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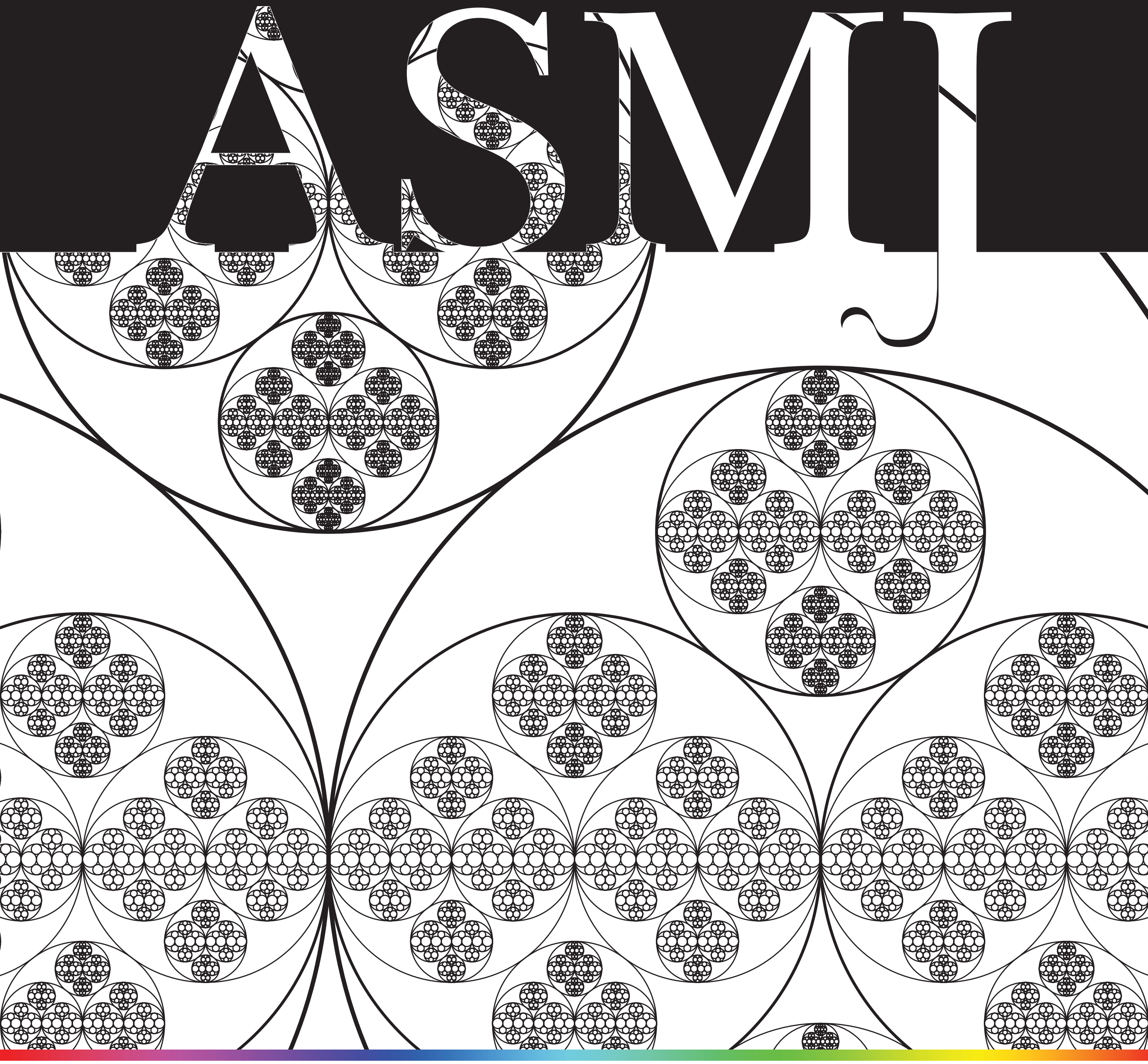

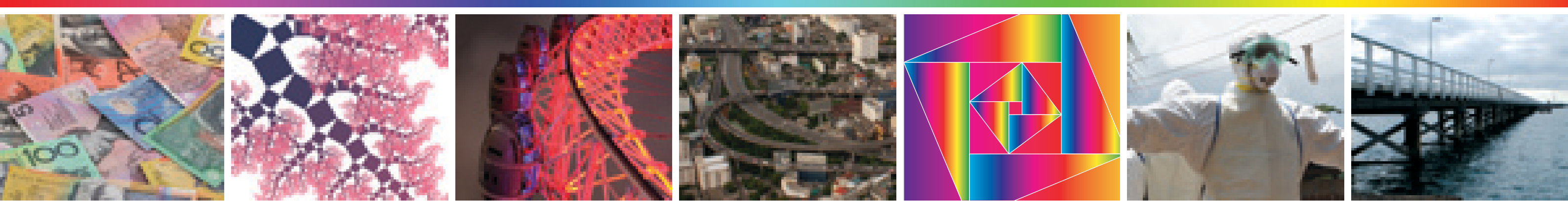

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All articles are subject to peer review. The following procedure is undertaken:

1. On receiving a paper, the editors decide if the paper is within the province of the journal, which is "the teaching and learning of mathematics in Years 11 and 12 of secondary school and the early years of tertiary education." The author(s) of the paper is notified and appropriately advised if not.
2. A potentially publishable paper is sent 'blind' to two referees with expertise related to the paper; the referees will usually be selected from the Editorial Panel and practising teachers and academics.
3. The Editors act in line with the referee reports in one of the following ways:
   (i) accept the paper for publishing 'as is';
   (ii) reject the paper outright, in which case the author(s) is notified;
   (iii) inform the author(s) of changes which need to be made before the paper is accepted for publication;
   (iv) make minor changes without prior reference to the author(s).

The gasket of circles: A fractal of circular nature

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Subdividing an equilateral triangle into four congruent triangles, then doing likewise to each of the three non-central triangles, and then again and again, leads to the Sierpinski gasket, from which the chaos game originated. An analogous procedure is hereforth applied to a circle, where a subdivision consists of two pairs of inscribed circles, with each circle tangential to the ones adjacent to it.

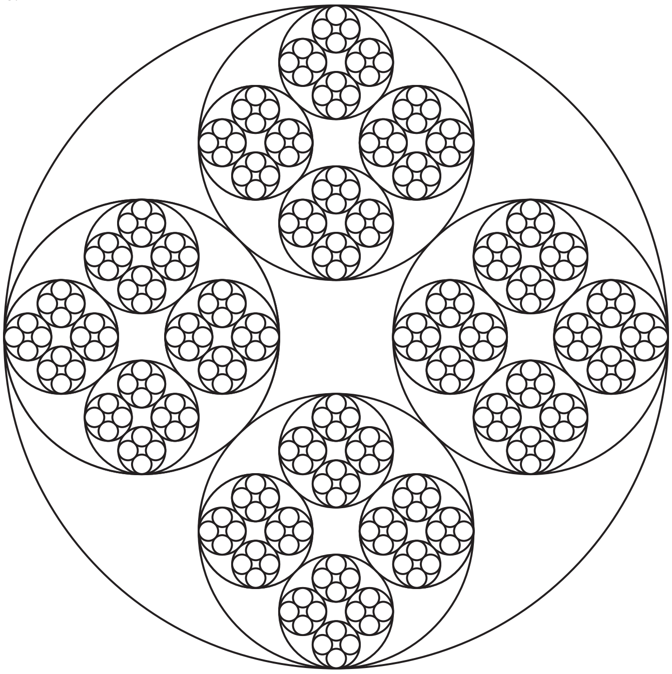

Figure 1

The first subdivision

The first subdivision of the circle Γ: $x^2 + y^2 = 1$ consists of the circles at A, B, and their reflections along the y-axis, x-axis respectively, as shown in Figure 1. All subsequent first subdivisions of circles in the interior of Γ: $x^2 + y^2 = 1$ are scaled down repetitions of the above.

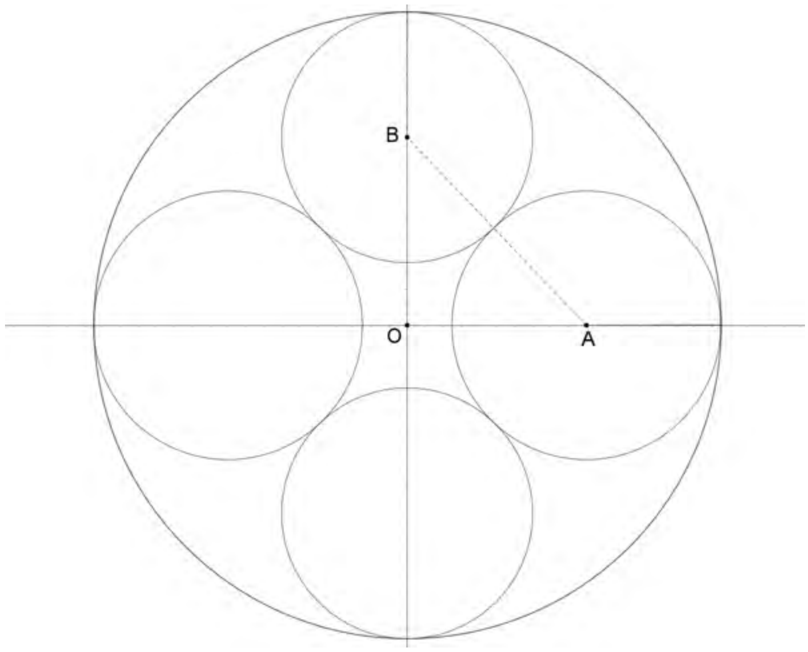

Figure 2. The first subdivision of the circle.

Since the circles at A, B are tangential, we have $OA^2 + OB^2 = AB^2$, hence their radii r, s, are related by

$$(1 - r)^2 + (1 - s)^2 = (r + s)^2$$

simplifying to

$$r + s + rs = 1 \qquad (1)$$

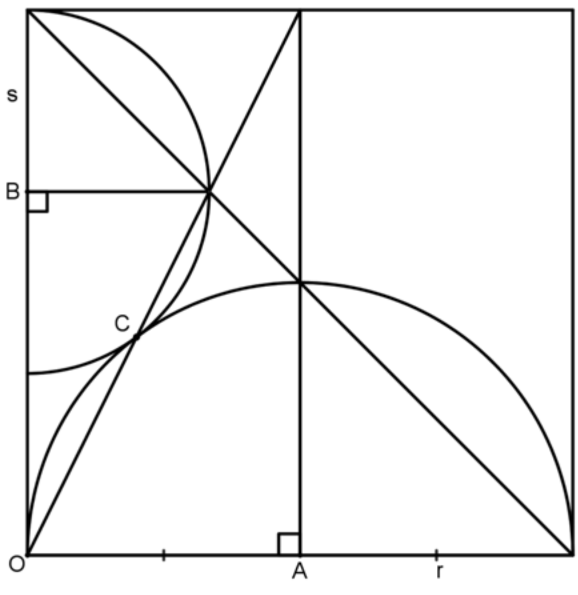

Figure 3. $r = \frac{1}{2}, s = \frac{1}{3}$

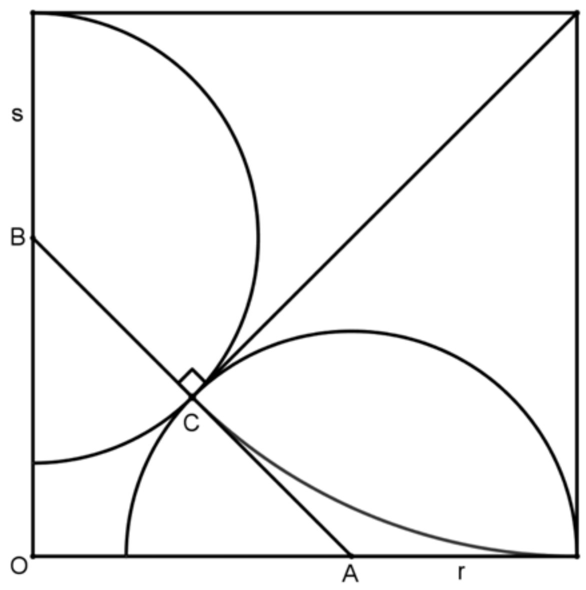

Figure 4. $r = s = \sqrt{2} - 1$

Let $s = \left(\frac{1-r}{1+r}\right) \le r$ then $r^2 - 2r - 1 \ge 0$, hence

$$\sqrt{2} - 1 \le r \le \frac{1}{2} \qquad (2)$$

$$\frac{1}{3} \le s \le \sqrt{2} - 1$$

$$\frac{d}{dr}(r+s) = \frac{d}{dr}\left(r + \frac{1-r}{1+r}\right) = \frac{r^2+2r-1}{(1+r)^2} = 0, \; r = -1 \pm \sqrt{2}$$

$$\frac{d}{dr}\left(r^2+s^2\right) = \frac{d}{dr}\left(r^2 + \left(\frac{1-r}{1+r}\right)^2\right) = \frac{2\left(r^2+r+2\right)\left(r^2+2r-1\right)}{(1+r)^3}, \; r = -1 \pm \sqrt{2}$$

therefore

$$2\left(\sqrt{2}-1\right) \le r + s \le \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \qquad (3)$$

$$2\left(\sqrt{2}-1\right)^2 \le r^2 + s^2 \le \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{13}{36} \qquad (4)$$

where $r = s = \sqrt{2} - 1$ corresponds to four equal circles, and $r = \frac{1}{2}, s = \frac{1}{3}$ to the case where the circle at A passes through the origin.

The combined area of these four circles is $S_1 = 2\pi(r^2 + s^2)$, and the region Ω_1 in their exterior has area $\pi - S_1$ and perimeter $2\pi + 4\pi(r + s)$.

The k-th subdivision

The second subdivision of Γ: $x^2 + y^2 = 1$ involves each of the existing circles being divided as in the manner of the first subdivision. It consists of 4 circles of radius r^2, 8 circles of radius rs, and 4 circles of radius s^2 with a combined area

$$S_2 = 4\pi(r^4 + 2r^2s^2 + s^4) = 4\pi(r_2 + s_2)^2$$

The region Ω_2 in their exterior and the interior of the four circles in the first subdivision has area = $S_1 - S_2$ and perimeter $4\pi(r+s) + 8\pi(r+s)^2$.

Repetitions of this step lead to the k-th subdivision of Γ with $2^k \binom{k}{i}$ circles of radii $r^{k-i}s^i$, $i = 0, 1\ldots k$:

A total of $2^k \sum_{i=0}^{k} \binom{k}{i} = 2^k \times 2^k = 4^k$ circles of combined area

$$S_k = 2^k \pi \sum_{i=0}^{k} \binom{k}{i} \left(r^2\right)^{k-i} \left(s^2\right)^i \tag{5}$$

$$= \pi\left(2\left(r^2 + s^2\right)\right)^k$$

$$= \pi\left(\frac{S_1}{S_0}\right)^k, \ S_0 = \pi$$

and

$$\frac{S_1}{S_0} = 2\left(r^2 + s^2\right) \le \frac{13}{18}$$

$$\therefore \lim_{k\to\infty} S_k = 0$$

Their combined perimeter is

$$C_k = 2^{k+1} \pi \sum_{i=0}^{k} \binom{k}{i} r^{k-i} s^i \tag{6}$$

$$= 2\pi\left(2(r+s)\right)^k$$

$$= 2\pi\left(\frac{C_1}{C_0}\right)^k$$

and

$$\frac{C_1}{C_0} = 2(r+s) \ge 4\left(\sqrt{2} - 1\right)$$

hence

$$\lim_{k\to\infty} C_k = \infty$$

By analogy, the corresponding ratio in the Sierpinski gasket is

$$\frac{\text{perimeter of 3 triangles of side } \tfrac{1}{2}}{\text{perimeter of triangle of side 1}} = \frac{3}{2}$$

The fractal dimension of a self-similar figure is

$$D = -\frac{\log(\text{number of self-similar pieces})}{\log(\text{scale factor})}$$

thus for the Sierpinski triangle it is

$$D_T = \frac{\log 3}{\log 2} \simeq 1.585$$

In the circular gasket, there are two scale factors: r along the x-axis, and s along the y-axis, each applied to two circles.

Its fractal dimension

$$D_C = -\frac{\log 2}{\log r} - \frac{\log 2}{\log s}$$

has boundary values

$$D_C = -\frac{2\log 2}{\log\left(\sqrt{2}-1\right)} \approx 1.573 \ \left(r = s = \sqrt{2}-1\right)$$

$$D_C = 1 + \frac{\log 2}{\log 3} \approx 1.631 \ \left(r = \frac{1}{2}, s = \frac{1}{3}\right)$$

corresponding to $r = s = \sqrt{2}-1$ and $r = \frac{1}{2}$, $s = \frac{1}{3}$ respectively, and $D_C = D_T$ if (r, s) ≈ (0.454, 0.376).

Locating the circles

Locating the 4^k circles in the k-th subdivision of Γ is no easy task: consider the 1024 circles for $k = 5$!

The workload is vastly reduced by allocating these circles to $k + 1$ classes, say $i = 0, 1\ldots k$, according to some criteria as follows:

A circle at $(X_{k-i,\ i}, Y_{k-i,\ i})$ of radius $r^i s^{k-i}$ generates two circles at $(X_{k-i,\ i} \pm (1-r) r^i s^{k-i}, Y_{k-i,\ i})$ of radii $r^{i+1} s^{k-i}$ along the x-direction and two circles at $(X_{k-i,\ i}, Y_{k-i,\ i} \pm (1-s) r^i s^{k-i},)$ of radii $r^i s^{k-i+1}$ along the y-direction.

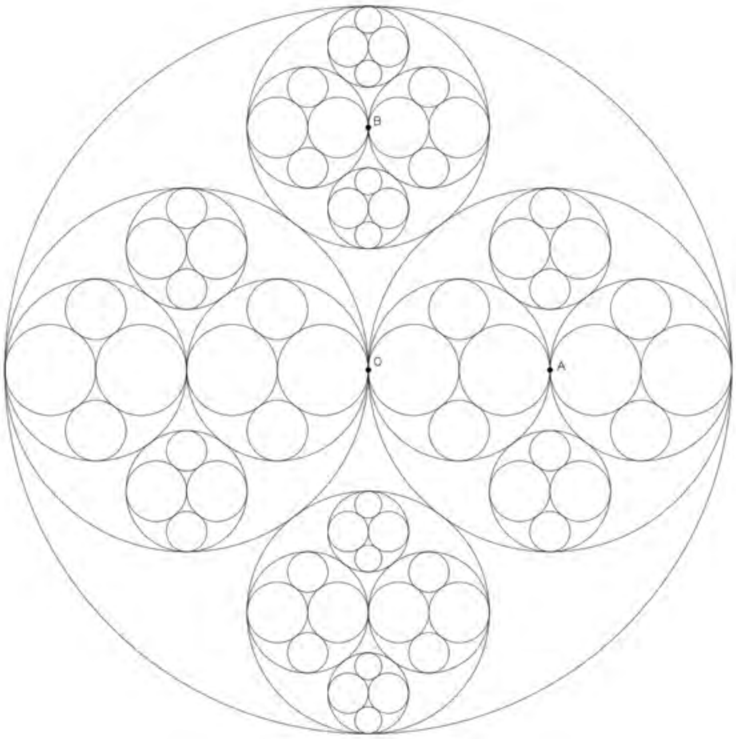

Figure 5. $r = \frac{1}{2}$, $s = \frac{1}{3}$, $k = 1, 2, 3$

For simplicity, let

$$(X_{k-i,\,i},\ Y_{k-i,\,i}) = ((1-r)\,u_{k-i,\,i},\ (1-s)\,v_{k-i,\,i}) \tag{7}$$

where $k-i$, i represents the number of terms in $u_{k-i,\,i}$, $v_{k-i,\,i}$ respectively, equivalently the number of iterations in the x, y-directions respectively.

Similarly to the random walk, where each term represents a constant shift in either the horizontal or vertical direction, starting from $(0, 0)$, each term in $u_{k-i,\,i}$, $v_{k-i,\,i}$ represents a shift by a factor of either r or s of its predecessor, or the radius of a circle in the preceding subdivision. We have:

$$\begin{aligned}
k=1:\quad & (u_{10},v_{10}) = (\pm1,0)\\
& (u_{01},v_{01}) = (0,\pm1)\\
k=2:\quad & (u_{20},v_{20}) = (\pm1\pm r,0)\\
& (u_{02},v_{02}) = (0,\pm1\pm s)\\
& (u_{11},v_{11}) = (\pm1,\pm r),(\pm s,\pm1)\\
k=3:\quad & (u_{30},v_{30}) = (\pm1\pm r\pm r^2,0)\\
& (u_{03},v_{03}) = (0,\pm1\pm s\pm s^2)\\
& (u_{21},v_{21}) = (\pm s\pm rs,\pm1),(\pm1\pm rs,\pm r),(\pm1\pm r,\pm r^2)\\
& (u_{12},v_{12}) = (\pm1,\pm r\pm rs),(\pm s,\pm1\pm rs),(\pm s^2,\pm1\pm s)\\
k=4:\quad & (u_{40},v_{40}) = (\pm1\pm r\pm r^2\pm r^3,0)\\
& (u_{04},v_{04}) = (0,\pm1\pm s\pm s^2\pm s^3)\\
& (u_{31},v_{31}) = (\pm s\pm rs\pm r^2s,\pm1),(\pm1\pm rs\pm r^2s,\pm r),\\
& \qquad\qquad (\pm1\pm r\pm r^2s,\pm r^2),(\pm1\pm r\pm r^2,\pm r^3)\\
& (u_{13},v_{13}) = (\pm1,\pm r\pm rs\pm rs^2),(\pm s,\pm1\pm rs\pm rs^2),\\
& \qquad\qquad (\pm s^2,\pm1\pm s\pm rs^2),(\pm s^3,\pm1\pm s\pm s^2)\\
& (u_{22},v_{22}) = (\pm s\pm rs,\pm1\pm r^2s),(\pm1\pm rs,\pm r\pm r^2s),(\pm1\pm r,\pm r^2\pm r^2s),\\
& \qquad\qquad (\pm1\pm rs^2,\pm r\pm rs),(\pm s\pm rs^2,\pm1\pm rs),(\pm s^2\pm rs^2,\pm1\pm s)
\end{aligned}$$

Thus, given $(u_{k-i,\,i},\ v_{k-i,\,i})$, two new centres at $(X_{k-i+1,\,i},\ Y_{k-i+1,\,i})$ and two centres at $(X_{k-i,\,i+1},\ Y_{k-i,\,i+1})$ are defined recursively by

$$\begin{aligned} u_{k-i+1,\,i} &= u_{k-i,\,i} \pm r^{k-i}s^i \\ v_{k-i+1,\,i} &= v_{k-i,\,i} \end{aligned} \tag{8a}$$

and

$$\begin{aligned} u_{k-i,\,i+1} &= u_{k-i,\,i} \\ v_{k-i,\,i+1} &= v_{k-i,\,i} \pm r^{k-i}s^i \end{aligned} \tag{8b}$$

Equations (8a) and (8b) satisfy the relations

$$u_{k-i,i} = \tilde{v}_{i,k-i} \qquad (9)$$
$$v_{k-i,i} = \tilde{u}_{i,k-i}$$

where the tilde ~ denotes the interchange $(r, s) \to (s, r)$, reducing the number of computations by half.

It is sufficient to apply (8a) with

$$i = 0, 1\ldots\left[\frac{k}{2}\right]$$

and (8b) with

$$i = 0, 1\ldots\left[\frac{k-1}{2}\right]$$

and then (9) with

$$i = \left[\frac{k+1}{2}\right], \left[\frac{k+3}{2}\right]\ldots k$$

where [ ] denotes the greatest integer less than or equal to, in order to generate a complete set of circles in the $(k+1)$-th subdivision.

In particular, if k is odd then half the centres in the $(k+1)$-th subdivision, corresponding to

$$i = k - i + 1 = \frac{k+1}{2}$$

are defined by

$$\left(u_{\frac{k+1}{2},\frac{k+1}{2}}, v_{\frac{k+1}{2},\frac{k+1}{2}}\right)$$

as in (8b) with $i = \dfrac{k-1}{2}$, and the remaining by

$$\left(\tilde{v}_{\frac{k+1}{2},\frac{k+1}{2}}, \tilde{u}_{\frac{k+1}{2},\frac{k+1}{2}}\right)$$

These $\left(\begin{array}{c} k+1 \\ \dfrac{k+1}{2} \end{array}\right) 2^{k+1}$ circles of radii = $(rs)^{\frac{k+1}{2}}$ are in a class of their own as they are reached from $(0, 0)$ by equal numbers of horizontal and vertical shifts.

The case r = s

Let $(c_{i0}, c_{i1}\ldots c_{i,\,k-i-1})$, $(c'_{i0}, c'_{i1}\ldots c'_{i,\,i-1})$ be two mutually exclusive ordered permutations of $k-i$, i elements from the set $\{0, 1\ldots k-1\}$. There are $\binom{k}{i}$ such pairs of sets, and each of these define $2^{k-i} \times 2^i = 2k$ centres in the k-th subdivision of Γ: $x^2 + y^2 = 1$ by

$$u_{k-i,i} = \sum_{j=0}^{k-i-1} \varepsilon_{ij} r^{c_{ij}}, \ \varepsilon_{ij} = \pm 1, \ r = \sqrt{2} - 1 \qquad (10)$$

$$v_{k-i,i} = \sum_{j=0}^{i-1} \varepsilon'_{ij} r^{c_{ij}'}, \ \varepsilon'_{ij} = \pm 1$$

The c_{ij} (or c'_{ij}) are solutions of the partition equations

$$\sum_{j=0}^{k-i-1} c_{ij} = \sum_{j=0}^{k-i-1} j + n = \frac{(k-i)(k-i-1)}{2} + n, \ n = 0, 1 \ldots i(k-i) \qquad (11)$$

$$0 \le c_{i0} < c_{i1} \ldots < c_{i,k-i-1} \le k-1$$

Equivalently

$$\sum_{j=0}^{i-1} c'_{ij} = \sum_{j=k-i}^{k-1} j - n = \frac{i(2k-i-1)}{2} - n \qquad (12)$$

$$0 \le c'_{i0} < c'_{i1} \ldots < c'_{i,i-1} \le k-1$$

There are $\binom{k}{i}$ solutions as n runs through 0, 1… $i(k-i)$.
Also, since the c_{ij}, c'_{ij} are distinct, any power of r is in either $u_{k-i,\,\mathrm{i}}$ or $v_{k-i,\,i}$, but not both, hence

$$X_{k-i,i} + Y_{k-i,i} = (1-r)\sum_{j=0}^{k-1} \varepsilon_j r^j, \ \varepsilon_j = \pm 1, \ \forall i$$

$$\therefore \left| X_{k-i,i} + Y_{k-i,i} \right| \ge (1-r)\left(1 - \sum_{j=1}^{k-1} r^j\right) = 1 - 2r + r^k \qquad (13)$$

$$\left| X_{k-i,i} + Y_{k-i,i} \right| \le (1-r)\sum_{j=0}^{k-1} r^j = 1 - r^k \qquad (14)$$

where $1 - r^k$ is the distance covered from (0, 0) to any centre $(X_{k-i,\,i}, Y_{k-i,\,i})$ by a combined total of k horizontal and vertical shifts.

The lines $y = \left(\pm\sqrt{2} \pm 1\right)x$

These lines divide the circle Γ: $x^2 + y^2 = 1$ into eight equal parts. In order that a centre be on either of the lines $y = \pm rx$, $y = \pm \frac{x}{r}$, $r = \sqrt{2} - 1$ it is necessary and sufficient that $(u_{k-i,\,i}, v_{k-i,\,i})$ have the same number of terms, i.e., $k - i = i = \frac{k}{2}$ for even k, and that

$$\left(c_{\frac{k}{2},j}, c'_{\frac{k}{2},j}\right) = \left(2j, 2j+1\right), \ j = 0, 1 \ldots \frac{k}{2} - 1$$

by substituting (10) in $y = \pm rx$, or

$$\left(c_{\frac{k}{2},j}, c'_{\frac{k}{2},j}\right) = \left(2j+1, 2j\right)$$

by substituting (10) in $y = \pm \frac{x}{r}$.

Hence the centres

$$\left(X_{\frac{k}{2},\frac{k}{2}}, Y_{\frac{k}{2},\frac{k}{2}}\right) = \left(1-r, \pm r(1-r)\right)\sum_{j=0}^{\frac{k}{2}-1} \varepsilon_j r^{2j},\ \varepsilon_j = \pm 1$$

are on the lines $y = \pm rx$ respectively, and the centres

$$\left(X_{\frac{k}{2},\frac{k}{2}}, Y_{\frac{k}{2},\frac{k}{2}}\right) = \left(r(1-r), \pm(1-r)\right)\sum_{j=0}^{\frac{k}{2}-1} \varepsilon_j r^{2j}$$

are on the lines $y = \pm \frac{x}{r}$ respectively. There are $2^{\frac{k}{2}}$ circles of radii = r^k on each of these four lines, adding to the visual appeal of the gasket of circles.

Many questions can be asked such as:

- Is a line passing through a circular gasket comprising all subdivisions of Γ, up to the k-th, tangential to at most 2^k circles in the gasket?
- How many colours are required to colour in the gasket in Figure 4 so as no two adjacent sub-regions have the same colour?