack: I don't know. We just are pleased that we are allowed to use CAS.
> Jai: … cos we are very familiar with it 'cos we have been using it since Year 9.
> Jack: … and yeah, we learned very quickly how to use it.
> Jai: at Year 9 the teachers taught us how to do stuff, but by Year 10 we knew more than the teachers did.
> Jack:… and after we got over putting all these games on it, we started to use it for Maths. (p. 84)

In consultation with a colleague about an upcoming assessment task for her Year 9 mathematics class, we discussed the three components of increasing difficulty of a VCE Application Task in Victoria. Teacher C wrote a task mimicking a VCE Application Task, using the theme of speed cameras. I observed Teacher C's class, with a particular interest in whether the students could work cooperatively while completing the task, and if, and when, they picked up and put down their calculators. I had the students' and the school's permission to observe, take notes and photographs.

Year 9 Application Task

Here is a summary of the Application Task that was given to students.

Speed cameras

There has been a large amount of debate about speed cameras in the news recently. Graham and Caroline have frequently been pulled up for speeding and have decided to do some research into the different types of speed cameras currently used. They discovered there are three different types of speed cameras: point-to-point, fixed speed and mobile speed cameras.

Question 1

There are a large number of point-to-point speed cameras on the Hume Freeway. Graham regularly travels along the Hume Freeway. Graham estimates that it takes 90 minutes to cover the distance from Wodonga to Benalla. Using the road sign, calculate Graham's average speed in km/h. If he travelled at an average speed of 90 km/h from Wodonga to Melbourne it would take three hours and 20 minutes. How far is it from Wodonga to Melbourne?

Figure 1. Road sign.

Question 2

The scale of the map is 1: 2 000 000. After measuring the distance on the map, find the distance from Seymour to Benalla in kilometres. There are point-to-point speed cameras in Seymour and Benalla. Caroline passes the camera in Benalla at 1:10 pm and again at Seymour at 2:30 pm. Decide whether Caroline is speeding, given that the speed limit along the freeway is 110 km/h.

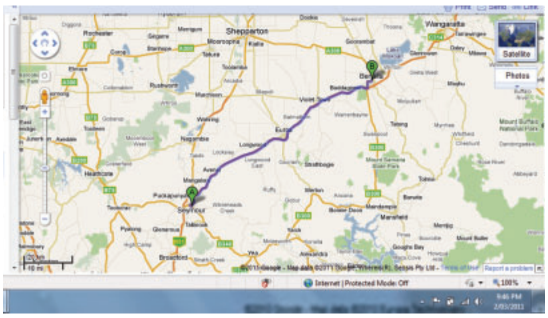

Figure 2. Map of Victoria from Seymour to Benalla.

Question 3

Graham and Caroline drive from Wodonga to Canberra, a distance of 350 km. They divide the driving in the ratio of 3:4 (Caroline to Graham). How far does Caroline have to drive? Along the Hume freeway from Wodonga to Canberra there are numerous point-to-point cameras so they record their distance from Wodonga every 45 minutes. Display this information graphically. Describe the journey from Wodonga to Canberra stating average speeds and break times.

Time	Time in hours from trip start	Distance from Wodonga (km)
10 am	0	0
10:45 am	0.75	80
11:30 am		160
12:15 pm	2.25	160

Figure 3. Part table with 45 minute splits.

Question 4
Sue is a terrible lead-foot and lucky to have her licence. The distance-time graph below shows a sixty-minute journey. A tangent line is drawn at 45 minutes.

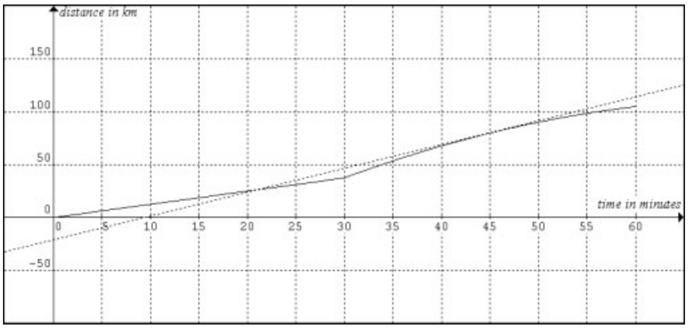

Figure 4. Sue's journey.

Find the average speed in km/h for the first and second 30 minutes, given that the tangent line passes through the points (30,37.5) and (60,105). Is Sue speeding up or slowing down in the second 30 minutes? A mobile speed camera clocks Sue's car. She has been travelling for 45 minutes and has gone 80 km into her journey. She is in a 110 km speed zone. Is she speeding?

Year 9 application task observations

All students immediately took out their *ClassPad* appearing ready to use them. The questions they asked showed a good level of awareness of the size of expected answers. Questions asked were:

- Is Seymour to Benalla that far away?
- Why have I got 10 000 kilometres distance? That can't be right. Do I just write it down because that is what we are told?

At first, there was not much calculator work. Mathematical talk was evident but two tables were quite distracted. One group of students engaged in discussion while attempting to recreate the graph in Figure 4. Some class work had previously been done solving linear equations with a *ClassPad*. The students were stumped by their inability to link that experience to the curved function on the graph. The statistics facility was suggested as a mode of investigation, but was not pursued. The concept of a tangent was understood and some students investigated the equation of the line using the symbolic facility. In defence of their limited use, these students have only had their calculators since the beginning of 2011, but compared with some other Victorian schools, this is quite early. Teachers often discuss when schools should introduce CAS into the maths classroom. Some schools are concerned that, if they start using CAS in Year 11, ostensibly in preparation for Year 12, they will be behind the schools that start with

CAS in Year 7. What is the correct approach? Most of what the students did could be done with a scientific calculator. This is somewhat surprising given that CAS is used in this classroom, which is taught by a teacher who is competent with CAS. Given that students of today are extremely technology literate, can we expect the same facility, or importantly interest, with a calculator? Garner (2007, p. 99) writes of "an ambivalence amongst students in their use of CAS at Years 9 and 10… the initial student experience of CAS can be very much a reflection of the individual teachers' approach".

CAS in the middle years

Students attempting the Application task showed limited use of the symbolic aspect of the CAS, not yet appreciating the potential of the technology. The teacher is a vital link between the technology and the students, and the modelling of the use of technology is an important step for students' future independent use. Teacher C began the year with teaching about the Theorem of Pythagoras, using dynamic geometry; Teacher C used the CAS to explore the concepts, as against teaching the formula and using CAS afterwards to check the answers. Pierce, Ball and Stacey (2009) write

> we are … mindful of the little use that these teachers … actually made in the first year. However, for schools that have good access to CAS technologies and teachers who are already familiar with their use *(for example through teaching at senior levels) (my emphasis)*, CAS can support skill development and contribute to deeper learning in the middle years of secondary school. (p. 1166)

Teacher C is an experienced, well qualified mathematics teacher who was one of the team of teachers in the VCAA CAS Pilot Study. She no longer teaches in the Year 12 classroom but, because of this experience, has a respect for the technology and the ability to adopt CAS in any classroom setting. It is difficult to say the same for all other Year 9 and 10 staff. Some do ask if the extra effort is worth it.

Jill Adler, in her keynote address to the AAMT 2009 biennial conference, stated that mathematics teacher education is being challenged by the "professional knowledge base of teaching": "The profound insight of Shulman's work (Lee Shulman, mid-1980s) was that being able to reason mathematically was necessary but not sufficient for being able to teach others to reason mathematically" (Adler, 2009, p. 3). There is a constant call for better educated maths teachers. Adler argues that "strengthening our understanding of the mathematical work of teaching, (*mathematics for teaching)* is a critical dimension of enhancing its teaching and learning" (Adler, 2009, p. 4). Yet the reality of textbooks and classrooms falls short of this ideal. Adler adds "despite the longevity and consistency of elementary algebra in school mathematics curricula worldwide, large numbers of learners experience difficulty with this powerful symbolic system" (Adler, 2009, p. 4). Classroom teachers have only begun to realise the potential of a handheld symbolic manipulator in elementary algebra.

Technology in the Australian Curriculum: Mathematics

The *Australian Curriculum: Mathematics*[1] describes successful learners as having "essential skills in literacy and numeracy" and being "creative and productive users of technology". Also as an intended educational outcome, the general capability of ICT

[1] http://www.acara.edu.au. Box 2(a): Educational goals for young Australians. Box 3: Intended educational outcomes for young Australians, 71.

competence states “students develop ICT competence as they learn to use ICT effectively and appropriately”. Technology is described as central to Australia’s skilled economy, providing “crucial pathways to post-school success” (ACARA, 2011).

Framing paper consultation report: Mathematics

The public consultation on the mathematics framing paper released by the National Curriculum Board ended on 28 February 2009. Points of discussion about the “Application of technology and incorporation into the curriculum” were, in brief:

1. Respondents commented that digital technologies should be used purposefully as a tool to support learning in mathematics, not to replace knowledge of the basics.
2. The interaction between technology and curriculum is changing. *New technologies provide possibilities for new pedagogical approaches* (my emphasis).
3. The equity of access, teacher training and resource funding was addressed.
4. Teacher training would be required if proposals in the Paper were to be adopted.
5. There was uncertainty around the assertion that the curriculum needs to “embed digital technologies so that they are not optional extras”. A number of respondents perceived the framing paper proposal to imply a mandate for CAS calculators.[2]

ACARA sought advice from expert mathematicians on key issues raised through the consultation. The position on inclusion of technology in the mathematics curriculum has been affirmed and “strengthened advice to reiterate the position and appropriate use of technology” is embedded in the curriculum (ACARA, 2011).

The embedding of CAS has happened in the VCE in Victoria. It has been reported that the use of CAS has changed how teachers and students viewed their mathematics.

> The unrestricted use of CAS has led to dramatic changes in pedagogy and assessment … Unrestricted access to CAS has challenged us, as educators, to start inventing new paradigms for the teaching and learning of senior mathematics. (Garner, McNamara & Moya, 2003, p. 271)

Despite this view, in one VCE classroom a student commented, “why should I have to rely on this futuristic piece of technology when I will never carry it with me for the rest of my life?” (Garner, 2009, p. 89) And a Queensland study noted, “it is not clear whether teachers already convinced of the benefits of technology simply embraced… calculators when they became available” (Goos & Bennison, 2008, p. 124). Do we expect this to happen with CAS throughout the school? Will there be delineation in teachers who do and teachers who don’t?

Herein lays the dilemma. Will the teacher who embraces change in pedagogy do so easily, or are there particular features of teaching with CAS that open up possibilities never before considered? Have all teachers taken up teaching and learning with CAS as an integral part of their teaching, or as some sort of add-on?

Conclusion

The use of CAS with students in the middle secondary school years is a complex issue. Pierce, Ball and Stacey (2009) write of teachers reporting their perception of success, or not, of CAS use, rather than hard evidence of success: “However perception data is important because teachers’ beliefs about the potential costs and benefits of any

[2] http://www.acara.edu.au. Application of technology and incorporation into the curriculum, 7.0

initiative will contribute to whether they incorporate it in their teaching" (p. 1173). This echoes earlier results that the use of CAS "sinks or swims with the teacher" (see Garner & Leigh-Lancaster, 2003; Garner, 2004; 2009). The teachers in Pierce, Ball and Stacey's (2009, p. 1174) study recognise that "adequate access to the CAS, and recognition that learning to use CAS syntax and commands is a substantial task" are barriers to students gaining benefit from CAS. Teacher C's classroom seems to meet these constraints: she has had teacher training using CAS; there is adequate access to the CAS for her students; and she is skilled at teaching CAS syntax. But, still it appears that the students have not yet absorbed CAS into their middle school classroom.

Hughes-Butters pleas for technology use to be genuine. She writes of the importance of embedding technology in the lesson, echoing the *Australian Curriculum: Mathematics*, creating an authentic learning experience. "Technology should be used when its application enhances … the contrived application of technology will actually take away from the meaning of a lesson as the students focus on the technology and not on the learning" (Hughes-Butters, 2009, p. 239).

The teacher's challenge

Wander and Pierce (2009) studied two approaches to a lesson, highlighting the choice that a teacher makes between by-hand or CAS-enabled algebra. Teachers now need to decide upon their approach. "In a CAS-enabled mathematics teaching environment several approaches to a lesson may be possible. The class teacher must now make even more decisions in order to choose the best path for her or his students" (p. 173). A state of "constant conflict between CAS and pen and paper methods" is described by Geiger, Faragher and Redmond (2007, p. 13).

To answer the initial question about whether all staff have taken teaching and learning with CAS as an integral part of their teaching, I would say, "No." We are yet to see where this process fits into the *Australian Curriculum: Mathematics*. I suspect much discussion is forthcoming.

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VR ELEMENTS: A 3D SPATIAL VISUALISATION TOOL FOR MATHEMATICALLY GIFTED STUDENTS

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Spatial visualisation is an important skill required in the study of geometry. Most mathematical software used in mathematics classrooms in Singapore is non-immersive in nature. *VR Elements* is a three-dimensional immersive computer software to assist students in perceiving three-dimensional geometrical figures from their two-dimensional representations and envisioning geometrical properties within them. This paper will elaborate on how *VR Elements* was developed in tandem with curriculum materials on the topic of three-dimensional trigonometry. In particular, the constructivist approach was used to develop these materials to meet the needs of mathematically gifted students under the School Based Gifted Education Programme.

Spatial ability and spatial visualisation

The definition of spatial ability has been discussed and re-defined by various scholars in the last century, with the first definition coined by McGee in 1979 as the ability to mentally manipulate, rotate, twist, or invert a pictorially presented stimulus object. Lohman (1979) later related spatial ability to that of arranging the pieces of an object to complete paper folding or overall shape, while Linn and Petersen (1985) defined spatial ability with mental processes being used in perceiving, storing, recalling, creating, arranging and making related spatial images. In more recent years, Noraini Idris (1998) has defined it as the ability to perceive the essential relationships among the elements of a given visual situation, and the ability to mentally manipulate one or more of these elements, and Spock (2010) has defined spatial ability as skill in perceiving the visual world, transforming and modifying initial perceptions, and mentally recreating spatial aspects of one's visual experience without the relevant stimuli.

Others, however, have tried to divide spatial ability into various independent components. Both Guilford (1967) and McGee (1979) divided spatial ability into two elements: spatial visualisation and spatial orientation. Lohman (1979) classified spatial ability into three components: spatial relation, spatial visualisation and spatial orientation. Linn and Peterson (1985) divided spatial sense into spatial perception, spatial rotation and spatial visualisation, and Spock (2010) has distinguished three categories of spatial abilities: (1) spatial orientation, the ability to keep track of objects or locations in space even after a rotation or movement to a new location; (2) spatial perception involving determining spatial relationships with respect to gravity or one's own body in spite of distracting information; and (3) spatial manipulation involving the ability to mentally rotate two- or three-dimensional figures rapidly and accurately. Amongst the various definitions and elements that define spatial ability, these researchers seemed to have identified one common component, which is spatial visualisation, and this element is the focus of this paper.

Importance of spatial visualisation in the learning of geometry

Geometry is a branch of mathematics concerned with the study of spatial properties of various figures abstracted from the concrete world of physical objects (Noraini Idris, 1998). The geometry curriculum constitutes many of these visual components that require students to have the ability to visualise spatially (Noraini Idris, 1998). Hence, this ability is a critical skill to have and develop in all students.

Yakimanskays (1971) believes that visualisations are used as a basis for assimilating abstract geometrical knowledge and individual concepts. Hoffer (1983) adds that the lack of basic visualisation skills sometimes results in insecurity that causes many students not to do well in geometry. Many students who have not had ample prior concrete experiences with solid objects tend to have problems when it comes to visualising three-dimensional objects from a two-dimensional perspective, especially so for visualising cross sections of solids (Ben-Chiam, Lappan & Houang, 1989). Due to their limited geometrical experiences, some students may not have had enough opportunities to develop and exercise their spatial thinking skills to help them learn geometry effectively. This inability to visualise can be regarded as an obstacle to the learning of geometry as there are concepts in geometry which require the student to visually perceive the objects in three dimensions and identify their properties by comparing them with their previous experiences involving similar objects. These geometrical concepts also require visual interpretations as many geometry problems are presented in a two-dimensional format on paper. Thus students who are not able to extract geometrical information about solid objects which are three-dimensional and that are drawn on paper will face difficulty in interpreting and solving questions involving solid geometry (Lappan, 1984).

Achievement in geometry

Presmeg (1986) has performed significant research on spatial visualisation ability in mathematics for various groups of students, including mathematically gifted students. She notes that specific research on spatial visualisation of the mathematically gifted seemed to be insufficient. According to Greenes (1981), mathematically gifted students differ from the general group of students studying mathematics because they are better at, or show more creativity and flexibility in, the following abilities: spontaneous formation of problems, flexibility in manipulating and analysing data, mental agility with ideas, mental data organisation and re-organisation, originality in interpreting possible solutions, transferring from one situation to another, and being able to generalise.

Kruteskii (1976) says that mathematically gifted students tend to view the world through a "mathematical lens" and uses the term "mathematical cast of mind" to describe this characteristic. He further identifies mathematically gifted students into three types, namely, the analytic, geometric, and harmonic. The analytic type tends to think in verbal-logical terms. This is the student who is able to think abstractly and does not rely on visual supports for visualising objects or patterns in problem solving. Geometric thinkers, however, strive to solve a problem using visual supports and to interpret abstract mathematical relationships visually. These students tend to relate problem solving to the analysis of spatial concepts. The harmonic type displays both characteristics and is successful at using both approaches to solving problems.

Studies by Diezmann and Watters (1996) showed that spatially gifted students may underachieve in classrooms due to the typical emphasis on analytical tasks and may experience significant difficulty verbalising their reasoning. Researchers such as Diezmann and Watters (2002) urged educators to increase sensitivity to the types of tasks that will allow spatially-gifted students to demonstrate their ability and the provision for appropriate tasks and procedures. A study by Ryu, Chong, and Song (2007) used *Geometer's Sketchpad* to examine the spatial visualisation ability of mathematically gifted students using geometrical tasks that require distinction of parts of a solid figure. Conducted at Konkuk University and Gyeogin National University of Education on seven Grade 7 mathematically gifted students in Korea, it was found that some had difficulty imagining a three-dimensional object in space from its two-dimensional planar representation. These students were confused when trying to distinguish the edges of a spatial object from the depicted picture, and were unable to distinguish planes of a three-dimensional object from its two-dimensional representation. These researchers were surprised to find such dependency on the visual facts represented in a plane picture.

Aim of the study

This study depicts the exact problems that I faced when teaching my group of mathematically gifted students on the topic of three-dimensional trigonometry. When dealing with two-dimensional representations of three-dimensional objects, comments like, “I can’t see it” or “How can you tell if that (pointing to an angle) is a right angle?” are frequently heard when taught using the traditional ‘chalk and talk’ method. Because they are unable to grasp mentally geometrical concepts in three dimensions, these mathematically gifted students are ‘bored’ and ‘restless’ in the classroom and do poorly as a result. Hence, the challenge for me as a mathematics teacher would be to develop or use existing spatial visualisation tools to aid the teaching and learning of geometry, so that these mathematically gifted students would be able to see and understand geometrical problems in three dimensions and so help them solve these problems more efficiently and easily.

Non-immersive and immersive technologies

Since the implementation of use of information communication technologies in the mathematics classroom, numerous studies, both local and overseas, have been conducted from elementary to tertiary level. Backed by constructivist learning theories, learners’ prior experience and their ability to build their own cognitive structures during the course of their learning experience have been emphasised, especially for the optimal understanding of mathematics (Battista, 1999; Greeno et al., 1996). Lim and Hang (2003) found that students using technological tools have higher learner autonomy.

Another study by Lim and Chai (2002) found that students using information communication technologies are positively engaged in higher order thinking activities. Researchers like Al-Rami (1990), Fodah (1990), and Renaud (1997) found that the use of computers has positively influenced the achievements of students in mathematics. Because of this widespread use of computers in education there has been a proliferation of many powerful technological tools such as *Geometers' Sketchpad* and *Cabri 3D* that have been designed to facilitate the learning of geometry. These tools have been described by Patsiomitou (2008) as "computational environments that can link symbolic and graphical representations" thus allowing students to explore the various solution paths, make decisions and receive feedback on their ideas and strategies individually or in small groups.

However, some researchers have questioned whether it is still applicable in today's world, where students are such 'digital natives'. Some other researchers have looked at technologies that are immersive in nature, allowing the user to access external information (e.g., the actual source code) without leaving the environment and the context of the representation (e.g., using a palmtop or laptop). Such immersion has been found to allow the user to take advantage of their stereoscopic vision which helps the viewer to judge relative size of objects and distances between objects. The work of Hubona, Shirah and Fout (1997) suggests that users' understanding of a three-dimensional structure improves when they can manipulate the structure. One of the defining features of virtual reality representations is the ability of the user to manipulate the visualisation, by being immersed in the environment. The work of Ware and Franck (1994) also indicates that displaying data in three dimensions instead of two can make it easier for users to understand the data. With this background, it is therefore pertinent that suitable immersive computer software be selected in order to develop spatial visualisation skills for our mathematically gifted students. For this study, new three-dimensional spatial visualisation software, *VR Elements* (Zepth Pty. Ltd.) would be used. Curriculum materials on the topic of three-dimensional trigonometry will be developed jointly by existing teachers from Hwa Chong Institution teaching mathematically gifted students under the School Based Gifted Education Programme and researchers from Zepth.

VR Elements

The hardware consists of a stereographic and an interactive sub-system (Figure 1). The interactive sub-system has (1) a control pad to input commands and numbers; and (2) a 3D pen to create, edit and manipulate the 3D elements. It has 6 degrees-of-freedom (6DOF) tracking and 3D digitising, and allows various modes; e.g., insertion, selection, and transformation. In insertion mode, basic geometrical elements can be created. In selection mode, each geometrical element can be selected for modification. If the selection contains two elements, their relationship can be derived. In transformation mode, objects can be rotated, translated or scaled. Figure 1 shows a user in front of the 3D screen.

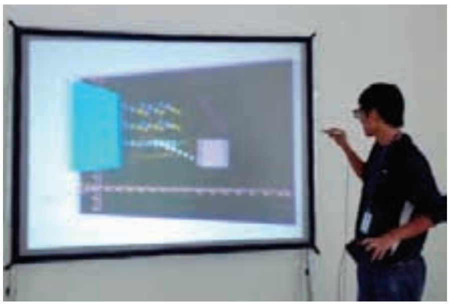


Figure 1. VR Elements.

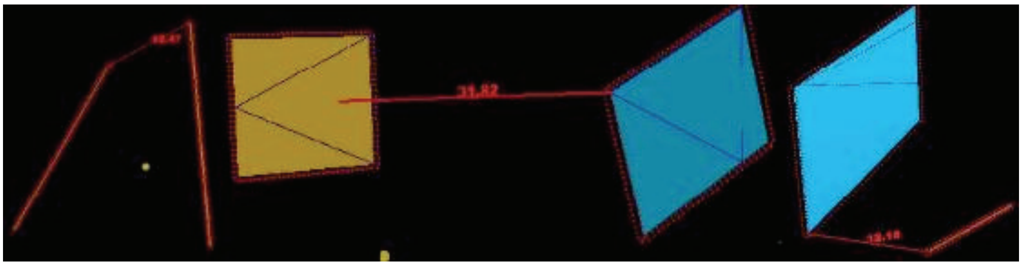
The software comprises a core layer and an application layer. The core layer is for users to develop the application. The application layer is designed for the learning of fundamental geometry topics in both 2-dimensional and 2-dimensional space. The basic geometrical elements of *VR Elements* are points, lines, planes, cubes and spheres (Table 1). The relationship between two elements can be obtained with the VR Elements.

Table 1. Properties of basic geometrical elements of VR Elements.

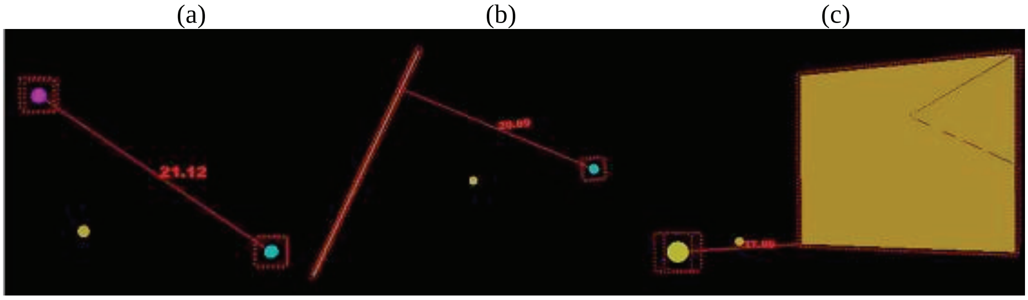
Elements	Properties		
Point	Position	Point thickness	Point color
Line	2 end points	Line thickness	Line color
Plane	3 locations	Four sides	Plane color
Cube	Centre	Width, height, and depth	Cube color
Sphere	Centre	Radius	Sphere Color

Distance

In *VR Elements*, the distance between two elements can be measured with the 3D pen (see Figure 2).



(a) (b) (c)



(d) (e) (f)

Figure 2. The distance between two elements can be measured in VR Elements: (a) point-point distance; (b) point-line distance; (c) point-plane distance; (d) line-line distance; (e) plane-plane distance; and (f) line-plane distance.

Angle

In 3-dimensional space, two lines may or may not intersect. A line can either intersect with or be parallel to a plane; two planes always intersect. The angle formed by two intersecting elements can be similarly measured using the input pen.

Dynamic feature

VR Elements has a feature which allows the users to dynamically transform geometric entities. For example, a line segment can be rotated in 3-dimensional space. During its rotation, its associated distance or angle will be dynamically updated.

Curriculum materials for virtual reality elements

There is a general consensus (Battista, 1999; Greeno et al., 1996) that for optimal learning of mathematics, ideas must be constructed by the learner. The National Council of Teacher of Mathematics (NCTM) *Standards* (2000) suggest that students should be given opportunities to engage in scientific inquiry and in problem solving, and in order to do this, students should be given the necessary scaffolding and allowed to collaborate with their peers. This is in line with the constructivist theory of learning. Constructivist-based theories are well suited for use in a digital classroom. Combined with appropriate use of pedagogy, a rich learning culture can be established: one where learning is authentic and learner centred, encourages students to explore and discover ideas and concepts and to share them in collaborative projects. Curriculum materials, based upon the syllabi of the School Based Gifted Education Programme (Table 2), have been developed by a team of teachers and researchers to provide the necessary scaffolding for the students to use the software. Students will also be given opportunities, both individually and collaboratively, to build this new knowledge.

Table 2. Topic and specific instructional objectives of school based gifted education programme for a topic on three-dimensional trigonometry.

Topic and Specific Instructional Objectives
Trigonometry Unit 2: 2D and 3D Problems At the end of the unit, students should be able to • solve triangles through Sine Rule & Cosine Rule • find area of triangles • understand concept of bearings, solve problems in 2D and 3D including those involving angles of elevation and depression and bearings • derive Heron’s formula • relate geometry to concepts on longitude and latitude • calculate angles between two planes or the angle between a straight line and a plane.

A sample teaching plan using *VR Elements* is shown in Table 3.

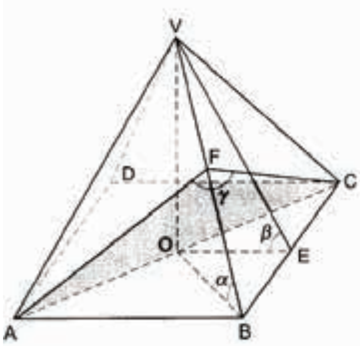
Table 3. Sample teaching plan.

Teaching Plan		
Lesson Objectives: At the end of the lesson, students should be able to calculate angles between two planes calculate angles between a straight line and a plane		
Lesson Aims: 1. Introduce *VR Elements* as a tool for classroom teaching and learning 2. Using *VR Elements* to support the learning of Geometry through a visual and interactive manner		
Time	Plan	Remarks
5 mins	Introduction	Generate students' interest by finding out their personal experience with Virtual Reality for e.g. Omni-Max Shows, 3D movies Teacher to show real life products created using *VR Elements*
10 mins	Teacher demonstration	Teacher to demonstrate the use of *VR Elements* with aid of 3D pen Teacher to define the following geometrical concepts and to emphasise that seeing is NOT believing with use of examples and non-examples. Line – Plane Relationship Plane – Plane Relationship Rotation of Plane or Points Measurement of Distances and Angles
15 mins	Student Participation	Some students invited to try out using the *VR Elements* software
10 mins	Teacher demonstration	Teacher to use some examples from *PowerPoint* slides and worksheets to show students how to calculate angles between two planes or straight line and plane
15 mins	Problem Solving	Students to solve problems on worksheet involving angle calculation Students invited to the whiteboard to share their answers
5 mins	Lesson Closure	Teacher to gather feedback from students

Geometry achievement test

To evaluate the effectiveness of the *VR Elements* software, a geometry achievement test has been designed by the same team of researchers aforementioned. This test has been formulated according to specific instructional objectives as prescribed by the syllabi of the School Based Gifted Education Programme. The professional judgment of senior mathematics teachers at Hwa Chong Institution has been sought to ensure content validity and to evaluate the level of difficulty of the questions. Necessary revisions to the test will be made before its use in the study. Designed conscientiously by the team, the test can be said to be of high face validity. This test is made up of six questions, consisting of four short and two long questions that test students' understanding of the geometrical concepts taught during the study. Students will need to spend an estimated half an hour on the test and the scores they achieved for the test will not be taken into account in their assessment for their promotion to the following year. A sample question has been included. (Figure 3)

The figure below shows a right pyramid *VABCD* with a square base.



All the edges are 4 cm long. Find the angle between

a. the slant edge *VB* and the base.

b. the face *VBC* and the base

c. two adjacent lateral faces.

Give your answers correct to the nearest 0.1°.

Figure 3. Sample question from geometry achievement test.

Conclusion

The use of such immersive technology would allow learning to be more learner-centred and can be used for collaborating, retrieving information and expression of ideas, leading to higher levels of cognition, building on previous knowledge and deeper understanding and ownership of concepts. However, the key to the success of such digital classrooms still lies with the teachers to use an effective pedagogy to combine the power of new digital tools available in a rich multimedia learning environment with the learning style of today's students. Informed by constructivist-based theories, it would be up to educators to use these platforms to merge the new learning styles of today's students with the power of emerging digital tools to produce a new generation of independent problem solvers. It is hoped that the *Virtual Reality Elements* software used in this study, merged with the high-end three-dimensional immersive technology and curriculum materials designed based on constructivist theories, would be the first of its kind to lead the way for this new learning climate for the mathematically gifted students, and for the rest of the learning community.

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TOWARDS EXCELLENCE IN MATHEMATICS TEACHING: FORGING LINKS BETWEEN NATIONAL CURRICULUM AND PROFESSIONAL STANDARDS INITIATIVES

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In 2011 many Australian mathematics teachers will 'meet' for the first time two key national initiatives: *The Australian Curriculum: Mathematics* (ACARA, 2010) and the *National Professional Standards for Teachers* (AITSL, 2011). This paper will explore the practicalities of coming-to-know these initiatives, their relationship to each other, and links to related documents such as the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2006). The intent of the authors paper is to encourage teachers to reflect on current classroom practices, and consider future visions and new practices that might emerge from their interaction with these initiatives. The discussion will focus specifically on what excellence in primary mathematics teaching might look like.

Introduction

Over the next few years, the work of mathematics teachers in Australia will be shaped by the implementation of two key national initiatives: *The Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2010), and the *National Professional Standards for Teachers* (Australian Institute of Teaching and School Leadership (AITSL), 2011). While these initiatives have different purposes, with one describing a national mathematics curriculum and the other articulating what teachers are expected to know and be able to do at different career stages, teachers will need to embrace both simultaneously as they do the work of teaching students mathematics.

The purpose of this paper is to begin conversations about ways the two initiatives relate to one another and ways primary mathematics teachers might effectively embrace them. The conversations that are presented follow a trajectory typical of 'first meetings', to include discussions related to: where the initiatives come from; how they relate to the current context; what they look like and how they work; how the initiatives might promote and support excellence in primary mathematics teaching; and, what's involved in moving towards this kind of excellence.

Where the initiatives come from

The National Professional Standards for Teachers

Over recent years, there has been accumulating national and international evidence signalling the powerful impact that a teacher's effectiveness has on students (see, for example, Hattie 2003; Hattie 2009; Jensen 2010). Recognising that teachers "have the greatest impact on student learning, far outweighing the impact of any other education program or policy" (Jensen, 2010, p.5), education systems across the world are developing professional standards for teachers as a mechanism for attracting, developing, recognising and retaining quality teachers (AITSL, 2011, p.1).

Work on the *National Professional Standards for Teachers* (AITSL, 2011) commenced in 2009, under the auspices of the Ministerial Council for Education, Early Childhood Development and Youth Affairs (MCEEDYA). That work continued during 2009-10 through the Australian Education, Early Childhood Development and Youth Affairs Senior Officials Committee (AEEYSOC), and in July 2010, responsibility for validating and finalising the Standards was assumed by the Australian Institute for Teaching and School Leadership (AITSL).

The *National Professional Standards*, endorsed by MCEEDYA in December 2010, articulate what teachers are expected to know and be able to do at four career stages. They are intended to guide professional learning, practice and engagement, facilitate improvement of teacher quality, and contribute positively to the public standing of the profession (AITSL, 2011, p.1).

The Australian Curriculum: Mathematics

The development of the *Australian Curriculum: Mathematics* (ACARA, 2010) is guided by the principles outlined in the *Melbourne Declaration on Educational Goals for Young Australians* (MCEEDYA, 2008). This policy statement, adopted by the council of state and territory ministers in 2008, emphasises the importance of knowledge, understanding and skills of learning areas, general capabilities and cross-curriculum priorities as the basis for a curriculum designed to support 21st century learning. The focus of the *Australian Curriculum*, developed by ACARA in consultation with education authorities, professional education associations, academics, business, industry, parent bodies and community groups, is to provide each Australian student with a foundation for successful, lifelong learning and participation in the Australian community.

The *Australian Curriculum* sets out what students should be taught through the specification of curriculum content, and the learning expected at points in their schooling through the specification of achievement standards (ACARA, 2011). Mathematics is one of the initial learning areas of the Australian Curriculum to be developed. The *Australian Curriculum: Mathematics* was published in December 2010 for implementation in 2011 and 2012.

How the initiatives relate to the current context

Links with the AAMT Standards for Excellence in Teaching Mathematics in Australian Schools

The *National Professional Standards for Teachers* are the first professional standards to be nationally endorsed by the Education Ministers from all States and Territories (MCEEDYA). However, mathematics teachers in Australia have had the opportunity to be guided by the Australian Association of Mathematics Teachers (AAMT) *Standards for Excellence in Teaching Mathematics in Australian Schools* since they were first adopted in 2002. How are the two sets of Standards related?

The AAMT *Standards*, describe the knowledge, skills and attributes required for good teaching of mathematics as specified "by the profession for the profession" (AAMT, 2006). They are intended to provide "targets to which all teachers of mathematics can aspire and work towards in their professional development". Aspirant teachers wanting to be acknowledged for reaching the high standards they describe, can participate in an established program of assessment that allows them to be awarded the AAMT's *Highly Accomplished Teacher of Mathematics* credential; this is the only program of assessment and accreditation against the *Standards for Excellence* endorsed by the AAMT (AAMT, 2006).

The *National Professional Standards for Teachers* "represent an analysis of effective, contemporary practice by teachers throughout Australia" (AITSL, 2011, p. 1). Their development included "a synthesis of the descriptions of teachers' knowledge, practice and professional engagement used by teacher accreditation and registration authorities, employers and professional associations" across Australia (AITSL, 2011, p. 1). The purpose of the *National Professional Standards* is threefold. They are a public statement of what constitutes teacher quality, providing a framework which makes explicit the knowledge, practice and engagement of teachers across career stages. They inform the development of professional learning goals, providing a framework by which teachers can judge the success of their learning and inform reflection and directions for future achievements. And, they provide the basis for professional accountability, helping ensure that members demonstrate certain levels of performance across four defined career stages. The purpose of the *National Professional Standards*, is therefore broader than that of the AAMT *Standards for Excellence*, covering registration and accreditation for career staging in addition to professional development. A question of interest is how the two will work alongside one another in the future.

Links with existing State and Territory curriculums

There are four stages in the development of the *Australian Curriculum*. The first stage, The Curriculum Shaping Stage, involved the development of a paper titled, *Shape of the Australian Curriculum.* This presented a broad outline of the curriculum K–12 and curriculum design advice for each of the learning areas. The second stage, The Curriculum Writing Stage, involved teams of writers, supported by expert advisory panels and ACARA curriculum staff, developing the *Australian Curriculum*, which includes content descriptions and achievement standards K–12. Writers were guided by information presented in ACARA's *Curriculum Design Paper*, and advice from the ACARA Board. Writers were expected to refer to national and international curriculum and assessment research, State and Territory curriculum materials, and research on the general capabilities described within the *Shape of the Australian Curriculum* paper. The draft *Australian Curriculum: Mathematics* was released for public consultation and subsequently modified in the light of feedback. The third stage, The Implementation Stage, has seen the *Australian Curriculum: Mathematics* delivered in an online environment for school authorities, schools and teachers to implement. ACARA will work with State and Territory curriculum and school authorities to develop implementation plans. Still to come is the fourth stage, The Evaluation and Review Stage, where processes will be put in place to monitor and review the *Australian Curriculum* based on implementation feedback.

What the initiatives look like and how they work

The National Professional Standards for Teachers

The *National Professional Standards for Teachers* are organised into four career stages that reflect the continuum of a teacher's developing expertise—"from undergraduate preparation through to being an exemplary classroom practitioner and a leader in the profession" (AITSL, 2011, p. 2). They comprise seven "interconnected, interdependent and overlapping" Standards that outline what teachers should know and be able to do (AITSL, 2011, p.3). These Standards are grouped into three "Domains of Teaching": Professional Knowledge, Professional Practice and Professional Engagement. Within each Standard further illustration of teaching knowledge, practice and engagement is provided in "Focus" areas and these are separated into "Descriptors" specific to each of the four career stages: Graduate, Proficient, Highly Accomplished and Lead. To support implementation of the *National Professional Standards*, content-specific elaborations of the "Descriptors" (including mathematics elaborations) are currently being prepared to provide detail about what each one looks like.

The AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* are organised into three domains: Professional Knowledge, Professional Attributes and Professional Practice. Within each domain there are three or four Standards that describe what excellent teachers know or do with respect to specific aspects of the Domain. The Standards, therefore, are all "high standards" providing targets to which all mathematics teachers can aspire.

Table 1 highlights the relationship between the structure of the *National Professional Standards for Teachers* (AITSL, 2011) and the *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2006).

The Australian Curriculum: Mathematics

The aims of *The Australian Curriculum: Mathematics* are to ensure that students:

- are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens
- develop increasingly sophisticated understanding of mathematical concepts and fluency with processes, and are able to pose and solve problems and reason in Number and Algebra, Measurement and Geometry, and Statistics and Probability
- recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study (ACARA, 2010, p.1).

Table 1. Comparison of the two Standards documents.

Domains of Teaching	***National Professional Standards for Teachers* (AITSL, 2011)**	***Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2006)**
Professional Knowledge	1. Know students and how they learn 2. Know the content and how to teach it	1.1 Knowledge of students 1.2 Knowledge of mathematics 1.3 Knowledge of students' learning of mathematics
Professional Practice	3. Plan for and implement effective teaching and learning 4. Create and maintain supportive and safe learning environments 5. Assess, provide feedback and report on student learning	3.1 The learning environment 3.2 Planning for learning 3.3 Teaching in action 3.4 Assessment
Professional Engagement (Professional Attributes, AAMT, 2006)	6. Engage in professional learning 7. Engage professionally with colleagues, parents/carers and the community	2.1 Personal attributes 2.2 Personal professional development 2.3 Community responsibilities

The *Australian Curriculum: Mathematics* is organised around the interaction of three content strands and four proficiency strands. The content strands describe what is to be taught and learnt, and include *Number and Algebra*, *Measurement and Geometry*, and *Statistics and Probability*. The proficiency strands, *Understanding*, *Fluency*, *Problem Solving* and *Reasoning*, describe how the content is to be explored or developed—that is, the thinking and doing of mathematics. They have been incorporated into the content strand descriptions to provide the language to build in the developmental aspects of the learning of mathematics (ACARA, 2010, p.2). Although the curriculum is described year by year, the curriculum provides advice on the nature of learners and the relevant curriculum for the following four groupings: Foundation–Year 2; Years 3–6; Years 7–10; Years 11–12.

"Content descriptions" are included at each year level, describing the knowledge, concepts, skills and processes that teachers are expected to teach and students are expected to learn. These are grouped into "Sub-strands" to illustrate the clarity and sequence of development of concepts through and across the years of schooling. Other aspects of the curriculum include: "Year level descriptors" (statements that provide an overview of the relationship between the proficiencies and the content for each year level); "Content elaborations" (to illustrate and exemplify content) and "Achievement standards" (comprising a written description and student work samples that indicate the quality of learning that students should typically demonstrate by a particular point in their schooling) (ACARA, 2010).

The *Australian Curriculum: Mathematics* is designed to provide students with carefully paced, in-depth study of critical skills and concepts. Its design encourages teachers to support students to become self-motivated, confident learners of mathematics through inquiry and active participation in challenging and engaging experiences. The following section explores ways this curriculum initiative and the *National Professional Standards* initiative might promote and support excellence in primary mathematics teaching.

How the initiatives can promote and support excellence in primary mathematics teaching

The two initiatives described in previous sections offer opportunities for teachers to reflect on their current practice and focus on improving the effectiveness of their work with students. The *Australian Curriculum: Mathematics* identifies key important areas of mathematics that students need to learn, the types of mathematical activity students should engage in as they learn these important areas (the proficiency strands), and how the key content identified should be sequenced across the years of schooling. Teachers can consider the extent to which their existing practice aligns with the key content and proficiency strands outlined in the Curriculum, and over time refine their practice to meet the expectations of the 'new' Curriculum. The *National Professional Standards for Teachers,* provides a framework to support teachers to identify professional development goals and focus improvement efforts for the specific purpose of professional learning or for accreditation associated with the career stages. A potential 'marry', therefore, exists between the two initiatives.

To begin the conversation about how these initiatives might work effectively together to promote and support excellence in primary mathematics teaching, the authors have selected one aspect of content and practice to focus upon. The discussion that follows focuses on ways the two initiatives could prompt teachers in the early years to: (i) engage in reflection about current practice related to supporting students' developing *number sense,* and (ii) select goals for increasing their effectiveness in teaching this area.

The *Australian Curriculum: Mathematics* emphasises that the early years, Foundation-Year 2, lay the foundation for learning mathematics. It states that in these years, children should "have the opportunity to access mathematical ideas by developing a sense of number" (ACARA, 2010, p.5). Howden (1989) described number sense as "a good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualising them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (p. 11). One aspect of supporting children's developing number sense involves helping them connect different meanings, interpretations, and relationships to the four operations of addition, subtraction, multiplication, and division. Helping children develop a highly integrated understanding of the four operations and the many different but related meanings these operations take on in real contexts enables them to develop *operation sense* (Van de Walle, Karp & Bay-Williams, 2010).

In Foundation-Year 2, the focus of work on the operations is on addition and subtraction. Table 2 displays the Proficiency statements, Content descriptions, and related components of the Achievement standards associated with addition and subtraction across these years in the *Australian Curriculum: Mathematics*. Teachers of the early years, need to effectively organise and support students' learning of addition and subtraction so that they successfully reach the stated Achievement standards. What do teachers need to 'know and do' to do this well?

Table 2. Australian Curriculum: Mathematics, addition and subtraction across Foundation-Year 2.

	Foundation	Year 1	Year 2
Proficiency Statements	**Problem Solving** includes using materials to model authentic problems... discussing the reasonableness of results	**Understanding** includes... partitioning numbers in various ways **Problem Solving** includes using materials to model authentic problems... discussing the reasonableness of results	**Understanding** partitioning and combining numbers flexibly, identifying and describing the relationship between addition and subtraction... **Problem Solving** includes formulating problems from authentic situations, making models and using number sentences that represent problem situations... **Reasoning** includes... comparing and contrasting related models of operations
Content Descriptions	Represent practical situations to model addition and sharing (ACMNA004)	Represent and solve simple addition and subtraction problems using a range of strategies including counting on, partitioning and rearranging parts (ACMNA015)	Explore the connection between addition and subtraction (ACMNA029) Solve simple addition and subtraction problems using a range of efficient mental and written strategies (ACMNA030)
Achievement Standards		By the end of Year 1... they solve simple addition and subtraction problems...	By the end of Year 2... represent problems involving addition and subtraction by number sentences

Both the *National Professional Standards*, and the AAMT *Standards for Excellence*, include Domains of Professional Knowledge. Among those aspects of Professional Knowledge considered crucial to effective teaching are knowledge of the content, how to teach it, and how students learn it. In the *National Professional Standards*, Standard 2, Focus 2.1, *Content and teaching strategies of the teaching area*, the description of the Highly Accomplished teacher includes “current and comprehensive knowledge of content and teaching strategies to develop and implement engaging learning and teaching programs”. How deep is our own knowledge of addition and subtraction, and how might this impact the learning opportunities we provide students?

Researchers and educators suggest that teachers need a deep understanding of addition and subtraction situations and structures in order to properly sequence programs to support students’ full grasp of the meaning of these operations (see for example, Ma, 1999; Van de Walle et al., 2010). This might seem quite obvious, however the complexity associated with understanding these operations in a deep way, is often not fully understood. Van de Walle et al. (2010), for example, explain that researchers have separated addition and subtraction problems into four categories based on the kinds of relationships involved. These include:

- *join* problems – involving an initial quantity, a change amount (the part being joined) and a resulting amount

- *separate* problems – involving an initial quantity, an amount being removed (the part being separated) from the initial value and a resulting amount
- *part-part-whole* problems – involving two parts that are combined into one whole
- *compare* problems – involving the comparison of two quantities, where the third amount doesn't actually exist but is the difference between the two amounts.

Each of these problem structures involves a number "family" such as 4, 8 and 12, and any one of the numbers can be unknown in a story problem. Table 3 provides examples of the *join* problem structure with different numbers unknown. Van de Walle et al. provide similar examples for each of the other three problem structures.

Table 3. Examples of "join" problems – source Van de Walle et al., 2010, p. 146.

Join: Result unknown	Join: Change unknown	Join: Initial unknown
Sandra had 8 coins. George gave her 4 more. How many coins does Sandra have altogether?	Sandra had 8 coins. George gave her some more. Now Sandra has 12 coins. How many did George give her?	Sandra had some coins. George gave her 4 more. Now Sandra has 12 coins. How many coins did Sandra have to begin with?

Van de Walle et al. suggest that although students would not be expected to master knowledge of all of the structures, teachers are expected to learn them, as they are part of the *Pedagogical Content Knowledge* (Shulman, 1986) needed to teach addition and subtraction effectively. They point out that the overwhelming emphasis in most curricula is on the easier join and separate problems with the result unknown, and that these become the "defacto definitions of addition and subtraction: Addition is 'put together' and subtraction is 'take away'" (2010, p. 147). These limited definitions pose problems for students later when they need to use other structures, so it is important that children be exposed to all forms within the four problem structures, and that they are guided towards accurate definitions for addition and subtraction. Van de Walle et al. also highlight that there are many more things to know about teaching and learning addition and subtraction. For example: how contextual problems support learning; how to choose appropriate numbers for problems; when and how to introduce symbols; using model-based problems; and, properties of addition and subtraction.

When we begin to explore mathematical content in a deep way, it becomes apparent that teaching even 'the basics' is not as simple as it might seem. This may be highlighted if we reflect on our own knowledge and practice of the content just discussed. For example, are we able to: Write problems similar to those displayed in Table 3 for the other problem structures? Use counters to model or solve the problems as we think children in the primary grades might do? Identify the difficulty levels of the various types of problems? Design learning opportunities that draw on these different problem structures, and combine the use of contextual problems and models to help students construct a rich understanding of the two operations? And, does our current practice reflect a depth of understanding about addition and subtraction that will best support students to develop operation sense? If the answer to all of these questions is "yes", then we might be considered well equipped to provide students with rich and balanced learning opportunities related to addition and subtraction operation sense. If, however, some answers to the questions above are "no" and our understanding of this area is not as deep as it could be, there are implications for future professional learning in this area.

In the conference presentation associated with this paper, the conversation about what good tasks and lessons built around addition and subtraction might look like will be continued. As emphasised in the *Australian Curriculum: Mathematics*, the Proficiency Strands focus on the thinking and doing of mathematics, and provide the language to build in the developmental aspects of the learning of mathematics. When students not only solve problems, but also use physical materials, words, pictures and numbers to explain how they went about solving the problems and why they think they are correct, they develop understanding and reasoning, and are able to make connections between the operations. These ideas will be explored further in the conference presentation.

Next steps in using the initiatives to move towards excellence

The *Australian Curriculum: Mathematics* focuses attention on key important areas of mathematics learning across the stages of schooling, and the *National Professional Standards for Teachers* provides a framework for making judgements about current teaching practice and future goals. The two initiatives offer opportunities for teachers to reflect on their current practice and focus on increasing the effectiveness of their work with students.

Over coming months, teachers will need to get-to-know these initiatives, develop shared understandings of them, and begin to use them. In doing this, teachers are encouraged to take advantage of professional learning opportunities and contribute to research programs to determine 'what works'. The authors are interested in forging collaborations across organisations and locations to support the implementation of these initiatives. An invitation is extended to interested groups and individuals to continue conversations related to the initiatives via projects and networking. Possibilities for this kind of professional engagement will be explored further in the conference presentation associated with this paper.

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WORKING WITH RAGGED DECIMALS

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In developing rational number understanding the use of an activity focussing on ragged decimals provides important experiences which can assist students to develop sound place value concepts. Samples of student work are analysed showing a range of conceptions. This material is linked to research in the area.

Introduction

Rational number is one of the 'problem' areas in middle school mathematics. Rational number includes both the decimal and common fraction forms of numbers which describe combinations of whole numbers and parts of one. A rational number is any number that can be written as

$$\frac{a}{b} \text{ where } a,b \in Z \text{ and } b \neq 0 \; (Z = \{\text{integers}\})$$

This paper focuses specifically on the connection between fraction and decimal representation and the use of a classroom activity that can provide a rich source of assessment data.

Rational number in the Australian Curriculum

The *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2010) has a substrand under the *Number* strand called *Fractions and decimals* which begins in Year 1 with recognising and describing a half as two equal parts of a whole, introduces decimal notation in Year 4 with "Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation" (Year 4) and in Year 6 has the decimal content:

- Add and subtract decimals, with and without digital technologies, and use estimation and rounding to check the reasonableness of answers (ACMNA128)
- Multiply decimals by whole numbers and perform divisions that result in terminating decimals, with and without digital technologies (ACMNA129)
- Multiply and divide decimals by powers of 10 (ACMNA130)
- Make connections between equivalent fractions, decimals and percentages (ACMNA131).

After Year 6 the substrand *Fractions and decimals* ceases and rational number knowledge continues under the substrand *Real numbers* with Year 7 covering multiplication and division of decimals, rounding and simple conversions between fractions, decimals and percentages. Year 8 content for decimals is to "Investigate terminating and recurring decimals" and in Year 9, while decimals are not specifically mentioned, scientific notation is introduced.

Fractions to decimals

The concept of fraction has a number of facets. At one level a fraction is a part of a whole and the part-whole aspect has dominated much of the early fraction learning. Many students, including those in secondary school and adults, have an image of a fraction as part of a whole. While a fraction can be considered a part of a whole, many take this to mean that fractions are all less than one and consider fractions as *only* part of a whole. One of the understandings that students need to develop in relation to fractions is a conception of fractions that includes fractions beyond one, fractions of collections and not just of a continuous whole, fractions as representing division and fractions as numbers on a number line including whole numbers. Probably the most used framework for fractions is Kieran's (1980) five-part model with the sub-constructs *Part-Whole* where a fraction is understood as a number of parts (numerator – the number) of a total number of equal parts (denominator – 'de name'), *Measure* where a fraction can be located as a number on a number line, *Quotient* where a fraction can be seen as a division, *Operator* where a fraction operates on a number or object such as two fifths of 24 cakes and *Ratio* where a fraction compares the size of two sets or measures. Kieran (1988) further developed his model incorporating the Part-Whole into the other four sub-constructs and including another layer beneath consisting of *Unit Forming*, where fractions can be seen as constructed of the sum of other parts so that, for example, ¾ is ½ and ¼, *Partitioning*, where students were able to partition and recognise equal partitions, and *Equivalence*.

One aspect of fractions that many students have not incorporated into their connected understandings is that of quotient or fractions as division. The connection between division and fractions is made by the very symbols we use since the division sign ÷ represents a fraction with the number before it becoming the numerator and the following number the denominator. This quotient concept of fraction is strongly connected to decimals as the numerical process of division often produces a decimal number.

There are many misconceptions that arise in the area of fractions and decimals. In this paper the particular focus is on comparisons of the magnitude of fractions and decimals as well as the transformation of rational numbers in fraction form to decimal form. Steinle and Stacey's work in the area of decimals has long been recognised as leading in the field (Steinle, Stacey & Chambers, 2006; Steinle & Stacey, 1998, 2004, 2011). Their documentation of student errors and written assessment techniques have been followed by others who have used interview assessments as well to gain greater understanding of students' strategies in tackling problems such as decimal comparisons (Roche, 2011; Roche & Clarke, 2004).

The following activity is one that can be used in a class to provide a learning activity but also to provide assessment information for the teacher on students ordering of fractions, decimals and transformation from fractions to decimals. The description presents the activity with a brief story of implementing it in a class.

Activity

In the initial introduction, the aspect of changing fractions to decimals using division was discussed and made explicit with the division sign and its connection to fractions demonstrated.

$$5 \div 6 = \frac{5}{6}$$

The activity then proceeded by modelling with a student the first few 'turns' of the activity demonstrating what the students were required to do and modelling what they were to record on the board to provide a sample. Part of the activity is a game. The materials required for each pair of players are a record sheet, a pack of ordinary cards with the picture cards removed (leaving 40 number cards with the ace being one), a calculator, a large sheet of paper at least A3 in size, a pair of scissors and a glue stick.

The record sheet is best as an A4 sheet with two columns and ten rows (no margins) labelled A1 to A10 on the left and B1 to B10 on the right. The first time this activity was used the cells were not labelled but labelling like this simplifies assessment.

Play was between two players. Five cards were dealt to each player. Each player then each chose two of their 5 cards to use. The two cards chosen were used to make a fraction with the aim being to make the smallest fraction. They had to make the decision without referring to the calculator or seeing which cards the other player had chosen. The column on the left, labelled with the A's was for one student to record while the other player records in the column on the right labelled with the B's. This fraction was written on the left of the cell on the paper followed by the calculation that should be done to change the fraction to a decimal. The calculation was then done, using the calculator where necessary, and the decimal written fully (if they already knew or could easily change it to a decimal in another way they were allowed write it first then just use the calculator to check). The students should write the complete decimal down as it appears on the calculator and not round it. The pair of students then decided which of them had the smaller number and that person scored a point which they wrote in the cell on the sheet either as a tick or a ①. If the two fractions were equal they each gained a half point. An example of recording is shown here in Figure 1.

A1. $\frac{3}{4}$ $3 \div 4 = 0.75$	① $\frac{5}{8}$ $5 \div 8 = 0.625$ B1.

Figure 1. Sample of recording demonstrated.

Students made errors in doing this but the errors were a part of learning. This provided rich assessment data enabling the teacher to see misunderstandings that could be followed up with the whole class in later discussion.

Once a row was completed each player picked up two cards, taking turns to replace those used. The used cards were placed to one side. For the last two turns there were not enough replacement cards and for the very last turn there was no choice of cards – just a choice of whether to make a card the numerator or the denominator.

When a number of players had completed their sheets, play paused, and there was a class discussion about how they decided how to make the smallest fraction. One student answered that they had to put a small number on the top (in the numerator) to make the fraction small. Another then said that the larger the number on the bottom (denominator) the smaller the fraction. A short argument followed about which was the better strategy to use, with a number of the students joining in. Another student then suggested that perhaps it would be best to use both strategies. The question was then asked, “Once you have the two fractions how do you know which one is smaller?” This led to another discussion which raised a number of ideas and finished with the teacher asking them to decide which was smaller, 3/8 or 5/7. Students were asked once they had decided on an answer to show which one by clasping hands for the first and hands on the floor (or desk) for the second. A discussion followed with students putting their arguments for their answer with the proviso that they were allowed to change their answer if another student’s argument convinced them.

Some gave the argument that one fraction was smaller than a half and the other larger, using a benchmarking approach. Another said it was easiest to compare them if they were changed to decimals or percentages. This discussion was not prolonged but once there was general agreement about which of the two was larger the activity continued.

The students then returned to the game. Once the students had completed each row on the grid and decided who “won”, they cut the grid into the twenty separate cells and then worked together to order the twenty numbers from smallest to largest, gluing them on to the A3 sheet when they were happy with the order.

When the students completed their sheets they returned to the group for a discussion on how to decide which is the larger of two decimal numbers.

Strategies and misconceptions

Many of the usual misconceptions such as *longer is larger* and *shorter is larger* were raised. Apart from the standard misconceptions, two main strategies were suggested by the students. One required them to add zeros to make the numbers the same length but they found this hard to do with these numbers because they were very ragged and there were twenty numbers to order. As Roche (2011) has found, students who appear to be expert when comparing two decimals can have difficulty ordering larger groups of decimals. The strategy of adding zeros is one that encourages whole number thinking as most students refer to the number after the point as a whole number so the point is often seen as separating two whole numbers. The other strategy students suggested drew on their understanding of ordering whole numbers, where you compare the largest parts first so if the numbers are in the thousands and the number of thousands is larger “you don’t have to look any further”. They argued that, “first you look at the place on the left then only use the next place if those two are equal”. These students explained to the rest of the class how to use this strategy with decimals, and the place value headings were written up with the *...ths* part of the words being emphasised.

Students need to really think about the place value and the real size of the decimals rather than a memorised process. This requires them to have some experience with materials that model the relative magnitudes of decimals and these may be area models such as Decimats (Roche, 2010) or length models such as Deci-tubes (also known as LAB).

The sheets that the students produced in this exercise provided a rich source of assessment data. The first time I used the activity, the cells in which the students recorded were not labelled to enable later identification, but this was rectified in subsequent uses. The samples of student work shown in Figure 2 are not well identified or recorded but illustrate student understandings. They are also segments of the final ordering of the twenty numbers rather than complete records.

The section of student work ordering the decimals on the left of the figure illustrates that they have been able to order but have not understood that division is not commutative as they have used expressions such as 8 ÷ 2 and 2 ÷ 8 interchangeably. The students were asked to record the division exactly as they pushed the keys on the calculator. In this case 8 ÷ 3 = .375 and 8 ÷ 2 = .25.

When students are introduced to commutativity, it is with addition and multiplication; however, many students have generalised this to subtraction and division. Our teaching needs to make explicit the times when commutativity does not work as well as when it does.

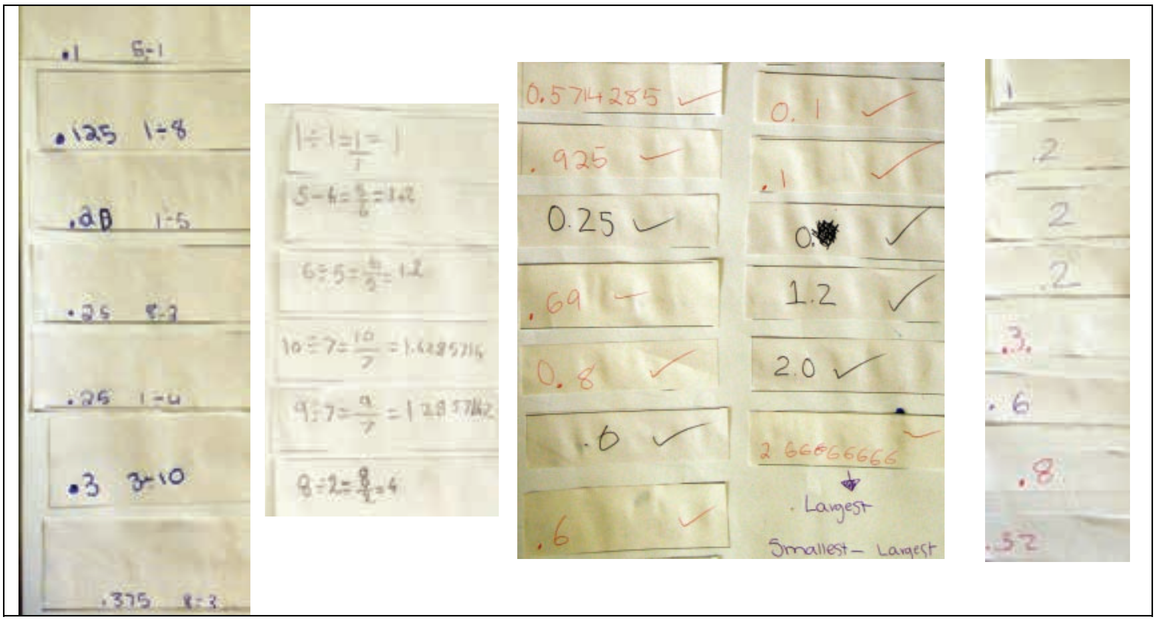

Figure 2. Samples of student work ordering ragged decimals.

The section of work in the left centre shows a similar problem with the second expression but also demonstrates that when trying to make the smallest fraction they had not fully grasped the idea of the larger of the two numbers needing to be in the denominator. This occurred with many of the students and can be seen in three of the four samples above. Student pairs with numbers larger than 1 in their lists were indicating to the teacher that they did not have a number sense which included estimation of the magnitude of a fraction.

Some students were able to order small groups but had trouble with the whole collection and the students who completed the work in the sample on the right of figure 2, showed very much whole number thinking moving from the whole number 1 to .1 then back to a whole number 2 then .2 followed by .3, .6, .8 and .32. For them, the decimal point really did not exist. These students also did not record as they were asked. The 0.32 > 0.8 answer is indicative of a *longer is larger* misconception. This is often because of *string length thinking* which enables success at ordering whole numbers since for them longer is indeed larger.

Students, and sometimes teachers and other adults, may treat decimals as whole numbers but just to the right of the decimal point. Whole number thinking is one of the major misunderstandings of decimals. The use of money to develop decimal understanding does not help this because money is seen as a whole number of dollars and a whole number of cents. Many children think that $7.6 means 7 dollars and 6 cents. The way we say the numbers can also contribute to this as children counting by 0.1 will say, "... point eight, point nine, point ten, point eleven" which they would write as 0.8, 0.9, 0.10, 0.11.

The pair of students, whose work is just right of centre in Figure 2, started in the left hand column with the numbers increasing then moved from .925 to .25, demonstrating a *shorter is larger* misconception. Once they reached numbers larger than 1 they demonstrated correct ordering. Shorter is larger misconceptions include *denominator focussed thinking*, where students think tenths are larger than hundredths so 0.3 > 0.54, *reciprocal thinking*, where 0.3 > 0.54 because thirds are larger than fifty-fourths and *negative thinking*, where the numbers to the right of the decimal point become negative so that 0.2 > 0.54 because $^{-}2 > {}^{-}54$, although this is not the way a student might describe it. It is not possible to tell from the students' work the reason for their *shorter is larger* thinking but it signals to the teacher the possible misconceptions.

This activity raises the issue of the teaching approaches we use with decimals. For example, as mentioned above, while the procedure of adding zeros to make the decimals the same length does allow the children to obtain the correct answers to some questions, it fosters whole number thinking rather than the understanding of place value. This illustrates the importance of allowing students to experience ragged decimals right from their early experiences with decimals and the importance of stressing place rather than procedure.

A fairly extensive classification of misconceptions and strategies in understandings of place value with decimal numbers has been completed by Steinle, Stacey and Chambers (2006). Their broad classification included *longer is larger*, *shorter is larger*, *other* strategies and *apparent experts*. Each of these had sub-categories with some excellent descriptions.

Even the way we say decimals is important in early understanding. When we say the words for whole number, the place is part of the saying: three thousand four hundred and fifty three. In introducing decimals to children we should use the same approach specifying each place so that 1.42 is one and four tenths and two hundredths. Later work with renaming can extend this to four tenths and two hundredths also being forty-two hundredths but that should also be done so that 1.8 is one and eight tenths but also eighteen tenths. In doing this it is useful to make fun of the ...*ths* part of the words, overemphasising it as Australians often swallow the ends of their words and children do not hear the ...*ths* part of the words.

Final comments

Students come in to any class with a large range of past experiences and hence very different knowledge and understanding. The use of formative assessment through rich tasks enables a teacher to collect data on the students' understandings and misconceptions while at the same time providing an activity through which students can learn. In the discussion and with the pairs working together the students were learning from each other.

By setting up the record sheet so that it is easy to identify the particular student in the pair (A or B) and match the fractions they were comparing at each stage through the number on the cell, it is possible to see a range of understandings and misconceptions in both fractions and decimals while at the same time providing opportunity for them to learn more about the equivalent forms of rational numbers.

The final component of such a task is the analysis of the student errors and planning for teaching to provide the students opportunity to gain greater understanding and number sense and move beyond these misconceptions.

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CONNECTING WITH THE AUSTRALIAN CURRICULUM:MATHEMATICS TO INTEGRATE LEARNING THROUGH THE PROFICIENCY STRANDS

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A review of the rationale of the draft *Australian Curriculum: Mathematics* revealed that it has strong links with numeracy, making sense of mathematics, and making connections, both within mathematics itself, and between mathematics and everything else. There is a clear intent for teachers to use a constructivist approach with an emphasis on rich conceptual understanding and by teaching the content strands through the Proficiency strands, as opposed to solely the acquisition of procedural knowledge. This paper provides teachers with strategies for using an investigative approach to develop rich understandings of key number concepts and to enable their students to make connections with and within their mathematical knowledge.

Introduction

The *Australian Curriculum: Mathematics,* developed by the Australian Curriculum, Assessment and Reporting Authority (ACARA) (2011) builds on the good work and sound intentions of strands such as Working Mathematically and Appreciating Mathematics from previous state curricula. It has strong links with numeracy, making sense of mathematics, and making connections, both within mathematics itself, and between mathematics and everything else. Today, there is much pressure on teachers to be ruled by NAPLAN results and to ensure that their school's comparative standing is strong, and there is a real danger that such pressures will lead to a narrowing of teaching strategies and practices, if this has not already occurred. NAPLAN test scores can greatly assist teachers if they are used appropriately. But it is important that teachers avoid having both eyes fixed on solely improving NAPLAN scores and following the Content strands, at the expense of the Proficiency strands. The intent of the *Australian Curriculum: Mathematics* is to teach the Content strands through the Proficiency strands and the former should be used as a guide about what and when we teach. Ultimately, if we teach through the Proficiency strands, with the aim being the development of deep conceptual understanding, our students' NAPLAN results will take care of themselves.

Australian Curriculum: Mathematics—What's the intent?

A quick review of some of the key words in the rationale of the draft *Australian Curriculum: Mathematics* may reveal the real message intended for teachers. A word count of the eight-page rationale revealed interesting frequencies of use of particular words, as shown in Table 1 and Table 2.

Table 1. Key word count from Australian Curriculum: Mathematics rationale- constructivist.

understand/understanding	31	connect/connection	8
problems/problem solving	20	pattern	6
concept/conceptual	13	mental	6
relate/related/relationship	13	variety/various	5
reason/reasoning	13	communicate/communication	4
interpret/interpretation	12	confident/confidence	4
represent/representation	12	link	4
investigate/investigation	9		

Table 2. Key word count from Australian Curriculum: Mathematics rationale- traditional.

skill	10	rote	0
procedure/s	2	algorithm	0
written	2		

It is interesting to note firstly the frequency of use of certain key words often associated with a constructivist approach (Table 1), and the corresponding lack of mention of other key words that could be described as indicating a more traditional approach to teaching mathematics (Table 2). What, then, is entailed in a constructivist approach?

Constructivism is about sense-making and connecting various concepts to one another. When mathematics makes sense to learners, it will have meaning and be "understood as a discipline with order, structure, and numerous relationships" (Reys, Lindquist, Lambdin & Smith, 2009, p. 23). Constructivist teaching has several features acknowledging that students actively build and construct their understanding, rather than passively receive knowledge. This is achieved through reflections on their physical and mental actions, aided by purposeful and focussed teacher questioning, and as such, learning is essentially a social process where learning occurs through interaction, dialogue and discussion (Reys et al., 2009).

What does this brief linguistic analysis tell us? It appears to suggest that teachers are being encouraged to use a constructivist approach to teaching mathematics where there is an emphasis on rich conceptual understanding as opposed to the mere acquisition of procedural knowledge. This, in turn, should take place in an environment that is rich in problem solving opportunities, dialogue and interaction, and opportunities for questioning and reflection. Indeed, this is precisely what is embedded in the four Proficiency strands of the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2010), these being Understanding, Fluency, Problem Solving and Reasoning. These strands describe "how content is explored or developed, i.e., the thinking and doing of mathematics" (ACARA, 2010, p. 2). Within the brief descriptors of each Proficiency strand, there are statements such as "students build robust knowledge of adaptable and transferable mathematical concepts, make connections between related concepts" (Understanding); "choosing appropriate procedures, carrying out procedures flexibly" (Fluency); "make choices interpret, formulate, model and investigate problem solutions, and communicate solutions effectively" (Problem solving); and "sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising" (Reasoning) (ACARA, 2010, p. 3). These are exciting ideas and give teachers a clear message about how mathematics is expected to be taught, consistent with the new *Australian Curriculum: Mathematics*.

Relational understanding

Within these points is the strong suggestion that the development of relational understanding is of paramount importance. Skemp (1976) described relational understanding as being conceptual by nature and allowing for connections to be made between fundamental ideas so that the broad application of mathematical principles can be appreciated and used. Smith (2006) also noted that the relational teacher is one who provides an appropriate context in which the learning of key concepts is embedded, so that children may work collaboratively to generate solutions to problems based on connections to previous experiences and understanding.

To illustrate this idea further, consider the notion of 'number sense'. In discussing number sense, Anghileri (2000) noted the importance of children being aware of relationships that exist between mathematical concepts so that they can generalise about patterns and processes and be able to link new and existing knowledge. She also made it clear that rote learning or drill and practice without the associated understanding will no longer serve children (if it ever really did!) and that teaching needs to "focus on the links that demonstrate the logical structure underlying numbers and number operations" (Anghileri, 2000, p. 2) Three key ideas emerging from Anghileri's discussion of number sense and they can be summarised as follows:

- recognition of patterns and relationships;
- recognition of links and connections between operations;
- emphasis on relational understanding (Anghileri, 2000).

An orientation of this kind certainly seems to be reflected in the *Australian Curriculum: Mathematics*.

Mathematical and pedagogical content knowledge (MCK and PCK)

If teachers are to teach successfully the various key concepts through the Proficiency strands, there are some essential prerequisites. The first of these is the teacher's own mathematical content knowledge (commonly abbreviated to MCK). A robust and deep understanding of the mathematics needed by children is a non-negotiable attribute of effective teachers. It goes without saying that for a teacher to expect children to explore and investigate concepts at a meaningful level, that teacher must know what questions to ask in order to constructively guide children's thinking so that they may analyse, infer, evaluate, explain and make appropriate generalisations as part of developing the deep conceptual understanding sought. Completely open investigations have some value but focused investigations and problem solving, based on specific aspects of a concept that have been identified as a need by the teacher, have far greater value. Guidance of this kind depends on the teacher having a deep conceptual understanding and being able to recognise what to embed in an investigation or problem solving task.

Secondly, a teacher needs to be able to make professional judgements based on identified needs. Their MCK will aid in identifying the mathematics that the children know and need to know next and informed judgements must then be made about how best to help the children learn what is needed. This is where a teacher's pedagogical content knowledge (PCK) comes to the fore with reflective questions such as the following being asked: "How do I best teach what I have identified as a need?", "Which aspect of the Proficiency strands will I use as the vehicle for teaching it?" and "How will I know that they have learned it?". The latter is reasonably simple if the target objectives are clear, unambiguous and assessable.

Thirdly, a constructivist classroom environment needs to prevail if teachers are to effectively teach concepts through the Proficiency strands. If children are to actively construct their knowledge and understanding, the learning of mathematics needs to be a social process where dialogue and interaction are the norm. Children need to be encouraged to take risks and understand that it is alright to make mistakes because they are avenues to learning. The posing of questions by teachers and students is a central plank in such a classroom where continual reflection is the norm. Similarly, there needs to be an acceptance that there is more than one way to a solution and that there may be more than one possible answer. In all of this, the teacher needs to be flexible and accepting of the fact that the direction of a lesson, no matter how well planned, may change or need to be changed at any point, depending on what understandings the children construct. Setting the agenda for using the proficiency strands to develop a deep conceptual understanding demands that teachers themselves have strong levels of MCK, that they know the abilities of the children they teach, and can ignite their learning through setting appropriate challenges.

Shulman's (1986) work on the knowledge required for teaching identified several categories of content knowledge required for effective teaching. The first of these was subject matter content knowledge that constituted "the amount and organisation of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9). However, this is much more than just 'knowing the facts' about mathematics. To this end, Shulman (1986, p.9) noted the following:

> To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures of the subject matter … and include[s] both the substantive and the syntactic structures. The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organised to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity are established.

For teachers of any level of experience, this entails knowing not only factual content but why certain facts are as they are, and most importantly, being able to explain such things to their students. (Shulman, 1986)

What does it look like in action?

The following is an example of how the Proficiency strands in the *Australian Curriculum: Mathematics* can be used as the vehicle for developing deep conceptual knowledge in children. The example is based on the Year Four level but can easily be adapted for other levels as similar skills are expected of children in those years and also, this task can underpin the development of deeper conceptual knowledge. The task described here aligns with the *Content Description* statement for Year Four which includes the following:

- Investigate number sequences involving multiples of 3, 4, 6, 7, 8, and 9
- Recall multiplication facts up to 10 × 10 and related division facts
- Develop efficient mental and written strategies for multiplication and for division where there is no remainder (ACARA, 2011)

As well, it aligns with the Year Four *Level Description*, which includes the following:

- *Understanding* includes making connections between representations of numbers
- *Fluency* includes recalling multiplication tables
- *Problem Solving* includes using properties of numbers to continue patterns
- *Reasoning* includes using generalising from number properties and results of calculation … communicating information (ACARA, 2011)

The task shown in Figure 1 is designed to be used with children following a range of preparatory tasks depending on the level of experience children have had with tasks of this nature. To complete the task, children need to decide what digits go in the boxes.

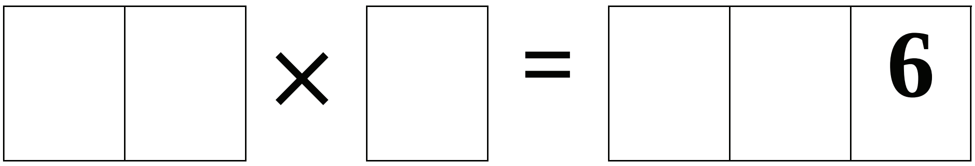

Figure 1. The main task

To illustrate some preparatory tasks, Figure 2 first shows a suitable entry level task, which is then extended along the lines suggested, eventually arriving at the task shown in Figure 1.

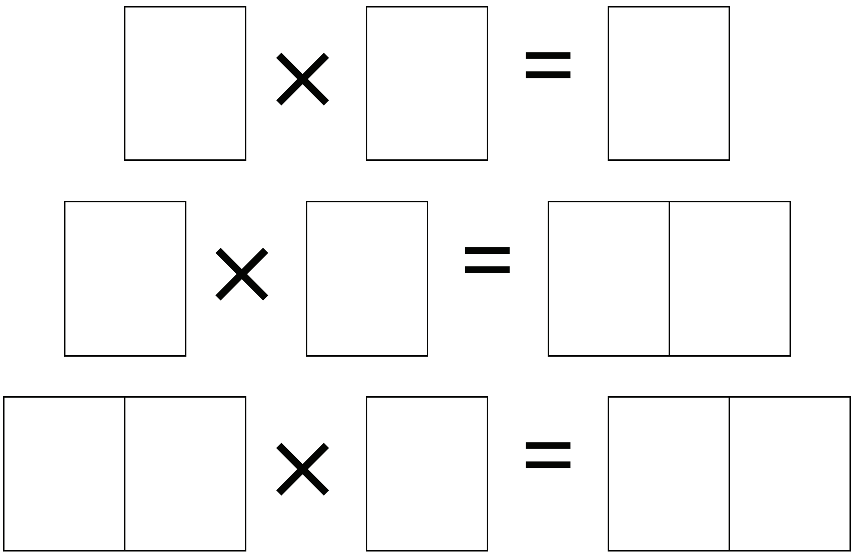

Figure 2. Possible stages of presentation leading to the main task

Key questions

Within a constructivist classroom, there are many questions a teacher could ask of their students, both to establish an understanding of the task as well as to begin to arrive at a solution. This needs to be done at all levels of the task development and a natural progression is to use the following questions as a guide:

- Can we find a solution?
- How many solutions do you think there might be?
- How will we know when we have found them all?

Regardless of the level at which the task is being tackled (Figure 1 or aspects of Figure 2), there are many other focus questions that could be posed by the teacher. The following are offered as examples and relate to the main task, but could easily be modified to accommodate the level at which the task is being done.

- What sort of a number sentence do we have?
- How big are the numbers?
- What do we know about the numbers?
- What numbers could be the ones digit in the first two numbers? Why?
- What numbers can we multiply together to make a number ending in 6?

Rich conceptual understanding

Within this task series, there are a number of key aspects of mathematical content knowledge required by teachers, in order to develop a rich conceptual understanding in children. These include:

- basic number facts;
- multiplication of one and two digit numbers;
- patterns in multiples;
- mental computation strategies;
- properties of multiplication (e.g., commutative).

Teaching points and more questions

In order to develop the rich conceptual understanding desired, the following are offered as teaching hints and possible questions to pose for children as they investigate this problem series.

- Have children list all the combinations that give a ‘6’ in the ones place: 1 × 6, 3 × 2, 6 × 6, 8 × 2, 8 × 7, 4 × 4, 9 × 4 ... and the commutations (i.e., 6 × 1, 2 × 3 etc.).
- Ask questions like: “What are some things that we can’t have?” — We can’t have a 1 in the single digit because we won’t be able to multiply it by anything big enough to make a three digit answer.
- Have them exhaust all the possibilities for each combination and tabulate the solutions.
- Ask questions like: “If we put a 2 and 3 in the ones places of the first two numbers, how big must the number in the tens place be to make a three digit answer?’ – We can’t have 12 × 3, 22 × 3, or 32 × 3 because the results will all be less than a three digit answer. Most will be able to do this mentally and through estimation. This promotes number sense.
- Have children work collaboratively and get them to document their solutions, perhaps on small card pieces. They could then arrange them in patterns.
- Pose some questions like ... ‘What would happen if we had a two digit number for the answer?’ or ‘How would it affect the number of solutions if we had a different number in the ones digit of the answer (e.g., a 3, 4, 7, 9 ...)?’

Representing solutions

Children should be encouraged to develop a system for representing the solutions they generate; this could be done initially by writing each solution on a piece of card, as suggested above, and then arranging them in some sort of pattern. Ultimately, a table as shown in Table 3, could be the end result. This might be developed as a collaborative class effort to conclude the investigation.

Table 3. Arrangement of possible solutions

Number combinations that give a number ending in 6 when multiplied	Possible solutions		Number of solutions
1, 6	No solutions possible for _6 × 1 Largest possible solution is 96 × 1 = 96	21 × 6 = 126 31 × 6 = 186 41 × 6 = 246 51 × 6 = 306 61 × 6 = 366 71 × 6 = 426 81 × 6 = 486 91 × 6 = 546	**8**
2, 3	42 × 3 = 126 52 × 3 = 156 62 × 3 = 186 72 × 3 = 216 82 × 3 = 246 92 × 3 = 276	53 × 2 = 106 63 × 2 = 126 73 × 2 = 146 83 × 2 = 166 93 × 2 = 186	**11**
6, 6			
2, 8			
7, 8			
4, 4			
4, 9			
Total possible solutions			**82**

Further questions and investigation

Once the solutions have been documented, there is opportunity for mathematical writing and further investigation of the results. Some possible focus points could be:

- Have children investigate patterns in the solutions. Ask them questions like ... 'How does the pattern of answers increase for the 1 × 6 solutions?' – The answers go up by 60 each time. Why is this?
- Ask questions like ... 'Which numbers occur more than once as an answer?' (e.g., 126 has already appeared three times: why?)

Conclusion

The *Australian Curriculum: Mathematics* clearly encourages teachers to adopt a constructivist stance to the teaching and learning of mathematics. The notion of developing a rich understanding of key concepts and content through the Proficiency strands is suggested in this paper as a preferred mode of operation. The example offered here is one that has many entry levels and which could be used with children in many different age groups and re-visited over a period of time to develop the task to more complex levels. Indeed, the main task as presented in Figure 1 could be extended further.

As children investigate this task series, they are provided with ample opportunity for developing those important skills embedded in the proficiency strands. Amongst others, they have an opportunity to engage in the following:

- the thinking and doing of mathematics
- building robust knowledge of adaptable and transferable mathematical concepts
- connecting related mathematical concepts
- choosing appropriate procedures and carrying them out flexibly
- making choices about strategies
- interpreting, formulating, modelling and investigating problem solutions
- communicating solutions effectively
- thinking and acting logically through analysing, proving, evaluating, explaining, inferring, justifying, and generalising (ACARA, 2011).

Of course, if this is actually to occur, it is dependent on the presence of a well-informed, reflective, and constructivist teacher in the context of a rich and positive classroom environment.

Author note

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AN OPEN-ACCESS ONLINE QUESTION GENERATOR WITH FULLY WORKED SOLUTIONS

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Mathematics staff at The University of Queensland have created an electronic question and solution generator that covers a wide range of fundamental mathematical, statistical and quantitative skills. This open-access system allows teachers and/or students of all grades to create an unlimited number of questions covering over 100 topics, ranging from order of operations through to calculus and linear algebra. The beauty and originality of this flexible electronic framework is that fully worked solutions are also included. This paper outlines the process of developing and testing the question generator and discuss evidence of its effective implementation.

Introduction

In recent years there has been a noticeable increase in the diversity of backgrounds, abilities and aspirations of students entering mathematics courses at Australian universities. Fewer students are studying higher-level mathematics in Australian high schools which is contributing to this increase (McPhan et al., 2008). As a result many students are increasingly struggling with understanding and applying mathematical and quantitative concepts at university (Kvyatkovskyy, Adams, & Zinchenko, 2007).

In order for mathematics education to enable more students to address the challenges they face in learning mathematics, appreciate the relevance of mathematics for their discipline, and experience positive and productive outcomes, better ways of providing supported engagement, feedback on learning and sustained practice need to be found. In addition, to prepare mathematics teachers to respond effectively to student diversity and learning difficulties, time-efficient yet mathematically sound resources need to be developed (Cobb et al, 1992; Gess-Newsome & Lederman, 1999).

Success at mathematics requires a combination of technical skills and intuition. Technical knowledge is important, but of equal or greater importance is the ability to use intuition, flair and elegance when solving problems. Teachers and lecturers are an important part of the learning process, as they provide students with the opportunity to observe an experienced practitioner applying these creative talents, and explaining and demonstrating how to do so (Entwistle, 2005). However, a common thread to mathematics learning experiences is that material cannot be absorbed and assimilated passively. It is learned by doing, not simply by watching. At every level, from primary school through to tertiary post-graduate study, inquiry- and discovery-based learning are essential, and students must work through many examples and problems in order to hone their technical skills and mathematical intuition (Kvyatkovskyy, Adams & Zinchenko, 2007). By thinking about what they are doing and observing the similarities and differences between various questions they become attuned to patterns and subtleties, thus improving their ability to choose what techniques to use and how to work creatively.

Traditionally, working through examples was sometimes regarded merely as rote learning. Certainly there is some need to commit mathematical facts to memory, but we are not suggesting that the primary reason for working through a number of questions is simply to learn how to recite facts. Instead, learning mathematics by practising is a genuine and necessary aid to improving understanding and enhancing creative abilities, in addition to learning technical skills (Bransford, 2000).

A multi-disciplinary team at The University of Queensland (UQ) has created an online question generator with fully worked solutions which gives students the opportunity to improve their mathematical understanding. In this paper we describe this new software package which has been used at UQ to assist students making the transition from secondary to tertiary mathematics.

The system

Overview

The development of *SmartAss* (which stands for Smart Assignments) was supported by the Carrick Institute for Learning and Teaching in Higher Education (now the soon to be defunct Australian Learning and Teaching Council (ALTC)). The system is based on a prototype question and solution generator designed by Professor Peter Adams from UQ.

SmartAss can be used to help with a variety of mathematical concepts and techniques, ranging from quite simple content to more sophisticated material. It has been used in courses that are primarily mathematical in nature, but is also very useful for students of science, engineering, business and agriculture who need to apply quantitative concepts in the context of their specific discipline. The main goals and features of the software include:

- automatically generating a suite of random questions and corresponding fully-worked, formatted solutions to every question, clearly and unambiguously reproducing the steps that students would typically take when correctly solving the problem;
- providing students with a mechanism for concentrating on those concepts which cause them difficulties, enabling inquiry-based learning and improving their technical and creative abilities;
- implementing a powerful learning aid that gives support for both introductory and advanced mathematical concepts and processes;
- allowing instructors to efficiently and easily create resources for illustrative examples, practice materials and individualised assessment;
- being directly usable in all discipline areas that require quantitative skills;
- free availability to the education community as open-source software, with a modular design allowing components to be easily redesigned and extended.

Randomization in computer-based learning resources is not new. However, successfully using it in effective aids for learning mathematics has previously proved to be problematic. Some well-known and excellent mathematics packages (such as *Maple*, *Mathematica* and *Matlab*) simply give the final answer to mathematical questions, with no indication of any intermediate steps or processes that are required in order to actually derive the answer. These are great tools, used very widely in research as well as tertiary teaching. However, they suffer from the disadvantage that they provide nothing more than the final answer. If a student makes a mistake, the only option is to go back and try again. Often the student simply repeats the same error, which quickly leads to loss of confidence.

There are commercial packages that include an attempt to format solutions step-by-step. Despite significant recent improvements in the scope and functionality of these packages, they still often cover only low-level material, or the variations in the questions are predominantly in superficial arithmetic, which is of limited use in helping students to improve their high-level reasoning skills. *SmartAss* overcomes these limitations through its careful design, power and flexibility.

In the following sections we describe the development of *SmartAss* in more detail, present some examples, and give some information on using the system at UQ.

Development

The key members of the development team were a mathematics academic (the second author of this paper), a high school mathematics teacher working at UQ (the first author) and a computer programmer who was quite proficient in mathematics. Using Adams' prototype and starting with the first chapter of content from a bridging mathematics course that is roughly equivalent to the Queensland senior secondary advanced mathematics course, the team set about writing questions with fully worked solutions.

The three members met weekly for a year to discuss questions and the appropriate setting out of solutions. Some topics were easy and required little discussion. The most time consuming topic was fractions. There are various ways of solving fractions questions and research has found that fractions are quite a difficult concept (Smith, 2002; Clarke, 2007). We eventually decided on a method that was not exactly how a mathematician would solve the question (e.g., doing many steps at once), but rather a step-by-step approach that allowed the student to see exactly what was happening in order to build understanding and confidence. A fully worked solution to a fractions question can be seen in Figure 1.

$$\begin{aligned}
\frac{4}{10} \div \frac{6}{-42} \times \frac{28}{-24} \times \frac{-27}{49} &= \frac{4}{10} \times \frac{-42}{6} \times \frac{28}{-24} \times \frac{-27}{49} \\
&= \frac{\cancel{2} \times 2}{\cancel{2} \times 5} \times \frac{\cancel{6} \times (-7)}{\cancel{6}} \times \frac{28}{-24} \times \frac{-27}{49} \\
&= \frac{2}{5} \times \frac{-7}{1} \times \frac{28}{-24} \times \frac{-27}{49} \\
&= \frac{2 \times (-7)}{5 \times 1} \times \frac{28}{-24} \times \frac{-27}{49} \\
&= \frac{-14}{5} \times \frac{28}{-24} \times \frac{-27}{49} \\
&= \frac{\cancel{2} \times (-7)}{5} \times \frac{\cancel{4} \times (-7)}{\cancel{4} \times \cancel{2} \times 3} \times \frac{-27}{49} \\
&= \frac{-7}{5} \times \frac{-7}{3} \times \frac{-27}{49} \\
&= \frac{-7 \times (-7)}{5 \times 3} \times \frac{-27}{49} \\
&= \frac{\cancel{49}}{15} \times \frac{-27}{\cancel{49}} \\
&= \frac{1}{15} \times \frac{-27}{1} \\
&= \frac{\cancel{3} \times (-9)}{\cancel{3} \times 5} \\
&= \frac{-9}{5} \\
&= -1\frac{4}{5}
\end{aligned}$$

Figure 1. SmartAss solution to a fractions question.

SmartAss comprises of Java and LaTeX files which are reasonably straightforward to make. (Kvyatkovskyy, Adams, and Zinchenko (2007) have published on the more technical aspects of *SmartAss*.) There are more than 100 different question templates and with the randomness allowed in the questions, there are more than 1000 questions and corresponding fully worked solutions.

While most textbooks contain exercises with the same questions just with different numbers, *SmartAss* has variations on questions which allows students to improve their mathematical understanding and high-level reasoning skills. Examples of different equation questions are shown in Figure 2.

1. Find y, if $1 = -4y - 4$

2. Find y, if $-5y + 6 = 1$

3. Find z, if $4 = \frac{3z}{-5} - 5$

4. Find z, if $\frac{3}{-6z} = 2$

Figure 2. SmartAss equation questions.

While some of the system's fully worked solutions contain only numbers and mathematical symbols, others include written explanations such as those in the simultaneous equations questions. Figure 3 shows the answer to "Do the lines $-15y + 5x = -5$ and $-2y - 2x = 18$ intersect? If so, find the point of intersection."

We need to find a solution for two simultaneous linear equations.
First we number the equations for convenience:

$$-15y + 5x = -5 \qquad (1)$$
$$-2y - 2x = 18 \qquad (2)$$

It's probably easier to solve these using elimination. Multiply equation (1) by 2 and equation (2) by 5 , giving

$$-30y + 10x = -10 \qquad (3)$$
$$-10y - 10x = 90 \qquad (4)$$

We add both sides of equations (3) and (4) , giving

$$-10y - 30y - 10x + 10x = 90 - 10 \qquad (5)$$

Simplifying equation (5) gives

$$-40y = 80 \qquad (6)$$
$$y = -2 \qquad (7)$$

Next we substitute the value for y into equation (1) to obtain the value for x, giving

$$-15 \times (-2) + 5x = -5$$
$$5x = -35 \qquad \text{so}$$
$$x = -7$$

Hence the simultaneous solution to equations (1) and (2) is $(-7, -2)$.

(As good boys and girls always do, check your answers by substituting into equations (1) and (2):

$$(1) \quad -15 \times (-2) + 5 \times (-7) = -5 \qquad 30 - 35 = -5 \qquad -5 = -5$$

$$(2) \quad -2 \times (-2) - 2 \times (-7) = 18 \qquad 4 + 14 = 18 \qquad 18 = 18$$

Both equations turned into true statements, as required. Hence the answer is correct.)

Figure 3. SmartAss solution to a simultaneous equations question.

Figure 4 shows the solution to the question, "Determine the range of $f(w) = \dfrac{-11}{\sqrt{w} - 9}$."

$$f(w) = \frac{-11}{\sqrt{w} - 9}$$

When evaluating the range, we need to keep in mind the following (starting with variable w):

- square root is always positive or 0, so $0 \le \sqrt{w}$;
- fraction can be 0 only if its numerator is 0 (which is not the case), denominator cannot be 0 ;
- so $-9 \le \sqrt{w} - 9$ and $\sqrt{w} - 9 \ne 0$.

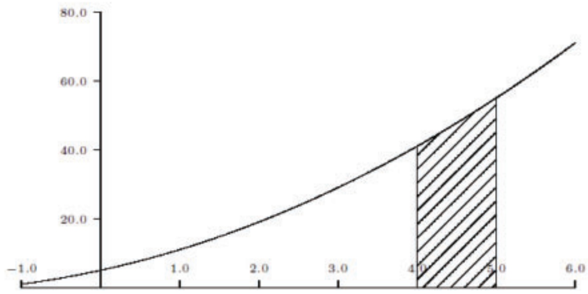
Hence, the range of this function is $(-\infty, 0) \cup [\frac{11}{9}, \infty]$.

Figure 4. SmartAss solution to a functions question.

SmartAss can also produce graphs, as seen in Figure 5, to answer the following question:

> *Find the area under the curve $y = x^2 + 5x + 5$ over the interval [4, 5].*

Function graph:



The area is given by:

$$\int_4^5 (x^2 + 5x + 5)\,dx = \left[\frac{x^3}{3} + \frac{5x^2}{2} + 5x\right]_4^5 =$$
$$= \left(\frac{1 \times 5^3}{3} + \frac{5 \times 5^2}{2} + 5 \times 5\right) - \left(\frac{1 \times 4^3}{3} + \frac{5 \times 4^2}{2} + 5 \times 4\right) \approx 47.83$$

Figure 5. SmartAss solution to an area under a curve question.

How to use

SmartAss is very user friendly and it takes only a few minutes for teachers or students to produce a set of questions. *No programming knowledge is required*. Users visit http://smartassignments.virtual.vps-host.net/index.htm and enter the title of their set of questions. A concise description of each question, along with sample questions and solutions, are available for viewing before inclusion. Multiple versions of one question can be included using the repeat function. It is also possible to create an assignment with several randomly chosen questions out of the available templates on a particular topic. Once the desired questions have been chosen, the file is executed.

The system produces three small PDF files: questions, fully worked solutions, and final answers only. Users can set up an account to save their work, meaning all that is needed to produce another set of questions on the same topics sometime in the future is to run that file again. Should the user have knowledge of *LaTeX* then extra information can be included outside of the title area (e.g., comments from teachers).

Implementation

The system was first used in one of UQ's bridging mathematics course that is roughly equivalent to the Queensland senior secondary advanced mathematics course. From 2007 to 2011 there have been 350 to 500 students enrolled each year across two campuses. Topics in the course which are covered by *SmartAss* include: manipulating fractions, order of operations, simple algebra, manipulating square roots, absolute values, solving equations, summation notation, inequalities, linear functions, simultaneous equations, quadratic equations, functions, graphs, logarithms, exponentials, simple differentiation and integration.

The system is also being used in the University's rough equivalent to the Queensland senior secondary specialist mathematics course. Over 700 students study this course each year, with *SmartAss* topics including differentiation, integration, matrices, vectors, sequences, series and complex numbers.

UQ staff use *SmartAss* on a weekly basis to prepare both tutorial and assignment questions for students. Ten to 20 different questions are usually chosen for tutorial sheets, while a smaller number are used for assignments. As staff are proficient in

LaTeX they can also add their own questions (e.g., worded problem solving questions). In addition, lecturers have placed sets of questions on course websites for students to access for revision throughout the semester, and before mid-semester and final examinations. Lecturers of smaller courses have given each student a different assignment by executing the file multiple times, thereby reducing plagiarism.

As mentioned above, *SmartAss* is not just used in mathematics courses. Biology, chemistry, physics, and Python programming questions have also been designed and are currently used in first-year science courses. In particular, Leslie matrix and Python questions are used in a first-year core science course with enrolments in excess of 600 students.

Feedback

Several Queensland high school teachers and a New Zealand mathematics academic have also trialled *SmartAss*, providing useful feedback. They are impressed with the power of the system, particularly with regard to the fully worked solutions. Suggestions have been made to make the system more visually appealing and also to remove the computer language that the ordinary user does not need to see.

We are impressed with how well *SmartAss* is working in tutorial classes, as students can concentrate on exactly those areas which cause them difficulties, rather than only passively observing an instructor presenting material on the board. If students require more practice then more questions can be generated, tailored to their specific needs. In addition, the questions and fully worked solutions have been greatly appreciated by external post-graduate students studying a Graduate Certificate of Education or Master of Educational Studies. These students generally work full-time or live outside south-east Queensland and therefore do not have the face-to-face contact that internal students have. Comments from students indicate that *SmartAss* is making a difference; for example, “As a post-graduate external student it’s hard to do maths via email or phone so I really appreciate having the fully worked solutions to see where I have gone wrong.”

Part of the Carrick Institute’s mission was to disseminate project results to the relevant stakeholders, so we now wish to promote *SmartAss* to schools and universities, both in Australia and overseas. Feedback on the system and ideas for more questions are most welcome.

Conclusion

It is clear that the current system is functioning very effectively from the students’ perspective. Discussion, feedback, observation and monitoring of assessment results all demonstrate that students are benefiting from having access to a large body of additional practice material. Comments such as the following indicate that not only are students’ technical skills improving but so is their mathematical intuition.

> I really need to practise maths, and the heaps of revision questions on the website are great! The questions are not all the same. I used to hate doing lots of the same questions at school. With these questions each one is slightly different; sometimes the x is in the numerator, sometimes it’s in the denominator. I am not scared of algebra or equations anymore as I understand what I have to do. The solutions are really clear! Thanks!

A number of extensions are planned for the system. The advanced mathematics bridging course covers many junior mathematics topics (e.g., manipulating fractions, order of operations, simple algebra, solving equations, and linear functions). Other high school topics are readily programmable. In addition, content from most first-year university mathematics courses, including discrete mathematics, could easily be included. Discrete mathematics topics would include truth tables, logic, simple circuit design, quantified statements, simple number theory, modular arithmetic, proofs by induction, graph theory, set theory, relations and group theory.

In addition to accessing the solutions in both printed and electronic forms, many students benefit from being able to access solutions one step at a time (so each step acts as a suggestion or hint as to the next step). Hence a web-based interactive component will be developed, allowing users to access partial solutions and key steps, but still work through the remainder of each question themselves. Also, the breadth, depth and variety of material covered must be greatly expanded. That will be achieved by the efforts of UQ staff and involvement of other interested programmers, teachers and students.

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MATHEMATICAL TASKS THAT ADVANCE REASONING AND COMMUNICATION IN CLASSROOMS

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Mathematical tasks with high cognitive demand often require students to make explicit their thinking. These tasks are necessary for the advancement of reasoning and communication in classrooms. Therefore teachers face the challenge of crafting suitable high-cognitive demand tasks for use in their lessons to engage their students in reasoning and communication when necessary. In this paper we demonstrate how close-ended textbook mathematical tasks can be transformed into tasks suitable for reasoning and communication in classrooms via the use of some "What?" strategies such as What's wrong?, What if?, and What's the question?

Mathematical tasks

A mathematical task is defined as a set of problems or a single complex problem that focuses students' attention on a particular mathematical idea (Stein, Grover, and Henningsen, 1996). From the TIMSS Video Study (NCES, 2003), in which Australia, Czech Republic, Hong Kong, Japan, Netherlands, Switzerland, and the United States participated, it was found that students spent over 80% of their time in mathematics class working on mathematical tasks. According to Doyle (1988), "the work students do, defined in large measure by the tasks teachers assign, determines how they think about a curricular domain and come to understand its meaning" (p. 167). Hence different kinds of tasks lead to different types of instruction, which subsequently lead to different opportunities for students' learning (Doyle, 1988). Boston and Smith (2009) report that research has consistently indicated that teachers' selection of instructional tasks is largely based on lists of skills and concepts they need to cover. Textbooks are often the main source of such tasks (Doyle, 1983; Kaur, 2010).

The works of Boaler and Staples (2008), Stein and Lane (1996) and Tarr et al. (2008) have all shown that the greatest student learning gains occur in classrooms in which mathematical tasks with high-level cognitive demand are used and the demand is consistently maintained throughout the instructional episode. Boaler and Staples (2008) in their longitudinal study comparing three high schools over a period of five years, found that the highest student achievement occurred at the school in which students were supported to engage in high-level thinking and reasoning. Tarr et al. (2008) and Stein and Lane (1996) have both found that learning environments in which teachers: (i) encourage multiple strategies and ways of thinking; (ii) support students to make conjectures and explain their reasoning, were associated with higher student performance on measures of thinking, reasoning and problem solving.

Table 1 shows a simplified version of Stein and Smith's (1998) task analysis guide that may be used to establish the cognitive demands of mathematical tasks. From the table it is apparent that tasks with high levels of cognitive demand require students to engage in explaining their thought processes.

Table 1. Levels of cognitive demand.

Levels of cognitive demand	*Characteristics of tasks*
Level 0 – [Very Low] Memorisation tasks	- Reproduction of facts, rules, formulae - No explanations required
Level 1 - [Low] Procedural tasks without connections	- Algorithmic in nature - Focussed on producing correct answers - Typical textbook word - problems - No explanations required
Level 3 [High] Procedural tasks with connections	- Algorithmic in nature - Has a meaningful / "real-world' context - Explanations required
Level 4 – [Very High] Problem Solving / Doing Mathematics	- Non-algorithmic in nature, requires understanding and application of mathematical concepts - Has a "real-world" context / a mathematical structure - Explanations required

Mathematics textbooks often lack tasks that are suitable for instruction to advance reasoning and communication in mathematics lessons. Therefore teachers face the challenge of crafting suitable high-level cognitive demand tasks for use in their lessons to engage their students in reasoning and communication. This challenge is not a formidable one as the works of Silver, Kilpatrick and Schlesinger (1990), Carroll (1999), Krulik and Rudnick (1999) , Yeap and Kaur (1997), and Kaur and Yeap (2009a, 2009b) show that closed-ended textbook mathematical tasks can be transformed into high-level cognitive demand tasks for use in lessons to advance reasoning and communication.

Silver, Kilpatrick and Schlesinger (1990), emphasize the need to look for appropriate opportunities for thinking and communication in the material teachers already use and are comfortable with. They suggest modifying common textbook problems to make them more open-ended as a plausible entry point for introducing speculation, group discussion and problem posing. Carroll (1999), found that one way of engaging students in the reasoning process is to have them examine and explain an error. This strategy simply involved turning closed-response questions into open-ended reasoning questions. Krulik and Rudnick (1999), in their work with mathematics teachers have also shown that standard textbook questions may be transformed into mathematical tasks capable of engaging students in critical and creative thinking, reasoning and communication (individual as well as group). They have used the following strategies: What's another way?, What if?, What's wrong?, and What would you do?. Yeap and Kaur (1997) and Kaur and Yeap (2009a, 2009b) have explored the use of problem-posing activities, drawing on typical mathematical tasks from textbooks, to promote reasoning and communication amongst students.

"What…?" strategies

Kaur and Yeap (2009a, 2009b) in their work with teachers in Singapore have used several "What?" strategies to engage students in reasoning and communication during mathematics lessons both in primary and secondary schools. In this paper three of the strategies, namely: "What's wrong?", "What if?", and "What's the question?" are presented. The mathematical tasks used in these strategies are crafted from closed-ended textbook tasks.

What's wrong?

In "What's wrong?" tasks students are presented with a problem and an erroneous solution that may be conceptual or computational. The student has to recognize the error, correct it and then explain what was wrong, why it was wrong and what was done to correct the error (Krulik and Rudnick, 1999). Such tasks demand higher order thinking, namely critical thinking. Students may be asked to complete the task in small groups or individually. The teacher must ensure that students are engaged in class discussion after completing the task so that they get the opportunity to see ways of solving problems that differ from their own. Furthermore, these discussions often lead to deeper mathematical understanding (Krulik and Rudnick, 2001). Teachers are in a good position to craft tasks like this as they are constantly exposed to errors students make in class and in their written assignments. Figure 1 shows one such task. The mathematical task in Figure 1 was crafted from the following textbook question on the topic of inverse variation:

> If 8 students take 2 hours to wash dishes, how many hours would 12 students take to wash the same number of dishes?

Washing dinner plates

During the school camp, Sally was in charge of Kitchen duty. After each meal, she had to get a few students to wash the dinner plates. The students were very cooperative and they all washed at the same rate. On the first day, she asked 8 students to wash the plates and they took 2 hours. On the second day, she asked 12 students to do the washing and told the camp commander that the students will complete the washing in 3 hours.

Sally's solution:-

8 students	take	2 hours
1 student	takes	$2 \div 8 =$ 0.25 hours
12 students	take	0.25 x 12 = 3 hours

There is something wrong with Sally's solution.
Show how you would solve the problem.
Explain the error in Sally's solution.

Figure 1. An example of a "What's wrong?" type of task.

What if?

In “What if?” kinds of tasks students are presented with a mathematical task following which aspects of the given information are modified, one at a time. This modification provides students with an opportunity to re-examine the task and see what effect these changes have on the solution process as well as the answer. The next part of the task requires the generation of “What if?” questions by the students. This process engages students in problem posing (Brown and Walter, 1985). Such tasks are non-routine and demand higher order thinking, namely critical and creative thinking, by the students. Whole class discussion must precede individuals working on such tasks because students need to share the “what if” tasks they created with others and also make their thinking visible. Teachers are in a good position to craft tasks like this as they merely need to extend typical textbook types of questions with ‘what if’ conditions. Figure 2 shows one such task. The mathematical task in Figure 2 was crafted from the following textbook question on the topic of: Mensuration - areas and volumes of cylinders.

> An open cylindrical tank with diameter 28 cm and height 50 cm contains water to a depth of 20 cm. Find
> i) the volume of the water inside the tank, giving your answer in litres;
> ii) the total surface area of the tank that is not in contact with the water.

<u>Cylindrical Tank</u>

An open cylindrical tank with diameter 28 cm and height 50 cm contains water to a depth of 20 cm. Find

i) the volume of the water inside the tank, giving your answer in litres;
ii) the total surface area of the tank that is not in contact with the water.

What if the cylindrical tank is closed?
What if the dimensions of the cylindrical tank are doubled?
What if the dimensions of the cylindrical tank are halved?
What if the depth of the water is reduced by 10 cm?
*What if the orientation of the cylindrical tank is changed such that it is lying on its curved side?

Generate another 3 “What if?” tasks and answer them.
Look out for any interesting observations / patterns.

* This “what if” requires students to make sense that when the cylinder is open and lying on its curved surface, the water will no longer be in the cylinder!

Figure 2. An example of a “What if?” type of task

What’s the question?

In “What’s the question?” kind of tasks, students are presented with a context and data but with question/s missing. Students are asked to write a question that matches a given answer or a partial solution. These tasks provide students with opportunities to engage in higher order thinking, namely critical thinking. Teachers are in a good position to craft tasks like this from typical textbook questions by using their context and data. Whole class discussion must precede individuals working on such tasks as it is important for students to learn that several questions may have the same answer, but certainly different solutions.

Figures 3 and 4, show two examples of such tasks. The mathematical task in Figure 3 was crafted from the following textbook question on the topic of probability.

> Eleven cards numbered 11, 12, 13, 14, …, 21 are placed in a box. A card is removed at random from the box. What is the probability that the card has an even number?

<u>Just one card</u>

Eleven cards numbered 11, 12, 13, 14, … , 21 are placed in a box. A card is removed at random from the box.

1. What's the question if the answer is $\frac{5}{11}$?
2. What's the question if the answer is $\frac{4}{11}$?
3. What's the question if the answer is $\frac{9}{11}$?
4. What's the question if the answer is $\frac{6}{11}$?
5. What's the question if the answer is $\frac{3}{11}$?

Figure 3. "What's the question?" type of task – example 1.

The goal of the mathematical task in Figure 4 was to facilitate students' review of some aspect of mathematical content knowledge. The task in Figure 4 provides students with a stimulus to review the topic: area of plane figures. Teachers are in a good position to craft such tasks as they usually have a good overview of the exercises on a topic in the textbook.

The area of a plane figure is 154 cm^2.

What could the question be?
[*Guiding prompts: what is the shape, dimensions of the plane figure?*]

Write 5 questions and work their solutions.

Question:

Solution:

Figure 4. "What's the question?" type of task – example 2.

Concluding remarks

Teachers who wish to advance reasoning and communication in their mathematics classrooms must not only have at their disposal mathematical tasks with high cognitive demand but also the knowledge and skill to maintain the cognitive demand of the tasks. Teachers may need to engage their students in varied forms of seatwork, e.g., individual, pair or group. They may need to shift their role from a disseminator or assessor of knowledge to that of a facilitator. They may also need to cultivate a classroom environment where mistakes are welcomed and all students are part of the classroom discourse. Most importantly they must provide their students with sufficient time to think through their attempts in resolving the tasks and also opportunities to explain and justify (through both oral and written communication) their solutions because in so doing students clarify their own thinking and often self-correct their errors if any.

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VISIBLE THINKING: YOUNG CHILDREN'S SHARED REASONING IN THE MATHEMATICS CLASSROOM

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This paper argues for classroom practices that support a mathematical learning environment for young children where models of reasoning and inquiry skills as well as concepts are shared and practised through dialogue. Some core aspects of collective dialogue and community of inquiry learning are addressed. Results are reported from a South Australian study of five year old children engaged in data-modelling activities incorporating collaborative learning as part of their mathematics curriculum. Findings include children's abilities to recognise differing ideas, to share reasoning through dialogue and to draw from each other's reasoning in their own problem solving and decision making.

Introduction

The notion that reasoning is an important component of mathematical competency that should be part of mathematical learning and supported by classroom pedagogy is neither foreign nor novel. Reasoning is generally seen as central to mathematics and mathematical learning, understanding and application (Goswami, 2004: Young-Loveridge, 2008). The theoretical valuing of reasoning in children's mathematical learning from the beginning of their schooling is clear from its inclusion in mathematics curriculum and standards documents (National Council of Teachers of Mathematics (NCTM), 2000; Australian Curriculum, Assessment and Reporting Authority (ACARA), 2010) and through the work of many researchers in children's mathematics (e.g., English, 2004; Garfield & Ben-Zvi, 2007; Lehrer & Schauble, 2005; Perry & Dockett, 2008). The NCTM (2000) calls for children from prekindergarten upward to recognise, understand and be actively engaged in mathematical reasoning. More immediately, the newly created *Australian Curriculum: Mathematics* (ACARA, 2010) has included 'Reasoning' as one of the four proficiency strands for Foundation Year to Year 10.

Yet despite this, recent mathematics research has shied away from exploring young children's mathematical reasoning. In Fox and Diezmann's 2007 survey of early years mathematical research carried out between 2000 and 2005, literature addressing "young children's ability to problem solve, reason and converse mathematically" (p. 301) accounted for only 1.3% of available literature. However, from an educator's point of view, there is a link between understanding reasoning as a way of thinking that has value and the pedagogy of the classroom which can help or hinder reasoning development which deserves attention. This link has been explored by researchers across disciplines who have an interest in the role and function of language in learning through social interaction (e.g. Lipman, 2007; Mercer & Littleton, 2007; Sfard, 2008).

This paper firstly argues for the need to review the theoretical foundation and practical considerations for establishing an environment that is conducive to mathematical reasoning and to enable young children to begin to access, value and build on each other's ideas. The results of a South Australian study with children in their first term of school (Reception Year) undertaking data modelling problems in both whole and small groups are addressed. Children were provided with lessons based on the 'Thinking Together' approach (University of Cambridge, 2011) that aimed to provide skills and specific language to support productive, and effective collective dialogue. Whole class discussions with the teacher utilised 'community of inquiry' pedagogy. In reporting the findings, the focus is on children's demonstrated capacities to share reasoning through dialogue and to draw from each other's reasoning and ideas in their own problem solving and decision making about data.

Reasoning and the classroom

There is limited research support for enabling educators to work out how children may go about accessing *mathematical* reasoning in the classroom environment and what a classroom that is conducive to developing reasoning might look like. Reasoning and particularly, how it can be supported, encouraged and developed in children in classrooms, is not however, the exclusive preoccupation of mathematics educators and researchers. There is a body of literature in other domains and disciplines which explores conditions for reasoning through activities where children are explicitly taught to use language to talk together and through the establishment of 'communities of inquiry'. Both of these approaches are inextricably linked to socio-cultural theory (Vygotsky, 1978) where there is an increased emphasis on the role of language as a mediator for cognitive development and as a primary organiser of cognitive activities, positioning the social environment as a principal catalyst for cognitive change (Garton, 2004). Language in interaction with others in this context is seen as providing "tools for reflection and reasoning" (Garton, 2004, p. 159) within the community of the learning environment.

The idea that children's understanding generally can develop through discussion is supported by Dockett and Perry (2001) and is found in literature focused on the conditions that are postulated as best supporting or influencing critical aspects of mathematical development and learning, in particular, the classroom context that enables dialogue to occur. There is a need for children to have opportunities to engage in dialogue where their mathematical ideas can be explained, clarified and revised (Diezmann, Watters & English, 2001; Ryan & Williams, 2007).

A reasoning culture

Dialogue is critical to community engagement. Cullingford (2006) argues that children are deeply motivated to talk to each other as they seek to make sense of their experiences and their world and that they need to share, explore and test their ideas with others. Central to sense making in this way is dialogue, which is the principal means by which children participate in communication, negotiation and decision-making with other people. This is a concept at the core of the social constructivism perspective, which from Vygotsky's (1971) view, sees children as acquiring their knowledge and understanding the world through interaction with others. Children's learning then is a combination of both their individual efforts and their communication and social interactions within the cultural context of the learning environment (Mercer & Littleton, 2007). Those who argue for learning to be embedded in classrooms that engage in pedagogy that actively values dialogue for facilitating collaboration, thinking, and reasoned discussion persuasively and effectively link these approaches to a social constructivist perspective, (e.g., Davey, 2005; Mercer & Littleton, 2007). It is this combination of ideas that provides a foundation for creating a genuinely inclusive classroom culture that is built on appreciation of the role of dialogue and social interaction in learning.

Communities of inquiry and valuing collective dialogue

A *community of inquiry* is analogous to a habitat that facilitates thinking development, a *cognitive ecology* where models of reasoning and inquiry skills as well as concepts are modelled (Lipman, 2007), where there are shared but explicit classroom expectations about participation, and where reasoning, and being reasonable, is practised through dialogue. For Lipman (2007), Sprod (2001) and others, both the disposition and ability to be reasonable are essential human qualities which should be at the heart of both education and democracy. Such reasonableness, it is argued, is best fostered in a classroom culture built around a community of inquiry, where inquiry processes lead students to "listen to one another with respect, build on one another's ideas, challenge one another to supply reasons for otherwise unsupported opinions, assist each other in drawing inferences from what has been said, and seek to identify one another's assumptions" (Lipman, 2007, p. 20).

The underlying principle that children need specific activities that teach how to talk and how to listen in order to be in a position to engage in collaborative learning is also central to the work of Wegerif, Mercer, Littleton, Rowe and Daws (2004) in their 'Thinking Together' project. They argue that children do not have a natural propensity for reasoned dialogue, and that the opportunity for instructional learning to be able to do so is needed for equity of access to education. Learning to reason through dialogue can also mark the beginning of children being willing to take intellectual risks. This occurs when language is not only valued as a tool for discussion and communication, but it takes place in an environment that also has a collective framework of expectations for listening and speaking behaviour (Mercer & Littleton, 2007).

Creating a community culture

It is clear from the literature that setting up a classroom to engage in a community of inquiry, including establishing the expectations, rights and responsibilities for the children's participation, falls to the teacher. This means creating a context for inquiry (Kennedy, 2009) and building trust in the community (Groves & Doig, 2004). The goal of creating an inclusive environment, with a common purpose for all participants also requires the development of shared understanding of how the community functions as a 'frame of reference' (Andreissen & Scharrz, 2009, p. 149). The establishment of expectations, rights and responsibilities for participation in effective dialogue in a community of inquiry is essential (Hayes, 2008). Providing clear mutually agreed upon procedures that serve to guide and positively encourage participation is particularly important for young children, as this supports their need for autonomy, belonging, competence and fairness (Nucci, 2009). Such procedures focus on regulating behaviours that will be helpful in promoting discussion, listening and turn-taking and thinking (Dawes & Sams, 2004; Haynes,2008; Nucci, 2009).

Mathematics and community of inquiry

Considering a reasoning culture also raises the idea that mathematical dispositions and capabilities, including reasoning, are shaped by both classroom and mathematical dialogue (Walshaw & Anthony, 2007) and so in mathematics, attention should be paid to how classrooms that encourage effective dialogue and create a culture of mathematical practice are established and managed. In everyday mathematics classrooms, the social and mathematical communication children engage in with others and how children reason and justify to each other is arguably critical to their learning (Goos, 2004). Supporting children to focus on the investigation, analysis, processing and collaborative negotiation of mathematics will provide a very different experience from the mathematics teaching children usually experience, where product and reasoning processes are isolated from each other (Kennedy, 2009).

Mathematics is increasingly seen as a discipline that relies on critical thinking skills and not rote memorization and there is concern to move to engaging children in problem-solving that will develop their mathematical thinking and reasoning and engage them in the discourse and practices of mathematics (Clarke, Goos & Morony, 2007). Problem-solving tasks that would move towards reasoning would stimulate and provide a reason to think, reason and engage in dialogue. Such tasks must be conceptually rich and genuinely and inherently problematic (Groves, 2009), focused, robust and accessible (Groves & Doig, 2004).

Methodology

Participants

One class of fifteen children in their Reception Year as a mid-year intake in their first term of school (2010, mean age 5 years 2 months) and their teacher participated in this study.

Design

The reported research is from a classroom based design study undertaken over a 10 week school term. The study research questions included investigating the context and conditions that support young children’s engagement in statistical learning and reasoning. Data modelling activities as instructional innovations were used to engage children in working with data in both whole class and small group activities in ways that would require the public sharing of ideas and thinking in the problem-solving process. Data modelling activities are designed specifically to support the creation of data by children that requires organising, quantifying and transforming in order to find a problem solution (English, 2007). Preliminary activities were implemented prior to the data modelling activities to provide specific language to support positive and reasoned collective dialogue. A series of lessons were adapted from a classroom activity book for 6-8 year olds ('Talk Box') based on the 'Thinking Together' approach (Dawes & Sams, 2004). The teacher also employed a community of inquiry pedagogy, based on the work of Lipman (2007) in whole group discussions as an integral aspect of classroom practice. These approaches and activities were implemented in order to:

1. actively work to provide the participating children with explicit language and collaborative skills for speaking and listening to support reasoned discussion;
2. provide a community of inquiry environment for the curriculum activities that had been designed for the study to optimise the chances that children's thinking and knowledge would be made visible; and
3. provide a framework of agreement with the participating teacher in the research study that would inform collaborative dialogue with the researcher about the conditions and contexts for the children's learning.

Whole class discussions and small group activities were video- and audio-taped and wholly transcribed. The teacher's shared educational philosophy and pedagogical knowledge were an important aspect of the collective negotiation processes, responsibility and involvement in the research.

Activities

The data modelling activities incorporated children's story book literature as a stimulus for posing a problem that was able to be resolved through the creation and processing of data, including generating and selecting attributes for classification and structuring and displaying data for analysis. Story books addressing the themes of recycling were used to engage the children in resolving problems requiring sorting, predicting and representing different types and amounts of rubbish.

Prior to the introduction of the data modelling activities, five adapted 'Talk Box' (2004) lessons were implemented. The lessons included the establishment of expectations for community behaviour. Through these lessons, the children were introduced to the terms 'talk', 'listen', 'I think', 'because', 'I agree' and 'I disagree' and given the opportunity to work in small groups with activities that focused on these ideas. Large cards with each of these words and a representational symbol (e.g. picture of an ear with 'listen') were displayed in the classroom and referred to. The use of these words were continually modelled by the teacher and encouraged during both whole and small group discussions.

Results and discussions

This whole group discussion occurred at the beginning of the sixth week of the children's formal schooling. The children sorted real objects which replicate the rubbish found in the room of the main character from the story, a dog named Baxter Brown. The children have been asked to help Baxter Brown clean up his room by deciding whether the objects should be recycled, reused or thrown away. The teacher both models and reminds the children to work with the reasoning language from the 'Talk Box' lessons. Children are using justificatory language ('because') and witness to the existence and acceptability of differing ideas, and changing your mind. Here the children are deciding what to do with dog biscuits.

Teacher: Can you tell me why you think we should throw it in the bin, remembering to use your words 'I think' and 'because' (pointing to the classroom cards).
T: I think we should throw it in the bin because...I just don't know.
Teacher: Can someone help T with an idea?
Y: I know where you really should put them. You should put them in reuse because dogs eat biscuits.
Teacher: So you've got a different idea: you think you should reuse them because dogs eat biscuits?
Y: (nods)
Teacher: Do you agree with that idea T?
T: Yes.
Teacher: So you've changed your mind? (T nods) So you think we should put them in reuse, because dogs use them. Do people agree with that idea? Does anyone have a different idea about where the biscuits should go? Should we throw them away, recycle them or reuse them? What do you think K?
K: Umm, you should recycle 'em, because you can get more ones.

As the whole group discussion continues, children begin to explore and engage in the collaborative language, and demonstrate the ability to recognise differences between ideas and common ideas. Here, one child believes the object, an empty drink can, should be thrown away and two children believe it should be recycled, although there are different justifications as to why.

G: I think we should recycle because...I think that's where they go.
Teacher: Tan you tell us why you think that's where they should go?
G: Ah, because my mum puts cans in recycling.
Teacher: Does everyone agree with it going in the recycling? Does anyone have a different idea about where it might go? What's your idea T?
T: You could put it in the bin.
Teacher: So you would throw it away, can you tell us why you would throw it away?
T: because it's rubbish, when you have drinked all of it, it has to be thrown in the bin.
Teacher: B, what's your idea?
B: I would get more money
Teacher: So you can get more money? How do you get more money?
B: Um, because ... because ... Those (points to can) can be turned into money.
Teacher: How can we do that? Do we throw them away, recycle them or reuse them?
B: Reuse them. No … recycle.
Teacher: So we have two different ideas.
Y: I agree with G and B.
K: Me too.

In small groups of three, children are sorting the same objects as pictures, in this case a rubbish bag. Although the children are not offering justifications, there is awareness that differences exist and that resolving these will require discussion.

C: I think it goes in the rubbish.
Y: (Points to paper) This one?
C: Yeah.
Y: Me too.
D: I think that rubbish bag ... goes (taps finger at 'throw away' section on the paper).
C: No, recycle I think it goes in.
Y: Yeah. Me too.
D: I think it goes in there (taps paper).
Y: So two people think it goes in recycling.
C: We need to talk.

At the end of their eighth week of school, the children were engaged in a problem requiring them to read and interpret a completed data table of six columns, with each column for a single story character and five rows for the items that had been recycled in a competition, with the sixth row providing a total of items each character had recycled. The data table was on the class *SmartBoard*, and children were able to make marks on the table as they worked through their justifications. When one child was given the opportunity to explain and demonstrate his thinking, other children were able to use this model to support their own thinking with similar questions. This reasoning was reflected later in small group work where children used finger movements to find the intersecting point in working out quantities attributable to different characters.

Teacher: We're trying to find out how many glass jars Charlie recycled. C can you show us what you think the answer is?
C: Glass jars (places pen on the glass jar picture/word in the left hand column) that says glass jars and that's (places his finger on a picture of a person on the top row of the table) Charlie (circles the number '4'where the row and the column intercept)
Teacher: can you explain to us how you got that answer?
C: Because that says glass jar (points) and Charlie's there (points) and the glass jar's pointing at the number 4.
Teacher: I think you're right. It's in the column with Charlie and in the row with glass jars. That this is the number. The 4, so four glass jars. So how many food scraps did Morton recycle? I wonder if we can use the same idea? G?
G: Zero.
Teacher: Can you explain why G?
G: Um, because um ... because, it's pointing to zero (puts the pen on zero and moves her hand back along the row to the food scraps picture/word) and it's on Morton
Teacher: So how many food scraps did Lotta recycle J?
J: (Circles a number) Nine of them.
Teacher: She recycled nine food scraps? (J nods) Can you tell us why you got that number? How did you figure it out?
J: I figure it out like C.

Conclusion

The children in the study were in their first term of formal schooling following a period of non-compulsory pre-school education. As the children participated in the data modeling activities, the specific language support and the expectations for collaboration and justification is reflected in their language and interaction. Children demonstrate their capacity to listen, recognise, categorise and respond to one of the most abstract concepts: other people's ideas. They are able to explain and justify their own thinking. There is evidence that they are beginning to build on each other's ideas when engaging with problem-solving tasks when working with mathematical concepts.

Although this study is in the early stages of data analysis, the findings thus far provide encouraging evidence of a quality and substance in children's expressed thinking, reasoning, and knowledge that exceeds what many may consider possible at this point in their schooling. Although the sample in the study is limited, further research is needed with respect to the context and conditions for effective collaborative reasoning that supports both mathematics learning in classrooms and children's reasoning in the early years of schooling.

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MATHEMATICS EDUCATION AND THE iPOD TOUCH

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Apple's iPod Touch is a personal digital device, essentially a version of the popular iPhone, but without telephone capabilities. Although not designed expressly for education, software has been developed for the device for mathematical and educational purposes, while some of its other capabilities (such as those for podcasts, videos and the Internet) can be used for mathematics education. This paper provides an analysis and evaluation of some of these various opportunities for 21st century mathematics education. While some elements of the iPod Touch offer attractions for mathematics education, some educational limitations are also identified.

Introduction

This paper explores a close relative of Apple's popular iPhone, the iPod Touch (here abbreviated simply to iPod), which has been used in some educational settings because of its significant digital capabilities. The most critical differences between the iPod being considered here and the iPhone is that the former has neither telephonic capabilities nor digital camera capabilities In an age in which digital technologies abound, it is appropriate to consider the potentials of a device of this kind for teaching and learning mathematics. Indeed, recent national curriculum initiatives, (Australian Curriculum, Assessment and Reporting Authority, 2011) are designed with modern digital technologies in mind, at least in part. Successive (annual) generations of iPods have become more sophisticated and powerful, as frequently happens with new technologies; this is quite problematic in a paper of this kind, which does not thus claim to be entirely up to date when published.

A focus on the iPod is not intended to suggest a preference for this particular product over others, but is made partly on financial grounds. Apple's more recent iPad series of devices have also created a good deal of excitement and commercial interest, but are presently much too expensive for typical schools.

Applications

From the perspective of mathematics education, the most likely way in which an iPod might be of value is through the use of applications, commonly abbreviated to apps, developed especially for mathematical purposes. Apps are available via Apple's free software, *iTunes*, which links directly to their iTunes Store. Details are available at Apple Corporation (2011). Apps can be downloaded via computer from the iTunes store or can be downloaded directly to an iPod via the Internet. While some apps are free, others must be purchased, usually for relatively small prices. (At present, the most common purchase price is $1.19, although a few are more expensive.) It is necessary to have an account to download apps, whether or not the apps are free. Updates are free.

There is a very large number of apps available in the App Store, with various classifications used to organise them. For example, apps are classified into categories, including Education, Productivity, Games, Utilities, Reference (in all of which I have found some interesting mathematical examples). A search engine allows for an app to be searched for by name (so finer-grained reference details of the apps mentioned in this paper are not provided), and other searches will allow a number of apps to be identified. The search engine is not the friendliest, so that it can sometimes be hard to navigate all the results and explore all the choices efficiently. For example, when 'math' was used as a search term recently by the author, a little under 6000 apps for an iPod were identified, including examples classified in each of the above categories (as well as some others), with both free and paid apps. Consequently, a paper of this length does not claim to cover the territory exhaustively, but rather intends to provide a perspective on some of the possibilities presently available, with a few examples chosen to illustrate these. There are reviews available online for many apps, especially those that have been around for a while, although it is problematic to place too much reliance on these, with educational interests in mind, without a sense of who the reviewers are.

Many of the free apps (but not all) are in fact reduced or slightly disabled versions of paid apps (and hence are often described as 'lite' versions, increasingly commonly but no more grammatically correctly), offered to provide potential customers with sufficient experience of the approach taken to encourage them to purchase the paid version, sometimes even with irritating and frequent messages in the form of advertisements to do so. This is of course understandable, as those producing the apps rely on their sales to support their businesses. While data are not available (to the author, at least), it is easy to get the impression that many of the app developers do not have much mathematics educational background or expertise, so that what is offered is not always pedagogically sound or even mathematically interesting. In addition, with the recent emergence of the iPad and iPad2, a good deal of energy is directed at making new versions of iPod apps to run on the new devices, or making apps only for the iPad, not the older and less sophisticated iPod, not a surprise for commercial organisations.

What follows is a brief and unavoidably personal account of some of the kinds of offerings presently available, especially in relation to school mathematics, mostly with an eye to material likely to be of interest to secondary schools. As noted earlier, no claim to exhaust the territory is provided here.

Graphing

Many apps allow users to graph and explore functions and other sorts of graphs. While secondary students might usually have access to a graphics calculator for this purpose, some of these apps offer some nice features that allow for easy manipulation of both 2D and 3D graphs, and using more colours and a higher resolution than a typical graphics calculator. As the iPod is operated mostly by finger movements on a touch-sensitive screen, this adds a new sort of experience including moving (two or three) axes around, zooming in and out by stretching the coordinate plane or 3D space with two fingers at once and locating points of intersection by just touching them. Figure 1 shows two good examples, the first from *GraphCalc* and the second from *Quick Graph*.

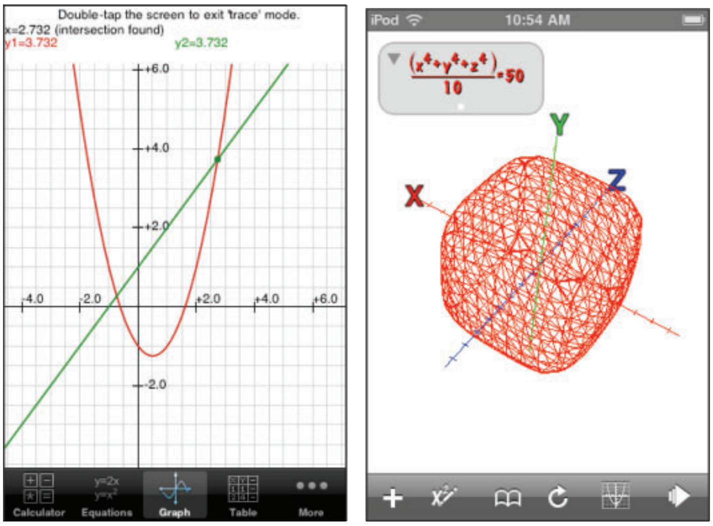

Figure 1. GraphCalc and Quick Graph, two examples of graphing apps.

With apps of these kinds, students might reasonably be expected to get a different and even sensory experience of graphing functions than a graphics calculator can offer. For example, many teachers have referred to tracing as being like moving one's finger along a graph; with a touch-screen of this kind, this is precisely what a user does.

Figure 2. Two other graphing apps, GraphBook and iTrig.

As well as providing graphical capabilities that in some ways match what students' graphics calculators might provide, some apps offer different graphing experiences. For example, *SpaceTime* is a recent and very expensive app ($23.99 at present, around the same price as a scientific calculator, but much more than typical apps) that claims to provide high-end and programmable features such as some those involved in *Mathematica* and *MatLab*. Figure 2 shows the *GraphBook* app, which seems to have been constructed to provide some animated and manipulable examples of the *SpaceTime* capabilities, as an inducement to purchase the complete app. Nor are graphical apps restricted to graphing functions. Figure 2 shows a screen dump from the *iTrig* app, which provides a unit circle around which a point can be moved with one's finger and associated graphs and values are shown.

These examples do not exhaust the possibilities; many other apps have a graphical element. For example, *4D Spin* addresses the nature of the fourth dimension, *Polar Sweep* is concerned with relationships between rectangular and polar coordinates and many apps such as *Fractals* allow students to explore fractal images of various kinds.

Calculator

A very large number of apps offer a calculator of some kind, and the standard iPod even comes with a calculator app. There are many kinds of specific calculators. An example is the right triangle solver in *iTrig*, to calculate lengths of sides and angles of a right triangle from partial information. The standard iPod calculator is both an arithmetic and scientific calculator (although some users may not realise this unless they turn or shake their iPod). The scientific calculator displays more places of decimals than a standard scientific calculator and in that sense is an improvement, as shown in Figure 3.

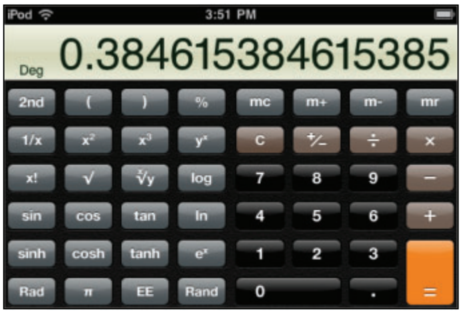

Figure 3. The standard iPod Calculator app in scientific mode.

There are iPod apps for just about any kind of calculation likely to be needed in secondary school, including unusual tasks such as those shown in Figure 4.

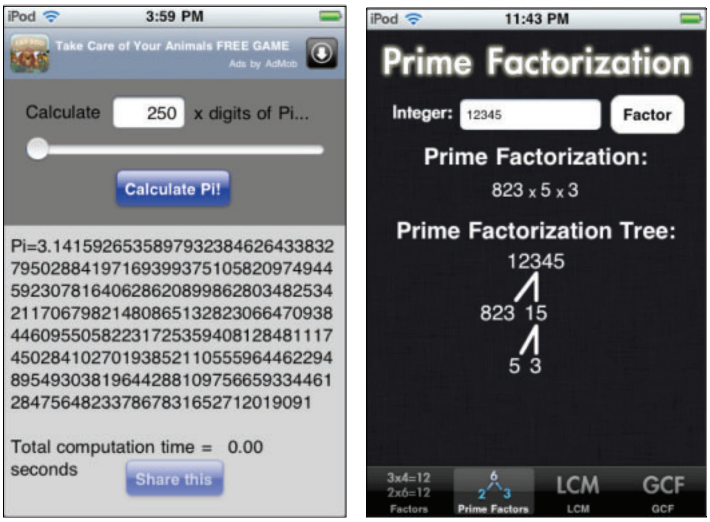

Figure 4. Calculators are involved in both iFactorization and Pi apps.

Others include algebraic calculations with *PocketCAS*, normal distribution tables with *GaussPad*, and unit conversions with *Units*. There are even old-style calculators such as an *Abacus*, *iSlideRule* and *Longhand Division*, each of which operates successfully. At least collectively, the large suite of calculator apps available will make it clear to students that many mathematical calculations can be automated for machines to do, and that they need to continue to choose the right tool for the job (which includes mental and approximate calculations some times, of course).

Reference source

A surprising number of apps seem to function as mathematical reference works, with tables of formulae, diagrams, theorems, and other items. It can be quite useful to have references of these kinds available when needed, especially the slightly more esoteric ones (which of course varies from person to person). Apps like these might reflect an image of those who have constructed them of mathematics as a discipline in which there are many formulae to memorise (or look up). Figure 5 shows three typical examples.

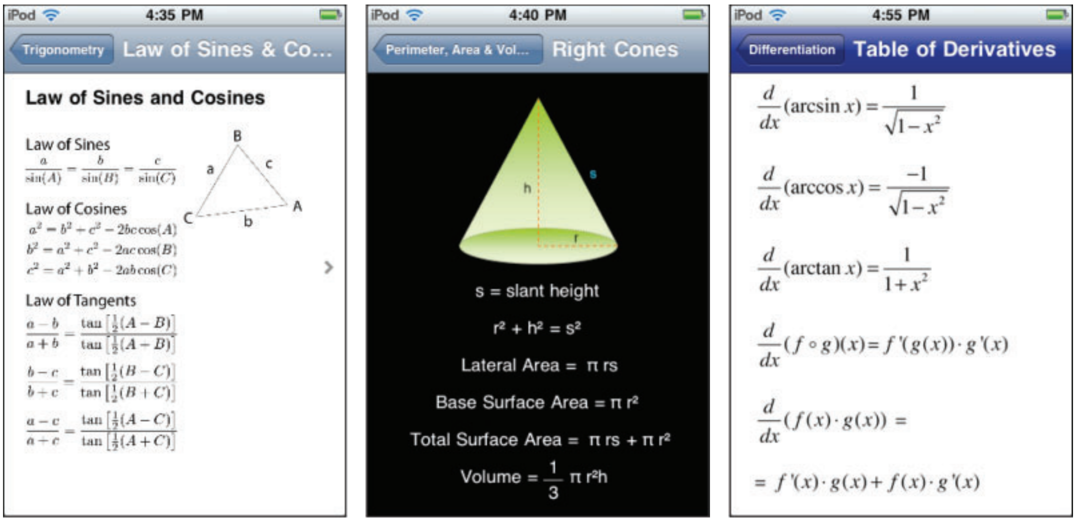

Figure 5. Pages from Math Reference Free, Math Pro and Formulas apps.

Measuring

A number of apps have been designed to handle various measurement tasks, often found in mathematics, although it is questionable whether an iPod version of these is a superior tool to the original measuring tool.

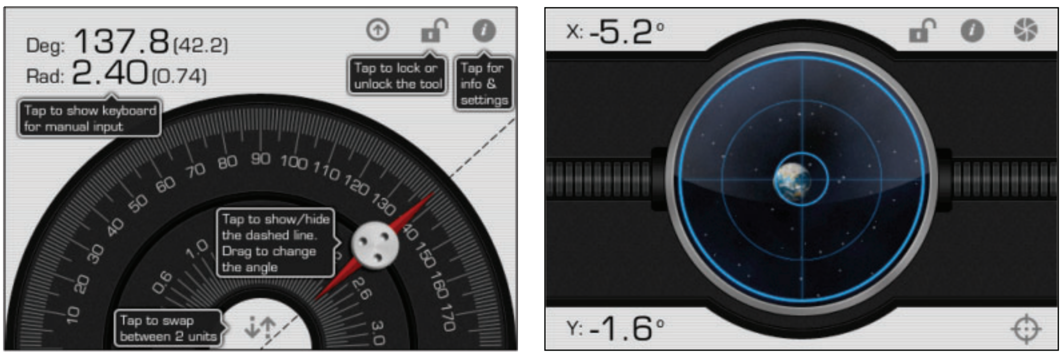

Figure 6. Protractor Deluxe and Measures apps seem to offer spurious accuracy.

Indeed, it seems that there are examples here of computer programmers making something in order to prove that it can be done, rather than to produce a genuinely useful tool. Figure 6 shows two of many possible examples. The protractor, surprisingly, seems to measure angles only in a clockwise rather than anticlockwise direction and to display an accuracy that is substantially beyond what is reasonable with the actual device in practice. Similarly, the spirit level (one of a suite of tools in the same app), in measuring a surface to the nearest tenth of a degree seems to overstate the accuracy of the measurement of extent to which my desk is horizontal. Other examples, such as *RulerPlus*, *Tape Measure* and *iHandy Carpenter*, can be similarly criticised.

Drill and practice

At first glance, the available apps for mathematics or for education seem to suggest that the most useful tool for iPods involves lots of practice of mathematical skills, especially those related to computation, with many of them focussed on the primary years of schooling. Practice certainly has an important place in school mathematics, and a device that uses colour, entertainment and novelty effects to engage students in practice at a range of levels may be a useful supplement to other experiences. Despite enthusiastic claims to the contrary by the designers, many of the apps I examined in this category seem to offer not much more than heaps of practice, often timed and speeded and generally with feedback; overall there seems to be a limited case to use an expensive piece of digital technology in such a mundane way. Indeed, Pelton (2011) suggested that around 40% of the designated 'top' apps for mathematics fell into the category of drilling basic facts. Some of these apps are designed in the form of flash cards, some as games and others differently, but fundamentally many offer little of lasting conceptual value, and it is questionable that students would be attracted to them for very long, once the novelty had worn off.

Some apps that essentially provide a form of practice do so in a slightly more engaging and interesting way, however. Figure 7 shows two examples, *Motion Math* and *Number Line*. The *Motion Math* app makes use of the motion sensor devices that are an integral part of the iPod and iPhone®, so that the device knows when and how it has been turned.

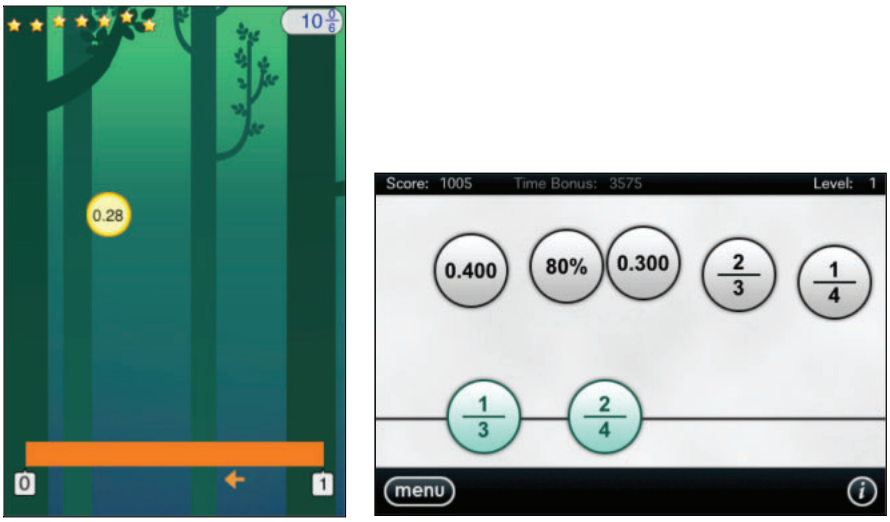

Figure 7. Motion Math and Number Line.

The screen shows a bouncing ball with a fraction or decimal inside. The user must tilt the device to make the ball land on (or close to) the appropriate point at the bottom between 0 and 1, in order to get a new number and to try again.

Similarly, *Number Line* requires the user to drag the numbered balls onto the number line in the correct numerical order. Each of these games is fundamentally concerned with understanding and comparing the sizes of numbers, in different representations, and seems to exploit better the educational possibilities than do many of the other apps which seem merely to automate what could as easily have been placed on a worksheet.

Miscellaneous

There are many other kinds of apps that might find a place in secondary school mathematics, and even be of interest to teachers themselves - too many to easily classify. Figure 8 shows two examples.

Figure 8. Samples from the MathFunFacts and Is That Prime? apps.

The *MathFunFacts* app, created by Francis Su of Harvey Mudd College in USA has a large number of mathematical snippets, many with fairly recent mathematics, that teachers may find of interest. *Is That Prime?* provides quick information about the primality (or the factors) of integers. While such apps might be regarded as a little quirky, they may still find a place in a modern classroom.

Figure 9 shows three quite different examples, each of which may have some kind of appeal to secondary students, while being out of the mainstream mathematics curriculum. The app, *Discover the Magic of M.C.Escher* contains many of Escher's famous images as well as other written information and activities. *Polyhedron* contains very many images of rotating polyhedra, while *Mathemagics* contains a large collection of number tricks, fertile ground for algebraic thinking.

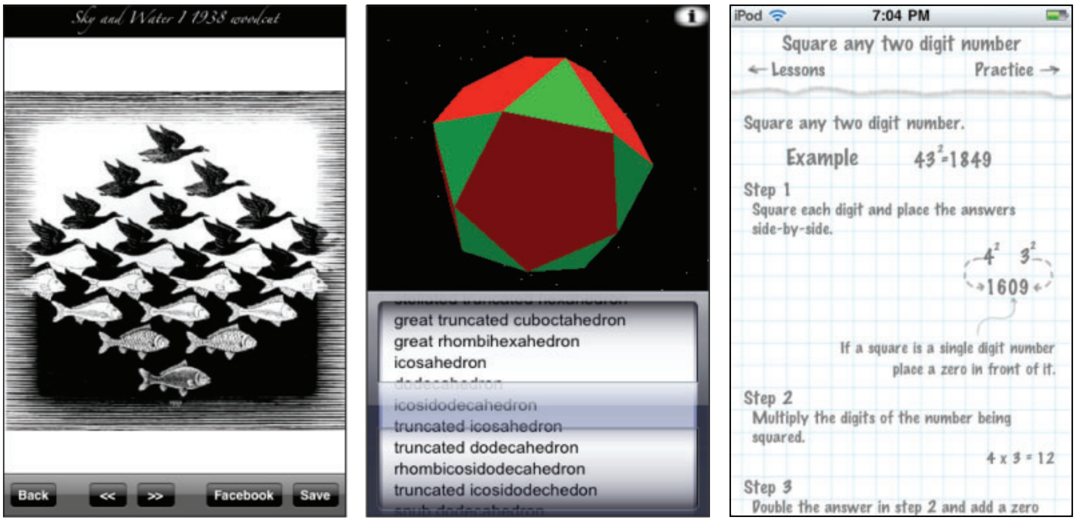

Figure 9. The Magic of Escher, Polyhedron and Mathemagics apps. All M. C. Escher works © 2011 The M.C. Escher Company, the Netherlands. All rights reserved. Used by permission. www.mcescher.com

As well as these, other apps such as *iBooks* provide a mechanism to read electronic books (some of which are free, courtesy of the Gutenburg Project, while others cost close to normal book prices), some apps such as *Chromatics* contain a great deal of visual mathematics for browsing, others such as *Collatz* explore particular bits of mathematics, while yet others, such as Pearson's *Trigonometry*, contain direct instructional materials. There are many other games, puzzles, patterns, spatial and numerical environments that contain elements of mathematics, all too difficult to classify here, but many seem worthy of a second look.

Internet

While applications offer the most likely use of an iPod, the capability of accessing the Internet on a wireless network at school or home leads to other possibilities as well. Kissane (2009) suggested a number of ways in which Internet access might be helpful for mathematics education, and many of these can be used with an iPod. In some cases, the iPod web browser is not needed, as a special app has been constructed for a similar purpose. A good example of this is *WolframAlpha* the very sophisticated search engine with the awesome power of Stephen Wolfram's *Mathematica* behind it, aiming to make all systematic knowledge immediately computable, accessing all available data. Another good example is *Wikipedia*, which has good entries related to mathematics.

A major limitation, however, of Internet use with the iPod is the lack of either Java or Flash capabilities. A consequence of this is that many excellent interactive websites of value for mathematics education (such as the *National Library of Virtual Manipulatives*, the NCTM's *Illuminations* and many parts of the *Nrich* site, as well as interactive software like *GeoGebra*) are rendered inoperative. These limitations might be removed in the future, but it seems that Apple is at present resolutely opposed to these platforms, preferring other approaches to interactive web-based materials. Until a solution to this problem is found, a major potential use of the iPod as a web browser will be severely curtailed, unfortunately.

Podcasts

While many iPod users regard their device as essentially a personal music machine, there are potential uses of the audio and video capabilities for mathematics education as well. Podcasts and video podcasts made locally (for example, by a teacher at a school, or by university staff for external students) or externally (disseminated through the Internet) can carry powerful and interesting mathematical messages. Excellent examples are available via the iTunes site on iTunesU or via the podcasting links on the website. From the UK, the regular *PLUS* podcasts, various Open University series, Marcus Du Sautoy's *A Brief History of Mathematics* from the BBC and the *Travels in a Mathematical World* collection from the Institute of Mathematics and its Applications are all good examples of contemporary mathematical materials that would be of interest to both teachers and older secondary students. The Swede Hans Rosling's *GapCasts* (using the wonderful *GapMinder* software) also offer excellent stimulating materials on an iPod related to the use of statistics to understand modern social and health issues internationally and don't require a live Internet link after downloading.

Projection

A major limitation of the iPod is the present inability of teachers to use it to communicate to a class, through a data projector or large television set. This is a consequence of the design of the device, which is thus only able to be used by one or two students at once. An exception is that some videos and podcasts can be shown on a television set with the appropriate cables. However, the lack of capacity to show apps to a wider audience is a significant educational limitation, which needs to be overcome.

It is possible, but sometimes a little difficult, to use a web camera or other visualisation device to project an iPod screen to a computer and thence to a data projector, but it would be much preferable for there to be a direct link.

Conclusion

While there are some nice apps for the iPod Touch, and some interesting potentials, there are also a lot of uninteresting apps as well as significant practical limitations for the use of the device in mathematics education. As a fairly expensive device (at present), the iPod Touch may be of limited lasting value as a mathematics education learning device, with substantial Internet limitations, while the lack of projection capability is a severe constraint on use for teaching in most cases. Hopefully, refinements to devices of these kinds in the future will address such shortcomings and provide a form of mobile technology that meets the needs of students and schools, and offers significant teaching and learning opportunities.

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FOCUS ON EFFECTIVE NUMERACY TEACHING TO IMPROVE STUDENT OUTCOMES

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Due to an increased focus and accountability in relation to improving student outcomes, many Victorian primary schools provide their teachers with professional development that promotes effective numeracy teaching. This paper describes two schools' mathematical professional development programs facilitated by professional officers from the Mathematical Association of Victoria. It reports on both schools' experiences, and the similar, but different, approaches aimed at strengthening effective numeracy teaching practices in mathematics. These teachers shared their experiences, building on what they knew and what they did in the numeracy classroom. In particular the role of the professional officers was important for assisting teachers to develop a collective understanding of teaching and learning primary mathematics.

Introduction

The Professional Officers at the Mathematical Association of Victoria (MAV) provide a unique range of services for mathematics teachers and mathematics education. Both authors are experienced primary teachers, currently employed as MAV Professional Officers. Both have a passion for guiding teachers to provide purposeful numeracy lessons for the students they teach. The role of the Professional Officers is special: supporting members by promoting interest in mathematics, as well as providing services such as delivering professional development programs, presenting at conferences and contributing to mathematics education journals. The primary school Professional Officers complete the majority of their work in schools and this year (2011) have already travelled across many Victorian school regions from Timboon P–12 School to Nhill College; the smallest school visited so far was Zeerust Road Primary School, with an enrolment of 15 students. All Professional Officers support mathematics education across many sectors, including primary, secondary and tertiary education.

The purpose of this paper is to report on how teachers from two schools worked to improve lesson structure and planning of their mathematics curriculum throughout 2010. The explanation of these experiences will provide a snap shot of the 'custom made' professional learning provided to two primary school MAV members.

Background

The MAV is a membership-driven not-for-profit association which provides a voice, leadership and professional support for mathematics education. Its mission is to advocate for the continual review and improvement of mathematics education and the profession of mathematics teaching. The MAV does this by being a leading voice in mathematics education. It also supports key priorities as outlined in the Blueprint for Victorian Government schools, recognising the need to improve numeracy performance for all students by providing quality teaching and learning (DEECD, 2008).

The Professional Officers deliver a range of professional development workshops for mathematics educators. These experiences provide teachers with opportunities to explore and improve their capacities of knowledge needed for teaching: pedagogical content knowledge, mathematical content knowledge as well as their curriculum knowledge (Schulman, 1987). This work has been initiated with primary and secondary teachers through many programs in the past two years, for example the Professional Learning Assistance Team (PLAT) project, Effective Numeracy Teaching, Primary Mathematics Specialists and Numeracy Coaching.

Effective numeracy teaching

There are many skills an effective teacher of numeracy will use and draw on as they work with students. Askew, Rhodes, Brown, Wiliam and Johnson's (1997) study of effective teachers provided three categories for approaches to teaching. Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke and Rowley's (2002) study of effective numeracy teachers (and effective schools) identified 10 practices that effective teachers demonstrate. A more recent list was provided through the Scaffolding Practices for Effective Numeracy Teachers (Table 1) which has been used by Victorian teachers to identify a range of practices that can be drawn on to assist students' learning needs (DEECD, 2004). The twelve Scaffolding Practices for Effective Numeracy Teachers have been used by teachers to assist them to identify important attributes of an effective mathematics teacher (DEECD, 2009b).

Table 1. Scaffolding Practices for Effective Numeracy Teachers (DEECD, 2004).

Excavating	Drawing out, digging, uncovering what is known, making it transparent
Modelling	Demonstrating, directing, instructing, showing, telling, funnelling, naming, labelling, explaining
Collaborating	Acting as an accomplice, co-learner/problem-solver, co-conspirator, negotiating
Guiding	Cuing, prompting, hinting, navigating, shepherding, encouraging, nudging
Convince Me	Seeking explanation, justification, evidence, proving
Noticing	Highlighting, drawing attention to, valuing, pointing to
Focusing	Coaching, tutoring mentoring, flagging, redirecting, revoicing, filtering
Probing	Clarifying, monitoring, checking
Orienting	Setting the scene, contextualising, reminding, alerting, recalling
Reflecting/ Reviewing	Sharing, reflecting, recounting, summarising, capturing, reinforcing, reflecting, rehearsing
Extending	Challenging, spring boarding, linking, connecting
Apprenticing	Inviting peer assistance, peer teaching, peer mentoring

Professional development

When a school approaches the Professional Officers to discuss the needs of their numeracy program, a Professional Officer will then make an appointment to meet with the principal and leadership team to identify and prioritise professional development needs at their school. During this visit, a school tour will be conducted and classroom teachers are encouraged to teach a numeracy lesson in order to provide an overview of the types of mathematical activity the teachers implement across the school. The principal might also provide an overview of student numeracy data across the school and a demographic description of the school. The Professional Officer will then work with the leadership team to design a professional learning program to meet the needs of the school.

Throughout 2010, two MAV Professional Officers were each invited to work with one primary school: School A and School B. Both schools focused on effective numeracy teaching to improve student outcomes and workshops were facilitated with classroom teachers throughout the year. During these school visits the teachers spent time as a whole staff, in small groups and individually, meeting and working with the Professional Officer assigned to their school. Teachers were engaged in professional conversations relating to their needs and the needs of their students. These sessions focused on awareness of the DEECD hyperlinks within the Key Characteristics of Effective Numeracy Teaching P–6 (DEECD, 2010) as well as structuring numeracy lessons, which led to the planning and provision of a differentiated numeracy program.

Both schools had ongoing support from leadership teams. Sometimes casual relief teachers released classroom teachers which provided opportunities for teachers to work in groups with the Professional Officer. The principals also attended some of the professional development and provided valuable ongoing support and feedback.

Effective numeracy teaching: School A

School A was located in Melbourne's eastern suburbs and was well resourced. There were 30 teaching staff and an enrolment of 412 students. The school consisted of 15 classes: four Prep, four Year 1/2 composites, three Year 3/4 composites and four Year 5/6 composite classes. The school was not funded for a numeracy coach. School data from mathematics assessment tools such as On Demand Testing (VCAA, 2009) and NAPLAN (ACARA, 2010) identified these students as having a diverse range of mathematical achievement. The program commenced in Term 2 and concluded during Term 4. The Professional Officer visited School A 10 times. These visits included six days working with teachers in classrooms and four after-school staff workshops.

At the commencement of 2010, seven new staff were appointed to the school. This provided an appropriate opportunity to review and plan teaching goals for the school's numeracy curriculum. The school's Strategic Plan included a learning goal to improve levels of achievement in numeracy and to promote each student's best performance. The leadership team and the Professional Officer agreed to focus on strengthening the school's delivery of the mathematics curriculum with an emphasis on developing teacher practices that supported students in their learning, while also meeting the needs of the range of learners in classrooms.

In order to focus thinking about their numeracy program, teachers were provided with professional development sessions on current theories of how children learn using many of the resources suggested in the Key Characteristics of Effective Numeracy Teaching P–6: Differentiating support for all students (DEECD, 2010). The program aimed to assist teachers to improve their pedagogical practices by referring to the Mathematics Developmental Continuum P–12 and Prep to Year 10 resources links on the Mathematics Domain site (DEECD, 2006).

After-school professional learning

As a staff, all teachers, including specialist teachers, attended after-school workshops to explore what constituted high quality instruction and to describe what effective teachers do in the classroom to engage students working mathematically. The first workshop promoted professional conversation, focusing on the teacher's delivery of lessons and the Scaffolding Practices for Effective Numeracy Teachers (DEECD, 2004). Each teacher identified two scaffolding practices they wished to improve in order to make more informed decisions about specific learning needs of their students and to promote sustained mathematical thinking. The majority of teachers chose to focus their learning around 'guiding' and 'focusing' (see Table 1, DEECD, 2004). These two key characteristics then became the foundation of the remaining program.

Other staff workshops focused on what a primary numeracy classroom should look like and teachers agreed that mathematics should be taught every day for an hour during the morning. Many mathematical resources were explored. Activities were demonstrated and the teachers were set 'between training session tasks' to trial with their students; for example, Ten New Preps (Downton, Knight, Clarke, & Lewis, 2006). Work samples were then shared in subsequent sessions.

Whole-day professional learning

Two whole-day coaching sessions for teaching teams across each Victorian Essential Learning Standards (VELS) level (VCAA, 2007) enabled teachers to watch demonstration lessons taught by the Professional Officer. Before and after these lessons, teams participated in small group conversations to discuss the lesson features using the Scaffolding Practices for Effective Numeracy Teachers (DEECD, 2004). Teachers were then encouraged to take the same lesson with their students as a follow up and to consolidate these teaching experiences. Subsequent to these days, the four teaching teams met with the Professional Officer to review a range of mathematical resources for use in planning future lessons at each level.

Two further full-day coaching sessions were planned where teachers participated in a micro-lesson with seven students of the same ability and focused on rich tasks for extending the learners. These sessions were taught in the same room by three teachers, each with six of their own students, so the Professional Officer could observe these lessons. Each teacher used a Flip video camera to record their lesson and was encouraged to use the video to reflect on their own teaching, either individually or with peers during a later team meeting. The teachers and Professional Officer met before and after these lessons, discussing the lesson structure—elaborating, guiding and focusing.

Observation of teacher outcomes

For the final day's workshop, 12 teachers from across the VELS levels volunteered to teach numeracy during the morning (Prep to Year 6). Four other teachers volunteered to form a consultative committee and conducted a 'numeracy walk-through' with the

Professional Officer. Throughout the morning, approximately 15 minutes was spent in each class to view a snap shot of the 12 numeracy lessons being conducted. The leadership team had agreed that the consultative committee would record field notes relating to the scaffolding practices of ‘guiding’ and ‘focusing’. All classroom teachers were aware of the focus the consultative committee was observing in action.

The numeracy walk-through provided an opportunity to celebrate the teachers’ journey. Teachers who were teaching engaged their students by implementing a range of numeracy tasks. The consultative committee was impressed by the range of lessons they saw and congratulated their peers. After the visits, the consultative committee provided a summary of what they had seen and reported back to the staff during a brief lunchtime meeting. This is a summary of some of the learning outcomes and comments that were noted during the final day of the program:

- Guiding—asking students to explain and justify their answers, teachers observing and listening.
- Focusing—providing group work, differentiation, students were engaged and enjoyed their learning, use of real world examples: students using the Internet to calculate the cost of a round-the-world flight.
- Orienting—focusing students during the introduction of the lesson: using a poster, “Today we are learning how to tell the time.” Posing a problem for the class to solve then sharing strategies by exploring more than one method.
- The same topic was being taught across the same year levels, providing evidence of teachers planning together.
- Some classes chose to use ICT for small group activities or interactive whiteboard with the whole class.
- The principal noted that teachers would not have opened their doors at the beginning of the year inviting their peers to watch them teach.
- The consultative committee enjoyed the opportunity to view teaching across the school and valued the opportunity see a snapshot of student development of mathematics skills from Prep to Year 6.
- The consultative committee recommended that all teachers should be given the opportunity to view numeracy lessons across all primary year levels.

Effective numeracy teaching: School B

School B worked with a different MAV Professional Officer. This school was going through a period of change with a newly appointed principal and was situated in the south-eastern growth corridor of Melbourne. This school catered for students from a wide range of social, economic, language and cultural backgrounds and had an enrolment of 500 students with representation from over 40 countries. There were 20 composite classrooms across four VELS levels. The school employed 40 teachers whose experience and knowledge varied from recent graduates to those nearing the end of their teaching careers. The school’s strategic plan and learning goals had identified a focus of school improvement for numeracy with an emphasis on the e5 Instructional Model (DEECD, 2009a) teaching practices, as well as ensuring teachers planned together and shared resources.

School B did not receive funding for a numeracy coach but released two teachers to train as PLAT leaders. PLAT training had been conducted by MAV Professional

Officers within the Dandenong Region for the past two years. To complement the PLAT program, a Professional Officer was invited to work with classroom teachers to assist with ongoing support for the teaching and learning of mathematics school-wide, and specifically to work with teachers on their pedagogy and approaches to mathematics teaching.

Features of the professional learning of School B

The leadership team and Professional Officer met to discuss School B's focus. They agreed to build on teachers' understanding of the five phases of the e5 Instructional Model through the use of rich tasks and open-ended questions. School B had been recognised for its outstanding curriculum innovation and the introduction of the e5 Instructional Model: engage, explore, explain, elaborate and evaluate (DEECD, 2009a). The teachers at School B were all developing and deepening understanding of what constituted high quality teacher practice in their classrooms by implementing the e5 Instructional Model. The five phases were embedded across the school's planning documents, including Mathematics. As a means of celebrating and sharing this work, four teachers had presented their mathematics planning documents at the annual MAV conference at La Trobe University in 2009.

The Professional Officer usually met with four teachers from a teaching team, based on VELS levels, for two hours every week for four weeks. During the teaching teams four week program, teachers were introduced to an open-ended investigative approach to mathematics. They observed the Professional Officer teaching model lessons, engaged in professional discussions, explored mathematical resources and were assisted with numeracy planning. Teams were revisited through whole-school professional development and informal discussions over lunchtime breaks as well as email support. Teachers debriefed after the lessons, focusing on the e5 Instructional Model. They discussed what good mathematicians do, explored attributes of powerful numeracy teaching, as well as the lesson features which are elements of Maths300 (Education Services Australia, 2010). Three after-school workshops complemented the classroom experiences by focusing on effective numeracy teaching and implementing the Mathematics Continuum (DEECD, 2006).

During the model lessons, teachers were introduced to a range of rich classroom tasks, including assessment tasks (Downton et al., 2006), investigations from *Working Mathematically with Infants* (Williams, 2010) and from Maths300 (Education Services Australia, 2010). During the lesson debrief, teachers commented that the rich tasks fostered positive learning and engaged students in making mathematical connections. These experiences were purposeful to the students, as all learners could make a start; students were active and worked together to explore different strategies and solutions.

A second focus of the modelled lessons was 'good' questions, linking to the phase of 'elaborate' as a technique for cultivating higher-order thinking and monitoring students' progress (DEECD, 2009a). Encouraging teachers to use higher-order thinking and 'good' questions aimed to enhance student learning. These questions require more than remembering a fact or reproducing a skill: students can learn by answering the questions, and the teacher learns about each student from the attempt, also noting there may be several acceptable answers (Sullivan & Lilburn, 2004).

During the model lessons the teachers were able to take notes observing what the students knew, noticing common misconceptions and errors. Using rich tasks provided an opportunity for teachers to evaluate and collect data during the lessons while the students were engaged in learning. During the lesson debrief the teachers brainstormed the tools they could use to collect data about their students' mathematical understanding: anecdotal notes, observation rubrics, check-lists for individual students, check-lists for whole classes, and a class 'big book' of reflections which was suggested for the early years.

Impact of the professional learning programs

The effective numeracy teaching undertaken by School A and School B aimed to improve student outcomes. The professional development focused on what and how classroom teachers delivered their numeracy program. Neither school's programs were formally assessed. Comparing pre- and post-program data with respect to student achievement could be completed at a later date to evaluate student outcomes and provide feedback of both programs.

Evidence of the success of School A and School B programs was provided by observations conducted by leadership teams, as well as verbal and written comments or reflections provided by the participating teachers and students. A summary of the outcomes includes:

- implementation of dedicated, regular daily numeracy lessons at all VELS levels;
- improved structure of lessons with focused introduction and lesson debrief with the students to conclude lessons;
- engaged learners working on tasks that catered for different abilities;
- implementing rich tasks that promoted whole-class investigations of mathematics;
- regular and systematic use of open-ended questions, games, authentic problems and extended investigations;
- commitment from teachers to meet and plan numeracy together for each VELS level within the schools;
- teachers learnt from each other through the opportunity to view peers in action.;
- the teachers reported that they had increased awareness of the range of quality resources available to assist them with planning and implementing their numeracy teaching;
- teachers explored websites and other resources for engaging and scaffolding learning such as the *Teach Maths for Understanding CD* (MAV, 2009);
- the overall experience promoted positive teaching teams for planning and working across the VELS levels.

Providing teachers with an opportunity to work in teams to observe and discuss numeracy lessons in action was a valued experience for both schools. The success of these programs was also due to the commitment by the entire school staff and their willingness to identify areas of concern, and to work with peers and the Professional Officer to explore and extend their own teaching and learning experiences.

Both principals attended some workshops with their staff and provided ongoing feedback to the leadership team and Professional Officers. Teachers made decisions regarding the structure of the program and how they wished to use their time when working with the professional officers. Regular staff meetings were allocated time to discuss and follow up on ideas presented and to foster a collective understanding of how to continue to improve and plan numeracy lessons.

The Professional Officers also gained knowledge from the teachers they worked with during these school visits. For example, one Prep teacher used a mathematics poster to “tune” her students into the lesson and explain what good mathematicians do during their work: think for themselves, estimate by having a best guess, count to work out the answer, check their answer, try hard and never give up (Figure 1).

Figure 1. Poster used by prep teacher (School A) for “tuning in” to mathematics lesson.

Conclusion

The Professional Officers provided two programs that assisted teachers to reflect on their own teaching and to develop a collective school approach to numeracy. This was achieved through promoting a range of experiences to develop a shared philosophy of effective numeracy teaching and learning of mathematics. Both schools worked with a Professional Officer to improve their pedagogical practices and planning for numeracy across all VELS levels. School A worked together to establish a common, understanding of effective practice using the scaffolding practices (DEECD, 2004). School B drew on the five phases from the e5 Instructional Model as a focus for improving numeracy teaching through rich tasks and good questioning techniques.

Customising professional development to promote effective numeracy met the needs of each school. This format encouraged all staff to implement shared structures for their whole-school numeracy program while also promoting each teacher’s individual teaching of primary mathematics. In particular, teachers were provided with opportunities to focus on their own classroom teaching and planning with reference to lesson structure, using open-ended tasks and providing differentiated lessons. Allowing teachers to work together promoted opportunities to reflect on practices and for teachers to justify what they do, and contributed to enhancing their knowledge of teaching. All teachers valued the opportunity to view each other in action and agreed that modelled lessons should continue across all Year levels to foster effective numeracy teaching.

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DEVELOPING THE PATTERN AND STRUCTURE ASSESSMENT (PASA) INTERVIEW TO INFORM EARLY MATHEMATICS LEARNING

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The Pattern and Structure Mathematical Awareness Program (PASMAP) stems from a 2-year longitudinal study on students' early mathematical development. The paper outlines the interview assessment the Pattern and Structure Assessment (PASA) designed to describe students' awareness of mathematical pattern and structure across a range of concepts. An overview of students' performance across items and descriptions of their structural development are described.

In the Australian *Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2010), the Number and Algebra strand highlights the importance of mathematical patterns, relationships, abstraction and generalisation, as well as the roles of Problem Solving and Reasoning Proficiency strands. Further, the integration of measurement and geometry, and statistics and probability brings new opportunities to develop a structural approach to mathematics learning.

The Pattern and Structure Mathematics Awareness Project has investigated the development of patterning and early algebraic reasoning over a series of related studies since 2001 (Mulligan, 2011). The project aims to promote a strong foundation for mathematical development by focusing on critical underlying general features of mathematics learning much earlier than previously thought possible. We suggest that an awareness of mathematical pattern and structure enables real mathematical thinking and simple forms of generalisation from an early age (Mulligan & Mitchelmore, 2009). From 2009 to 2010 we evaluated the effectiveness of a school-entry year-long mathematics program promoting patterning and structural awareness.

Pattern and structure in mathematical development

Young children learn mathematical ideas by seeing patterns in an organised way: looking for sameness and difference. We call this ‘pattern and structure’. A mathematical pattern can be:

- a simple repetition such as a ‘unit of repeat’—ABC, ABC, ABC;
- spatial patterns such as 2D and 3D designs, tessellations, transformations;
- a growing pattern such as a systematic increase or decrease, e.g., triangular number pattern 1, 3, 6, 10, 15; or
- a function where relationship between variables are formed, e.g., table of values.

Mathematical structure refers to other features such as:

- numerical structure, e.g., counting in multiples and equal groups;
- spatial structure, e.g., row and column array; similarity ‘same shape, different size’ and congruence ‘same shape, same size’;
- structure of units of measure; and
- structural features that lead to abstraction and generalisation, e.g., $a + b = b + a$

Our goal is to develop an assessment and pedagogical framework. In this paper we describe the development of the Pattern and Structure Assessment (PASA) interview and the broad findings, with some examples of students’ responses.

Background to the research

Structure has been a growing theme in research on children’s development of mathematical concepts. Mason, Stephens & Watson (2009) believe that the roots of mathematical thinking lie in detecting sameness and difference, in making distinctions, in classifying and labelling, or simply in “algorithm seeking” Studies of young children’s mathematical reasoning have provided complementary evidence of the importance of early patterning skills, analogical reasoning and the development of structural thinking (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; English, 2004; Papic, Mulligan, & Mitchelmore, 2011). Recent initiatives in early childhood mathematics education, for example ‘Building Blocks’ (Clements & Sarama, 2007), ‘Big Maths for Little Kids’ (Greenes, Ginsburg, & Balfanz, 2004), and ‘Curious Minds’ (van Nes & de Lange, 2007) provide research frameworks to promote ‘big ideas’ in early mathematics education.

Recent initiatives in early childhood mathematics assessment instruments highlight patterning and spatial skills moving beyond early numeracy (van Nes & de Lange, 2007). Thus in designing PASMAP and an accompanying assessment, we focussed on the relationships between children’s patterning skills, structural relationships and the big ideas in mathematics.

Assessment of early mathematical development

One of the limitations of traditional early mathematics assessment is the use of standardised instruments such as *I Can Do Maths* that do not enable the depth of analysis reflected by current research (Doig & de Lemos, 2000). Several effective assessment instruments and programs have been developed such as Mathematics Recovery (Wright, 2003) or interventions (Gervasoni, 2005). At system level, for example, the *Count Me in Too* Learning Framework in Number (NSW Department of Education and Training, 2002) provides support for the assessment and development of children's counting, arithmetical and measurement strategies. Broader measures of mathematical achievement for four to eight year olds have been developed including patterns and geometry, measurement and data (Clements & Sarama, 2007). However, no assessment instruments incorporate aspects of pattern or related features of mathematical structure.

Method

A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from diverse socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. At the follow-up assessment in September 2010, 303 students were retained. Two different mathematics programs were implemented: in each school, two Kindergarten teachers implemented the PASMAP and two implemented their standard program. The PASMAP framework was embedded within but almost entirely replaced the regular Kindergarten mathematics curriculum. The program focused on unitising and multiplicative structure, simple and complex repetitions, growing patterns and functions, spatial structuring, the spatial properties of congruence and similarity and transformation, the structure of measurement units and data representation. Emphasis was also laid on the development of visual memory and simple generalisation (for details see Mulligan, Mitchelmore, English, & Robertson, 2010). A researcher/teacher visited each teacher on a weekly basis and equivalent professional development for both pairs of teachers was provided. Incremental features of PASMAP were introduced by the research team gradually, at approximately the same pace and with equivalent mentoring for each teacher, over three school terms (May-December 2009). Implementation time varied considerably between classes and schools, ranging from one 50-minute lesson per week to more than 5 one-hour lessons per week.

Students were pre- and post-tested with *I Can Do Maths* (ICDM) (Doig & de Lemos, 2000) in February and December 2009, and September 2010; from pre-test data two 'focus' groups of five students in each class were selected from the upper and lower quartiles, respectively. These 190 students were interviewed by the research team using a new version of a 20-item Pattern and Structure Assessment (PASA1) in February 2009, a revised 19-item PASA2 in December 2009 (n=184), and the PASA2 and "extension" PASA in September 2010 (n=170).

Focus students were monitored closely by the teacher and the research assistant collecting detailed observation notes, digital recordings of their mathematics learning and work samples, and other classroom-based and school-based assessment data. These data formed the basis of digital profiles for each student. The Appendix presents an abridged version of two of three PASA assessment instruments.

The PASA assessment instrument

The assessment interview sought to complement interview-based numeracy assessment instruments such as the Schedule for Early Number Assessment 1 (SENA) (NSW DET, 2002) by extending counting and arithmetic strategies (addition and subtraction) to multiplicative reasoning. Thus many of the items (4, 5, 6, 9, 10, 11, 12), focused on multiple counting and patterning, the development of composite units and unitising, base ten structure, partitioning and multiplicative reasoning, and combinatorial thinking

(English, 1993; Mulligan & Mitchelmore, 1997; Thomas, Mulligan & Goldin, 2002). Related to these items were those on the structure of 2-dimensional and 3-dimensional arrays (Items 7, 8, 18) and measurement units (Outhred & Mitchelmore, 2000). The patterning tasks (Items 1, 2, 15) were based on simple repetitions and were extended to include an item integrating multiple counting and emergent functional thinking (Blanton & Kaput, 2005; Papic et al., 2011; Warren & Cooper, 2008). The subitizing tasks extended those in the SENA 1 (NSW DET, 2002). Items 13 and 14 were based on the notion that there are strong links between analogical reasoning and spatial patterning. Further, several items required students to draw and explain representations such as the structuring features evident on a clockface.

Discussion of results

In summary, both groups of students made substantial gains on the ICDM and PASA1 and 2 across the three assessments with PASMAP students' overall mean scores higher than the regular group. We focus here on the PASA item difficulty and the growth between pre- and post-assessment; Figure 1 shows the percentage of correct responses by item by assessment.

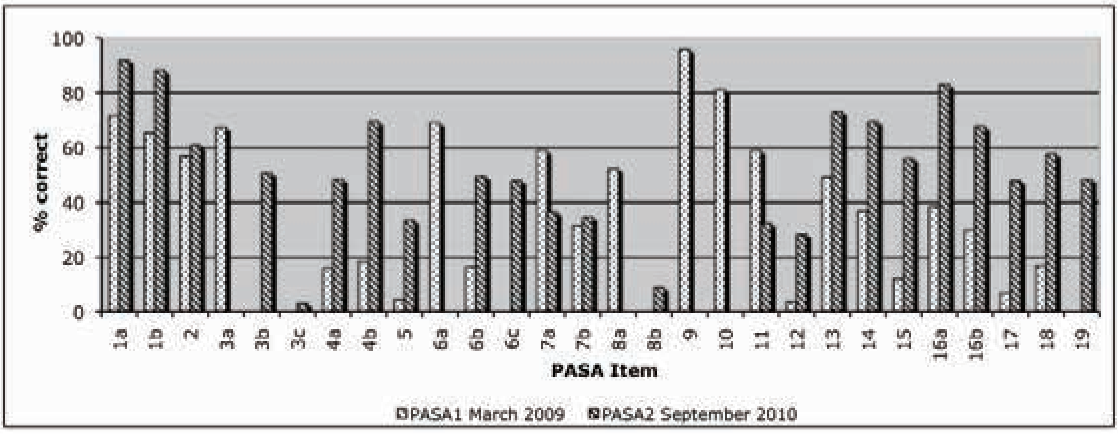

Figure 1. Performance on PASA1 and PASA2.

At the beginning of Kindergarten, 50% or more of students could correctly solve eleven of the 19 items. Most impressive was the students' ability to construct simple repetitions (Items 1 and 2), subitize (Item 3a), demonstrate halving (Item 6a), represent a 2×3 grid from memory (Item 7b), visualise units of volume (Item 8a), share by dealing and reformulate a share (Items 9 and 10) and use analogy to reason (Item 13). Items 9 and 10 assessed students' sharing strategies, which proved too easy for most students because of the simple context (6 between 2). Consequently the item was removed in PASA2. However, a more difficult partitioning item might reveal more complex strategies and provide further insight into students' development of multiple counting. The most difficult items were those items involving counting by twos (Item 4a), partitioning 50 (Item 5), quotition, representing with drawing from memory a triangular pattern (Item 15), a clockface (Item 17) and a grid pattern (Item 18).

At post-assessment there was marked growth in the responses to most items, particularly multiple counting (count by threes) and the related item (16) using count by twos and fours. Students participating in PASMAP accounted for much improvement shown in multiplicative tasks, possibly because the emphasis on skip counting and border patterns encouraged the development of composite units. Increased AMPS was reflected in their drawn responses to Items 15, 17 and 18. PASMAP students had opportunities to develop visual memory and the representation of structured units in many classroom activities. These students produced more structured representations earlier than students in the regular program. Further it was unexpected that students would use and explain the structure of units of measure for Item 19, i.e., the smaller the unit the more required. At post- assessment Item 3c proved unusually difficult. It was apparent that the students relied on unitary counting of a 5×5 square and ignored the structure. The patterning items were not sufficiently challenging, but it was important to examine evidence of understanding the unit of repeat because these patterning tasks were critical to assess children's underlying understanding of pattern and structure. We found that solution strategies were similar to those found by Papic et al. (2011) with pre-schoolers. At PASA1 the majority of children used direct comparison by copying the pattern model and matching blocks one to one from 'top down' or 'bottom up'. This strategy was replaced by an alternation, or unit of repeat, strategy by the second or third interview.

Students using an alternation strategy focused on successive items regardless of the complexity of the unit of repeat. The alternation strategy proved successful with simple repetitions, but with an increase in task complexity (e.g., an ABB repetition in PASA 2), it became ineffective. Students who identified the *unit of repeat* in the pattern model constructed the unit repeatedly showing some form of chunking (i.e., AB or ABB). They could then use the unit of repeat to extend the pattern; those who were successful typically first identified the unit of repeat and calculated the number of repetitions using the language of multiplication. For example, "I need blue, red, red, three times".

Categorising responses for stages of structural development

Analysis of qualitative data, tracking of the 'focus' students, indicated marked differences between groups in students' levels of structural development, Awareness of Mathematical Pattern and Structure (AMPS.) Students participating in the PASMAP program showed higher levels of AMPS than the regular group at post-assessment 3, made connections between mathematical ideas and processes, and formed emergent generalisations. Broadly, students' responses to particular items were categorised as in previous studies (Mulligan & Mitchelmore, 2009) as follows:

- *Pre-structural*: representations lack evidence of numerical or spatial structure.
- *Emergent (inventive-semiotic):* representations show some relevant elements but numerical or spatial structure is not represented.
- *Partial structural:* representations show most relevant aspects but are incomplete.
- *Structural:* representations correctly integrate numerical and spatial structural features.

We looked for evidence that a student had connected pattern and structure. An exemplar of students' developing structural features is now described. We drew on the qualitative analysis of a total of 600 drawn responses (Item 7) including approximately 10% as 'second attempts'. An independent coder categorised each response for level of structural development with reference to each interview script.

Figures 2 to 5 show typical examples of developmental features of students' AMPS in response to Item 7. In Figure 2 the student guesses the number of squares as "15" and draws single unit squares in a row (with some replication of shape) without 2-dimensional structure. Interestingly Figure 3 presents the groupings of 3 and 4 units of the grid as a border. Figure 4 shows the structure of the grid but additional units are provided, again showing "crowding". Figure 5 presents accurate alignment of a 3×4 array as the student explains the representation as "3 by 4" rows sequentially drawn.

Figure 2. Pre-structural

Figure 3. Emergent

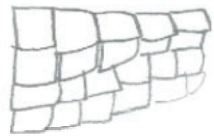

Figure 4. Partial

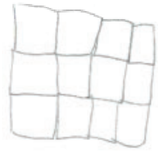

Figure 5. Structural

Conclusions and implications

The study shows that a program such as PASMAP that explicitly focuses on the promotion of students' awareness of pattern and structure (AMPS) certainly can achieve its aims. Particular gains were noted in the related areas of patterning, multiplicative thinking (skip counting and quotition) and rectangular structure (regular covering of circles and rectangles). It is not difficult to see how such understanding will be of value to students in their mathematics learning in Years 1 and 2.

As expected, a focus on pattern, structure, representation and emergent generalisation advantaged the PASMAP students. The advanced structural representations elicited at post-assessments reflected the learning that occurred during the program implementation. However, students in the regular program were also able to elicit structural responses but had not been given opportunities to describe or explain their emergent generalised thinking that may have been developing. It was not possible to determine whether more advanced examples of structural development could be directly attributed to the program impact. One of the most promising findings was that the focus students categorised as low ability were able to develop structural responses over a relatively short period of time.

Another aim of the project was to enhance teachers' mathematical content and pedagogical knowledge bases, including an understanding of young students' development, skills in assessing and documenting their learning. The participating teachers played a crucial role in the review of the PASA and the analysis of stages of development. Collaborative, sustained, and productive working relationships among school leaders, teachers and the researchers were pivotal to program implementation and the quality of the assessment and learning process. The underlying concepts and pedagogy required to implement a program of this kind are complex and these have not been central to traditional mathematics syllabuses or early mathematics learning programs. The PASA can enable a deeper and broader approach to assessment and serve to inform a much more challenging framework of mathematical ideas commensurate with young children's potential. It is anticipated that professionals will take on this approach with flexibility so that structural relationships across mathematical concepts will be considered seriously, resulting in more holistic and meaningful mathematics learning.

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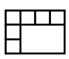
Appendix

Table 1. PASA1 and PASA2 assessment instruments.

Task	*PASA1 version*	*PASA2 version*
1	*Pattern: simple repetition* Show tower in an ABABAB pattern. Provide additional cubes. *Make a tower exactly the same as this one. What do you think comes next?*	*Pattern: complex repetition* Show tower in an ABBABBABB pattern. Provide additional cubes. *Make a tower exactly the same as this one. What do you think comes next?*
2	*Border pattern (ABAB)* Provide diagram of 3x4 border and 10 cubes (5 each of 2 colours). *Make a pattern on the border using these cubes.*	*Border pattern (ABCABC)* Provide diagram with 4x4 border and 12 cubes (4 each of 3 colours). *Do you have the right cubes to make a border pattern?Make a pattern on the border using these cubes.*
3	*Number/subitising* Flash 5 dot pattern card for one second. Hide from view. *How many dots did you see?*	*Number/subitising* Flash (2x5) array card for one second. Hide from view. *How many dots did you see?* Flash (5x5) array card for one second. Hide from view. *How many dots did you see?*
4	*Counting: multiples of 2.* *Count aloud by twos….two….* Provide numeral track (1-20). *Now count again and put a circle round the numbers as you go.*	*Counting: multiples of 3.* *Count aloud by threes….three….* Provide numeral track (1-21). *Now count again and put a circle round the numbers as you go.*
5	*Ten as unit* Show opaque box containing five 10c coins. *I've got some coins in this box. They are all 10c coins like this.* Show one 10c coin. *There is 50c in the box altogether. How many 10c coins are in the box?*	*Ten as unit* *Show opaque box containing ten 10c coins. I've got some coins in this box. They are all 10c coins like this.* Show one 10c coin. *There is $1 in the box altogether. How many 10c coins are in the box?*
6	*Length: halves and thirds.* Show 50cm paper streamer. *I need to cut this streamer into 2 pieces the same size. Where should I cut it?* *Now I need to cut this streamer into 3 pieces the same size. Where should I cut it?*	*Length: thirds and quarters* Show 50cm paper streamer. *I need to cut this streamer into 3 pieces the same size. Where should I cut it?* *Now I need to cut this streamer into 4 pieces the same size. Where should I cut it?*
7	*Visual Memory Grid (2x3 in PASA1 and 3x4 in PASA2)* Provide Student Recording Sheet, pencil and eraser.	

	I'm going to show you a card quickly. Look carefully and tell me how many small squares there are. Show Grid Card face up for one second. Cover. *Draw exactly what you saw.*	
8	*Visualisation/Volume/Unitising* Provide net of open box (2x2x1 – PASA1 and 2x2x2 – PASA2) and one multilink cube. *Imagine this shape folded up to make a box. How many cubes like this would fill the box without any spaces left?*	
9	*Sharing* Provide 2 teddies and 6 counters. *Share all these biscuits between the 2 teddies. Make sure each teddy has the same.*	No equivalent task in PASA2
10	*Sharing: reformulation* Provide an extra teddy. Now share these biscuits between the 3 teddies.	No equivalent task in PASA2
11	*Combinatorial: multiplication* Provide a card showing outlines of 4 teddy bears, cut outs of 4 tops (in 2 colours) and 4 pants (in 2 colours). *How many bears can you dress so that no bears are dressed the same?*	*Combinatorial: multiplication* Provide a card showing outlines of 8 teddy bears, cut outs of 8 tops (eg. 3 red, 3 pink & 2 purple) and 8 pants (eg. 4 blue & 4 green). *How many bears can you dress so that no bears are dressed the same?*
12	*Quotition: division* *I have $10 in coins. I want to give some children $2 each. How many children can I give $2 to?*	
13	*Analogical Reasoning* *Your hand goes with your arm in the same way as your foot goes with your ….*	
14	*Analogical Reasoning and Transformation* Place the card with three arrows in front of the student. Make sure "TOP" is up. *Show me which way you think the arrow will go next? And which way after that? Can you tell me why you think that?*	
15	Provide Student Recording Sheet & pencil. *I'm going to show you a pattern on this card quickly*. Flash triangular 6-dot pattern for <u>one</u> second. *Draw exactly what you saw*. (PASA2 addition) - *Draw what you think comes next*.	
16	*Picture graph: functional thinking* Show student card with four dogs briefly. Ask, *How many ears altogether on 1 dog? 2 dogs? 3 dogs?* whilst uncovering each dog. Leaving 4th dog covered ask *a) How many ears on 4 dogs altogether?* Cover card. Repeat process with "legs" up to 3 dogs without revealing 3rd dog. *b) How many legs on 3 dogs altogether?*	
17	*Time: analogue clock face (hour)* Provide Student Recording Sheet and pencil. Someone started drawing a clock, could you finish it for me?	
18	*Area: unitising*  Provide Student Recording Sheet and pencil. *Someone has started to draw some small squares to cover this shape.* (Point to whole shape.) *Finish drawing the squares*. (Point to the space.)	
19	No equivalent task in PASA1	*Volume:* Show 3 cups of varying size. *How many small cups of water are needed to fill the big cup? Medium sized cups?Why?*

USING INSIGHTS FROM MYERS-BRIGGS TYPE PREFERENCES TO SUPPORT EARLY CHILDHOOD PRE-SERVICE TEACHERS' PERSONAL AND PROFESSIONAL MATHEMATICAL UNDERSTANDING

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This paper outlines an intervention to support pre-service teachers without prejudice in a first year mathematics unit, *Personal and Professional Numeracy*. The unit aims to develop and/or build on personal and professional numeracy understanding. Mathematics anxiety in pre-service teachers has been reported in the literature and might be linked to some students' low success rate or their withdrawal from the unit. An intervention is proposed to assist students gain an understanding of their personal learning preferences using descriptors identified by the Myers-Briggs Type Indicator (MBTI). This intervention aims to use self-understanding to change behavioural outcomes.

Introduction

The University of Tasmania (UTas) Faculty of Education has an objective of increasing the success and retention of first year students enrolled in its compulsory mathematics unit. This reflects the significance of mathematics performance as a national priority, and the need for pre-service teachers to be able to demonstrate required levels of mathematical understanding prior to their employment as teachers. Watson (2011) provides a comprehensive description of the design of ESH120 *Personal and Professional Numeracy*, a first year, first semester unit in a re-conceptualised four year BEd (EC/Primary) course introduced at UTas in 2010. The unit precedes two mathematics education units offered later in the course and targets specific mathematical understandings for personal and professional proficiency in mathematics. ESH120 is delivered both on-campus and online.

In response to a relatively high attrition rate, and low success rate of some cohorts of students studying ESH120 in its initial offering, a team outside of the ESH120 teaching team has designed an intervention being introduced into the unit in 2011. The purpose of the intervention is to raise students' self efficacy in mathematics by enhancing their levels of self-understanding. It is anticipated that the intervention will contribute to students' success in ESH120 and ultimately their capacity to teach mathematics in educational settings.

Delivered during the second week of semester, the intervention involves students self-selecting to engage in a series of tasks and reflections outside of their ESH120 lectures and tutorials. It provides students with the opportunity to determine their Myers-Briggs Type Indicator (MBTI) profile (Briggs & Briggs Meyers, 1998) and gain personal insights that may enhance their success in the learning activities required in ESH120. The students are directed to Reinhold (2006), which provides a free and online opportunity to determine their MBTI profile. The intervention, conducted on-line, delivers:

- information about MBTI and learning preferences for different personality types
- information on mathematics anxiety
- strategies to use MBTI to overcome mathematics anxiety
- access to an online facility for engaging with one another and with intervention team members.

Mathematics anxiety and pre-service teachers

Mathematics anxiety has been described by Smith and Smith (1998) as involving feelings such as intense frustration or helplessness when confronted with the need to undertake mathematics activities. This, they argue, is a learned emotional response. While mathematics anxiety is evident in the wider community, Gresham (2007) notes that high levels of mathematics anxiety occur among pre-service teachers. Bursal and Paznokas (2006) reported that a significantly larger percentage of pre-service teachers experienced higher levels of mathematics anxiety than other undergraduate university students.

Mathematics anxiety in pre-service teachers not only affects their learning in mathematics units throughout their course, but also leads to doubts as to their potential effectiveness in teaching mathematics to children (Gresham, 2007). These doubts are well founded. Uusimaki and Nason (2004) support the notion of a flow-on effect of mathematics anxiety after pre-service teachers graduate. They suggest that teachers' beliefs (about mathematics) play a major role in their students' formation of beliefs towards mathematics. Gresham (2007) notes that educators such as Vinson argue strongly that teachers transmit their avoidance and fear of mathematics to their students. There is a case then for taking action in pre-service teacher education courses.

The importance of taking action to alleviate mathematics anxiety is even more critical for particular cohorts of pre-service teachers. Perry (2000) singles out early childhood pre-service teachers as needing additional support. He notes that this cohort, who are mainly women, often have low level mathematics skills and hold negative attitudes to learning mathematics, and notes that teacher education programs do not alleviate these students' deficiencies. Perry recommends addressing the continuing low levels of competence in, and attitudes towards mathematics and mathematics education in the early childhood sector. He advises that it is during the early childhood years that children's foundational attitudes towards mathematics are nurtured. Confidence and competence in pre-service early childhood teachers' own mathematics abilities, Perry argues, will be carried through to enhance the in-depth teaching of mathematics to young children.

Addressing mathematics anxiety in pre-service teachers

Trujillo and Hadfield (1999) present the causes of mathematics anxiety in three categories: personality factors, environmental factors and intellectual factors. Research into these causes of mathematics anxiety can and has been used to inform the teaching of mathematics in pre-service teacher education courses (Breen, 2003). Specifically targeted interventions are often designed to reduce or alleviate particular aspects of mathematics anxiety. For example Gresham's (2007) pre-service mathematics teaching model addresses intellectual factors such as lack of persistence and lack of confidence in mathematical ability. Gresham adopted Bruner's learning theory to get students to attack mathematics problems symbolically by first having them engage with concrete, semi-concrete or pictorial activities. She argued that "while mathematics anxiety can be reduced by establishing a supportive classroom environment, using manipulatives to bridge concrete to abstract learning is important to address students' attitudes towards mathematics" (p. 183). Gresham provides a comprehensive range of teaching and learning strategies that can inform pre-service teacher mathematics educators.

Breen (2004) adopted the use of journals as a reflective process for students to address their fear of mathematics by tapping into their metacognition. This intervention follows on from his earlier (2003) discussion of theoretical principles underpinning an enactive approach to addressing mathematics anxiety. There is a role, he says for hermeneutic listening which involves both student and teacher being mutually engaged in a shared project, in this case, sharing fears and insights through journal writing. Callingham and Falle (2010) confirm that the development of language skills which focus on mathematics understanding and performance contributes to mathematics self-efficacy.

Interventions such as those described above add to the knowledge base and tools for tertiary mathematics educators. Tobias (1998) contends that the root of some mathematics anxiety lies in how one is taught mathematics. This is particularly significant since teachers are inclined to teach just as they were taught (Furner & Berman, 2005). Watson (2011) describes the range of approaches used in the teaching of ESH120. Some of these mirror the strategies used by Gresham and are aimed at constructivist learning. Constructivist teaching and learning strategies result in higher levels of student engagement with mathematics activities, concepts and problem-solving (Gresham, 2007). Gresham also suggests that students often take constructivist strategies used in their pre-service education units into the teaching of mathematics to children.

Trujillo and Hadfield (1999) provide identifiers related to mathematics anxiety causal factors. These include:

- personality factors such as reluctance to ask questions due to shyness, low self esteem, and, for females, viewing mathematics as a male domain;
- intellectual factors such as being taught with mismatched learning styles, student attitude and lack of persistence, lack of confidence in mathematical ability, and the lack of perceived usefulness of mathematics
- environmental factors such as negative experiences in the classroom, parental demands, insensitive teachers, and according to Idris (2006) the use of traditional teaching methods.

Such identifiers help to target the design of interventions for pre-service teacher mathematics education. The proposed intervention intends to use MBTI to address the causal personality and intellectual factors contributing to students' mathematics anxiety.

The Myers-Briggs Type Indicator (MBTI)

MBTI (Briggs & Briggs Meyers, 1998) is a well-known and readily available, personality style instrument that can help students identify their personality characteristics. The MBTI is based on four dichotomous preferences: extraversion/intraversion (E-I), sensing/intuition (S-N), thinking/feeling (T-F), and judgment/perception (J-P). These four preference scales describe focus of attention, acquisition of information, decision making and orientation towards the outer world. Sixteen different four-letter combinations result from these categories. In practice the determination of type consists of three stages which include using information from the scored inventory, participating in an MBTI facilitated session and confirming type by reading type summaries. (See the Appendix to this paper.).

Personality-type preferences can have a determining effect on learning and learning styles (Irani, Scherler, Harrington & Telg, 2000), and the assimilation of new knowledge (Kiersey & Bates, 1984). Overbay, Gable, Oliver, and Vasu, (2006) noted that studies by Lennon and Melear (1994) indicate that personality, as measured by the MBTI, can be used as a predictor of instructional preference.

MBTI has been used to support learning in universities. Lynch (2001) provided students with access to psychological type and learning style inventories prior to their participation in a discussion board. Students were asked to reflect on whether their problem-solving in response to scenarios was affected by their learning style and psychological type. Lynch describes positive outcomes from this meta-cognitive intervention, noting that students identified ways in which they were better able to take control of their own learning. Many reported that becoming aware of their learning style and psychological type increased their self esteem and their confidence as online learners (Lynch, 2001). Another university study by Irani, Scherler, Harrington and Telg, (2000) demonstrated that personality type affected student perceptions of the instructional techniques used in online learning and noted that personality type also affected student performance.

These studies suggest that the MBTI can be acknowledged as one way for adult learners to explore or become more aware of their personal preferences and learning styles.

The UTas ESH120 Intervention

In 2010 ESH120 *Personal and Professional Numeracy* had a total of 624 enrolments at the beginning of the semester: Of these 226 were internal students and 398 were external/on-line students. By the end of semester, 36 students had withdrawn from the unit (Downing, 2010). Downing notes that 68% of those who withdrew identified mathematics anxiety as a contributing reason. She observes that in their additional comments to her survey, students described their fear of mathematics, their lack of confidence and their feeling of being overwhelmed or confused by the learning materials. One response quoted by Downing represented the feeling of others.

> I felt there was no system in place to offer a bridge to people who haven't studied math for over 15 years and are not confident. Felt a lot of pressure was on MATHEMATICS being crucial to teaching (WHICH IT IS) but to have to perform to high school level when I only intend to teach kinder or prep made me feel sick and put me off [uppercase original]. (Downing, 2010, p. 11)

The instructional design for the ESH120 unit described by Watson (2011) reflects strategies necessary to address the environmental aspects of mathematics anxiety. The unit makes use of a variety of UTas specific online technology including *MyLO*, *PebblePad* and *Lectopia* recorded lectures, web-based mathematics activities and other learning related programs; Watson (2011) describes how these were used. Self-paced and scaffolded learning in authentic contexts was actively promoted throughout the conduct of the unit (Watson, 2011). Downing's (2010) study generally acknowledges the effectiveness of the learning processes in ESH120. However, her study also highlights the need to provide additional support for the personality and intellectual factors that cause mathematics anxiety.

The MBTI-based intervention is designed to be accessed by all students in ESH120. A folder set up in week 1 of the semester is located on the front page of the ESH120 *MyLO* site and contains *PowerPoint* slides introducing the intervention, reasons for its development, its benefits to students, an invitation to participate and procedures for participating the following week. In week 2, students were invited to:

- Complete an online MBTI instrument, taking note of their four letter type.
- Watch a facilitator-led *Lectopia* presentation. The presentation by an accredited MBTI facilitator assists students to interpret their inventory results and explain the relevance of personality type and learning styles to engaging in ESH120. The video also provided a range of insights into ways in which students can draw upon their preferred MBTI strengths and accommodate their least preferred dimensions.
- Access the MBTI *PebblePad* which provides a range of readings about MBTI and mathematics anxiety, and engage in a Discussion Blog where insights and conversations about MBTI learning styles, mathematics anxiety and other related learning matters can be shared.

Downing (2010) recommends that online students be scaffolded and supported and points to the need to create a community. She also highlights the need to act early in the semester to engage students. Participation in the intervention is voluntary and confidential. Lecturers in ESH120 *Professional and Personal Numeracy* are not given access to the *PebblePad* Blog in order to ensure that students could speak freely about their participation in the unit. The *PebblePad* also serves as a communication forum in which students can reflect on their progress in ESH120, learn from other students about their insights into their learning styles and strategies, and the researchers can post relevant readings and other timely support materials. The intention is that the intervention will proceed in a developmental transformational way throughout the semester.

Limitations

The MBTI relies on students self-reporting their behaviours and attitudes. In this respect, students may want to present themselves in the best possible light. However students were advised of the personal benefits of recording their MBTI answers and Blog postings honestly. Confidentiality requirements comply with the UTas ethics protocols.

The MBTI uses a dichotomous scale and has been criticised in the literature in this respect (Pittenger, (2005). Researchers such as Overbay, Grable, Oliver and Vasu (2006), note that there is debate over whether MBTI actually measures "type" which differentiates over time, or "traits" which can be modified through training. The ESH120 intervention is conducted over a semester and has well-targeted aims. It focuses largely on the descriptors of each dimension of personality type, giving participants the opportunity to assess for themselves which types they most resonated with. The inventory was used in a constructive manner in order to raise awareness amongst students of the many different ways in which people take in information and relate to the world. Students are given the opportunity and information to explore MBTI and its application to teaching and learning for themselves.

Conclusion

At the most basic level, this intervention is a call to attention. The premise of the intervention is that awareness of personal preferences and learning styles can reduce mathematics anxiety amongst pre-service teachers. This will enable them to engage with mathematics in ways that increase their mathematics self-efficacy. It targets largely unexplored mathematics anxiety causal factors and in so doing may provide additional insights for tertiary mathematics educators in pre-service education courses. During the semester it is expected that the intervention will take on the form of an action research project by documenting the continually changing strategies that are required to engage with, learn from, and support students in their reflective activity. The MBTI intervention in ESH120 draws upon some of the enactive principles proposed by Breen (2003). As the information base for its development expands, other theories and research will add further insights.

It is anticipated that particular student cohorts may require attention. Perry's (2000) quote from an early childhood pre-service teacher echoes that provided by Downing (2010) "What do you mean – I have to do two mathematics units! I chose early childhood teaching because I couldn't handle primary mathematics." (p. 32). Some students feel that there seems to be no point undertaking mathematics learning beyond a rudimentary level. The *Maths? Why Not?* study by McPhan, Morony, Pegg, Cooksey and Lynch (2008) has the potential to inform what needs to be addressed to enhance these students' willingness to continue to pursue mathematics learning at higher levels:

- self-perception of ability;
- interest and liking for higher-level mathematics;
- perception of the difficulty of higher-level mathematics subjects;
- previous achievement in mathematics; and
- perception of the usefulness of higher level mathematics.

The study focused on secondary students' unwillingness to pursue higher level mathematics subjects. In the case of some early childhood and other pre-service teachers, however, higher levels of mathematics are but one step beyond rudimentary. This may be a major contributor to mathematics anxiety in units related to personal and professional numeracy. Pre-service teacher education courses that address the points noted by McPhan et al., (2008) above may have more chance at alleviating mathematics anxiety among cohorts of students most in need of attention. The challenge for the MBTI intervention will be to incorporate and address further causal aspects of mathematics anxiety into its design and delivery.

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Appendix: Learning preferences associated with the four dichotomies of MBTI

Adapted from Silver, Strong and Perrini (2000).

Extraversion (E)

Direct energy outward

"E" learners prefer:

- Talking, discussion
- Psychomotor activity
- Working within a group

Intraversion (I)

Direct energy inward

"I" learners prefer:

- Reading, verbal reasoning
- Time for internal processing
- Working individually

Sensing (S)

Take in information through the five senses

"S" learners prefer tasks that call for:

- Carefulness, thoroughness and sound understanding
- Going step by step
- Observing specifics
- Recall of facts
- Practical interests

Intuition (N)

Take in information through hunches and impressions

"N" learners prefer tasks that call for:

- Quickness of insight and in seeing relationships
- Finding own way in new material
- Grasping general concepts
- Imagination
- Intellectual interests

Thinking (T)

Make decisions based on logic and objectivity

"T" learners prefer:

- Teacher's logical organisation
- Objective materials to study
- Depth and accuracy of content

Feeling (F)

Make decisions based on personal values and the effects on others

"F" learners prefer:

- Personal rapport with teacher
- Learning through relationships
- Personal connection to content

Judging (J)

Prefer structure, plans and achieving closure quickly

"J" learners prefer:

- Working in steady, orderly ways
- Formalised instruction
- Prescribed tasks
- Driving toward completion
-

Perceiving (P)

Prefer flexibility, spontaneity and keeping options open

"P" learners prefer:

- Working in flexible ways
- Following impulses
- Informal problem solving
- Disovery tasks

PROFESSIONAL REFLECTION AND DEVELOPMENT: MATHEMATICS TEACHER EDUCATION LECTURERS AND BEGINNING TEACHERS

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Jaworski (2003) extended Lave and Wenger's (1991) idea about learning to teach through being engaged in a 'community of practice' to include teachers and researchers collaborating in a 'community of enquiry' as they investigate their own practice. Most beginning teachers feel overwhelmed and it can be a very intense time of reflection as they develop the habits and skills of good teaching. This paper describes the professional reflections of four participants when two university lecturers each made weekly visits to the classroom of a former university student to observe a lesson, followed by a time of shared reflection and planning.

Introduction

In recent years, the work of Lave and Wenger (1991) has proved helpful to researchers in understanding how teachers come to know and learn about the practice of teaching. Learning is not so much concerned with replicating the performance of others or acquiring knowledge transmitted through instruction, but rather occurs through becoming part of the community and having access to a wider range of ongoing activity in its practice (Cavanagh & Prescott, 2007).

While university professional experience programs are set up to allow pre-service teachers to imagine possibilities beyond traditional norms and experiment with new ways of teaching, often their identity formation is compromised by the disjointed nature of their university and school-based programs. The tasks of engagement, imagination, and alignment (Wenger, 1998) become more complex and problematic. As well, supervising teachers often see their role as giving advice about the practical concerns of classroom routines and organisation rather than developing the pre-service teachers' reflective pedagogy (Prescott & Cavanagh, 2008a,b). As a result pre-service teachers sometimes struggle to engage meaningfully in what appear to be two separate communities of practice that are, in many respects, at odds with each other.

Consequently, when beginning teachers arrive in a school, they want to be seen as effective in the classroom by their supervisors and their students (Kardos & Moore, 2007) so they emulate their colleagues and adopt what they perceive as the safer option of relying on the textbook (the more traditional approach) rather than the reform-oriented approach encouraged in their university course.

Background

Mentors play a key role in supporting beginning teachers to become active agents in analysing and improving their own practice and, in doing so, develop their identity as teachers. Mentors play many different roles—model, coach, supervisor, helper, guide, supporter, facilitator, observer, critical friend—helping beginning teachers to develop their mathematical and pedagogical skills, while at the same time developing their own educative skills (Wang 2009).

Muir and Beswick (2007) note that critical reflection is unlikely among teachers in the absence of an external voice that can serve to challenge current ideas and practices. In the case of a teacher education academic mentoring their former students, the external voice is also external to the school. The academic's knowledge of the culture of the school and the community of practice at the school (Cavanagh & Prescott, 2007) is limited but the benefits are that the academic can be the 'honest broker' in discussing teaching and classroom interactions.

Waghorn and Stevens (1996) discuss the lack of communication between education research and teacher decision-making. Research reveals the complexities of what goes on in the classroom and therefore has much to offer the beginning teacher about current best practice. When the research is undertaken as collaboration between beginning teachers and university academics it becomes a powerful tool for improving theory and its implementation in practice (Potari, Sakonidis, Chatzigoula, & Manaridis, 2010).

Blase (2009) discusses various ways teachers can be mentored. These can be seen as:

- a model of transmission in which the mentor transfers his/her knowledge about teaching to the teacher;
- a model of transformation in which mentors assist teachers in understanding school culture and teaching in order to reform classroom instruction, school development and community work;
- mentors and teachers practising unquestioned teaching strategies;
- mentors and teachers taking a reflective stance in carefully considering and reconstructing their knowledge of teaching.

The last view of mentoring is the most relevant to this project, continuing our work with pre-service and beginning teachers (Cavanagh & Prescott, 2007, 2009, 2010; Prescott & Cavanagh, 2008a,b). The collaboration can be described as "partners in an enquiry process of learning and teaching mathematics, holding separate but not incompatible roles. In particular they are seen as insiders or outsiders to the teaching practice, both acting and reflecting on it, each informed by his/her own practice, both learning about teaching" (Potari et al., 2010).

This paper describes the early stages of a research project between two university academics and their former students where we undertake to close the gap between the current best practice that our students gain from their university course and what is happening in the classroom. We set out to answer the following questions. Can a beginning teacher benefit from mentoring by their teacher education lecturer? What does a teacher education lecturer gain from mentoring a former student?

Methodology

Tania is in her third year of teaching at a private girls' school in metropolitan Sydney. Anne began observing Tania in Term Four of 2010 with her Year 8 mathematics class and in 2011 with her Year 10 mathematics class. Frederic is in his fourth year of teaching. He taught for two years in a private coeducational school in Sydney's North West, and has been at his current school since the start of 2010. The school is a Catholic systemic school in the outer western suburbs of Sydney. Michael first visited Frederic in Term Four of 2010 to observe his Year 9 mathematics class, one of two classes in the top stream. In 2011, Michael has observed the same class, now in Year 10. Each academic makes regular visits, usually once per week, to spend time in the classroom, taking observation notes about what is happening in the lesson. After each observation lesson, the teacher and academic discuss what happened in the lesson, looking for patterns and differences, and seeking a focus for future observations and discussions.

Results

It is important to recognise that each of us interprets the observations and interactions differently so we will each provide our own reflections on our initial impressions of the collaborations we have begun.

Anne's comments

When I visit schools on practicum, I am well aware that many teachers know that I have not taught in a school for a while so they may feel that my knowledge of teaching is out of date. The implications, of course, are that I have little to offer my pre-service teachers and that teachers in the school are the ones who know what teaching is really about. Being a part of Tania's class allows me to be involved with the students' learning. My observations include classroom management, content knowledge, and questions that arise about students' learning, the mathematics, and, often, the errors that students have made. On more than one occasion, Tania has used a different method from me for teaching a topic and I have found myself looking at the benefits of each method. These observations have become the basis of our discussions after the lesson and form the background for her subsequent lessons with the class. They have also allowed me to think about my own teaching practice and I have been able to look anew at some topics and offer suggestions for a different approach with other student teachers.

I have found myself undertaking many of the roles described by Wang (2009) but probably the most important role has been in giving Tania the confidence to believe that she is an excellent teacher who has much to offer her students. Beginning teachers tend to obsess about the relatively small number of things that go wrong rather than the myriad of things that have gone well (Prescott & Cavanagh, 2008b).

Even experienced teachers know how difficult it can be to begin the year with a new class. Beginning teachers are particularly concerned about how to establish themselves in the classroom and Tania was no exception. We discussed how she might approach her Year 10 class and then I attended her first lesson with them this year. She had taken over a Year 10 class the previous year and struggled to establish a rapport with the students, so was keen to work positively with the class this year.

One of Wang's mentoring roles was as a critical friend. While Tania's school has been very supportive of her as a beginning teacher, and she has been mentored by one of the mathematics teachers, the role of critical friend allowed Tania to voice concerns and beliefs that she knew were private and would stay that way. Even with mentoring, beginning teachers feel alone. They are expected to be independent from the start and so find it hard to ask for help so they do not appear to be floundering (Kardos & Moore-Johnson, 2007). The tension between looking for support and being regarded as an effective teacher can be problematic for beginning teachers—a critical friend who is outside the school allows those concerns to be voiced.

Tania is very organised in class; she uses an electronic whiteboard to great effect and has an easy style of teaching with the students who respect her and enjoy her classes. Tania obviously cares for the students she teaches and the consequent rapport she has with the class is wonderful to see in action.

Tania's comments

As a beginning teacher I am on a very steep learning curve. Any opportunity of assistance is to be embraced, so I was excited to have Anne watch my classes and give me feedback on my teaching practice. In my third year of teaching I have gained some understanding of what I need to do to be an effective teacher. It is a much more complex job than I thought when I started which is both exciting and daunting. As I look back over the past two years I see how much I have learnt. As I look forward I see how much more there is to learn.

There have been several benefits of Anne sitting in on my classes, but the main one has been to receive detailed feedback on my teaching practice which has enabled me to improve. It is helpful to have an observer commenting on how I am connecting with the class. Am I getting through to them? Am I pitching at the right level? Am I going too fast or too slow? Anne had plenty of positive observations which was very encouraging. For example, when I introduced the quadratic formula, I showed a video so the students could learn to sing the formula making it easier to remember. Anne wrote in her comments "Videos are a fun introduction." and "While they're copying the formula, they're singing it!". She also gave constructive criticism. For example, she noted that I could have used the warm up questions already solved before introducing the quadratic formula to help the students see that it works. Her written observations are a good starting point for further reflection.

Teaching mathematics can be very intricate and I was looking forward to receiving technical feedback. What should I emphasise and what not? What is a good way to teach change of units, for example? During one class, I reviewed conversion of linear units, square units and cubic units using multiple diagrams, as students often have difficulty converting square and cubic units. Anne suggested I supplement this approach with a picture as well so that the students can understand more of the relationship between length, area and volume. This would also aid the visual learners. Another question which arose is what are the common errors of my students and how can I help correct them? During a lesson on quadratic equations, Anne made a list of common errors she observed as she moved around the class. A few weeks later I used this list to help the students revise the topic by creating a "Spot the Error" exercise. It was very fruitful revision and it was interesting to observe that some students knew what their typical mistakes were.

Anne was also a second pair of eyes and ears in class. There have been incidents in class I have been completely unaware of. For example, a student got up out of her seat and walked to the door and I didn't notice because I was helping another student. It is excellent to become aware that I don't see and hear everything that happens. I have tried to be more vigilant during the times when the class is working independently by doing a quick scan of the room regularly and helping students to stay on task.

I am striving to become an excellent teacher and so have high expectations of myself. I want my lessons to run smoothly with the students learning as much as possible. At times it doesn't work out that way. Anne has been a great source of support and has helped to build my confidence. I am fortunate to work in a very collegiate staffroom and supportive school. Nevertheless, it has been extremely helpful to have an external person helping me deal with the inevitable difficult situations which arise from time to time. I feel like Anne is in my corner of the ring.

The only difficulty has been that it is nerve wracking to have someone watching me teach. As time goes by I am more relaxed with Anne in the classroom, but I am still conscious that she is there. At first I made very sure that I was thoroughly prepared for each class with Anne. Now, I do my usual preparation, which at times is not as thorough due to other classes or school pressures. I feel that after two terms, Anne is seeing the real me without any (or at least not much) difference from how I teach all my classes.

Michael's comments

My primary goal in collaborating in professional dialogue with Frederic has been to learn more about how beginning teachers develop their classroom practice and their capacity to reflect on the lessons they have taught (Bean & Stevens, 2002). In particular, I am interested in the extent to which early-career teachers are able to implement student-centred approaches in the classroom. I clearly remember observing Frederic's lessons while he was still at university and noting his deliberate attempts to design meaningful activities that challenged students to think mathematically. I also remember talking with him during his first year after graduation as he shared some concerns about the extent to which he was able to engage students in their learning. Now I am struck by his confidence in the classroom and the careful planning that is evident in the way he structures his lessons.

As an academic working in teacher education, I have observed many mathematics lessons from pre-service teachers and participated in numerous post-lesson discussions with them, but observing and discussing Frederic's lessons is quite different. Whereas pre-service teachers tend to dwell almost exclusively on their own actions, Frederic is far more concerned with student learning. Frederic's lesson preparation, his classroom practice, and his personal reflections have a strong focus on improving student learning outcomes. I have also been struck by the quality of Frederic's reflections on his lessons. His comments demonstrate how adept he is at noticing (Mason, 2002) how students are working and he can clearly articulate probable causes to explain why students sometimes do not progress in the way he had anticipated. Frederic generally takes the lead in our deliberations. He points out where students have experienced difficulties trying to understand a new concept and how he has adapted his lesson to deal with unexpected outcomes. It is interesting to compare Frederic's reflections with my own observation notes to see how we sometimes consider different causes for student misconceptions and proffer different remedies for them.

The other interesting story that has begun to emerge is how Frederic's career has evolved in the just over three years he has been teaching. There have been significant highs and lows and Frederic has reported that there were times earlier on when he considered leaving the profession because he did not feel he was achieving some of the goals he had set for himself. I am looking forward to exploring these ideas further and examining in sharper detail how he overcame his doubts and developed his resilience.

The research partnership with Frederic has helped me to reflect on my own teaching and has provided me with rich descriptions of classroom incidents that I can share with my own students. I have found it particularly advantageous to present my pre-service teachers with an episode from one of Frederic's lessons and ask them to consider what they might have done in a similar situation. I am then able to report how Frederic has acted and the results of his actions. These reports add a certain authenticity to my tutorials.

Frederic's comments

The first few years as a teacher are very hard for many reasons. You have to learn and prepare the content, but also need to learn the whole relationship part of the profession. University does not prepare you for the classroom and the fact that teaching can be very challenging mentally, physically and psychologically. I am always looking for opportunities to improve and become better at the art and science of teaching. When Michael contacted me to be part of the study, I was really pleased and at once realised that accepting the proposal would help me become a better teacher.

I am used to team-teaching and having someone observe my lessons. However, having a visitor such as Michael always adds a little bit of pressure. I did not change the way I taught or the content of my lessons but I clearly wanted to do well and rise to the occasion. The first lesson that Michael observed was a real success.

My lessons often start with quick questions in order to warm up and be ready for the new content which will be learnt during the lesson. I have started to link these quick questions more explicitly with the new content. As an example, if the key idea of the lesson is to solve a quadratic equation like $x^2 + 3x + 2 = 0$ by factorising, the quick questions will contain a specific example of factorising a quadratic expression such as $x^2 + 3x + 2$. Michael really liked the idea and his approbation and praise encouraged me to use, refine and extend this method. It worked really well in the most difficult part of the algebra topic because, after the quick questions, students only had to focus on the new content for that lesson.

I have also noticed that I can transfer the ideas that I discuss with Michael into lessons with other classes. One of these is a more generous use of praise. After a particularly good lesson, Michael started a debriefing by praising my teaching and it made me feel good. A few minutes later, he introduced the fact that everyone needs a little praise. I could relate to it straight away. It also made me reflect on the fact that Michael used the idea of praising me in order to teach me that praise is important.

Having an external view on my lessons has proved to be very helpful too. Michael made the remark that my class was not particularly strong (I have one of the two high ability Year 10 classes). He also said that I was very patient with weak students. Michael has also praised the scaffolding I have used on the board and this has encouraged me to keep it and improve it.

I have seen noticeable improvement from my middle and top students, using all of the strategies described before. However, I am still looking forward to helping my weaker students in a better way and that is an area I want to discuss more with Michael.

Michael has told me that the way I teach now is as good as an experienced teacher. I know that he is looking forward to me trying new ways of engaging the class, including lessons more focused on Working Mathematically and more student-centred. We had a very interesting discussion on this subject. During university, pre-service teachers learn and create lesson plans and activities which are more focused on Working Mathematically because engaging lessons can limit classroom misbehaviour. It all makes sense in theory and I was therefore very enthusiastic when I started my first year of teaching four years ago. However, when starting as a teacher, I found that this is not really the case. It took me a long time to realise that classroom management, respect and good rapport with students must come first if I want to have a chance of successful student-centred lessons. Michael noticed that I have such a good relationship with my students in Year 10 and that it could be time to come back to using such approaches in my lessons. I am looking forward to practising these ideas in the new topic on Measurement. I have decided to use an open-ended assignment in order for my class to learn the formulae to find the surface area of pyramids, right cones and spheres. Students will have to work individually or in pairs for about ten days to create a *PowerPoint* presentation, a movie or a *Prezi* presentation. At the time of writing, students have only just begun the project, but already I can see that they are motivated by having a choice and engaged by a new way of studying a mathematics topic.

Discussion and conclusion

There are some common themes which have begun to emerge from our collaborations. Anne and Michael have noted that their classroom observations have provided a unique opportunity to reflect on their own practice. The quality of the teaching is superior to what is typically seen from pre-service teachers so we can focus on the subtleties of teaching and examine student learning in greater detail. The research project has provided a rich source of authentic classroom episodes and teacher reflections that have already proven to be an extremely useful resource for discussions in university classes.

Anne and Michael have been reminded of some of the struggles that beginning teachers must deal with as they learn to become effective classroom practitioners. As university academics they have both been struck by how even minor setbacks can have a disproportionate impact on the self-confidence of a novice. Their work in the project has shown how an external voice of encouragement and support can help overcome some of the doubts that new teachers will inevitably experience when a lesson does not go according to plan or when a teacher-student relationship becomes strained.

Both Tania and Frederic have commented on the significant learning that is entailed in becoming a teacher and have suggested that their university studies did not fully prepare them for the classroom. Both teachers are dedicated to self-improvement and want to develop into the best teachers that they can be. So, despite some anxiety about having their lessons observed they both eagerly accepted the invitation to join the project because they recognised the benefits of participating in a regular cycle of action, discussion and reflection.

In discussions following their lessons, Tania and Frederic have demonstrated a remarkable ability not only to identify critical learning and teaching incidents but also to analyse some likely causes and suggest remedies for dealing with them. The nature of the comments and questions posed by Tania and Frederic are an indication of a developing ability to truly reflect on their classroom practice.

This work is still very much in its early stages. To some extent, we have all been feeling our way as to how the relationships we have established will progress but each of us has identified particular areas to explore. Together, we hope to highlight some important issues faced by early career teachers and better understand how university programs can be designed to support the transition from the lecture theatre to the classroom.

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A THREE-HOUR TOUR OF SOME MODERN MATHEMATICS

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This paper describes some important modern mathematics, including problems as yet having no solution, and a three-hour workshop for sharing them with young children. The activities are so self-contained that children understand what to do with very little instruction, even in places where we do not speak the language, such as India or Norway. We describe the activities, the *Computer Science Unplugged* Project, how we present the workshop, and some reactions from children, teachers and parents.

Introduction

The *Computer Science Unplugged!* Project, based at the University of Canterbury, NZ, has a wealth of materials designed to demonstrate the foundational mathematical ideas of computing, but without using a computer. For example, in a playground treasure-hunt game, children work out a pirate map that is actually a demonstration of how computers are fed information (finite-state automata). For our workshop, we chose five activities with a specific theme in mind (algorithms and complexity) which will be described later. First, we describe the *Unplugged* project.

The *Unplugged* project is devoted to providing activities and hands-on materials to expose students to the ideas and ways of thinking used by computer scientists, all without having to use computers. (Using computers may sometimes be a distraction, or even misleading—encouraging children to falsely think everything about computer science involves programming, or computers). Free teacher resources are available on the *Unplugged* website (University of Canterbury, 2011), with ready-to-copy handouts, background and curriculum materials. Additional activities are available from Fellows (2011). Videos have been helpful in demonstrating the kinaesthetic activities, and these are available at *YouTube* and *TeacherTube.com*. The activities are easy to implement for outreach (workshops, school assemblies or science festivals), non-programming competitions, as a complete course in computing, or as supplementary topics in mathematics, computer science or other subjects. For example, the Sorting Network (Activity 1) is often kept on the classroom floor for use in many subjects: ordering distances from planets to the sun (science), molecular weights or densities (chemistry), fractions (maths), notes and scales (music), eras or events (history), or priorities (social studies). A session on encryption methods (Activity 5) can fit into a history session studying World War II.

The *Unplugged* website supports educators in sharing teaching methods, curriculum plans and developing new formats and activities. The website and materials are recommended by the ACM K-12 Curriculum and the CSTA (Computer Science Teachers Association, an international organisation aimed at school teachers), and NCWIT (the US National Center for Women and Information Technology), and there are sister projects worldwide (see Blum 2008). The project is backed by a 28-member Advisory Group ensuring appropriate vision and direction, consisting of a broad range of international educational institutions, science museums, industries including Microsoft, Google (which has provided financial support), and professional associations. The origin of the project was the book, *Computer Science Unplugged! Off-Line Activities and Games for All Ages* written by Tim Bell (Univ. Canterbury, NZ), Ian Witten (Waikato Univ., NZ), and Michael Fellows (now at Charles Darwin Univ., AU). The book has been adopted by two different school districts in Vancouver in support of their information technology curricula. It has won science communication awards, and has been translated into 12 languages (including Chinese, Korean, Spanish, Japanese, German, and Italian), and the videos demonstrating activities have sound tracks translated into five languages, including Swedish and Maori. Access details for versions of the book are available from the *Unplugged* website (University of Canterbury, 2011).

The activities

Children often believe that all the interesting mathematics has been done, and that any problem can be solved with a fast enough computer. As the same time, many countries are facing a severe decline in Computer Science enrolment, while governments and industries are seeking a knowledge-based economy that depends on innovative problem-solving in a wide range of areas such as text imaging, data compression, networks, parallel computation, security, programming, human-computer interface design, and many others. Computer Science is a rich subject concerned with what computers can and cannot do, how to approach problems, and how to make computers more valuable to the user. The *Unplugged* activities try to overcome the misconception that everything about computing is already known, and is all about programming, spreadsheets or web page design.

The activities demonstrate some of the problem-solving strategies and ways of thinking (recursion, randomness, or very slightly changing the problem requirements) that computer scientists use to solve important, real-world problems. In a highly-regarded paper, Jeanette Wing (2006) advocates ‘Computational Thinking’ as a way of approaching problems that is valuable to all students, regardless of whether or not they intend to study Computer Science as a specialty. The following example from Hromkovic (2009) illustrates the wide range of innovation needed. Suppose we manage a medical emergency centre with mobile doctors. Our aim is to deploy doctors efficiently, although nobody knows when or where the next emergency will occur. The control centre might try to minimise the average (or maximum) wait time of patients, or minimise the overall length of routes. A strategy might be to have doctors wait at strategically chosen locations after finishing one patient, before being called to the next. Without knowledge of the future, there are many online situations that occur in real life that may not even have a reasonable strategy. However, using ideas from computer science, we can often come up with clever solutions.

For our workshops, we chose activities that focus on the notion of 'algorithm'—a list of instructions for completing a task so that the task can be done in an automatic way, which is what computers do. Perhaps surprisingly, there are many problems that computers cannot solve, whereas others, that appear to be almost the same, can be solved quickly. The length of time it takes for a computer to solve a problem generally increases with the size of the input. (Searching 50 entries in a database takes longer than searching through 10.) For example, if searching the database takes time "linear in the input size" of "2 × input", then searching 50 entries versus 10 takes time 100 versus 20 which is still a ratio of 5:1, very reasonable. Unfortunately, most important problems of today take time that increases *exponentially* with the input size. The database example using base 2 rose to the input size, would require time of 2 raised to the 50th versus 2 raised to the 10th. This ratio is dramatic—over a quadrillion (a one followed by 15 zeros, in the US system) compared to just over a thousand (2 raised to the 10 is 1024). If one counts one number per second, it will take almost 40 years to count to a quadrillion.

Everyone wants their problems to be solved quickly, so linear (or polynomial) time is preferred and these are called 'efficient' algorithms, as opposed to those taking exponential time, which are likely to take longer than anyone's lifetime to complete—too long to be of any practical value. Figuring out whether problems or groups of problems currently requiring exponential or worse time could instead have a polynomial time algorithm is called the '*P* versus *NP*' problem, and a million dollars for its solution has been offered by the Clay Mathematical Institute (2011).

Our workshops begin outdoors, introducing the notion of 'algorithms' using a sorting network. Following that, we contrast two problems that at first appear almost identical from a class of 'graph colouring' problems. However, children readily find a fast algorithm for graph 2-colouring, and experience first-hand the difficulty of 3-colouring, for which there is no known efficient algorithm. Our society is linked by many networks: mobile phone networks, roads, utility systems. Usually there is some choice about where the signal links, roads or cables can be placed. In Activity 3, children design a network with a minimal total length. Almost every student will find a fast algorithm for *Muddy City* (connect every house in the city by a paved road at minimal cost). However, nobody knows a fast algorithm for finding a cheap plan to connect up only a few priority sites (such as: school, hospital, water tower). In fact, it is not even known if a fast algorithm exists. Activity 4 is the *Ice-Cream Stands* problem: Where should the city place ice-cream stands so that no one has to walk more than one block to get an ice cream? This problem represents a general class called 'dominating set' problems, which has important applications in resource allocation. Again, for any of these important problems, computers are unlikely to produce solutions in a reasonable time. Some beautiful mathematics has shown that all of these problems rise or fall together—if a fast algorithm can be found for any one of them, then a fast algorithm can be found for the others. But, most people believe finding a fast algorithm is not likely. Finally, Activity 5 describes an area in which everyone is happy with problems having no fast solution—cryptography—sending and receiving secret messages, which is a very active area of computer science and mathematical research, important in economics, banking and security.

Activity 1: The sorting network

A schematic of the sorting network is shown in Figure 1. Values flow through the network from left to right. They are mixed up on the left, but come out on the right in order. For example, the six values on the left from top down in the figure are 5, 1, 6, 3, 4, 2. At each comparator node of the network (circle), two values enter from the left and exit to the right, with (by agreement) the larger value exiting below the smaller value, both becoming inputs for the next comparison. Notice that at the beginning, three sets of values are being compared simultaneously (in parallel).

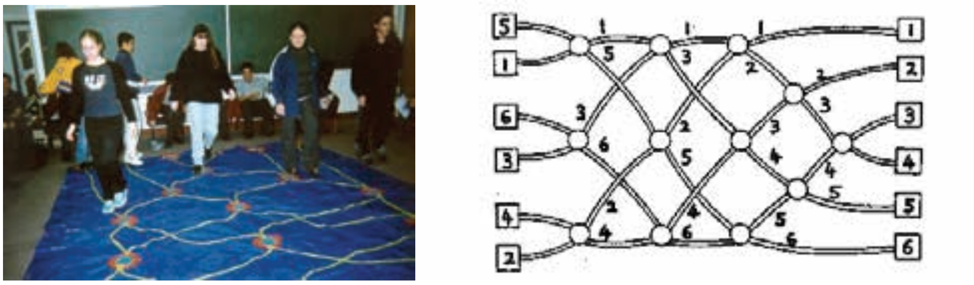

Figure 1. A diagram of a 6-input sorting network, and the tarp.

After much experimentation with string, spray paint and other ideas, we now build durable and beautiful sorting networks using coloured tape and aluminium pie pans on blue tarpaulin (the type used for covering a car or boat). We like to start workshops outdoors, with the children walking along the tape paths on the network. They meet two at a time at a comparator node. They compare the values that they are holding (say, each has a sheet of paper with a fraction written on it) and exit the node, taking separate paths towards the next comparator. Sometimes, the paths cross. Almost always, we have to start again because some children have little understanding of maps or how to follow a path.

Individuals must have patience to wait at a comparator node until they are joined by someone else so that they can compare their values. No single individual can progress. Progress can take a while if some couples further back cannot reach agreement—often agreement is made with the help of mates surrounding the tarp. Finally, it is understood that nobody wins alone; all win together as the sorting resolves the different values into a clear order.

Be aware that six children lined up on one end of a tarp facing six goals on the other is an invitation to race to the other end. The sorting network, however, is a model of cooperative learning, more of a dance or a series of conversations than a race. We ask individuals entering a node to greet each other before comparing values, which has led to pleasant surprises as children in some countries salute or charmingly bow in greeting. The sorting network algorithm is thus:

Input: Six children holding values
1. Walk forward along the path
2. Enter a comparator node
3. Greet the person there (if nobody is there, wait)
4. Compare values
5. Exit the comparator node (in the agreed upon directions, "Larger value towards the school, smaller towards the trees").
6. Repeat (Walk, Enter, Greet, Compare, Exit)

Output: Values are sorted in order

When the children reach the far end of the sorting network, they often spontaneously try to use the sorting network in reverse. This does not work, and many questions are raised by teachers and children. Can a sorting network be designed that works in both directions? Except for putting two sorting networks back-to-back, the answer to this question is not known. What is the minimum number of comparisons for a given size input? Is there only one unique way to design a sorting network with *n* inputs? A department chair decided to use the sorting network at an organisational retreat to help faculty prioritise objectives—but, as staff members walked through the tapestry of dots and lines, they realised that not every staff member met every other to evaluate each objective. How does the transitive property relate here? Were all objectives evaluated, or were some comparison possibilities missed?

Another topic of discussion is what sorts of activities can/cannot be done in parallel—digging a hole? Driving from Alice to Darwin? One of the main tasks of computers is sorting: putting lists into order, whether alphabetical, numeric, or by date. Fast sorting algorithms are very important, but even though computers are fast, there is a limit to how quickly any single computer can solve problems. One way to speed things up is to break a job into pieces and have each piece processed by a different computer simultaneously, a strategy called parallel computing. A sorting network is an example of a parallel algorithm.

We have sorted on integers, fractions, distances of planets from the sun, whose birthday is closer to Christmas, heights, weights, ages, brightness of colours. We have even put cord under the tape and had people sort with their eyes closed, feeling along the cord path with their toes, each holding a little bell and sorting on the higher/lower tone. Some teachers keep the net permanently on the classroom floor, using it for lessons in history, social science and other subjects as well as mathematics.

A group of six-year-olds at an elementary school in Wellington, NZ delighted in going through the sorting net repeatedly. They did not want to stop—even for snack—and kept finding more items to put in order. We were in the school library, and when I asked how the librarian sorted the books to go on the shelves the students chanted: "By the first letter of the surname of the author" and began sorting the books (see Rosamond 2006).

Wondering if they were really learning anything, or just having a lot of fun, I asked the six-year olds to design a three-input sorting network. One small group figured out all the permutations of three numbers that would have to be checked (to see if they came out in order), and they went around the room offering to check the other children's networks. In other words, they had learned logic notions of "*for every* (permutation of inputs) *there exists* (a correctly sorted output)."

The sorting network offers an experience of an algorithm through whole body movement, turning abstract ideas of computer science into actions that become part of the child's physical memory. Through educational kinesiology we've learned that certain physical movements help strengthen connections between the two hemispheres of the brain, thus aiding the process of learning (see Aigen, 2006). The manifestation of parallel computing on the sorting tarp incorporates:

- Contemporary mathematics
- Social and cooperative learning
- Engagement of multiple senses

- Interdisciplinary facts and concepts.

The question might be asked about the influence of a brief three-hour workshop on a child's impression of computer science, or on a child's future choices. Research in this direction is available on the *Unplugged* website and other sources. We report two anecdotes. Although we travel extensively for a different scientific purpose, inevitably a colleague will ask: Would you please come to my child's school and put on your workshop? (In Bergen, a parent has asked us three times, once for each of his children.) The second anecdote has to do with walking up the sidewalk to the Wellington school, where one of us had given a workshop two years previously. We were immediately surrounded by a cacophony of children's joyous greetings: "I remember you. Are you going to put on the sorting network again?" The workshop had indeed made a two-year positive impression.

Activity 2: Graph 2-colouring versus graph 3-colouring

A graph is 'properly coloured' if any two vertices that are connected by an edge receive different colours. 'Graph colouring' is a class of problems that models scheduling or allocation of resources. For example: the school is having an event and various committees (e.g., decorations, lighting, food, tickets, music, photos) must schedule meetings. These committees can be represented on a graph. The vertices of the graph represent committees, while an edge between two vertices means there is a conflict (e.g., Jane is on both committees, so those committees cannot meet at the same time.) The colours are the time blocks (1.00 pm – 2.00 pm, etc.). The goal is to properly colour the graph using as few different colours as possible. Of course, scheduling may be for ships to pick up coal, or the scheduling of micro-timing where the 'committees' are various processors and upload/download time within our computers.

Figure 2. Graph Colouring: colour with a minimum number of colours. The graph has been coloured using three colours. Find a solution using only two.

Finding the fewest number of colours rapidly turns into a fun competition, easily understood: "Sam has done it with 6 … Tina has done it with 5 … Can anyone colour the graph with 4?" Children surround us with their colouring and we loudly and carefully check every pair of vertices. "This vertex is connected to this other vertex. Do they have different colours? Yes. Check. Check … Check." We are demonstrating a key feature that problems may be very hard to solve, but easy to check. (Like a picture puzzle that is hard to put together, but it is easy to notice if a piece is out of place.)

We have tried using coloured tokens to designate the vertex colours, but these slide off easily. Crayons all turned the same brown if one changed their mind often. What we do now is to hand out lots of sticky pads in various colours, and children tear off bits of the sticky edge to mark the colour of each vertex.

Eventually, someone will discover that the graph on the left in Figure 3 can be coloured with two colours. Even young children can articulate their algorithm.

1. Start with any vertex; call it the "Start" vertex. Colour it any colour; say, blue.
2. Colour each of Start's neighbours red.
3. Colour the neighbours of the neighbours blue. The neighbours of those, colour red.
4. Continue. If a graph can be two-coloured, then the colouring has been found.

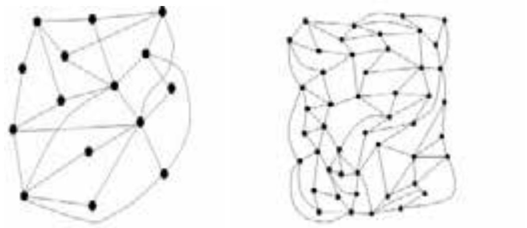

Figure 3. Children quickly find a fast way to properly 2-colour the graph on the left. The graph on the right, although similar in appearance, requires three colours, and there is no fast algorithm for finding a 3-colouring.

We contrast Graph Two-Colouring (and its fast solution) with the graph on the right in Figure 3, which can be coloured with three colours, but we let the children know that it will take a lot longer to find a solution. There is no fast algorithm for finding a three-colouring, and it is not likely that there ever will be one. Three-Colouring is one of those *NP-hard* problems that take exponential time. Children quickly experience the difference between the two graphs. If they make a mistake, there is nothing to do except start over.

We ask children, "Would you like to make a puzzle for your parents for which you know the answer, but it will take them a long time to figure out?" The children have asked us, "How do you know the graph can be coloured with three colours?" We draw several vertices on the chalkboard and colour them using, say three colours (for a 3-colouring). Add edges so that the graph is properly coloured. Now, (dramatically) shade all the vertices with chalk so the colours are hidden. This is the puzzle for the parents.

Activity 3: Muddy City and priority paving

The vertices in a graph represent houses and the lines joining them represent roads in a very muddy town. The goal is to pave just enough streets in order to be able to go from any house to any other house along paved streets, but do this at minimal cost. Figure 4 shows the graph.

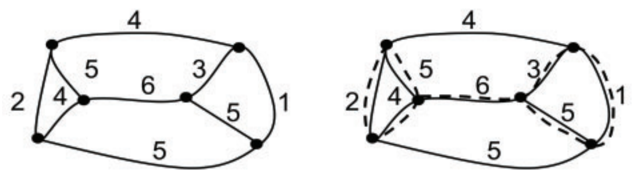

Figure 4. Muddy City problem. The number indicates the cost of paving that section. The dotted route shows a cost of 2 + 4 + 5 + 6 + 3 + 1 + 5 = 26.

Children call out a steady stream of improved solutions. They realise that length is not related to cost (twice a length does not mean twice the cost). Instead of roads, the lines could represent oil pipelines or wires. Children find a fast algorithm: never pave in a circle and always use the least expensive to go to the next node. We contrast *Muddy*

City with another graph where particular locations need to be linked (school, hospital, water tower). This is again *NP-hard* (sometimes called the *Discrete Steiner Problem*) for which there is no known fast solution.

Activity 4: Ice cream stands (dominating set)

No one will have to walk more than one block to get an ice cream if stands are placed on the three street corners indicated in Figure 5. Could we use fewer than three? Note that *any* solution requires two stands (they may be different sets of two). Other questions are raised. What if there are street corners that *have* to be used? Children construct valid arguments: *more than one* stand *must* be used (because only one would be more than one block away from some vertices). Again, no one knows if there is an algorithm for finding a minimum set of locations that is significantly faster than the "brute force" method of trying all possibilities (which for most important problems is longer than practical).

Figure 5. Can fewer than three ice-cream stands be placed at the intersections in the city map at left in order that no one has to walk more than one block for an ice-cream? How many for the graph at right?

The vertices that reach every other vertex in one hop (and also reach themselves) is called the 'minimum dominating set' of a graph. We have built the graph at left in Figure 5 in a special way with a "perfect code" (see Figure 6) as an example of a *one-way* function, sometimes called a *trap-door* function. Like falling through a trap-door, a one-way function is a process that is easy to do but perhaps impossible to undo. For example, it is easy to multiply two prime numbers together (fall through the mathematical trap-door). But, it can be very hard for someone to take the product and figure out the numbers you started with. One-way functions are essential for public key cryptography used in banking, telecommunications, personal identity and security applications everywhere in the world.

Activity 5: Coin flip over the telephone (cryptography)

We begin with a skit. A husband and wife are talking to each other on the telephone. H: We are getting a divorce. I want the red sports car. (Aside to children: I am willing to cheat to get it.) W (firmly): I want the car. H: OK, I'll toss a coin and you call it. Children: No, no. He can cheat. You cannot see the coin over the telephone.

As in Figure 6, a 'coin flip' can be created where H cannot cheat. H creates a special graph, for which he knows the size of the perfect code, and invites W to declare if the size is odd or even. In a perfect code, all solutions are of the same size. In Figure 6, the three solid vertices in the 'star centres' at left dominate all the other vertices (and themselves). Add a few disguising edges, but only between white vertices. In the resulting graph, every vertex can be reached by exactly one of the star centres. This special dominating set is called a 3-vertex 'perfect code'. With a longer workshop, children use perfect code to rescue a secret agent (this can also introduce solving simultaneous equations). Perfect Code was used for a school open day by Neal Koblitz (2011).

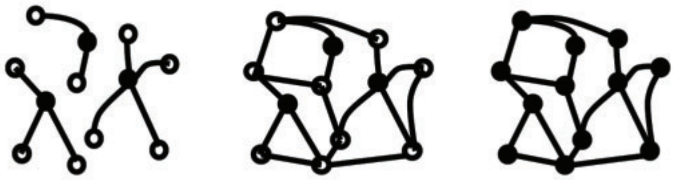

Figure 6. Constructing a perfect code.

Conclusion

We have described an easy-to-present workshop that introduces important modern problems of scheduling, facility allocation, network design, and cryptography. The activities demonstrate that many of these problems take exponential time to solve. Whether or not there is a fast method is one of the most important problems in computer science. Even children as young as six were able to understand the graph modelling of a problem, the concept of an algorithm, and experience the complexity difference between seemingly similar problems.

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ARE AIRPORT TAXI FARES FAIR ACROSS AUSTRALIAN CITIES?

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Which city/town in Australia has the cheapest (best value) taxi fares to and from the nearest airport? Which is the biggest 'rip off''? Are taxi fares to airports consistent in Australia (and overseas)? Using data correlation, linear analysis and a liberal dose of graphics calculator technology we attempt to answers all of these 'burning the money in our pocket' questions. Student samples are examined to question the merit of the investigation.

Introduction

Investigations liven up a mathematics lesson and discovering new investigations or new ways to look at mathematics topics is a tonic to refresh our teaching of mathematics. As an experienced teacher stated in Goos and Bennison (2007), "the role of the mathematics teacher is to provide students with activities that encourage them to wonder about and explore mathematics" (p. 320). Using technology with mathematics has become common with the ability to investigate data quickly, and so the emphasis has changed to focus more upon the analysis and interpretation of results rather than the 'number crunching'. Doerr and Zangor (2000) commented that teacher skills, experience and flexibility with technology leads to a classroom where calculators are freely used and should inevitably lead to increased exploration, confirmation and competence. The investigations that would make a good mathematics lesson would involve unlimited use of graphics calculator technology, especially for questions that were encouraging both exploration and interpretation.

On a flight from Hobart to Melbourne in December 2009 I was reading the in-flight Jetstar magazine and was drawn to some data rich pages near the end where information about the airports around Australia was given. The data included information about the distances from the airports to the nearest city, how long it would take to travel the distance by car and how much it would cost to travel in a variety of transport options. There was also information about parking costs in the airports. I could not help but consider whether the taxi fare to Hobart from the airport was expensive relative to other cities and towns around Australia with a nearby airport.

How it was done

The investigation (*Taxi fares—are they fair?*) shown in Figure 1 was initially given in 2010 (and recently in 2011) to a Mathematics Applied TQA315109 class of about 20 students. The subject Mathematics Applied TQA315109 is a university entrance subject administered in Tasmania for Year 11 and 12 students by the Tasmanian Qualifications Authority (TQA) and has algebraic modelling, applied geometry, applied calculus, finance and data and statistics as the five modules covered in a yearly program. Along with the investigation was a one page excerpt from the Jetstar magazine called 'Introducing our Airports' where all of the data were provided. The students were asked to complete the investigation in 90 minutes with minimal guidance.

There was an initial discussion for ten minutes during which students were able to ask questions about the investigation and to consider issues that may occur. Some students talked about methods and approaches that they might use. The notion of what was 'fair' was discussed with a broad agreement that consideration of cost against distance would be useful in comparing the different cities. The general acceptance was that 'fair' could be seen as points that were not expensive relative to other cities. It was apparent that the students were considering an average and looking at values either side as best value or worst value depending upon how they went about their calculations. Several groups were considering best value and worst value lists. They were told that they could use technology as little or as much as they thought appropriate in answering the investigation questions.

The students were then formed into groups of two or three students using birth months. Overseas data were available for students to extend their response in class if time allowed. The overseas data were not included in this paper but are available in the Jetstar in-flight magazine.

Taxi fares—Are they fair?

Are taxi fares from the cities to the airports in Australia fair? Is Hobart more expensive or less expensive (relatively) than other cities? This investigation should enable you to determine the answer to these burning questions.

1. Look at the information that is found in Jetstar magazines about taxi fares to the airports of various cities around Australia.
2. Create a table of values and enter these values into your graphics calculator or other technology.
3. Is there a linear correlation? Determine the line of best fit, correlation coefficient and coefficient of determination. Explain what they inform us about the data.
4. Are there any cities which are clearly better value than others?
5. Are there any cities which are clearly worse value than others?
6. Comment on the data including any assumptions and errors that you have noticed.
7. Answer the initial questions and hand in your work complete with tables, graphs and comments.
8. Extension: look at the overseas destinations and compare the results against both overseas destination and Australian destination.

Figure 1: The investigation: Taxi fares—are they fair?

Course documents

The Mathematics Applied course in Tasmania is a university entrance subject that is completed by a large cohort of students. Most of these students will not undertake degrees in Mathematics but will undertake degrees in other disciplines. Two of the eight criteria that are in the course were chosen as the most appropriate for the task. Criterion 2 is to use an investigative approach to collect data, analyse it and draw conclusions and Criterion 4 is to use algebraic or graphical linear and non-linear models to solve problems (TQA, 2008). The task involves the collating of the data and then making sense of the data by analysing graphs develop from the data. The final analysis involves conclusions and comments that should be made by students to indicate an understanding of what they have found. Charles Lovitt (1999) comments that the investigative process can possibly provide us with a unifying overview of mathematics in the hope that we can get universal agreement. A central theme of this is for the mathematics done in school to be close as possible as to what ‘real’ mathematicians do.

Sample solutions

Students addressed the investigation tasks well and submitted their group solutions. In this section, three of these are included to illustrate the range of approaches taken. The included solutions involve extracts from several pages of working completed and submitted by three of the class groups. The responses shown are typical of the responses of the entire class and all groups clearly displayed an investigative approach during the task. The conclusions and comments made by students were sensible. The responses included have been ordered from lower level responses to higher level responses.

Group A (see Figure 2) clearly identified the independent and dependent variables before commencing the investigation. What was revealing in the work sample in Figure 2 was that the group considered not only the distance against cost situation but also the time against cost situation. This was not considered by any groups when the investigation was undertaken with a class in 2010, and so was unexpected. By looking at the ratio of cost against distance, Group A was able to get an instant measure of value that they expanded upon in their answers. The fact that they actually calculated km/$ led them to incorrect conclusions in regard to the best value and worst value cities. It meant that, as would be expected, the answers were inverted, with best value cities they listed actually representing the worst value and vice versa.

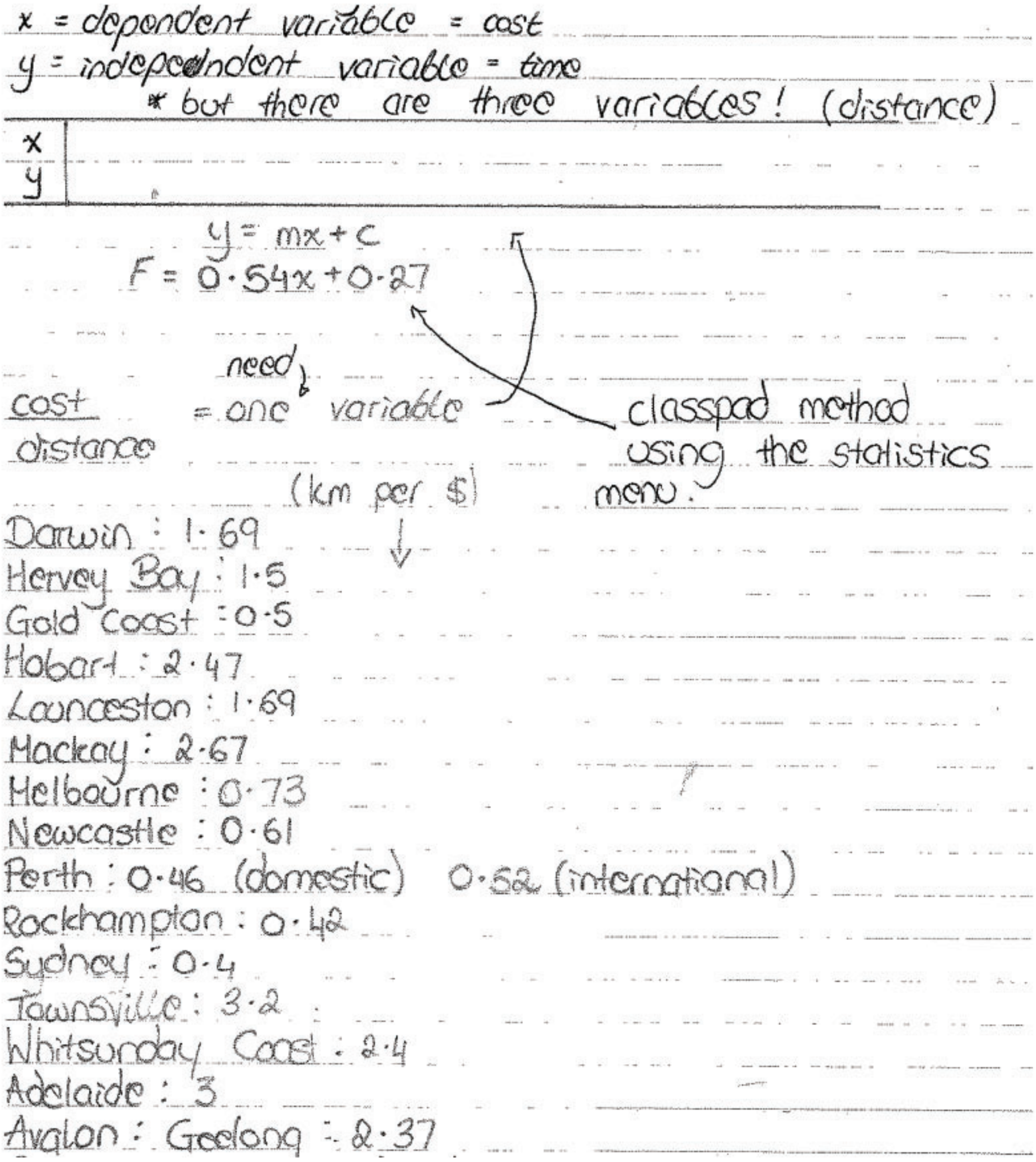


Figure 2. Sample of solution from Group A.

Group B (see Figure 3) also considered that cost was affected by both time and distance, which showed some insight into the investigation. The group was able to find a linear model in each case and obtain values for the correlation coefficient (r) and the coefficient of determination (r^2), which they explained later in the submitted work. They went on to make lists of the best and worst value for both the distance/cost and time/cost criteria. Brisbane was included in their best-value lists and Hobart was in their worst-value lists for both situations.

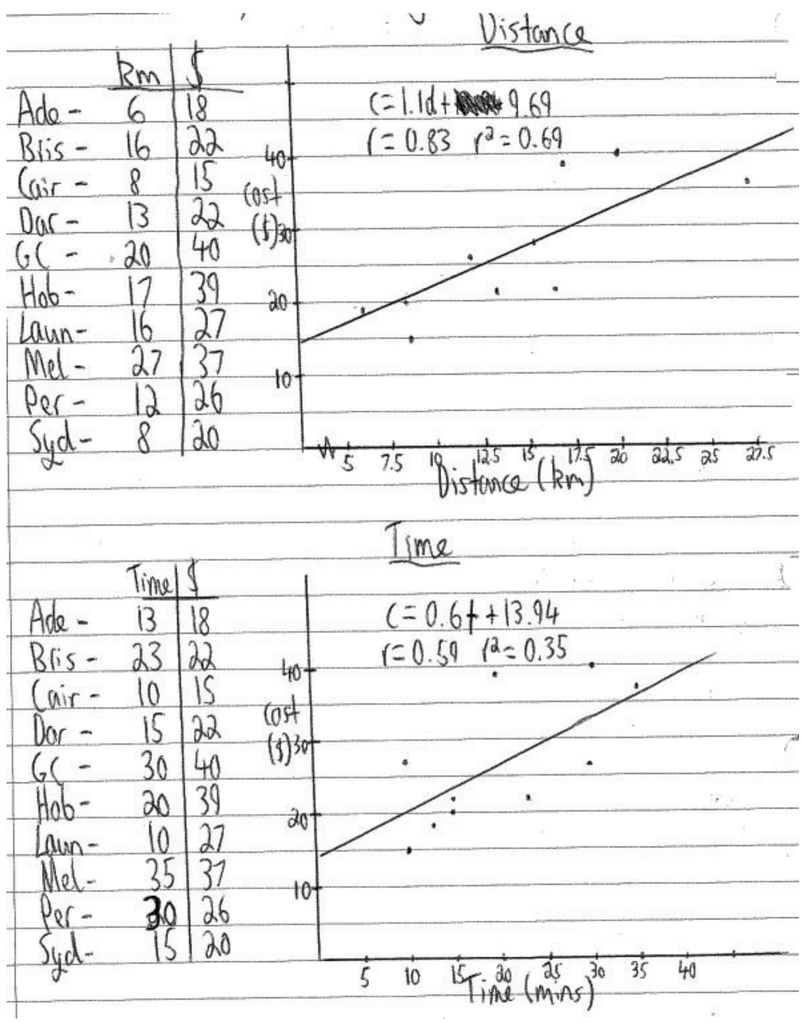
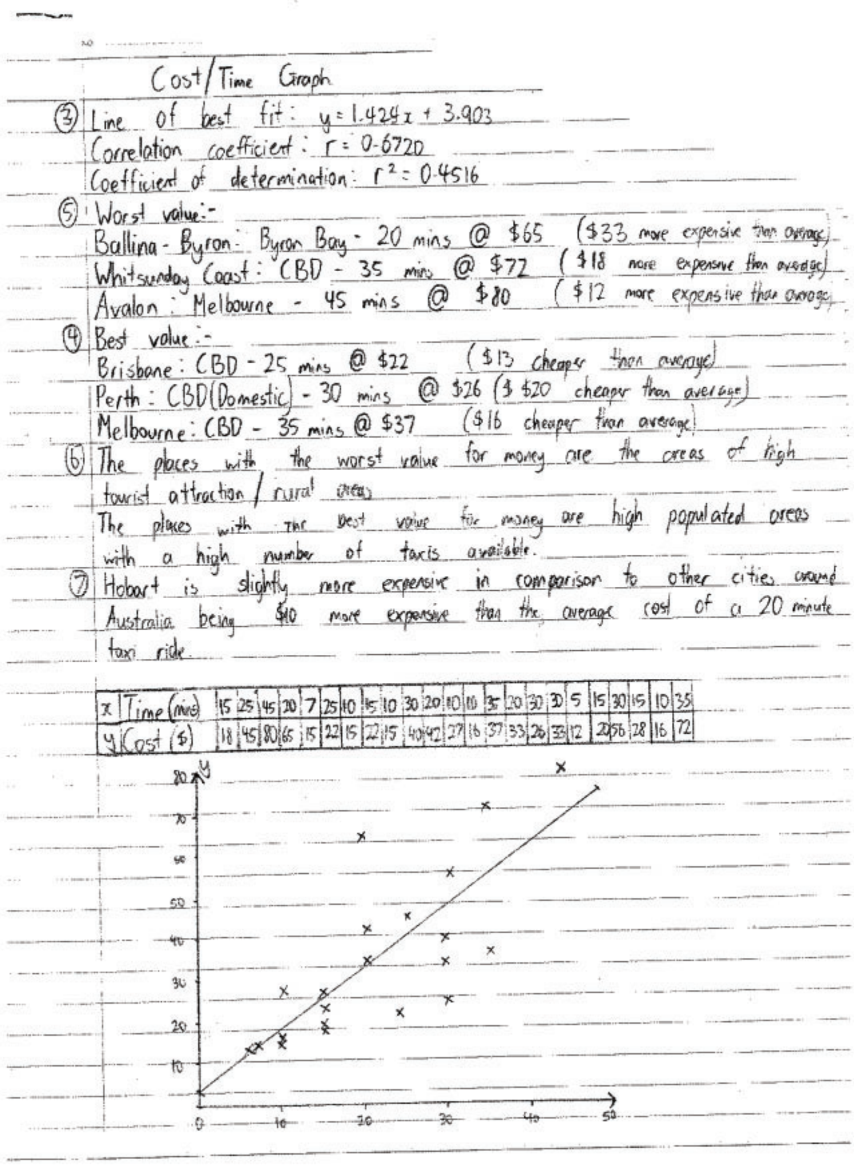


Figure 3. Sample of solution from Group B.

Group C (see Figure 4) calculated both Cost/Time and Cost/Distance after considering both scenarios like Groups A and B. The line of best fit was accurately obtained and their consideration of how much the actual value was less than or greater than the average (modelled value) was insightful, as it indicated an understanding of variation. The answers were clearly provided and the group’s comment that the best value was in high population areas and the worst values were in rural and tourist areas was insightful.



Cost/Time Graph

③ Line of best fit: y = 1.424x + 3.903
Correlation coefficient: r = 0.6720
Coefficient of determination: r² = 0.4516

⑤ Worst value:-
Ballina - Byron: Byron Bay - 20 mins @ $65 ($33 more expensive than average)
Whitsunday Coast: CBD - 35 mins @ $72 ($18 more expensive than average)
Avalon: Melbourne - 45 mins @ $80 ($12 more expensive than average)

④ Best value:-
Brisbane: CBD - 25 mins @ $22 ($13 cheaper than average)
Perth: CBD (Domestic) - 30 mins @ $26 ($ $20 cheaper than average)
Melbourne: CBD - 35 mins @ $37 ($16 cheaper than average)

⑥ The places with the worst value for money are the areas of high tourist attraction / rural areas
The places with the best value for money are high populated areas with a high number of taxis available.

⑦ Hobart is slightly more expensive in comparison to other cities around Australia being $10 more expensive than the average cost of a 20 minute taxi ride.

x	Time (mins)	15	25	45	20	7	25	40	15	10	30	20	10	10	35	20	30	30	5	15	30	15	10	35
y	Cost ($)	18	45	80	65	15	22	15	22	15	40	42	27	16	37	33	26	33	12	20	56	28	16	72

Figure 4. Sample of solution from Group C.

Reflections

One of the positive but unnerving things that can happen when a class is fully undertaking an investigation is the feeling of irrelevancy that a teacher can experience. In this instance the students were clear on how they were going to tackle the investigation and they were task focussed and were not really interested in discussion outside of their group. Open discussion at the start and at the end of an investigation are crucial to build the confidence to explore and investigate with minimal input from a teacher. The amount of discussion at the end of the investigation was limited and more scope for discussion at the end of the investigation was warranted. Technology questions and how to use the graphics calculators were much more common than questions about what they should be doing mathematically.

The overseas element of the investigation involved issues such as exchange rates—something that did not need to be considered in the comparison of taxi fares to airports of Australian cities. An investigative lesson to examine the taxi fares from international airports to their relevant cities and comparing them to taxi fares from Australian airports to their closest cities could be considered as a follow up to this investigation.

The investigation was successful in the sense that the students were fully engaged productively for a long session (105 minutes). The background knowledge of correlation and regression was limited amongst the class at the start of the investigation. The use of the graphic calculator assisted the students to look at a line of best fit for the data quickly and investigate what it actually meant. Students freely used the graphics calculator to present tables, graphs and equations in this investigation with little or no guidance.

Farrell (1996) found a shift in the role of a teacher from that of a task setter and explainer to a consultant, fellow investigator and resource. The taxi-fare experience was similar in many respects to this. Even though a defined task was set, the investigation did change the class dynamic. The teaching role became that of a guide and assisting investigator to the groups. Each group could tackle the investigation in the way that they chose and the end result was something that they had ownership over.

The responses and answers to the questions posed were varied in nature and presentation. Some responses were in tabular form and some were in sentence form but all attempted to answer the questions posed from their perspective. The fact that Hobart was slightly more expensive (in relative terms) was not entirely unexpected, as it had been the source of motivation for the investigation in the first place.

Conclusion

According to Lovitt (1999) open-ended investigative approaches may offer just the right structure in our endless search for a rich, balanced and appropriate mathematics curriculum. Using the graphic calculator as a powerful tool to assist in the investigation of whether taxi fares were ‘fair’ engaged the students and encouraged them to consider a situation where regression and linear correlation could be used to answer what seemed at first to be a relatively simple question.

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IMPROVING MATHEMATICAL FLEXIBILITY IN PRIMARY STUDENTS: WHAT HAVE WE LEARNED?

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Extra system support in numeracy has consistently focused on schools with high numbers of low achieving students, resulting in little attention directed towards schools in higher socio-economic areas. Most students from these schools perform well in NAPLAN and appear to be achieving successfully in their day to day mathematics learning activities, and so are not considered in need of extra support. The results of a pilot study, however, revealed that many of these students lacked flexibility in their thinking with reliance on standard algorithms masking misconceptions. Many teachers re-thought their teaching emphases and were able to improve their students' understanding and flexibility within one term.

As a direct result of the 2002–2009 *Getting it Right* Literacy and Numeracy Strategy, the level of specialised expertise in the teaching of primary mathematics has greatly increased in Western Australia (Meiers, Ingvarson, Beavis, Hogan & Kleinhenz, 2005, p. 124; Ingvarson, 2005). Most of these teachers, however, continue to be employed in schools with large numbers of students who overtly demonstrate high educational needs. While it is essential that those students continue to be the focus of system-wide concern and high quality support, the possibility that there are less obvious needs in some of the higher achieving schools requires serious consideration.

Most schools that produce results towards the top end of the scale in the NAPLAN numeracy tests are assumed to be serving their students well and therefore do not generally attract system attention, nor are they expected to need extra support. At the same time there is continuing and widespread concern that too few students are choosing to pursue studies in mathematics beyond the compulsory years.

The choice to disengage from mathematics learning is influenced by many factors, but it is generally acknowledged that students' experiences of learning mathematics in school, particularly in the upper primary and middle years, has a significant impact on their future educational decisions (Nardi & Steward, 2003). A wide ranging study in Victorian schools by Siemon, Virgona and Corneille (2001) included consideration of students' attitudes and responses to their mathematical experiences. An important conclusion was that students' engagement in mathematics is a consequence, not a cause, of understanding. This has important implications for many teachers who may successfully involve students in enjoyable mathematical activities, be skilled in teaching procedures for producing accurate results, and yet fail to notice that their students may not be making sense of the mathematics involved and so be at risk of future disengagement. Hiebert and Grouws advocated “explicit attention to connections among ideas, facts and procedures” (2007, p. 391) as essential for facilitating students’ engagement with and confidence in the learning of mathematics.

The rationale in the *Australian Curriculum: Mathematics* confirms a continuing emphasis on high level understanding and connectedness: “The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, logical reasoning, analytical thought and problem-solving skills” (Australian Curriculum, Assessment and Reporting Authority, 2010).

However, Siemon, Virgona and Corneille (2001) also claimed that trying to cover too much in too little time typically results in superficial procedural learning at the expense of developing the understanding needed for full engagement in mathematical generalising and conjecturing. The *National Numeracy Review Report* (2008) also draws attention to this issue.

> The mathematical knowledge, skill and understanding people need today, if they are to be truly numerate, involves considerably more than the acquisition of mathematical routines and algorithms, no matter how well they are learned.
> …The time, understanding and thoughtful action that deep mathematical learning requires must be acknowledged, and therefore both curriculum emphases and assessment regimes should be explicitly designed to discourage a reliance upon superficial and low level proficiency. (*National Numeracy Review Report*, 2008, p. xi)

While much is made of the need to develop deeper mathematical understanding, anecdotal evidence suggests that the mathematics programs that many students experience in some of the higher performing primary schools may well continue to be dominated by routine procedures and algorithms. This paper reports some early results from a short term pilot study aimed at exploring the degree to which schools in higher socio-economic areas emphasise procedural approaches when teaching calculation strategies. Teachers from those schools were provided with extra support in an attempt to engage students in the use of flexible calculation strategies that require a greater degree of number understanding. The longer term aim was to inform system decisions about the kinds of professional learning opportunities that would be more supportive of teachers of higher achieving students and promote a greater emphasis on mathematical understanding.

An overview of the pilot program

Eleven schools and a total of over two thousand students from Years 3 to 7 were involved in the pilot program. Three schools did not wish to engage in the professional learning program, but were willing for their students to be given the pre-test and post-test and so act as a control group to help determine the effects of the professional learning activities accessed by the focus schools.

The testing process

A pre-test administered at the end of Term 3 and then a post-test administered at the end of Term 3 were designed to test students’ flexibility in basic calculations. Several further items tested number and operational understanding. All of the items in the tests were chosen because they could be easily calculated using a simple mental strategy if the numbers were understood and students had experience in using such techniques. None of the items required the use of a standard algorithm. Some attitudinal questions were also included in both tests.

All students from Years 3 to 7 took the same test which required approximately one hour to complete. The post-test items were identical in form to the pre-test items and required similar thinking processes but the numbers were changed slightly to avoid the possibility that students might have remembered answers from the pre-test. The tests were administered by a team of curriculum officers using the same introduction and procedure. All students at each testing were encouraged to choose calculation strategies they thought would be the easiest to use, even if they had learned them out of school. They were explicitly told that they did not need to use a ‘setting out’ or ‘vertical’ method if they had another way to do it and that we were interested to know if they had shortcuts or personal methods for calculating. They were asked to use drawings or diagrams, numbers and symbols, or words to try and show how they carried out their calculations. If they tried and could not do a question, they were to write ‘too difficult’ or use a question mark to show they’d looked at a question but could not do it. Items that were left entirely blank were generally not included in the analyses.

Following the testing, a team of Numeracy Specialist Teachers were trained to code the strategies used by the students without knowing which schools were the focus schools and which were the control schools. Each paper was numbered to enable matching of pre-test to post-test responses. All codings were entered into Microsoft *Excel* spreadsheets to aid analyses. The test results from students who were only tested on one test date through enrolment changes or absenteeism have been excluded from the data. Thus the pre-test and post-test comparisons involve identical cohorts of students.

The professional learning program

All Years 3 to 7 teachers from the focus schools were committed to engage in the program through their principals for the first eight weeks of Term 4. Following the pre-testing of the students, teachers attended a two hour information session during which the requirements of the program were explained. They were shown video and audio clips of students displaying flexible calculation strategies based on partitioning and rearranging numbers in various ways. Some of the misconceptions surrounding the supposed value and necessity of learning standard algorithms and the rote learning of basic facts were challenged. Teachers were exposed to the idea that students can appear to recognise and use place value, but may not deeply understand the connection between the numerals and the quantities the digits represent, and so do not trust partitioning and rearranging numbers—preferring instead to use primitive counting strategies, rote learned procedures or a calculator when they needed to calculate.

To consolidate the session, the following three ideas were put forward as beliefs about students’ number learning that underpinned the approach:

- being able to partition numbers flexibly—with understanding—is essential to the development of fluent computational strategies;
- flexibility with number manipulations supports deep understanding of the properties of operations and is the basis for algebraic thinking;

- over-emphasis on memorising basic facts and standard algorithms can block students' mathematical development.

Teachers were asked to spend 10 minutes every day on a "How did you do it?" activity to explicitly teach informal calculation strategies based on partitioning, and to spend two full lessons each week using the *First Steps in Mathematics Number* resources (Willis, Devlin, Jacob, Powell, Tomazos & Treacy, 2004) which each teacher received as part of the support provided. They were asked to teach tables and basic facts through practicing understood calculation strategies rather than rote memorisation and encouraged to take a collaborative 'action learning' approach by working with a colleague to share strategies and planning.

The program also provided each teacher with the equivalent of two days teacher relief funding, paid to the school, for use in any way the teachers felt would be helpful. An experienced Numeracy Specialist Teacher trained through the *Getting It Right* strategy was seconded to the project to visit the schools in rotation and provide any forms of support requested.

Teachers were given a list of further support that they individually and collectively could access if they wished, including professional learning workshops, modelled lessons, individual or whole school planning sessions, diagnostic task reviews, across school visits, shadowing of other teachers and assistance with parent meetings.

Some schools and some teachers engaged in more of the professional learning opportunities than others. Overall, however, there was a low demand for the extra help listed. Detailed records of the support and input provided by the seconded Specialist Teacher were maintained. Teachers completed a survey at the end of the pilot detailing their experiences and preferences for the particular forms of support with which they engaged. This will enable further fine-grained analyses of students' results in relation to their teachers' engagement with professional learning opportunities offered. Currently the analyses are confined to the students' responses to the items at a broader level.

Results

The data for items 1a to 8b have been analysed to date. The results reported in this paper are from students in Years 5 to 7 in schools that have been separated according to their socio-economic status. Three schools fall below 105 SEI, while the remaining eight are between 105 and 120 SEI, which includes the three control group schools.

As the main focus is currently on the higher achieving schools, the analyses included in this paper are confined to students in the five higher SEI focus schools (a total of 600 students) and the three higher SEI control schools (286 students) and their responses to the calculation questions 1a to 8b.

The types of strategies used by each student were coded and then grouped into broader categories to reflect the focus of the program. The three groupings used were *Standard Algorithms, Flexible Strategies,* and *Other* (which included unknown strategies).

- *Standard Algorithms* were coded for all vertically set out methods that were clearly used for the calculation. Students who set out the numbers vertically, but clearly explained a different calculation strategy were not included in this group.
- *Flexible strategies* included all those that demonstrated manipulation of numbers in meaningful ways, the most common included:

  - partitioning and rearranging, (e.g., 137 + 26, I'll split 26 into 23 and 3 and add it to 137 to make 140 add 23),
  - rounding and compensating (e.g., 457 – 98, I can round up to 100, take 100 away to give 357, then add back the 2 to give 359),
  - place value (e.g., 15 × 3, I can make 15 into 10 and 5 and then multiply 10 × 3 and then 5 by 3 and add the answers)
- *Other or Unknown* strategies included those that did not demonstrate quantitative meaning, but also those that did not provide a clear explanation. Examples of the range of responses includes:
  - Counting (I counted on my fingers, or the marks on the page)
  - In head (I did it in my brain)
  - Face value (137 + 26, I said 6 and 7 is 13, and 3 and 2 is 5 and 1 more I put on the end)

Table 1. Percentage and number of Years 5 to 7 students using each type of strategy for the addition and subtraction items in the pre-test and post-test (as a proportion of students attempting each item).

		Focus Schools - extra support provided (n=600) Percentage number of students using strategy/students attempting item						**Control Schools - no extra support provided** (n=286) Percentage number of students using strategy/students attempting item						**All Schools** (n=886)	
Item Number	**Item content**	Students using a Standard Algorithm		Students using a Flexible Strategy		Students using other or unknown strategy		Students using a Standard Algorithm		Students using Flexible Strategy		Students using other or unknown strategy		Students attempting each item	
	Pre-test September Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December
1a	Pre- 28+7= Post 37+6=	32.5% 195/600	11.5% 69/599	42.5% 255/600	77.0% 461/599	19.5% 117/600	7.8% 47/599	29.0% 83/286	43.7% 125/286	46.9% 134/286	40.2% 115/286	16.1% 46/286	9.8% 28/286	100.0% 886	99.9% 885
1b	Pre- 137+26= Post- 127+48=	50.8% 304/599	19.6% 117/587	37.9% 227/599	71.2% 425/587	7.8% 47/599	3.4% 20/587	43.5% 123/283	56.8% 162/285	40.3% 114/283	30.5% 87/285	10.2% 29/283	5.6% 16/285	99.5% 882	99.5% 882
2a	Pre- 45–6= Post- 57–8=	37.5% 225/600	19.1% 113/593	42.8% 257/600	68.6% 407/593	12.0% 72/600	5.4% 32/593	31.8% 91/286	45.8% 131/286	41.3% 118/286	33.9% 97/286	12.9% 37/286	9.1% 26/286	100.0% 886	99.2% 879
2b	Pre- 457–98= Post- 376–97=	67.1% 390/581	32.6% 189/580	24.6% 143/581	60.9% 353/580	2.9% 17/581	1.9% 11/580	63.0% 170/270	70.8% 196/277	23.3% 63/270	17.7% 49/277	5.6% 15/270	2.9% 8/277	96.0% 851	96.7% 857
5a	Pre- 468+493= Post- 376+398=	72.6% 401/552	32.6% 184/564	18.5% 102/449	62.1% 350/564	4.2% 23/449	2.3% 13/564	70.5% 158/224	75.6% 198/262	17.0% 38/224	17.2% 45/262	7.1% 16/224	1.9% 5/262	87.6% 776	93.2% 826
5b	Pre- 45.6 + 29.9= Post- 26.7 + 39.8=	76.8% 400/521	38.4% 201/523	15.5% 81/521	54.3% 284/523	3.5% 18/521	2.5% 13/523	76.6% 160/209	79.8% 202/253	11.0% 23/209	11.1% 28/253	3.8% 8/209	2.0% 5/253	82.4% 730	87.6% 776
6a	Pre- 503–289= Post- 510–291=	78.3% 396/506	36.1% 193/535	12.8% 65/521	56.6% 303/535	2.0% 10/521	1.5% 8/535	80.4% 164/204	79.6% 199/250	10.3% 21/204	14.4% 36/250	2.9% 6/204	1.6% 4/250	80.1% 710	88.6% 785
6b	Pre- 138–9.8= Post- 257–9.9=	70.4% 321/456	34.4% 164/477	17.3% 79/456	53.5% 255/477	3.5% 16/456	1.0% 5/477	76.7% 138/180	73.9% 170/230	11.7% 21/180	14.3% 33/230	0.6% 1/180	0.0% 0/230	71.8% 636	79.8% 707

Tables 1 and 2 show the percentages and numbers of students in each group using one of the three strategy types[1] for each item as a percentage of students attempting each item. The overall percentage and number of students attempting each item in the pre-test and post-test across all schools is also provided in the tables. Whereas 80% to 100% of students attempted most addition and subtraction items and many of the multiplication items, the percentages reduced to 50% to 60% for division problems. Even though the decimal items involved easy-to-visualise numbers to one or two decimal places, they too reduced the number of students willing to attempt the items. While the main focus here is on the comparative use of strategies, many of the students' responses revealed misconceptions and misunderstandings that can be masked when

[1] Note that each student's response on each item was coded with one strategy. When more than one strategy was evident in a single item response, a decision was made as to the dominant strategy used.

students accurately apply standard procedures. For example, students had no difficulty using column addition for 45.6 + 29.9, though few noticed they could round to 30 and add 45.5. When confronted with 138 – 9.8 many using a standard algorithm made errors through incorrect placement of digits. Fewer than 20% used a flexible strategy, but of those who did, most demonstrated their understanding by subtracting 10 and adding back 0.2. Division of 9 by 1.5 was another item that revealed students' incorrect use of an algorithm and underlying lack of understanding, many producing answers such as 1r6, 1.6, or 9r5, while those using flexible strategies demonstrated understanding by doubling 1.5 and using the relationship between 3 and 9 to arrive at 6.

Table 2. Percentage and number of Years 5 to 7 students using each type of strategy for the multiplication and division items in the pre-test and post-test (as a proportion of students attempting each item.

		Focus Schools - extra support provided (n=600) Percentage number of students using strategy/students attempting item						Control Schools - no extra support provided (n=286) Percentage number of students using strategy/students attempting item						All Schools (n=886)	
Item Number	Item content	Students using a Standard Algorithm		Students using a Flexible Strategy		Students using other or unknown strategy		Students using a Standard Algorithm		Students using Flexible Strategy		Students using other or unknown strategy		Students attempting each item	
	Pre-test September Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December	Pre-test September	Post-test December
3a	Pre- 15x3= Post- 25x3=	40.1% 236/589	21.8% 127/583	46.2% 272/589	67.8% 395/583	5.8% 34/589	3.9% 23/583	41.9% 113/270	52.7% 146/277	44.1% 119/270	33.6% 93/277	8.1% 22/270	6.5% 18/277	97.0% 859	97.1% 860
3b	Pre- 63 x 4= Post- 54 x 4=	55.7% 308/553	21.9% 124/566	33.1% 183/553	71.4% 404/566	6.7% 37/553	3.9% 22/566	53.5% 137/256	60.1% 161/268	28.1% 72/256	28.4% 76/268	11.7% 30/256	5.2% 14/268	91.3% 809	94.1% 834
4a	Pre- 80÷4= Post- 56÷4=	46.3% 252/544	41.4% 211/510	29.2% 159/544	45.7% 233/510	15.4% 84/544	2.4% 12/510	40.2% 102/254	68.1% 175/257	29.9% 76/254	17.1% 44/257	16.9% 43/254	4.7% 12/257	90.1% 798	86.6% 767
4b	Pre- 350÷25= Post- 475÷25=	49.6% 242/488	30.3% 136/449	33.2% 162/488	53.0% 238/449	3.7% 18/488	0.9% 4/449	50.2% 110/219	53.1% 127/239	26.0% 57/219	26.8% 64/239	5.5% 12/219	0.8% 2/239	79.8% 707	77.7% 688
7a	Pre- 84x25= Post- 88x25=	74.7% 327/438	38.5% 176/457	12.3% 54/456	49.7% 227	2.7% 12/456	3.5% 16/457	84.7% 150/177	78.9% 179/227	6.2% 11/177	10.1% 23/227	4.0% 7/177	2.2% 5/227	69.4% 615	77.2% 684
7b	Pre- 13.25x4= Post- 21.25x4=	72.3% 289/400	32.0% 131/410	18.5% 74/400	57.3% 235/410	2.0% 8/400	0.7% 3/410	73.4% 113/154	70.8% 153/216	16.9% 26/154	17.1% 37/216	1.3% 2/154	0.5% 1/216	62.5% 554	70.7% 626
8a	Pre- 288÷24= Post- 286÷26=	72.6% 252/347	48.1% 163/339	11.5% 40/347	31.9% 108/339	2.9% 10/347	3.2% 11/339	79.1% 110/139	76.3% 145/190	7.9% 11/139	7.9% 15/190	1.4% 2/139	0.5% 1/190	54.9% 486	59.7% 529
8b	Pre- 9÷1.5= Post- 12÷1.5=	44.2% 155/351	24.3% 84/346	34.5% 121/351	51.2% 177/346	1.7% 6/351	0.9% 3/346	48.9% 65/133	40.5% 77/190	27.8% 37/133	33.2% 63/190	3.0% 4/133	2.6% 5/190	54.6% 484	60.5% 536

It is clear from the results that a considerable number of students do rely on the use of standard algorithms for these basic calculations that are relatively easy to process using an informal strategy. None are so difficult that they require the use of a standard algorithm or a calculator. To use informal strategies with confidence, however, requires a solid understanding of the numeration system and a range of partitioning strategies. A lower proportion of students used flexible strategies for many of the items, particularly in the pre-test. The effect of the support provided to the focus schools' teachers can be inferred from the large proportions of students in the focus schools who moved from using algorithms in the pre-test to flexible strategies in the post-test. Figure 1 illustrates this more clearly for the multiplication and division items and shows that for all items students in the focus group reduced their reliance on standard algorithms and increased their use of flexible strategies between the two test dates. The effect is strong and sufficiently consistent across all items to suggest the differences observed could be statistically significant. In contrast, the students from the control group schools appear to have demonstrated a similar choice of strategies across both tests with little variation observed.

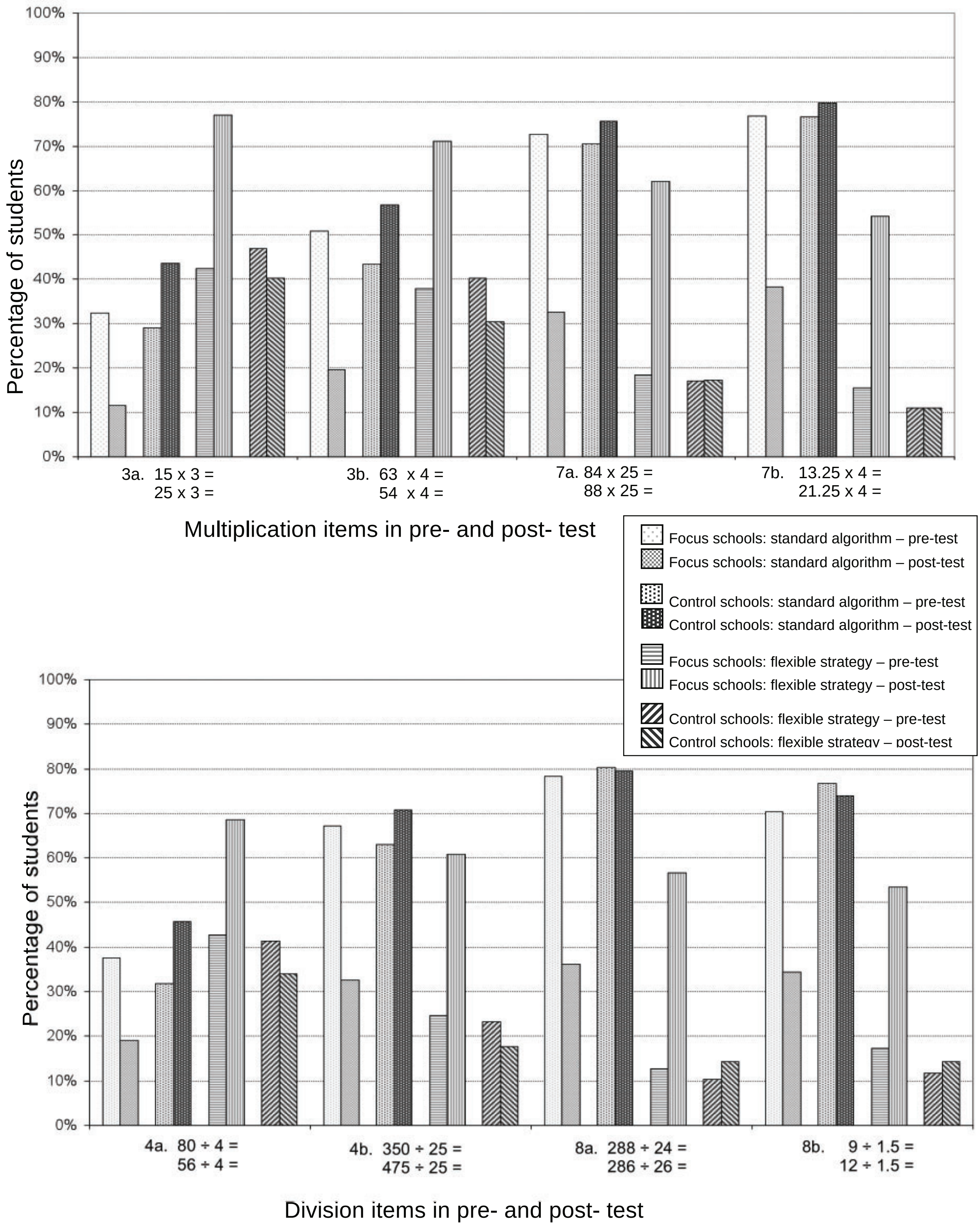

Figure 1. Comparison of strategies for multiplication and division items, pre- and post test for Years 5 to 7 students in the focus and control schools as a percentage of items attempted by each group.

Discussion

These results clearly demonstrate the effectiveness of the pilot program's approach for effecting changes in teachers' classroom practices in a relatively short period of time with a measurable impact on students' mathematical behaviour. While the teacher survey responses and the Specialist Teachers records have not yet been fully analysed at this stage, it is clear that teachers appreciated the choices provided and consistently expressed their belief that their professional learning needs had not previously been met.

While only the choices of calculation strategies have been analysed in detail to date, the accuracy of student's answers are also of interest. In general, accuracy appears to have reduced in the focus schools alongside the students' increased use of flexible strategies, while the control schools seem to have improved in accuracy. While this requires further analysis, it does present a dilemma. Is the drop in accuracy cause for concern, or is this a normal learning process—namely that increased error is expected when learning a new skill or technique? Would insistence on accuracy lead students to reject new learning and revert to a lower level of performance to maintain accuracy?

A closer look at one focus school's results for a single item illustrates the phenomenon. Figure 2 provides a profile for each student's choice of strategy and the level of accuracy attained in each test. While a large group of students moved from using an algorithm in the pre-test to using a flexible strategy in the post-test, an increase in accuracy did not follow. Six students used an algorithm correctly in the pre-test, but were incorrect when using a flexible strategy in the post-test and three more were incorrect in both tests. None in that group achieved higher accuracy in the post-test.

While the literature clearly supports the need for greater levels of understanding at all stages of learning mathematics, the complexities involved are often underestimated.

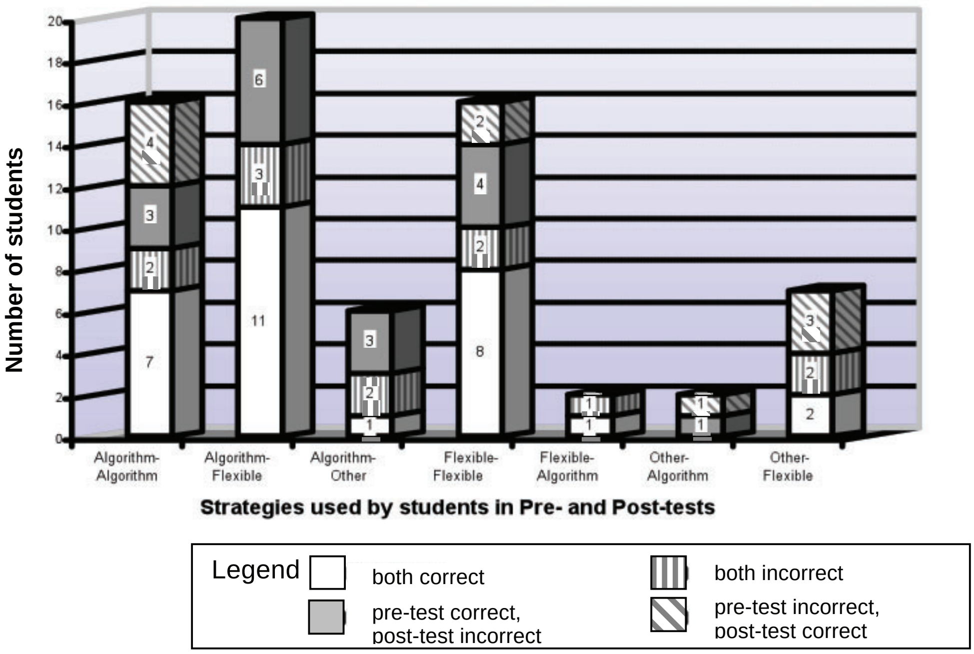

Figure 2. *Graph showing each student's type of strategy and the level of accuracy for Item 2B in the pre-test (457-98=) and the post-test (376-97=) for Years 5 to 7 in one focus group school.*

Implications

The data from this pilot are still being analysed. The students' attitudes to mathematics captured in both the pre- and post-tests, the teachers' responses to the post-program survey and the Specialist Teacher's records are yet to be considered with the rest of the data. However, even at this early stage, some implications can be considered.

Firstly, the results of the pilot program thus far provides evidence that, with relatively little system input, experienced teachers' classroom practices can be changed to the degree that they directly impact on many students' mathematical learning in a short period of time. This may be particularly true for these students due to their willingness to conform to their school's expectations. The team administering the tests noticed and remarked on the seriousness with which these students engaged with the test items on both occasions, in contrast to previous experiences trying to test students in some of the lower achieving schools. From a political viewpoint, a much lower investment in teacher support could result in relatively greater impact on student learning in these schools, with a greater effect on system performance. It is difficult to achieve significant measurable improvement across the system in mathematics learning by concentrating resources only on the very lowest achieving schools.

Secondly, there needs to be more discussion and reflection around the possible consequences of engaging higher-achieving students in mathematical behaviours that are 'risky' in the initial stages, because changes in approach may not immediately result in greater accuracy or understanding. Some students in the study were clearly very resistant to the change and expressed this in their comments on the tests. A relatively high proportion of students persisted in using standard procedures for simple calculations in spite of their teachers' efforts. Teachers may be willing to try new ways of working but for many students in higher achieving schools, learning mathematics is a high stakes activity, for which proficiency in carrying out procedures and getting high marks in tests may take precedence over more challenging mathematical activity that they don't recognise as helpful. They are yet to realise that their future mathematical learning could be at risk if they choose not to engage with meaning at every stage.

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WRITING CAS-ENRICHED MATHEMATICS LESSONS: OBSERVATIONS, CAUTIONS AND ENCOURAGEMENT

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As teachers become more skilled in the use of computer algebra systems (CAS) for presenting mathematical concepts and applications, they may be called upon by their colleagues to share their ideas for use by an audience beyond their own classrooms. In this paper, the author describes one view of the process of writing lesson materials for teachers and students. Insights gained through experiences as a member of a research team as well as being a presenter of professional development to teachers will be explored.

Introduction

Mathematics educators in Australia typically begin their careers in the classroom using resources such as textbooks, worksheets and assessment tasks provided by faculty colleagues. Graduates may also have acquired or written materials during their pre-service training which they will bring to their new positions. There can often be a key person amongst the mathematics staff at a school whose ability and willingness to produce (and share) such class- or school-specific resources is well known by her/his colleagues. Teachers in other countries, including the United States, may have entirely proscribed curriculum directives (see, for example Harris, Marcus, McLaren, & Fey, 2001) which in effect provide a daily script for the lesson as well as system-sanctioned assessment tasks. Some teachers, particularly those for whom mathematics may be only one of several disciplines they teach, continue their careers quite content to maintain this drip-feed of work prepared by others.

Increasingly however, mathematics teachers have been able to use technology to generate their own materials. Advances in word processing have meant that quality documents with accurate diagrams, precise mathematical notation, and illustrative screen shots from other technologies are easily produced by teachers to show algebraic, graphical and numerical representations of mathematical concepts. This phenomenon has been encouraged by the provision of mathematics education inservice activities whose explicit outcomes include expectations that participants write original lessons and units of student work, with the understanding that these be shared electronically in a forum such as Wikispaces.

In this paper, I describe my own journey of writing and sharing mathematics education resources as a secondary teacher, postgraduate student and university researcher. Four such resources, developed over the past five years, are used to illustrate stages of this journey. Finally, reflections on the value of the knowledge gained are shared.

Background—secondary teaching

As a secondary mathematics classroom teacher of over thirty years' experience, I have been asked many times to reflect on my initial motivations for this career choice. A genuine love of writing has emerged from these reflections as one of the strong influences, and hence many opportunities were taken to write worksheets and assessment tasks for specific classes. This became a collegial effort in the 1990s when, in co-authorship with two colleagues at a Melbourne school, each year we produced a unique application task (conducted over a two-week period) for the school-based assessment of the Victorian Certificate of Education (VCE) *Mathematical Methods* unit (see HREF1). We spent countless hours after school ensuring that the individualised versions (different parameters, similar problem) of the task were of a consistent standard amongst the three classes for whom we were writing. As these tasks were for high-stakes internal assessment procedures in Year 12, we did not recycle these for use in subsequent years, nor were they shared with other schools. We felt as a team that we could control the integrity of the assessment process in this manner and avoid answers being passed around the Year 12 cohort. The effort spent in producing these tasks was extensive and exhausting, and that experience partially contributed to a protectionist attitude within the team when thoughts of sharing the work arose.

Upon reflection, it can be seen that we also felt there was little available time and less perceivable confidence amongst the team to present this work to a larger, possibly more critical audience. The work thus remained on our floppy disks. I never considered that I had the time, or the presentation skills, required to put this or other work in front of my peers. Through hindsight I now realise we did indeed have ideas worth sharing, and that it would have broadened our professional experience to have been in contact with our neighbouring schools in the first instance, or to write descriptive articles for journals published by the Mathematical Association of Victoria (MAV) or The Australian Association of Mathematics Teachers (AAMT).

Postgraduate study

With computer algebra systems (CAS) technology being encouraged in Victorian school mathematics, I began M.Ed. studies in mathematics education at the University of Melbourne in 2006. The first two subjects of the course were technology-based, and it was imperative to become skilled in using technology for assignments and presentations. One particular assignment (see HREF2), concerned with the development of a technology-based unit of work for a Year 10 class I was currently teaching, presented many challenges. Amongst these was the incorporation of selected screen dumps from the CAS I was using then (Texas Instruments *TI-89*) to facilitate students' technical knowledge, to be kept in balance so that the mathematical content was not overshadowed by the "now press this button" mechanics of how the CAS could be used. The coursework I had already done presented me with the opportunity to explore the literature surrounding the always controversial balance between technology-active and technology-free mathematics instruction and learning (see Coffland and Strickland, 2004). I now needed to apply that knowledge, keeping in mind the needs of my own students.

In the process of writing this unit on Linear Functions and Measurement I was conscious of teaching myself the uses (and limitations) of the technology, as well as exploring a mathematical idea which had fascinated me for a while – using coordinate geometry to find the relationship between a polygonal region's area and the algebraic properties of its boundaries without reliance on differential calculus. (See Figure 1)

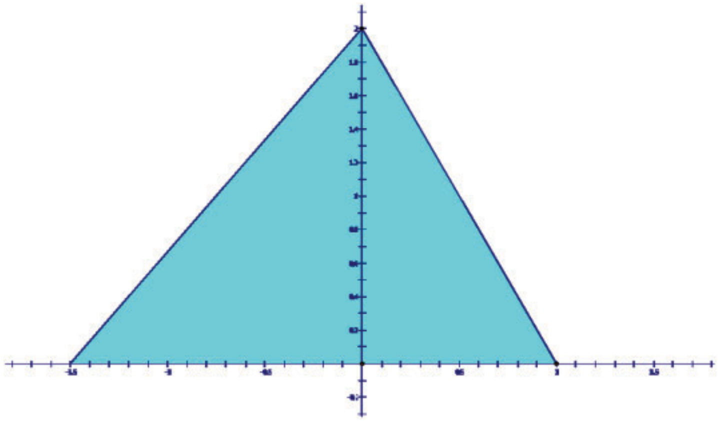

Figure 1. Finding the area enclosed by the graphs of $2x + y = 2$ and $4x - 3y = -6$ and the x-axis.

Technology enabled me to explore this relationship dynamically, and I felt students would be interested to do the same. As I was referring to the same type of area continuously (e.g., the region bounded by two oblique linear graphs and one of the coordinate axes), I needed to avoid excessive wording and hence created a variable (▲) to describe the area of such a region. Thus, I felt I was truly working mathematically in the creation of this concept, and that students could share in this experience.

The unit includes an overall unit plan, six specific lesson plans with accompanying student worksheets, and an investigative task with possible extension work. When presenting this work to fellow postgraduate students, there was genuine interest in the mathematical ideas involved as well as the overall structure of the unit. Their interest sparked confidence to share these ideas through a larger forum, resulting in my first presentation for the MAV December Conference in 2008.

From this exercise I became more aware of the need to let the written work reflect a reasonable pace and sequence of mathematical ideas. The lesson plans and student worksheets had to clearly convey my intentions for the interplay of teacher presentation and student exploration without being too verbose. If the unit was to be successfully used by other teachers, there had to be a blend of independent student investigation and collaborative student work to suit varying school learning environments. Finally, I felt there needed to be an assessment task to allow teachers to gauge the mathematical understandings of their students in this unit. In my MAV conference presentation I became conscious of trying to 'sell' these facets of my work; it was gratifying and confidence-building to receive positive feedback.

I would encourage teachers to experiment with whole-unit writing, as a fresh approach can often give colleagues welcome relief from textbook-influenced sequencing. Plans need to be made early in the year for this to happen, and the feedback received from colleagues on the merits of the new work will likely be valuable and confidence building. There is also a chance that others' creativity might be encouraged by these first steps. The great irony of this particular exercise is that I wrote it as a university assignment in isolation from my colleagues' work, left secondary teaching later that year and never had the chance to trial the unit at my school.

Research—Phase 1

Postgraduate study introduced me to *TI-Nspire CAS* technology, and in 2008 I joined a research team at the University of Melbourne. Our project, *New Technologies for Teaching Mathematics* (NTTM), was funded by Texas Instruments as a two-year study into the uses of *TI-Nspire CAS* for the teaching and learning of secondary mathematics. Our main brief that year was to design a technology-rich lesson and to observe its delivery by twelve teachers from two Melbourne-area schools within a modified lesson study format (see Pierce & Stacey, 2009) to classes of Year 10 students.

The lesson, *Marina's Fish Shop* (see HREF3; also Wander & Pierce, 2009), was written by the NTTM team in response to the teachers' requests for an application of quadratic functions. Once again, the interplay between geometric concepts and algebraic expressions and equations was being explored (see Figure 2).

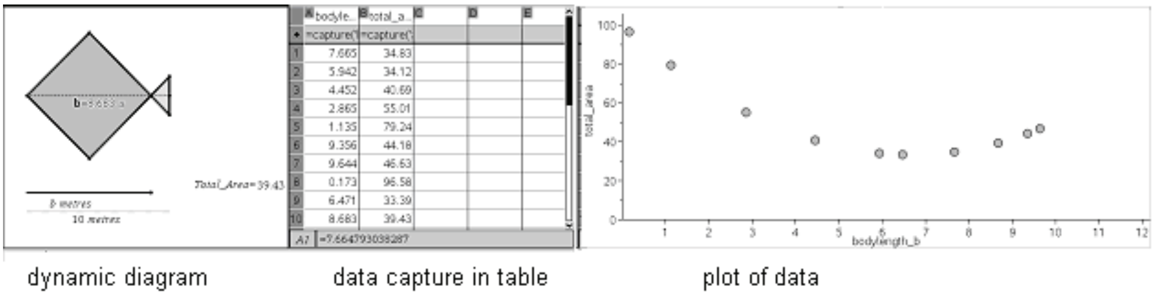

Figure 2. Screen shots from technology files of Marina's Fish Shop.

The writing of this single 80-minute lesson took several weeks, during which time there was fierce debate within the team as to how best to use the multiple representations (see Pierce et al., 2011) afforded by the technology to assist teacher presentation and student learning of the underlying mathematical principles. Questions related to how much space to leave on the worksheet for student written work, and whether to provide an undirected space or use a 'fill in the blank' style within a sentence approach. Also, as the students were those of the participating teachers (and most were unfamiliar with *TI-Nspire* technology), we took their advice regarding the extent to which we could assume that these students, through normal classroom discussion in groups, could generalise algebraically from a few specific examples.

I often reflect on the relative luxury of time we were given to prepare these materials, in contrast to preparing something "on the run" as I know I did when I had a full school teaching load. As a writer used to producing something for my own classes within a previous evening or at most a few sessions with two other colleagues, the transition to *Marina's Fish Shop* was eye-opening. Opinions were now backed by years of world-standard research experience from amongst my colleagues, and challenged and modified by those at the coal-face who were to use them with their students. Screen dumps and technically-precise diagrams were created, debated and modified. The entire lesson was also changed radically after we analysed its first presentation before we took it to the second school.

Along with the extensive lesson plan and student worksheets we created, we also produced supporting technology files to be used by teachers using *TI-Nspire CAS* teacher software for the computer, and by students using handheld *TI-Nspire* devices. The complexity of deciding what to include, the layout of each screen, which underlying geometric constructions to "hide" and which to show on the diagrams—all of these considerations make the writing of such lesson materials quite a complex task.

Through the publications which arose through our research, as well as presentations at the national and overseas conferences, *Marina's Fish Shop* has been seen, analysed and used by many educators. A major challenge for me as one of its authors is to ensure its continued usefulness by monitoring the technical instructions and screenshots in the documentation. As all mathematics application software (like *TI-Nspire CAS*) is continually updated, written publications quickly lose their relevance and attractiveness to teachers if they do not reflect the latest versions and best features of the technology used.

The prospect of preparing new materials and then being observed by colleagues while teaching, particularly when new technology is involved, is daunting for most mathematics teachers. However, its value was immediate for our research staff and the teachers, and many commented on how their involvement in the lesson design and delivery was highly rewarding. If individuals within teams of teachers take on various tasks –researching related ideas, writing initial material, proofreading, preparing technology and presenting – true ownership of the resource can result.

Research—Phase 2

The NTTM team decided to prepare two lessons in 2009 for the research schools. One of these, The *Surd Spiral* (see HREF4), was based on earlier work by Stacey and Price (2005). The spiral (seen below in Figure 3) is based on an isosceles right-angled triangle whose hypotenuse becomes the longer of the two perpendicular sides in the next successive triangle. The lesson's overall aim was to provide geometric and algebraic representations of the perimeters and areas of these triangles, which form a pattern of surd expressions. Through the lesson's activities, students were to acquire a better understanding of the process of surd simplification

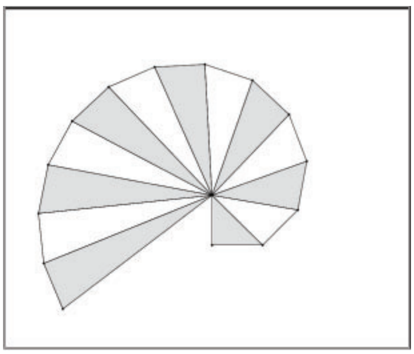

Figure 3. Screen shot from TI-Nspire CAS technology file of The Surd Spiral.

The motivations for writing this lesson were many. In our team's early experience with *TI-Nspire CAS* technology, we were interested in the capabilities of its algebraic spreadsheet. I had presented a demonstration lesson on surd patterns we had written for another Melbourne school's Year 9 cohort early in 2008, and through this experience we learned about the pitfalls of cognitive overload when new technology was being used to teach new mathematical content. We wanted the *Surd Spiral* lesson to reflect our acquired knowledge through this first experience. We were also keen to involve our lesson study teachers in exploring additional features of the technology we had not utilised the previous year, the spreadsheet being the most prominent of these.

Most of the teachers for whom we were writing this lesson were known to us from the *Marina's Fish Shop* work, and were much more confident with *TI-Nspire* technology than they had been the previous year. Their students were also more confident users, and knowing this enabled us to write a lesson which allowed for more independent student work, with only occasional technological step-by-step instructions.

Though it had been planned for the research team to once again use our modified lesson-study approach to observe our teachers, I was called on to teach the lesson twice at the second school when two of the teachers were absent on the observation day. That experience was invaluable for me, as this was the first time I had used *TI-Nspire* technology in front of students since the previous demonstration lesson 18 months ago. As one of the authors I knew the lesson very well, yet still found it challenging to link the detailed lesson plan with the student worksheets and the accompanying technology files.

From this experience I learnt I had become unrealistic in terms of estimating the time required for technology-based lessons. Though I had years of teaching experience, these highly-detailed lessons and their technological foundation required my complete concentration on delivery and did not allow enough student discussion time. We had written a lesson for a specified timeframe of 100 minutes, and it appeared too teacher-centred if the planned activities were completed. On a positive note, it was apparent through pre- and post-lesson testing that the students' knowledge of surd simplification had increased.

In schools where the classes of, say Year 10 Mathematics are blocked together on the timetable, teachers may be able to use the modified lesson study (research, write, observe and rewrite) approach described in the sections above. This involves high levels of commitment and requires the team to continually clarify the lesson goals for learning mathematics. Teachers should be prepared for at least two cycles of the process, allowing for the new unit to be taught in Year 1, modified through observational analysis, and retaught in Year 2. The benefits of this collaborative process are numerous, and may inspire some team members to report their experiences through journals and conferences.

Sponsored writing

In 2010 I was commissioned to do some writing for Texas Instruments within my role at the University of Melbourne. The resulting resources were intended for use by students and teachers of *Further Mathematics*, a Year 12 subject in the Victorian Certificate of Education (see HREF1). This is a subject which allows, but does not require, students to use CAS technology in all assessment tasks. The materials I was to produce included specific technical advice sheets regarding how to perform an isolated mathematical operation, detailed lessons and student investigative tasks. The brief was to produce 20 such documents and any associated technology files as required. Though colleagues gave valuable advice along the way, the final decisions regarding the various course content covered by the materials were mine.

As data analysis forms the core content of *Further Mathematics*, nine of the pieces were based on statistics. The others were centred on five of the six optional modules (number patterns, geometry and trigonometry, graphs and relations, business-related mathematics, and matrices). Thus, I had a wide selection of content from which to choose.

The independence (and resulting responsibility) associated with this endeavour were professionally enriching on many levels. Previous work with the *TI-Nspire CAS* had given me only a surface knowledge of the statistical capabilities of this technology, so I now had the incentive to explore these features in more depth. Though it was not specifically required in the brief, I wrote all activities based on an application or problem, hoping that creative teachers could modify a "how to…" document into a lesson using the associated problem as the vehicle for discussion. Also, I was given the opportunity to present some of the activities at the Mathematical Association of Western Australia (MAWA) and Mathematical Association of Victoria (MAV) conferences that year, which brought the resource to the attention of many teachers.

In producing this resource I have certainly learned the value of regularly seeking the opinions of colleagues in terms of content, layout and mathematical language. Having had the previous experience of writing materials in a team context, I had developed a good sense of knowing when to follow my own instincts and when to consult. I noted that trusted colleagues are often able to find layout and wording faults which are not so apparent when one is more intent on pursuing an interesting mathematical concept.

At the time of writing, this material is planned for inclusion on the Texas Instruments website (see HREF5).

Conclusion

The teacher who wishes to start writing mathematics education materials for others would be well advised to start by becoming familiar with the various journals of state based mathematical associations such as the MAV and its parent organisation AAMT. One will find lesson ideas ranging from the simple to the highly complex, and references for additional research. Becoming aware of what is already available will enable potential writers to identify what might be missing, and the writing can thus become more purposeful.

Looking for opportunities to start working in teams is essential. These may be formed at the school or network level, and with a deliberate division of labour to match abilities and interests it may be possible to create positive professional growth and a greater sense of ownership of the education process. Formal study and professional development workshops provide ideal venues for creative teachers to extend their skills and confidence in presenting work to peers.

In looking back at the five short years that have elapsed since I started my postgraduate studies, I am conscious of the opportunities which have allowed me to write mathematical education materials. I recognise that as a teacher I felt I was too busy to engage in this exercise, yet the input of practising teachers seems vital if these resources are to be worthwhile. Technology has played an overwhelmingly positive role in my journey and will continue to do so. Working within a community of educational researchers and practitioners has challenged and enriched my professional life. I encourage those teachers who feel they have creative, worthwhile ideas to share with others to consider career pathways similar to those described above.

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PERCENTAGES AS A CO-ORDINATION CLASS

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Co-ordination class theory describes processes and functions by which the learner co-ordinates fine-grained knowledge elements in developing mathematical concepts. This study investigated similarities and differences in the strategies used by 11 to 13 year-old students as they worked on percentage problems. The results suggest important knowledge elements and processes for co-ordination of these elements that are necessary for the development of a concept of percentages.

Introduction

Networks, hierarchies or a combination of the two are commonly used as metaphors for the way in which learners construct concepts (Hiebert & Carpenter, 1992). For example, Skemp's (1978) paper on relational and instrumental understanding advocated a connectivist view consistent with a network metaphor while Sfard (1991) used a hierarchical metaphor to describe the process by which a learner comes to treat a new concept as an object with which to think.

At the heart of theories about conceptual learning is the process of abstracting. Two perspectives dominate the literature. One view is that creation of an abstraction occurs as connections and common structure are perceived by the learner between situations. The abstraction is encapsulated as an object of thought, stripped away from the founding situations, that allows the learner to consistently apply the concept to new situations (Tall, Thomas, Davis, Gray, & Simpson, 2000). The other perspective is that transfer is influenced by learner (actor) perception and social interaction in situations (Lobato, 2006). Co-ordination class theory (diSessa & Wagner, 2005) suggests that abstraction occurs as learners transfer knowledge between situations rather than abstraction being the cause of that transfer. The theory provides the theoretical framework for this paper and reflects the latter, actor-oriented perspective about the construction of concepts.

Co-ordination class theory and percentages

Co-ordination class is a term first used by diSessa (1993) to describe a type of concept which requires the co-ordination of fine-grained knowledge elements. Learners have existing contextual knowledge elements about situations that are resources in the construction of concepts and have cueing preferences for the activation of knowledge in situations perceived by them as similar (Pratt & Noss, 2002). Well-developed co-ordination classes are signalled by the breadth of situations to which the learner applies the concept (span) and the consistency with which they apply it (alignment).

diSessa and Wagner (2005) described two features that comprise the architecture of co-ordination classes. *Readout* involves the learner attending to conditions in a situation that relate to the concept. The *causal or inferential net* involves the learner in mapping the readout data onto the concept to create and enact a sequence of sub-tasks to meet the demands of the situation. Co-ordination class theory anticipates both assistance and inhibition from existing knowledge in the development of new concepts, and accommodates learners holding contradictory knowledge at a given time. Recent studies have demonstrated the applicability of diSessa's theory to the learning of the law of large numbers (Wagner, 2003) and integers (Simpson, 2009).

Percentage is a concept that requires the co-ordination of many knowledge elements. A well developed procept, embodiment of process in a symbol (Gray & Tall, 2001), for percentage includes many interpretations described below in terms of Kieren's sub-constructs for rational number (Kieren, 1993).

- Measures: percentages as numbers or quantities that can be partitioned and combined, for example, 35% is 20% + 10% + 5%;
- Operators: percentages acting upon numbers or quantities as scalars, for example, 35% of $80;
- Quotients: percentages as shares, for example, four people sharing a quantity results in each person getting 25% of that quantity;
- Ratios: percentages describing equivalent constants of proportionality in both part-whole and whole-whole relationships, for example, 40% and 60% quantify the part-whole relationships in the ratio of 2:3.

Real-life problems usually involve the operators and ratios sub-constructs, though both measures and quotients are strongly connected in strategy use; for example, 35% of $80 as 10% of $80 + 20% of $80 + 5% of $80. Commonly, percentage problems are isomorphisms of measure situations under Vergnaud's classification of multiplicative structures (Vergnaud, 1994). For example, finding 24 out of 40 as percentage can be shown in the ratio table below although technically part and whole are not different measure spaces. The multiplier $2\frac{1}{2}$ is a scalar operating within each measure space while $\frac{6}{10}$ is the constant of proportionality between measure spaces.

Table 1. Ratio table for a part-whole percentage problem.

	Part		Whole	
$\times 2\frac{1}{2}$	24	$\times \frac{6}{10}$	40	$\times 2\frac{1}{2}$
	?		100	

Teaching experiments in Australasia and overseas have capitalised on students' real-life situational knowledge about percentages (Moss & Case, 1999; White, Wilson, Faragher, & Mitchelmore, 2007). Researchers used embodiments such as containers of water and double number lines with mixed success to develop students' knowledge of percentages as part-whole relationships and to link percentages to decimals and fractional numbers. Students in the middle school apply capably common benchmark fractions such as 50% and 10%, readily learned in real-life situations, but still have considerable difficulty with more complex percentage calculations (Dole, 2000). Co-ordination class theory suggests that real-life experiences provide some useful knowledge resources but these resources are inadequate for development of a robust concept for percentages. diSessa (2008) argued that the process of constructing particular concepts can only be understood through attention to the fine-grained knowledge elements that learners use in doing so.

Method

The data for this paper come from a teaching experiment in 2007 aimed at developing the multiplicative thinking and proportional reasoning of thirty students in a class of Year 8 students (11–13 years old). The setting was a medium sized middle school in a large town in semi-rural New Zealand. Data from a case study group of seven students, selected to provide a range of gender, ethnicity and achievement level, are presented in this paper. I worked with the students as both teacher and researcher for 16 weeks during the course of the year. The students received instruction about number during that time and the normal class instruction delivered by their teacher during the other 24 weeks of the school year.

The focus of the data analysis is on answering the following research question:

> What fine-grained knowledge elements do students co-ordinate in solving percentage problems, and what is the process of co-ordination?

For the purposes of this paper, I selected nine assessment tasks from a large bank of tasks attempted by students during the year. Selection was based on the tasks having a range of problem structures involving percentages. Seven tasks were from interviews carried out at four points during the year (21 February, 26 March, 16 August, 19 November). Each entire interview was videoed and transcribed. Interviews adopted a *teaching interview* protocol (Hershkowitz, Schwarz, & Dreyfus, 2001) during which the interviewer plays a more eliciting role than in traditional clinical interviews. In this protocol the interviewer offers assurance if needed, seeks explanations, provides alternative ideas and points out inconsistencies. Two tasks were items from a written test, chosen because student recording provided considerable detail about the strategies used. The rationale for task selection was validity of the data in terms of revealing the strategies and knowledge used. Since the interview and written tasks were tailored for each instructional group, and the selected students worked in two different groups, the set of students who attempted each task varied.

The tasks were as follows:

- Task One (interview): Remi has 30 calves to feed. Nine of the calves are Friesians. What percentage of the calves is Friesians?
- Task Two (interview): Remi has 45 calves to feed. 27 of the calves are Friesians. What percentage of the calves is Friesians?
- Task Three (written test): Jess is shopping for a pair of Levi jeans. She finds these deals. Which shop gives her the cheapest deal? Show how you worked out your answer. Jean City: Normal price $119.95, Discount 35%. Denim Dungeon: Normal price $79.95, 7% off for cash.

- Task Four (written test): Odette wants to buy this armchair for her Mother. It is on sale at 35% off. How much will Odette have to pay? (Price shown: $480)
- Task Five (interview): Rachel got 80% of her shots in during the netball game. She took 35 shots. How many goals did she get?
- Task Six (interview): Screecher shoes are 80 percent of the price of Petrol shoes. Nykie shoes are 75% of the price of Screechers. What percentage of the price of a pair of Petrol shoes is a pair of Nykies?
- Task Seven (interview): Odette buys a pair of jeans that usually cost $96.00. She gets 25% discount. How much does Odette pay for the jeans?
- Task Eight (interview): You sit a spelling test and get 18 words right. The teacher gives you 67% correct. How many words are in the test?
- Task Nine (interview): You sit a spelling test and get 18 words right. The teacher gives you 60% correct. How many words are in the test?

Student responses for each task were grouped into incorrect, partially correct, and completely correct categories. All responses were considered in developing a coding for the strategies used by students. This coding was then applied to all responses. Multiple codes were frequently applied to a single response.

- *SA* = Speculative algorithm – A calculation performed on the numbers in the problem with no apparent recognition of the problem structure
- *I* = Inertia – Inaction, no attempt
- *CPD* = Conversion percentage to decimal – Renaming a percentage as a decimal or vice versa
- *CPF* = Conversion percentage to fraction – Renaming a percentage as a fraction or vice versa
- *RA* = Additive Rate – Mapping of the given rate onto a rate with 100% by addition of incremental rates (distribution of the operating percentage)
- *RS* = Scalar Rate - Mapping of the given rate directly onto a rate with 100% by scalar multiplication
- *RU* = Unit Rate – Finding a unit rate by division then mapping it onto a rate with 100% by scalar multiplication
- *FS* = Equivalent Fractions (Common Factors) – Reduction of a fraction to simpler terms through dividing both numerator and denominator by a common factor
- *FO* = Fractions as Operator – Finding a fraction of a quantity by dividing by the denominator and multiplying by the numerator (or reverse order)

Results

Most student responses in the incorrect category consisted of speculative algorithms. That is, given a problem with numbers, students carried out a calculation in the hope that it sufficed. For example, asked to work out nine out of 30 as a percentage (Task One), on 21 February, Simon calculated 30.00 – 9.00 = 21.00 (21%) and Bob calculated 9 × 3 = 27 (27%). A few responses used potentially productive knowledge elements without awareness of how to co-ordinate them. For Task One Andrew knew 0.1 equals 10% so 0.3 equals 30% but then calculated 9 × 0.3 = 2.7 (2.7%). Odette recognised the applicability of common factors and 50% as one-half in her answer of 53 to Task Two.

O: It's weird.
I: What's weird about it? What made it hard?
O: I divided it by nine (common factor).
I: And what did you get when you divided by nine?
O: Fifty three. 45 is five and 27 is three.
I: So that is where you get 53% (5 and 3)? [Answer 53]

In her first interview on 21 February, in response to Task One, Rachel knew that percentage meant 'out of 100' but her self-perceived inability to solve percentage problems resulted in inertia.

A greater variety of strategy types and knowledge elements were evident in the partially correct solutions. Scaling based strategies were common and mostly involved additive build-up and multiplicative scalars in combination. Students treated the percentage as one measure space in a rate. For example, in answering Task Two, Bob reasoned that 100%: 45 calves so one-half (50%): 22.5 calves so 8%: ≈ 5 calves so 58%: ≈ 27 calves. Rachel used similar reasoning to find the price of a $119.95 pair of jeans at 35% discount for Task Three. Using division algorithms she created a table; $60 = 50%, $12 = 10%, $6 = 5%. Rachel then added $42 (7 × $6) and $60 to mistakenly find 85% of $120 rather than the required 65%.

There were four examples of students trying to find the multiplicative scalar by dividing 100% by the base amount or building up the base amount to 100%. These strategies founded on division calculations where the base amount did not map tidily into 100% with an integral scalar. For example, Bob mapped 30 onto 100 and got "33..." as the scalar (indicating recurring threes). He was unable to use this information to calculate nine out of 30 as a percentage.

There were also examples of fraction as operators and equivalent fraction based strategies among the partially correct solutions. In answering Task Three, Jess calculated
$\frac{3}{10}$ × $120 = $36 but added on one-fifth of $36 instead of 5% of $120. Similarly, responding to Task Five, Rachel knew that division by 100 and multiplication by 80 was a way to find $\frac{80}{100}$ of 35 and that 100 ÷ 35 gave the unit percentage rate for each shot. Figure 1 shows that her inability to express the quotient as a decimal halted her solution strategy.

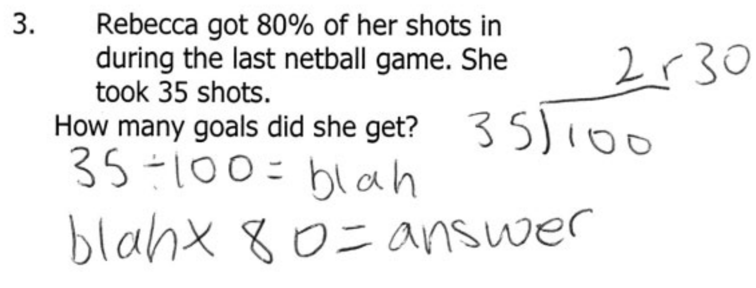

Figure 1. Rachel attempting division to find a scalar.

With interviewer prompting, Jason recognised that 80% = $\frac{8}{10}$ = $\frac{4}{5}$ but could not calculate
35 ÷ 5 = 7 correctly in order to apply his equivalent fraction knowledge.

The partially correct solutions contained a broad range of knowledge elements that were potentially productive for enacting correct solutions. These elements included additive and multiplicative partitions of 100, common fraction to percentage associations, halving and doubling, numerator as multiplier and denominator as divisor, and conservation of rate. The main problems in execution of strategies were fluency of calculation (particularly division) and co-ordination of the chain of inferences required, without succumbing to overload of working memory.

Correct solution strategies were characterised by successful co-ordination of multiple knowledge elements. Three main strategy types emerged; equivalent fractions, fractions as operators and unit rates. Combinations of strategies were frequently used. Typical of equivalent fractions based strategies was Jess's strategy for finding 27 out of 45 as a percentage for Task Two. She reasoned $\frac{27}{45} = \frac{3}{5} = 60\%$. Identification of common factors to simplify fractions and fraction to percentage associations were the knowledge elements employed commonly in complete solutions. Jess also used a fraction as operator strategy to find 80% of 35 netball shots (Task Five). She used the algorithm $35 \div 100 \times 80$ ($\frac{35}{100} \times 80$). Successful operator based strategies required knowledge of percentages as fractions with denominators as powers of ten, numerator as multiplier and denominator as divisor, and sufficient fluency with multiplication and division calculations.

Unit rate strategies were used by students on Task Eight where there was access to a calculator, thus removing the burden of calculation. Four students found the unit rate by finding $67 \div 18 = 3.72...$, meaning each word was equivalent to 3.72%. They then calculated $100 \div 3.72 = 26.88...$, and rounded the answer to 27 words. All four students checked their solution using fractional equivalence ($\frac{18}{27} = \frac{2}{3} \approx 67\%$), inverse operations ($27 \times 3.7 = 99.9$), or estimation (10 words = 37% so 9 words ≈ 33% so 27 words ≈ 100%). Students also used situational knowledge to confirm their estimates. For example, on Task Nine, Jason estimated that the number of words in a test as either 29 or 30 using rate build-up and opted for 30 "because that is what it's most likely to be in a test."

Flexibility of solution path and knowledge co-ordination also characterised the correct responses. Students appeared to have access to multiple strategies for solving the problem and these multiple strategies gave them certitude in their solutions. For example, in the interview below Simon responded to Task Six (Shoes problem). His solution demonstrated the use of a broad range of knowledge, and flexibility in combining strategies and unitising with different referents.

S: I pretended that was $10.00 (Price of Petrols). Eighty percent is four-fifths. One-fifth of ten is two. Two times four is eight…hold on. Eighty percent…

I: So you pretended this was ten dollars so what would this one be (pointing to price of Screechers) if it's eighty percent?

S: Eight dollars…oh I did this wrong but okay…Seventy-five percent is three-quarters. A fourth of eight is two. Three times two is six dollars.

I: So what percentage is that?

S: Six out of that is six and four-sixths (meaning $1\frac{4}{6} \times 6 = 10$)…four and two-thirds

I: Let me get this right (recording on answer sheet). You pretended this was ten dollars, this came out to be eight dollars and this to be six dollars. So what percentage is this of this price (pointing to $6 and $10]?

S: One and two-fourths…one and two-thirds.

I: I see…but what percentage of the cost of a pair of Petrols is a pair of Nykies, not the other way around?
S: Point six.
I: And what's that as a percentage?
S: Sixty percent.

Discussion and conclusion

The data show that possession and co-ordination of key knowledge elements is necessary for students to become proficient solvers of percentage problems. Students' strategies became more sophisticated as the year progressed. Knowledge gaps and unsuccessful co-ordination of the multiple inferences needed to enact self-chosen solution paths resulted in many instances of partially correct strategies. Poor readout, mapping of situational conditions to a concept for percentages, resulted in speculative algorithms. Possession of knowledge seemed to influence both the selection of possible solutions and the ease with which these strategies were enacted. The observations are consistent with predictions from co-ordination class theory and vindicate the significance of identifying fine-grained knowledge elements.

Some of the elements identified in this study were:

- fraction to percentage associations, particularly those involving halves, quarters, fifths, tenths and hundredths;
- identification of common factors to simplify fractions that worked in conjunction with associations;
- understanding of conservation of rate and ratio;
- use of multiplication and division to find scalars up and down that worked in conjunction with conservation;
- fraction as operator schemes that use numerator as multiplier and denominator as divisor; and
- estimation based on reasonableness, rounding, and adjusting numerators, denominators and measures in rates up and down.

All of these elements are built on other foundational elements. For example, fraction as operator is built on iteration of a unit fraction and common factors built on multiplication facts. Proficiency in solving percentage problems requires flexible connection of Kieren's sub-constructs. Percentage is one representation within a broader field for proportionality.

Students in this study who were able to consistently solve percentage problems called flexibly on multiple views of percentages as fractions, equivalent rates, operators and measures. The main implication for instruction arising from this study is that teachers need to assist students to connect the broad range of situations in which the need for proportional reasoning arises, the various representations and equivalent number forms used to express proportional relationships and the sub-constructs for rational number. While using students' situational knowledge about percentage to introduce ideas about proportionality seems plausible initially, a broader instructional brush appears to be required to help students to understand concepts such as fractions, ratios and proportions in a connected way.

From a research perspective the study is supportive of co-ordination class theory in highlighting the importance of attending to the fine-grained knowledge elements students employ in solving problems and the processes they use in co-ordinating that knowledge. Given the small number of students involved, a larger-scale study involving more students and a greater diversity of problem types is required to substantiate the coding used for classifying solution strategies. Such a study might also illuminate the extent to which knowledge elements applied to percentages are shared with other concepts.

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THE P–4 MATHEMATICS INTERVENTION SPECIALIST PROJECT: PEDAGOGICAL TOOLS AND PROFESSIONAL DEVELOPMENT RESOURCES

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A four-year project focusing on intervention in students' number learning in the P–4 (K–4) range is described. The project involves development and implementation of a year-long professional development program in up to 70 schools. Also described is the P–4 Learning Framework in Number, consisting of three bands and nine domains of learning. The framework is used to organize an elaborated set of pedagogical tools: eight assessment schedules, nine models of progressions of learning each with up to seven levels, and extensive tables of teaching topics and teaching procedures. Also described is the program of evaluation and its initial outcomes.

In the last decade there have been increasing calls for the development of specialized programs to support low-attainers' number learning (Bryant, Bryant, & Hammill, 2000; *Mapping the territory*, 2000; Milton, 2000; *National Numeracy Review Report*, 2008), with recommendations that such programs be based on reform approaches to general mathematics education (Rivera, 1998) and incorporate "effective teaching strategies that work for all rather than distinct routes based on diagnostic categories" (Gross, 2007, p.153). According to Dowker and Sigley (2010, p.77), "individually targeted interventions are effective for children with mathematical difficulties, and may work better than similar amounts of individual attention in mathematics that are not targeted to a child's specific strengths and weakness". The P–4 Mathematics Intervention Specialist Project (MISP) is a four-year (2009–2012) project operating in a large school system in Victoria that accords with the calls and findings just described. The purpose of this paper is to provide an overview of key aspects of MISP.

Origins and overview of MISP

The origins of MISP are in: (a) the Mathematics Recovery (MR) program (Wright, 2003; 2008), an established program of intervention in the number learning of low-attaining students in Year 1; and (b) the Numeracy Intervention Research Project (NIRP) (e.g. Ellemor-Collins & Wright, 2009, 2011; Wright, Ellemor-Collins & Lewis, 2007) which was funded by a Linkage Grant from the Australian Research Council (2004–2006) and the Catholic Education Commission of Victoria (CECV), and which focused on intervention in the number learning of low-attaining students in Years 3 and 4 (8- to 10-year-olds). Building on these origins, MISP seeks to develop a program of intervention in number and arithmetic learning across years P–4. MISP involves development at three levels: (a) developing an elaborated and coherent set of pedagogical tools for mathematics intervention; (b) developing an intensive, year-long program of professional development for teachers to become specialists in mathematics intervention; and (c) providing significant professional development for teachers. Ultimately, building teacher expertise related to the teaching of number and arithmetic across the P–4 years is a key goal of MISP.

MISP is planned over four years, 2009-2012. Table 1 shows the number of teachers involved in the project in 2009, 2010 and 2011, and projected numbers for 2012. In its first year (2009), MISP focused on intervention with Years 1 and 2 students. This involved adapting key features of the MR program. Most of the teachers in 2009 were involved in NIRP (2004–2006). After their initial year in MISP, many of the teachers continue their involvement in subsequent years.

From 2010, two of the teachers who participated in NIRP and the first year of MISP were established in a new role of Numeracy Intervention Tutors (NITs). The NITs are dedicated to professional development and teacher support work, under the guidance of the project leader. As the NITs' expertise develops, they are able to lead more of the professional development sessions, creating capacity for larger cohorts of new teachers in subsequent years. We anticipate that, by the end of the four years, at least 70 teachers will have had at least one full year of the professional development, and many will have been involved for at least two years.

Table 1. The number of teachers involved in each year of the project.

Cohort	*2009*	*2010*	*2011*	*2012 (projected)*
Cohort 1	10	7	2	2
Cohort 2	–	15	9	7
Cohort 3	–	–	15	15
Cohort 4	–	–	–	30

The intervention program

The intervention program has four stages: (a) school assessments of the whole cohort which are used to select 12 students for intervention; (b) individual pre-assessments of the 12 students; (c) a teaching cycle; and (d) post-assessments of the 12 students.

School assessments

In each participating school, several assessments are administered to the cohort of all students in the year level. The assessments used vary with year level. In the case of Year 1 students, a short one-to-one assessment is used. In the case of Years 2–4 students, the Westwood one-minute tests of basic facts (Westwood, 2000) are used. Year 2 students are given the addition and subtraction tests only, while Years 3 and 4 students are given the multiplication and division tests as well as the addition and subtraction tests. Other assessments commonly used include the Success in Numeracy Education (SINE) assessment (CECV, 2002) and the Progressive Achievement Tests (ACER, 2005). On the basis of the assessments administered to the cohort, 12 low-attaining students are selected for involvement in the intervention program.

MISP pre-assessments

The 12 students are given one-on-one MISP assessments (see below). Typically, Year 1 students are given Schedule 2, while Years 3 and 4 students are given Schedules 3A, 3B and 3C. Year 2 students may be given Schedule 2 and a selection of task groups from Schedules 3A, 3B and 3C, according to their responses on Schedule 2. Across all four year-levels, the assessment tasks on the schedules can be used in a flexible way, in response to the student's responses. The intervention teachers use the MISP assessments as the basis for tailored planning of the intervention teaching.

Teaching cycles and MISP post-assessments

Of the 12 students, eight students are taught in teaching cycles of 10 to 12 weeks. At least two are taught as singletons and typically up to six are taught as pairs or trios. The other four students serve as counterparts and enable comparison between participants and counterparts over the course of a teaching cycle. In many cases counterparts are provided with intervention instruction later in the school year or in the following year. The teaching cycles are intensive, involving four or five teaching sessions per week, each typically of 25 minutes' duration. At the conclusion of the teaching cycle, all 12 students are again given MISP assessments. All of the assessments and all teaching sessions with singletons are videotaped.

Pedagogy for intervention

The pedagogical approach involves: (a) a one-to-one interview-based assessment which is videotaped for analysis; (b) development of an individualized instructional program that is strongly informed by the results of the assessment; (c) an approach to instruction that is intensive and inquiry-based, and aimed just beyond the cutting-edge of the student's current knowledge and strategies; (d) a process of on-going observational assessment that enables modification of the instructional program; and (e) use of an elaborated set of instructional topics and teaching procedures. An extensive amount of detailed information about the pedagogical approach is readily available (Wright, Ellemor-Collins & Tabor, 2011; Wright, Martland & Stafford, 2006; Wright, Martland, Stafford, & Stanger, 2006; Wright, Stanger, Stafford & Martland, 2006). Thus this paper does not provide detailed information about this approach. Nevertheless, we believe it is distinctive and embodies a pedagogy that constitutes important professional learning for all teachers at the primary (elementary) level.

The professional development program

The professional development program involves supporting the teachers to implement two intervention cycles, one in each half of the school year. The first cycle involves working with students in Years 3 and 4, and the second involves working with students in Years 1 and 2. Eight professional development workshops are scheduled across the year for a total of 14 days. The first workshop in an intervention cycle focuses on learning to administer and analyse the MISP assessment schedules. Workshops preceding the teaching cycles address planning for teaching, the approach to teaching, and induction into the use of pedagogical tools specifically for instruction (described below). Workshops during the teaching cycles allow for further discussion and refinement of the teaching. An important part of each workshop is for each intervention teacher to present a case study using video segments drawn from assessment and teaching sessions. The case studies highlight student responses to instruction and progressions in student learning. As well as the workshops, each teacher receives mentoring and support via bi-weekly school visits by one of the NITs. In addition, NITs support teachers on an on-going basis via emails and blogs.

By the end of their initial year in MISP, teachers have completed: at least 14 days of professional development meetings; approximately 16 days of individual assessment and analysis of the number knowledge of 24 students; and approximately 50 days of intervention teaching of 16 students. In addition, they have worked extensively with most of the MISP tools and resources, across Years 1–4. In subsequent years, the teachers continue to implement intervention programs, and participate in up to six additional professional development workshops that focus on reviewing their intervention work and learning about updates of the pedagogical tools. An important goal of MISP is that, rather than adopt a standard form of a program of intervention, the teachers will develop sufficient expertise to further tailor the intervention program to the needs of their school. In this way, MISP teachers work with colleagues in their schools to develop specific implementation models. In addition, many of the MISP teachers, either formally or informally, become instructional leaders in their schools.

Pedagogical tools and professional development resources

Fundamental to the MISP approach is an extensive set of pedagogical tools and professional development resources. These include a P–4 Learning Framework in Number (P–4LFIN), assessment schedules, tables setting out progressions in student learning, and an elaborated set of teaching procedures. These are described below.

P–4 Learning Framework in Number

As shown in Table 2, the P–4LFIN is organized into three bands. Bands 2 and 3 are the main focus of MISP. Bands 2 and 3 are organized into, respectively, three and five domains of number knowledge. We find it very useful to construe number knowledge in terms of distinct bands and domains within each band (Wright et al., 2007). The bands and domains are used to organise assessment materials, student profiles, teaching materials, lesson planning, and so on. At the same time, we find it very useful to highlight the links among the domains in each band and between domains in different bands. For example, domains 2A and 2C link strongly with 3A and 3B respectively.

Table 2. The bands and domains of the P–4 Learning Framework in Number.

Broad bands of number learning	*Approx. ages*	*Domains of number knowledge*
1. Very Early Number (VEN)	2 to 5	Very early number
2. Early Number (EN)	4 to 8	2A: Early number words and numerals
		2B: Counting and early arithmetical strategies
		2C: Early grouping and structuring
3. Middle Number (MN)	6 to 10	3A: Number words and numerals
		3B: Structuring numbers 1 to 20
		3C: Conceptual place value
		3D: Addition and subtraction to 100
		3E: Multiplication and division

Assessment schedules, task groups and models

There are assessment schedules for each domain. The assessment schedules consist of groups of assessment tasks – task groups. Each task group consists of a set of closely related tasks and has the purpose of enabling the documentation of the student's current knowledge (Wright, Martland & Stafford, 2006). For example, Figure 1 shows Task Group 4 from Schedule 3C, which assesses the domain of Conceptual Place Value. Within each task group, after posing an initial set of tasks, the teacher chooses to pose easier or more difficult tasks, according to the child's responses on the initial tasks.

Task group 4: Incrementing by ten without materials
For each task, show the number on the card, and ask *Which number is ten more than this?*
Initial tasks in the range to 100:
40 90 79

If student is incrementing by ten without counting by 1s, try tasks in the range to 1000:
356 306 195

If student is incrementing by ten in the range to 1000, try a task across 1000:
999

Figure 1. A task group on incrementing by ten, from Assessment Schedule 3C on conceptual place value.

There are models of learning for each domain. A model is a table that sets out a progression of student's learning across up to seven levels. For example, Table 3 shows the model for the learning of Conceptual Place Value. Table 4 lists all nine models in the learning framework, and the number of levels in each model. By administering an assessment schedule and analyzing the video record, a teacher can profile a student's knowledge of a domain in terms of levels on relevant models.

Table 3. Model for learning of conceptual place value.

Level 0	Emergent inc/decrementing by ten
Level 1	Inc/decrementing by 10 off the decuple with materials
Level 2	Inc/decrementing by 10 formal to 100
Level 3	Inc/decrementing by 10 formal to 1 000

Table 4. The nine models of the learning framework.

Model name	*Acronym*	*Range of levels*
Forward number word sequences	FNWS	0 - 6
Backward number word sequences	BNWS	0 - 6
Numeral identification	NID	0 - 5
Stages of early arithmetical learning	SEAL	0 - 5
Structuring numbers 1 to 20	SN20	0 - 6
Conceptual place value	CPV	0 - 3
Addition and subtraction to 100	A&S	0 - 6
Multiplication and division strategies	M&D	0 - 6
Multiplication basic facts	M-BF	0 - 4

Table 5. Schedules and models for each domain.

Bands	*Domains*		*Schedules*	*Models*		
Early Number	2A	Early Number Words and Numerals		FNWS	BNWS	NID
	2B	Counting and Early Arithmetic Strategies	*Sch. 2*	SEAL		
	2C	Early Grouping and Structuring		SN20		
Middle Number	3A	Number Words and Numerals	*Sch. 3A*	FNWS	BNWS	NID
	3B	Structuring Numbers 1 to 20	*Sch. 3B*	SN20		
	3C	Conceptual Place Value	*Sch. 3C*	CPV		
	3D	Addition and Subtraction to 100	*Sch. 3D*	A&S		
	3E	Multiplication and Division	*Sch. 3Ea* *Sch. 3Eb*	M&D M-BF		

Table 5 shows how the domains, schedules, and models are linked. The table sets out for each domain, the schedule(s) which are used to assess knowledge of that domain, and the model(s) which are used to profile the levels of knowledge of that domain.

Teaching tables and topics

For each domain of learning a corresponding teaching table lays out, in summary form, all the teaching topics for the domain. Figure 2 provides an example of a segment of the teaching table for the domain of Conceptual Place Value. A topic is a small, important aspect of the learning domain. A segment of one lesson can focus on a whole topic and a student typically can make progress on a topic over just a few lessons. At the same time, each topic is sufficiently significant, that achieving facility in a topic entails real progress in number knowledge. Each domain has about 10–20 topics. The teaching table lists the topics in an order indicative of the progression of instruction.

Topic	**Teaching procedures**			
RANGE I: 0 to 130				
Inc/decrementing by 10s	*Bundling sticks shown* Say each number	*Bundling sticks screened* Say each number		
Inc/decrementing by 1s and 10s	*Bundling sticks shown* Extend to switched, multiple, & mixed units	*Bundling sticks screened* Extend to switched, multiple, & mixed units	*Sticks & arrow cards* Say numbers, and build with arrow cards	…
RANGE II: 0 to 1000				
Inc/decrementing by 10s	*Dot materials shown* Say each number.	*Dot materials screened* Say each number.		

Figure 2. A segment of the teaching table for Domain 3C Conceptual Place Value.

Teaching procedures and instructional settings

For each topic in a teaching table, the table lists the associated teaching procedures. A teaching procedure sets out in an exemplary way, a teacher's words and actions for a segment of a lesson. An extensive set of teaching procedures is available (Wright, Martland, Stafford & Stanger, 2006). Teaching procedures typically involve the use of an instructional setting consisting of instructional materials especially designed or customised. Coming to use the settings appropriately and judiciously involves significant learning on the part of teachers.

Each teaching topic typically has three or four associated teaching procedures. Each teaching table lists the teaching procedures in an order indicative of the progression of instruction. Thus, a teaching table lays out all of the topics and procedures for a domain, in one or two pages, such that the important instructional progressions through the domain of learning are clearly set out across the table.

Professional development resources

The pedagogical tools described above can be regarded as professional development resources. The professional development workshops focus on learning to use the tools and reflecting on their use, and include significant time allocated to demonstrations and discussions of the use of the tools. In this way the pedagogical tools become tools for teachers' learning. In addition, extensive use is made of video exemplars to support teachers' learning. These exemplars include video excerpts from assessment interviews and teaching sessions. Examples of this approach are the use of video excerpts (a) to practice assigning a student's level on a particular model in the P–4 LFIN and (b) to demonstrate a set of inter-related teaching procedures from the teaching tables.

Evaluation of the Mathematics Intervention Specialist Project

Connected to the development and implementation of the Mathematics Intervention Specialist Project is an evaluation of the implementation of the program within the school settings. The evaluation is being conducted, by Australian Catholic University, independently of the program. The evaluation undertakes to give feedback on what happens when MISP is used with yearly cohorts of teachers over a four-year period; the following central questions guide the collection of data:

> Did the teachers' pedagogical knowledge increase with regard to the variety, and appropriateness of strategies they can use for intervention?
> Did the teachers' depth of knowledge of number grow as they undertook the program?
> Did students learn how to 'cope' (change in attitude, confidence and demeanour) with mathematics both in the short and long term?
> Did students begin to learn some of the underlying values of mathematics (persistence, rationality and questioning skills)?
> Did students' knowledge of number improve? (Clarkson, 2009).

In evaluating MISP, the following methods of data collection (along with the student assessments referred to earlier) will be used to monitor the effectiveness of the program:

1. Questionnaires for teachers participating in the intervention program
2. Self reports in digital video format from the intervention teachers
3. Analysis of the teachers' written responses and observations during the professional learning sessions
4. Focus group interviews with the intervention teachers
5. Individual interviews with the intervention teachers
6. Written background profiles of students by the intervention teachers
7. Incidental classroom observations
8. Written background profiles of students by classroom teachers
9. Individual interviews with students who participate in the intervention program.

Usefulness to school systems

The Catholic Education Office Melbourne (CEOM) has been presenting various professional learning programs for primary and secondary schools in the last 12 years and clearly MISP is having a most positive effect. Still in its initial stages, the program is having an effect in increasing student performance in some schools. The evidence for this is schools' improvements in the National Assessment Program Literacy and Numeracy (NAPLAN). Importantly, there are a number of themes emerging in the program. This is articulated in the first-year evaluation report from Australian Catholic University. The following themes are emerging in mathematics education and professional learning for the Archdiocese of Melbourne:

1. Increase in teachers' enthusiasm
2. Teachers drawing on their previous experience to enhance their intervention teaching
3. Number intervention teachers feeling part of a learning community in number improvement
4. The fine-grain of mathematics is being covered in the professional learning. Teachers are learning about how to connect the number concepts in a stronger way.
5. Separating the logic of mathematics and praxis
6. The use of specific resources to assist in the teaching of number
7. The importance of number intervention teachers connecting with the classroom teachers
8. Some intervention teachers have seen the value of connecting with parents
9. Teaching contexts within schools and the importance of having an allocated space to teach the number intervention program.

Conclusion

From 2011, monitoring improvements in student performance will be important for the program. However, observing the above nine themes will also have important implications for future professional learning in mathematics education. We believe that MISP: (a) constitutes an important response to the need for intervention in the number learning of low-attaining students; (b) is a successful example of a long-term collaboration involving researchers, teachers and system leaders; and (c) has significant potential value and usefulness for school systems.

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UNDERSTANDING DIVISIBILITY: HOW CAN WE RECOGNISE IF A NUMBER IS DIVISIBLE BY NINE?

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This workshop explores understanding of divisibility rules as part of helping students become advanced multiplicative thinkers. It is based on recent research with Yr 7-8 teachers who were observed teaching a group of students a rule for divisibility by 9 and a way of proving why the rule works using grouped materials (NZ Numeracy Project Book 6 lesson p. 70). Many students (and teachers) knew the rule, but did not understand why it works. After the lesson, students' understanding of multiplication and division deepened. The workshop looks at divisibility rules and how they can be used in the mathematics classroom.

Introduction

Understanding multiplication and division is an important part of the mathematics curriculum. According to Baek (1998, p. 151), "understanding multiplication is central to knowing mathematics." The importance of multiplicative thinking for understanding later mathematics is well established (Beckmann & Fuson, 2008; Charles & Duckett, 2008; National Council of Teachers of Mathematics, 2000, n.d.; Young-Loveridge, 2011). It has been argued that students need to be multiplicative thinkers to engage with the formal algebra presented in secondary schools (Baek, 2008, Brown & Quinn, 2006; Lamon, 2007).

The term multiplicative thinking refers to a particular type of thinking used to solve a range of problems, including multiplication, division, fraction, ratio, and other mathematical concepts involving multiplication and division (Mulligan & Vergnaud, 2005; Vergnaud, 1983). There are many different definitions of multiplicative thinking (e.g., Clark & Kamii, 1996; Siemon, 2005; Steffe, 1992). The following definition is used in the Numeracy Development Projects (NDP) resource book for New Zealand teachers (Ministry of Education, 2008, p. 3):

> The construction and manipulation of factors (the numbers being multiplied) in response to a variety of contexts; [and] deriving unknown results from known facts using the properties of multiplication and division [e.g., commutative, associative, distributive, inverse].

Structure and pattern are at the heart of learning multiplication and division (Mulligan & Mitchelmore, 1997, 2009; Young-Loveridge & Mills, 2010).

Multiplication is a process involving groups of groups. Evidence has shown that students' appreciation of structure and pattern is very important, and is related to their understanding of mathematics (Mulligan & Mitchelmore, 2009). Low achievers do not seem to notice structure and regularity in mathematics the way that high achievers do. Hence is important for teachers to draw students' attention to structure and pattern because it can bring about substantial improvement in their mathematical understanding and learning.

Initially, students may count by ones to solve simple problems such as two biscuits on each of five plates. Children who know how to skip count by twos could use this skill to count the five groups of two, as in: two, four, six, eight, ten. Skip counting can then be linked to repeated addition (2 + 2 + 2 + 2 + 2 = 10). Eventually students come to realise that multiplication can effectively shorten the repeated addition process to 5 x 2 = 10 (5 groups of 2 make 10). This understanding forms the foundation for students to be able to apply simple multiplicative (part-whole) strategies to combine (using multiplication) or partition (using division) whole numbers. Eventually students' knowledge of basic facts and understanding of partitioning strategies enable them to choose flexibly from a broad range of different part-whole strategies to find answers to multiplication and division problems (Advanced multiplicative thinkers; see Ministry of Education, 2007).

Advanced multiplicative thinkers should be able to partition a dividend in various ways to enable them to use known multiplication/division facts to work out which parts comprise the quotient (the result of division) (see Young-Loveridge, 2011). For example, in solving 72 ÷ 4 = , they could halve the 72 and work out that each part of 36 consists of 9 groups of 4, then double the 9 to get the final quotient of 18. Alternatively, they could split the 72 into 40 and 32 to work out that 10 groups of 4 plus 8 groups of 4 give the final quotient of 18. Other possible ways to partition 72 include 48 and 24, 60 and 12, or 64 and 8. By rounding 72 up to 80, then taking 2 groups of 4 (i.e., 8) away from the 20 groups of 4 (20 – 2), the final quotient of 18 can be reached using a rounding and compensation strategy.

Understanding division is critical for being able to work in the domain of rational number, including fractions, decimals, proportions and ratios. Work with whole-number division is important before students go on to work with rational number. However, teachers frequently spend far more class time on multiplication than on division. Although division problems can be solved using a Reversibility strategy and building up the groups using multiplication, an understanding of division concepts themselves is necessary for students to be able to work flexibly with division strategies.

Mathematics has many little 'tricks' that can be used to 'work things out', such as whether or not a large number is divisible by single-digit values. Divisibility by nine can be determined by adding up the digits in a multi-digit number to check whether the sum is nine, or a multiple of nine. The lesson that is the focus of this workshop takes the students through a process for proving why the divisibility rule for nine works the way it does. Not only do the students learn why the trick works, but they also deepen their understanding of multiplication and division as part of the process.

The aims and focus of the workshop

The purpose of this workshop is to take participants through the activities that were the central part of the lesson that was observed as part of the research. A brief overview of the research is also provided to share some of the key findings of the research. Students' views about the lesson are also described.

The workshop activities

In the workshop ,we start by asking participants to determine whether or not the number 8514 is divisible by 9, and if it is, how you know. Next we ask whether or not the number 5142 is divisible by 9, and "how you know".

Next, we examine some multiples of nine from the "times nine" (x9) table: 18, 27, 36, 54, 81, and ask what these numbers have in common.

We then focus on the number 27, and ask participants to make this using plastic beans some of which are grouped in tens inside translucent film canisters. We ask how many groups of nine are in the number 27, and how participants work out their answers.

The next number we ask participants to make with the beans is 45. Again, we ask how many groups of nine are in 45, and how participants work out their answers. We want to know whether the rule or method used to work out the number of nines in 27 is the same rule that was used to work out the number of nines in 45. We draw attention to the way that for each group of ten beans, there is one group of nine, and one leftover "one." When the two leftover "ones" from each "ten" in 27 are combined with the seven single beans, the total forms a further group of nine. Likewise, with 45, the four leftover "ones" from each "ten" can be combined with the five single beans to create another group of nine.

We then ask participants to make the number 32, and again work out the number of nines in the number. This time there are some beans left over and we discuss why this happens and how this is related to the fact that the sum of the digits (3 + 2) does not equal 9.

The next number to be made with the beans is 135. Again we ask how many groups of nine can be made, and where they come from. Figure 1 shows the way the beans can be used to make the number 135, with ten canisters of ten beans on one ten-frame to show 100, three canisters of ten beans on the second ten-frame to show 30, and the 5 loose beans on the third ten-frame.

Figure 1. Representation of the number 135 using ten-frames to show 10 tens (100), 3 tens (30), and 5 single beans.

The next step involves taking one bean out of each of the canisters of ten beans to reduce the number of beans in each canister to nine beans. The tenth bean is placed on top of the canister, as shown in Figure 2.

Figure 2. For each of the groups of ten beans in a canister, one bean is removed and placed on top of the container, leaving a group of nine beans in each canister.

There are now 10 single beans on top of the ten canisters of ten beans representing 100. We can take nine of those beans and put them in a group beside the canisters to make an eleventh group of nine beans (99 beans altogether in 11 groups of nine), and there is one bean left over that is still on top of the group of ten canisters (see Figure 3). We draw participants' attention to the way that the leftover beans on top of the canisters correspond directly to the digits in the number itself; that is, 1 for the one hundred, 3 for the thirty, and 5 for the five single beans.

We can now take the leftover beans on top of the canisters and place them with the five single beans on the right-hand ten-frame to show one further group of nine that can be made (see Figure 4). The right-hand ten-frame contains the beans that correspond to the sum of the digits in the number 135 (1 + 3 + 5 = 9).

Figure 3. An eleventh group of nine can be made out of the 100 beans, leaving one bean on top of a canister to show the leftover "one" after the 11 groups of 9 (99) have been made out of 100.

Figure 4. The four leftover "ones" have been moved from on top of the canisters and combined with the five single beans in the right-hand ten-frame to show one further group of nine that can be made.

In Figure 5, we have placed the eleventh group of nine in a canister and labelled each of the groups of nine in a canister with the digit 9. Again, the right-hand ten-frame contains the beans that correspond to the sum of the digits in the number 135.

Figure 5. The eleventh group of nine from 100 has been placed in a canister and all the canisters of nine have been labelled with the digit 9.

For every 100 beans, there are 11 groups of 9 beans (99) and one bean left over. For every 10 beans there is a group of 9, and one bean left over. If we add up the leftover "ones" and the single beans (1 + 3 + 5), we find there are nine beans leftover, which makes one more group of nine. Hence the digit in any position in a multi-digit number from the tens upwards tells how many leftover "ones" there are after the groups of nine are made. For example, in the number 30, there are 3 groups of nine and 3 leftover "ones". In the number 100, there are 11 groups of nine and one leftover "one". In the number 200, there are 22 groups of nine and two leftover "ones". In the number 1000, there are 111 groups of nine and one leftover "ones". In the number 10 000, there are 1111 groups of nine and one leftover "one", and so on. Hence the digit in a multi-digit number not only tells how many groups of ten there are in a particular place-value position, but also how many leftover "ones" there will be after groups of nine are made.

Next, we try the number 162, asking participants to make the number using the ten-frame, and the beans in canisters. We ask how many groups of nine can participants make, where do they come from, and what happens this time. We then repeat the process for the number 132.

We conclude the workshop by stating the divisibility rule for nine: If the digits in a certain number add to nine (e.g., 324) or a multiple of nine (e.g., 37 224), then the number is divisible by nine.

We take some more numbers to check the divisibility rule: 279, 298, 345, 1467, 62316. We ask which of those numbers are divisible by nine, and how you can tell. We then ask participants to explain their reasoning. We go back to the number 132 that we had previously found was not divisible by nine, and ask participants to check whether or not it is divisible by three, and how they work it out. Finally we ask participants to comment on whether they see any links between divisibility by nine and divisibility by three.

The research

Method

Participants

Seven female teachers working at the year 7–8 level (11- to 13-year-olds) from four schools (three intermediate schools and one full primary school that served communities ranging from low to high socio-economic status) agreed to participate in the study. Teachers varied in years of teaching experience from approximately 1.5 years to 20 years. Likewise, the teachers' experience of working with the NDP approach ranged from one to seven years. Each teacher chose a group of students to work with on enhancing multiplicative thinking. A combined total of 46 students took part in the lessons and assessments used in the study.

Procedure

The researchers visited each teacher twice. At the first visit, the students were given written assessment tasks to complete before the lesson, with instructions to "explain how you worked out your answer. Where possible, draw a diagram to help show your thinking." The eight tasks included: three concerning whole-number multiplication; two that involved deriving answers from information given and known number facts (e.g., If $4 \times 30 = 120$, what is 4×28? If $5 \times 9 = 45$, what is 5×18?); and one multi-digit multiplication problem (e.g., What is 11×99?). The teacher then taught a lesson on multi-digit multiplication while the researchers observed. The following week, the teacher taught a lesson on proving why the rule for divisibility by nine works. Teachers adapted the lesson to suit the knowledge of the students in the group. When teaching the lesson, the teacher wore a digital audio recorder with lapel microphone to record as much as possible of the dialogue with the students. After the lesson, the researchers talked to the students, and later to the teacher, about their experiences in the lesson, in order to explore their perceptions of the lesson and any confusion that may have arisen during the lesson.

After the second lesson (on divisibility by nine), the students and their teacher were interviewed again and the students were given written assessment tasks related to the two lessons. In most cases, the interval between the two lessons was between two to three weeks. All teachers taught the same two lessons taken from the support materials for the NDP on teaching multiplication and division (Book 6). This workshop focuses on the second lesson, *Nines and threes* (Ministry of Education, 2008, pp. 70–72).

Findings

We observed during the lessons that most teachers used canisters of nine beans, and some also used ten-frames to show the structure of 10 tens making 100 in total. Most teachers began the lesson using multiples of nine from the "times nine" tables (e.g., 18, 27, 36, 45, 54, 63). They asked their students to make the number using the materials, and then focused on the "tens" digit to work out how many groups of nine there were and how many "ones" were left over. Figure 6 shows representations of the numbers 27 and 81 using the canisters of ten and counters for the "ones".

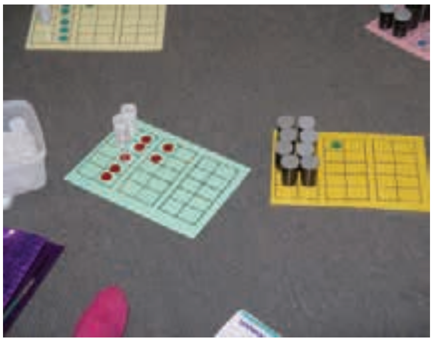

Figure 6. Representations for 27 and 81 using canisters of ten and counters for "ones"

Teachers drew diagrams in the group workbook to show the leftover "ones" coming out of the groups of ten to create the groups of nine. By recording the digits representing the number below the diagram, students could see how the leftover "ones" correspond to the digit in the original number (see Figure 7).

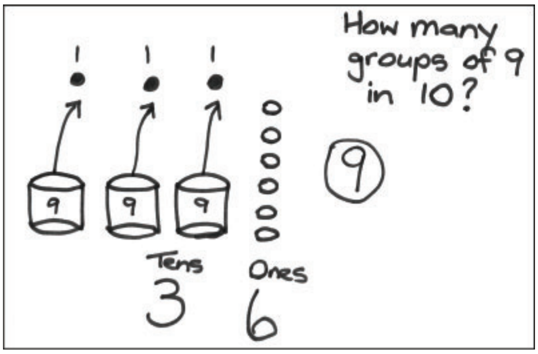

Figure 7. A copy of Ann's diagram in the group workbook showing a leftover "one" coming out of each group of 10, leaving a group of 9 in each canister, and the 3 leftover "ones" plus 6 "ones" totaling 9.

Figure 7 presents a copy of the diagram that Ann put in her group workbook showing the process of combining the 3 leftover "ones" from the tens with the 6 original "ones" to make a total of 9. At the top of the diagram, she had written: "How many groups of 9 are in 10?" Each drawing of a canister has the digit 9 inside it, and the digit 1 is written above each leftover "one" coming out of the canisters of ten to leave 9.
Figure 8 shows a diagram for the representation of 135.

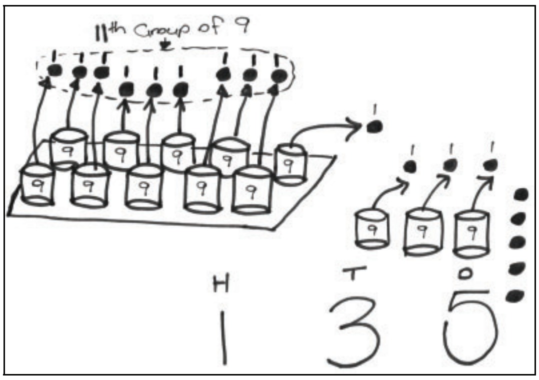

Figure 8. A diagram showing the representation of 135, with 10 groups of 9 in canisters, an 11th group of 9 made up of 9 leftover "ones", one leftover "one" for 100, 3 leftover "ones" for 30, and 5 "ones" for 5.

Some of our teachers told their students that it did not matter how many groups of nine there were. Our most experienced teacher, Ann, made a point of drawing her students' attention to the fact that with numbers in the hundreds, there is an eleventh group of nine that can be made for each hundred. We think this is an important idea for the students to understand and one which should help to deepen their understanding of multiplication and division.

Up to 99, the number of leftover "ones" on top of the canisters of ten corresponds to the number of canisters. However, once the number goes beyond 99 (i.e., 100 or higher), an eleventh group of nine must be made for each 100, eleven more groups of nine must be made for each 1000, and so on.

A few of our teachers did not spend enough time reading the description of the lesson in the NDP resource book, or trying out the materials to ensure that they understood how the lesson was to proceed. The lessons of those particular teachers tended to end abruptly, once the teachers realised that they could not complete the lesson, as their understanding of the proof for the divisibility rule was not solid enough. They had made the mistake of assuming that they could simply follow the lesson description in the resource book while they were teaching the lesson. Unfortunately, they discovered that the resource book was not prescriptive in the way that they had assumed.

Interviews with the students in Ann's group revealed how delighted they were at the end of the lesson, having understood how to prove the divisibility rule for nine.

Conclusions

This activity contains far more deep learning than at first appears. When an easy 'trick' with numbers is unpacked, it offers the opportunity for deepening students' understanding of multiplication and division. However, it is vital that teachers make sure that they understand the lesson fully, including how to use the materials, before they try it with their students.

Acknowledgements

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SOME PRINCIPLES AND GUIDELINES FOR DESIGNING MATHEMATICAL DISCIPLINARY TASKS FOR SINGAPORE SCHOOLS

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This paper is about the methods and process of designing disciplinary tasks, which have the distinctive features of their emphasis on contextual aspects. In order to make the development of further disciplinary tasks more efficient, it is helpful to have a set of useful guiding principles based on our first-hand experiences in the Singapore Mathematics Assessment and Pedagogy Project. These principles should also be useful for school teachers who want to design such tasks for themselves. In this paper, we use concrete examples from the completed tasks to illustrate the points delineated in the guidelines.

Introduction

Assessment is one of the most important components in education, in particular mathematics education. In the past decades, various alternative approaches to assessment have been considered for evaluation of learning (see Fan, 2002; Hargreaves, Earl, & Schmidt, 2002; Hogan, 2007; Kulm, 1994; Williams, 1998.)

In the last three years, the Singapore Mathematics Assessment and Pedagogy Project (SMAPP) team has successfully designed a number of assessment tasks and has tried them in five local schools. Fan et al. (2010) have given a summary of some of the tasks designed in the early stage of the project.

These tasks, referred to as 'disciplinary tasks' in SMAPP, are designed to provide a new format of mathematics assessment for Secondary One (grade 7) students. One of the distinctive features of these tasks is the emphasis on contextual aspects. The design of each task is based on a real-life scenario closely linked to the Singapore context. The assessment problems are then posed in progression from easier to more difficult ones. The initial version of each task is reviewed and refined several times, based on feedback from reviewers, school teachers and students. Schools can use either hard copies or online version of finalised tasks.

This paper is about the methods and process of designing such tasks. In order to make the development of further disciplinary tasks more efficient, it is helpful to have a set of useful guiding principles based on our first-hand experiences. These principles should also be useful for school teachers who want to design similar tasks on their own.

In this paper, we use a few concrete examples from the completed tasks to illustrate the points delineated in the guidelines.

The paper is organized as follows. In the next section, we list and explain some criteria for an ideal disciplinary task that serve as the general foundation for the guidelines. In the following two sections, we use some examples from the designed tasks to elaborate on suggestions for initiating and developing an idea into a satisfactory, implementable task. More information can be found from the SMAPP website (Singapore Mathematics Assessment and Pedagogy Project, 2011).

What is a good quality disciplinary task?

In order to design a good quality disciplinary task, we need to identify some specific criteria. We propose that a quality disciplinary task should have the following attributes.

Links to real life

A special feature of disciplinary tasks that distinguishes them from the traditional assessment problems is their contextualized content. We use local events, places, data or commonly encountered names and terms to make a scenario more realistic and relevant to students' daily life. The following are some examples taken from the introductions of designed tasks.

In the task *Malacca Trip*, Malacca and Yong Peng are two cities in Malaysia not far away from Singapore, and Ang Mo Kio and Bedok are towns in Singapore:

> Aziz and Bryan are planning to drive to Malacca during the June holidays. Aziz lives in Ang Mo Kio Avenue 1 and Bryan in Bedok South Avenue 1. On the way from their homes to Malacca, they plan to meet at the Woodlands Checkpoint and again at the rest point in Yong Peng. You are to help in the planning of the trip by working through tasks.

Singapore is a small island country that has shortage of water resources. The task *Water Water Water!* is based on Marina Reservoir which is the most recently built reservoir in Singapore:

> The following picture is obtained from *Google Maps*. It shows the Marina Reservoir near the Singapore River and the Kallang River. In this task, you are to estimate the area of the Marina Reservoir and make some calculations based on your estimation.

Real and relevant data

The data used in the problems should be realistic and obtained from reliable sources. Fictitious data should be avoided as far as possible. Real data provide students with a realistic sense of how mathematics can be applied in the real world.

For example, the data 1 262 000 m^3 in the following question in the task *Water Water Water!* are obtained from a government source (Public Utilities Board webpage):

> According to national statistics, water consumption in Singapore is about 1 262 000 m^3 a day in 2008. Assuming Singapore maintains these rates of water consumption, how long will the supply of water (as approximated in B3) from the Marina Reservoir last?

The following question is about the areas of different sizes sheets of papers. These are the actual sizes of the papers students are using daily. After doing this task, students will have a better understanding of the standard sizes of paper in daily use:

> Given that the size of an A0 sheet of paper is 0.841 m by 1.189 m, find the area of an A0 sheet of paper.

Curriculum connection

As developed tasks will be used by schools for embedded assessment, it is important that the tasks are connected to the school curriculum. It is also necessary to consider when the tasks in the semester will be implemented and make sure the students have all the core prerequisite content knowledge. It is thus necessary to list the prerequisite knowledge for doing the tasks. Furthermore, the terms and definitions used in the tasks must be the same as those given in the relevant national syllabus.

For example, the following is part of the teaching notes describing the prerequisite for the task *Malacca Trip*:

> Pre-requisite/Content are: Concept of average speed; knowledge of computation and conversion involving distance, speed and time.

In the following question, originally students were asked to solve the inequality, but later they are just required to form an inequality, as they still have not learned the solution of inequality at that stage.

> If Bryan wants to reach Yong Peng before 09 55, what should his average speed be? Write down an inequality in v using your answer from part (a).

Also, in denoting times, we changed from the original “09:55” into the current “09 55” to keep the notation consistent with the school text book being used.

Multiple competencies and content knowledge assessment

One advantage of disciplinary tasks (especially complex ones) is for assessing the multiple mathematical competencies of students and their comprehensive abilities to apply what they learn in the classroom. The task problems should then be designed to serve these purposes. The competencies we usually focus on include (i) understanding problems and extracting information from them; (ii) constructing mathematical models of real life problems; (iii) computation and reasoning; (iv) communication using appropriate representations and means such as graphs, tables, algebraic expressions and functions.

For example, the following are the competencies assessed in *Malacca Trip*: (i) basic skills involving speed, distance and time; (ii) ability to represent a situation (x minutes before 07 30) mathematically using algebra; (iii) ability in calculations and solving algebraic equations; (iii) ability to translate scenerio-based situation into algebraic expressions; (iv) ability to use the correct inequaliy sign to formulate linear inequalities; (v) ability to formulate linear equations and linear inequalities.

Another task, *Up Down, Up Down,* can be used to assess whether students can

- make a simple prediction based on the information provided by graphs;
- make reasonable judgements about solutions.

As an illustration, the following are some questions in *Up Down, Up Down*:

> The percentage increases in population for the two periods shown below are approximately the same. … Do you think that the actual population increases for these two periods are approximately the same? Give a reason for your answer.
>
> Based on the graph from question B3, mark the following statements with True or False with regard to the population of Singapore.
> (1) There are fewer residents of working age supporting each resident above 65 years old now compared to 10 years ago.

(2) The percentage of elderly (65 and above) in the population (15 and above) has decreased over the years.
(3) There will likely be more elderly residents than working age residents in 10 years time.

Enriching student experience

Besides practicing and learning mathematics skills and knowledge, attempting a disciplinary task may also provide chances for students, to gain some social experiences and learn more about the environment and society. The enriching experience will also increase the students' motivation and interest for attempting such task.

For instance, by going through the task *Water Water Water!*, students will have a better understanding of how precious water is to Singapore; which are the seven reservoirs and their locations in Singapore; and their sizes and roles in supplying water.

In the task *Up Down, Up Down*, questions are posed based on data including ethnic composition of residents, population growth rates and old age support ratio in Singapore. By doing this task, students will also learn many terms such as demography, ethnic, old-age support ratio, total fertility rate.

While in the task *Paper Recycling*, students will find detailed information of the actual sizes of different types of sheets of paper and their weights, as well as the number of trees and the amount of water needed to produce a given quantity of papers.

The actual *Google* map is used in the task *Malacca Trip*, so that students will have an accurate idea about the locations and distances between some of the places in Malaysia.

Scaled levels of difficulties

Assessment based on disciplinary tasks is very different from traditional forms of assessment. Most of the students are not familiar with it and need practice to get used to it. It is therefore necessary to arrange the questions in order of increasing difficulty. In our design, the first part usually consists of warm up questions, giving a chance for students to familiarize themselves with the scenario involved and to recall associated basic skills and knowledge. The parts that follow will usually consist of more difficult and challenging questions. Sometimes, open-ended questions are also included.

Getting started with designing a topic

From our experience, the crucial and difficult part is the initial stage. There are two suggested approaches: (i) start with a topic and then make up a suitable and interesting scenario; (ii) start with a rich scenario and then pose the relevant mathematics questions.

Start with a topic

Suppose we want to assess students' ability to solve speed and related problems. After considering a few possible scenarios, we found that the one concerning a trip to Malaysia is the ideal one. For many Singaporeans, travelling to the neighbouring country Malaysia is a favourite way to spend their holidays. So, planning a trip to a Malaysian city such as Malacca provides us with a real life scenario to pose mathematics questions. The scenario is rich in the variables that we need to consider in planning a trip: departing and arriving times, resting time, etc.

> Aziz wants to meet up with Bryan at 07 50 at Woodlands Checkpoint, 20 km away from Ang Mo Kio Ave 1. If he travels at an average speed of 60 km/h, find his departure time from Ang Mo Kio Ave 1.

Start with a scenario

Sometimes we can start with a scenario leading to good questions. In line with the special feature of linkage to real life described in the previous section, current affairs and local events provide a rich source of rich scenarios. With a local scenario, National Education elements can also be incorporated into the task. Below are some examples.

In the arithmetic task *Water Water Water!*, the idea of approximating the irregular shape of Marina Reservoir (Figure 7) using geometrical shapes was inspired by the completion of the Marina Barrage in Singapore. This task illustrates an interesting application of the mathematical technique of estimation.

In the statistics task *Up Down, Up Down* concerning Singapore's aging population, the National Day Rally 2008 by Singapore's Prime Minister provided a backdrop. This lead to questions on Singapore demographic issues. For example, the declining birth rate motivated questions based on the line graph of total fertility rate (TFR) shown in Figure 1, such as:

> There is a sharp decline in the TFR from 1972 to 1975. Search the web for possible reason(s) to explain this decline.

Notice that this sharp decline in TFR from 1972 to 1975 also allows National Education elements and the use of IT (searching the internet) to be incorporated into the task. The 'spike' feature of TFR in the Chinese 'dragon' years (e.g., 1988 and 2000) also inspired the following question:

> In 1976, there is a 'spike' in the TFR. Identify the other years in the line graph where there are spikes in the TFR. By observing the graph, can you predict the year for the next possible spike?

To deal with the aging population, one aspect is the adjustment of immigration policies. The associated immigrant issues then motivated question on the interpretation of the bar chart shown in Figure 2. This bar chart, a comparison of populations of citizens, permanent residents and non-residents, is a further illustration about how a rich scenario can generate good questions. As this chart is not commonly seen in school textbooks, it also serves the purpose of assessment as learning.

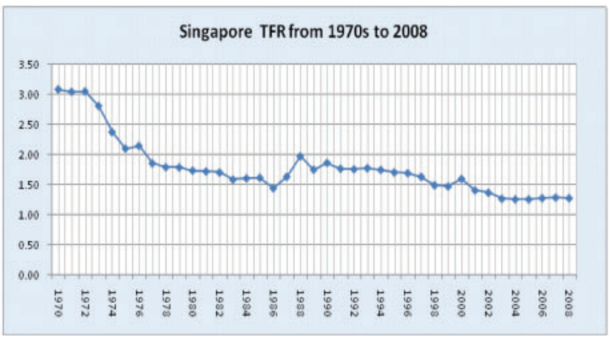

Figure 1. Singapore total fertility rate, 1970–2008.

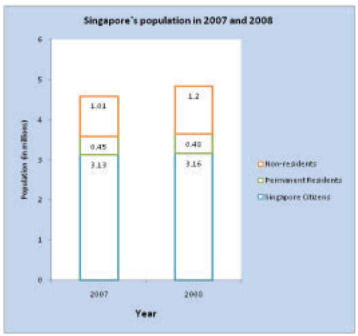

Figure 2. Singapore population, 2007–2008.

Posing warm-up questions

A typical disciplinary task is often divided into sub-tasks: Task A, Task B, and so on. The first questions in each sub-task should have straightforward solutions. These are warm-up questions that allow students to recall the required concepts, and get familiar with the scenario concerned. Below are some examples.

In the task *Malacca trip*, a natural setting for the warm-up questions is the journey from home to the Woodlands Checkpoint. In doing these questions, students would need to recall basic formulae like ‘speed = distance/time’. After some practice with the basic skills, more challenging questions are then set for the journey from the Checkpoint to Yong Peng and eventually to Malacca.

The pie chart is one of the basic graphical representations taught in the early secondary curriculum. In the task *Up Down, Up Down*, an understanding of a pie chart showing ethnic composition of Singapore residents is used as the starting point:

> The pie chart below shows the ethnic composition of residents for 2008. The total number of residents was 3,642,700. Estimate the number of Chinese residents in 2008, correct to the nearest hundred thousand.

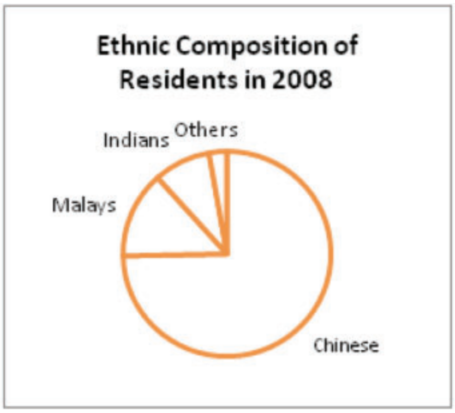

Figure 3. Ethnic composition of Singapore residents.

Notice that the pie chart utilizes the fact that about 75% of the Singapore residents are Chinese, thus infusing National Education into the question.

In the task *Water Water Water!*, we want students to get familiar with the scenario, and the idea of approximating the irregular shape of Marina Reservoir (Figures 7 and 8) using polygons. To this end, properties of triangles and quadrilaterals are recalled in the warming-up questions:

> The figures below shows the pieces used to approximate the area of the Marina Reservoir. Name the shapes of those figures.

Developing the task

As discussed in the previous section, the problem/topic or scenario for a task is often selected for its richness in the sense that it allows mathematical questions of different levels of difficulty to be asked to assess students’ multiple competencies and content knowledge. After the first few warm-up questions, the next few questions are usually less straightforward.

Consider the following question from Task A in the task *Water Water Water!*

The picture below shows an L-shape swimming pool. Its surface can be divided into two rectangular shapes, measuring 18 m by 16 m and 12 m by 10 m, respectively. Find the perimeter and area of the surface of the pool. (You may refer to the diagram shown below.)

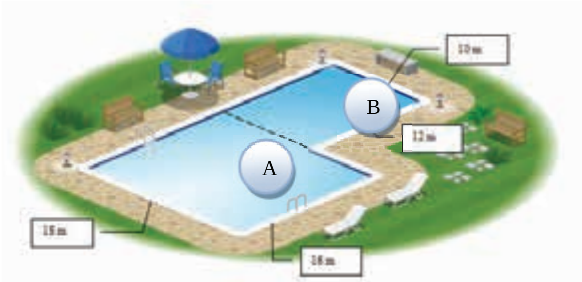

Figure 4. Swimming pool 1.

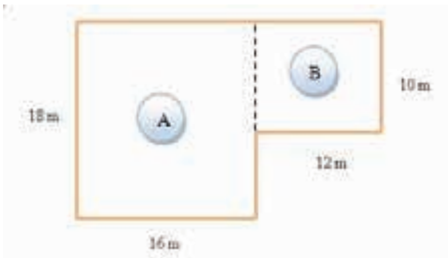

Figure 5. Swimming pool 2.

This is a straightforward question that requires only direct application of formulae for the perimeter and area of a rectangle. Instead of asking students to find perimeters and areas of swimming pools of different regular shapes, which is repetitive and involves the same skill, the context allows us to ask students to find the volume of the water in the pool. If the depth of the pool is a constant, then it is easy to calculate the volume of the pool water by finding the volumes of the two cuboids. In a real situation, however, the depth of the pool need not be a constant; for example, many pools are deep at one end and shallow at the other end. Therefore we can ask a slightly more demanding question for calculating the volumes of such pools, as in the following question in the same task:

The diagram below shows a three-dimensional view of the pool. The depth of water in Section A increases gradually from 0.9 m at the shallow end to 1.8 m at the deep end. The depth of water in Section B is 0.9 m throughout. Find the volume of water in the pool, assuming it is completely filled. Give your answer in litres.

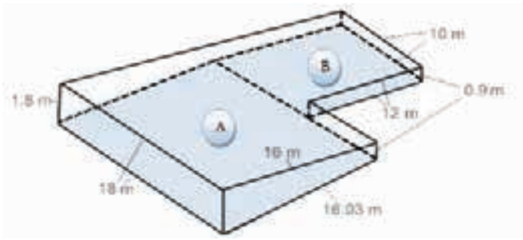

Figure 6. Swimming pool 3.

As a swimming pool needs to be drained and refilled once in a while, this leads to more challenging questions relating to the refilling of swimming pool:

Recall that the capacity of the pool is 496 800 litres.
(a) If the pool is empty, how much time is required to completely fill the pool at a rate of 800 litres per minute? Give your answer in hours and minutes.
(b) If water is pumped into the pool at a rate of x litres per minute, express the time needed to completely fill the pool in terms of x. Give your answer in hours."

In Task B of *Water Water Water!* the main objective is to introduce to the students the important idea of approximation in mathematics. At the beginning, students are asked to find the approximate area of the Marina Reservoir by using six different polygons that are provided:

> (a) Find the area of figures C to E (the areas of Figures A, B and F are given.)
> (b) Hence find the approximate area of the reservoir. Give your answer correct to three significant figures.

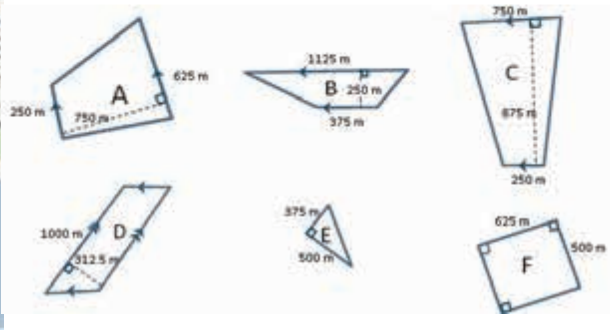

Figure 7. Marina Reservoir map. *Figure 8. Parts of Reservoir*

This question serves as a guide to help students solve the more challenging question of devising their own polygons to approximate the area of the reservoir, and the open-ended question of comparing the two approximated areas:

> (a) On the question paper provided, draw the shapes you would use (different from the diagram given in Question 1) to approximate the area of the Marina Reservoir.
> (b) Based on the shapes you have drawn, will the measure of the area of the Marina Reservoir be exactly the same as you obtained earlier? Give a reason for your answer.

The task *Water Water Water!* illustrates how a disciplinary task is developed from easy warm-up questions to the more challenging and open-ended questions. Because the scenario is rich, it gives rise to different types of questions that assess students' mathematical skills on different topics: mensuration, rates, and algebraic expressions.

The following is one the problems in this task:

> The bar graph below shows the amount of water consumed per person per day among households … in Singapore for four different years. The Public Utilities Board has taken a series of water conservation initiatives … targeted to reduce domestic water consumption per capita to about 155 litres per day by 2012. Based on the trend in the graph, do you think this target is achievable? Give an explanation for your answer.

The last part of a regular task generally contains some more challenging or open-ended questions. This is to facilitate the different needs of students and schools. They are not expected to be attempted by all schools.

Lastly, the following general points should be attended to when designing the tasks: Use plain words; avoid long sentences**;** make sure students won't require too much time to understand the questions; scaffold questions if necessary; where it is possible, pictures, tables or other forms of presentation should be used.

Conclusion

In this article we listed and illustrated some criteria for determining the quality of disciplinary tasks for Singapore secondary schools. Some of these criteria have already been applied in our design of the various tasks for the SMAPP project. At the beginning, we created a few tasks without any existing guidelines. Then when reviewers started to review and revise them, there were a lot of disagreements and arguments, and we needed several rounds of revision and modification before reaching an agreement. Having agreed criteria and guidelines helps to reduce the work of correction and revision of the tasks. In addition, more schools may choose to implement this type of assessment and consequently need to design new tasks by themselves. The criteria and illustrations shown in this paper could help with this work in the future. In one recent training workshop, some teachers already tried out these guidelines and used these to help with the design of some tasks.

Of course, our list of criteria is still incomplete and imperfect, so will need further refining and improvement. We hope this can serve as a good starting point for readers to use to frame the creation of their own tasks, based on the special needs and backgrounds of their students and curriculum.

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